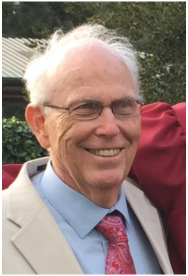
THE SPECIAL LAGRANGIAN POTENTIAL EQUATION

with Reese Harvey



SOME HISTORY

Calibrations

A **Calibration** on a manifold X is a smooth k-form $\varphi \in \mathcal{E}^k(X)$ with $d\varphi = 0$ such that

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for all oriented tangent k-planes P on X.

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An oriented k-dimensional submanifold $M \subset X$ is a **φ-submanifold** if

$$T_x M \in \mathbb{G}(\varphi) \qquad \forall\, x \in M.$$

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This Proposition extends from submanifolds to integral currents.

Kähler Manifolds

Herb Federer: (using Wirtinger's inequaity)

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The $\frac{1}{k!}\omega^k$-submanifolds are **complex submanifolds**.

The (locally closed) $\frac{1}{k!}\omega^k$-currents are **positive holomorphic chains**.

Jim King

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I will work in $\mathbb{C}^n$,

but we knew that everything carried over to **Calabi-Yau** manifolds.

The Special Lagrangian Calibration

$$\varphi \equiv \mathrm{Re}\{\mathbf{dz_1} \wedge \cdots \wedge \mathbf{dz_n}\} \qquad \text{in} \ \mathbb{C}^n$$

This is a sum of 2^{n-1} orthogonal simple vectors.

Nevertheless, for a **unit simple** vector $\xi = e_1 \wedge \cdots \wedge e_n$

$$|\{dz_1 \wedge \cdots \wedge dz_n\}(\xi)| \leq 1$$

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and $|\{dz_1 \wedge \cdots \wedge dz_n\}(\xi)| = 1 \iff \xi$ is **Lagrangian**.

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Thus for a **unit simple** vector $\xi = e_1 \wedge \cdots \wedge e_n$

$$\mathrm{Re}\{dz_1 \wedge \cdots \wedge dz_n\}(\xi) \;=\; \pm 1 \qquad \Longleftrightarrow$$

$$\xi \ \text{ is Lagrangian} \quad \text{and} \quad \mathrm{Im}\{\mathbf{dz_1} \wedge \cdots \wedge \mathbf{dz_n}\}(\xi) \;=\; \mathbf{0}$$

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Otherwise said, for an oriented real n-plane P

$$\mathrm{Re}\{dz_1 \wedge \cdots \wedge dz_n\}\big|_P \;=\; \pm d\mathrm{vol}_P \qquad \Longleftrightarrow$$

$$\mathbf{P} \text{ is Lagrangian} \quad \text{and} \quad \mathrm{Im}\{\mathbf{dz_1} \wedge \cdots \wedge \mathbf{dz_n}\}\big|_{\mathbf{P}} \;=\; \mathbf{0}$$

Lagrangian Graphs

Let $z = x + iy$, and consider a submanifold of the form

$$M^n \equiv \{(x, y) : y = F(x), x \in \Omega\}.$$

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The tangent space to $M^n = \mathrm{graph}(F)$ at x is just the graph of $(DF)_x$.

W.r.t. coordinates $(x_1, ..., x_n, y_1, ..., y_n)$ the matrix is just

$$\left(\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)\right)$$

Lagrangian Graphs

This matrix is symmetric, so under a change of variables (gx, gy) we get

$$\left(\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)\right) = \begin{pmatrix} \lambda_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \lambda_n \end{pmatrix}$$

The graph of this in $\mathbb{C}^n = \mathbb{R}^n \oplus J\,\mathbb{R}^n$ is spanned by

$$e_1 + \lambda_1 J e_1, \; \ldots, e_n + \lambda_n J e_n$$

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$$\xi \ = \ (e_1 + \lambda_1 Je_1) \wedge \ \cdots \ \wedge (e_n + \lambda_n Je_n).$$

The SL Potential Equation

$$\xi \;=\; (e_1 + \lambda_1 J e_1) \wedge \;\cdots\; \wedge (e_n + \lambda_n J e_n).$$

$$\{\mathbf{dz_1} \wedge \cdots \wedge \mathbf{dz_n}\}(\xi) \;=\; \prod_{\mathbf{k=1}}^{\mathbf{n}} (\mathbf{1} + \mathbf{i}\,\lambda_{\mathbf{k}})$$

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Example $(n = 3)$ $\quad \mathrm{tr}(D^2\, u) = det(D^2\, u) \quad$ i.e. $\quad \Delta u = MA(u).$

Circular Symmetry

Note that there is an S^1**-symmetry**

If we set

$$\varphi \equiv \mathrm{Re}\{e^{-i\theta} dz_1 \wedge \cdots \wedge dz_n\},$$

Then we get the equation

$$\mathrm{Im}\big[\{e^{-i\theta} dz_1 \wedge \cdots \wedge dz_n\}(\xi)\big] \;=\; 0, \quad \text{i.e.,}$$

$$\mathrm{Im}\left\{\mathbf{e}^{-\mathbf{i}\theta} \prod_{\mathbf{k}=1}^{\mathbf{n}} (\mathbf{1} + \mathbf{i}\,\lambda_{\mathbf{k}})\right\} \;=\; \mathbf{0}$$

A Different Way to Write the Equation

Caffarelli, Nirenberg and Spruck

$$\mathrm{Im}\left\{ e^{-i\theta} \prod_{k=1}^{n} (1 + i\,\lambda_k) \right\} = 0$$

Consider

$$1 + i\lambda_k \in \mathbb{C}.$$

Then

$$\lambda_k = \tan\theta_k \qquad \text{where} \qquad -\frac{\pi}{2} < \theta_k < \frac{\pi}{2}$$

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$$\Theta \;=\; \sum_{k=1}^{n} \arg(1 + i\lambda_k) \;=\; \sum_{k=1}^{n} \theta_k \;=\; \sum_{k=1}^{n} \arctan(\lambda_k).$$

The Special Lagrangian Potential Operator

$$\mathrm{Im}\left\{e^{-i\theta}\prod_{k=1}^{n}(1+i\,\lambda_k)\right\} \;=\; \mathrm{Im}\left\{e^{-i\theta}e^{R+i\Theta}\right\} \;=\; e^{R}\mathrm{Im}\left\{e^{-i\theta}e^{i\Theta}\right\} \;=\; 0$$

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$SL(A)$ **takes values in** $(-n\frac{\pi}{2}, n\frac{\pi}{2})$.

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There exist many explicit and many families of solutions to these equations

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A Lagrangian of mean curvature 0 is Special Lagrangian

The Dirichlet Problem

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$$\mathrm{tr}\left\{\arctan\left(D^2 u\right)\right\} \;=\; \theta$$

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What about solutions for other phases?

Viscosity Solutions

Consider the subequation

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Def. Fix a domain $\Omega \subset \mathbb{R}^n$ and $u \in USC(\Omega)$. By a **test function** for u at $x \in \Omega$ we mean a C^2-function ϕ in a neighborhood of x with

$$u \leq \phi \;\; \text{near } x \qquad \text{and} \qquad u(x) \;=\; \phi(x).$$

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Def. Fix a domain $\Omega \subset \mathbb{R}^n$ and $u \in USC(\Omega)$. Then u is **$\mathbf{F}_\theta$-subharmonic** if for every test function ϕ for u at any point $x \in \Omega$, we have

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Def. $u \in C(\Omega)$ is **$\mathbf{F}_\theta$-harmonic**, i.e., a **viscosity solution of our equation**, if

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Work of Wang and Yuan

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Graphing the gradients gives infinitely many Special Lagrangian Varieties each with an isolated singularity

Continuous Solutions to the (DP)

Definitions hold for any non-empty closed $\mathbf{F} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ satisfying

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one has that

$$u + v \ \leq \ 0 \quad \text{on} \ \ \partial\Omega \qquad \Rightarrow \qquad u + v \ \leq \ 0 \quad \text{on} \ \ \Omega.$$

Theorem. (Harvey-L. 2009) Existence for the (DP). *Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with smooth boundary $\partial\Omega$. If at each point $x \in \partial\Omega$ the boundary is both*

strictly $\mathbf{F}$-convex and strictly $\widetilde{\mathbf{F}}$-convex

then Perron existence holds for the (DP).

Strict $\mathbf{F}$-Convexity of $\partial\Omega$

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Def. The **asymptotic interior** of **F** is

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Compute $\mathrm{Int}\,\overrightarrow{\mathbf{F}}_\theta$

$$-n\frac{\pi}{2} \quad -(n-2)\frac{\pi}{2} \qquad\qquad 0 \qquad\qquad (n-2)\frac{\pi}{2} \quad n\frac{\pi}{2}$$

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This gives (probably) the best geometric condition on $\partial\Omega$ for existence to the (DP) for

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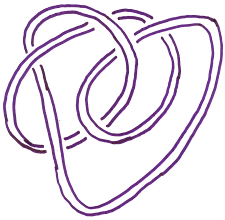

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Example: Results on Riemannian manifolds.

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$$\mathrm{tr}\left\{\arctan(D_x^2 u)\right\} \;=\; \psi(x) \qquad (*)$$

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Note 2. S. Dinew, H. Do and T. D. Tô have proved the continuous viscosity version of this result.

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Note 2. The SLP operator is tamable on $\mathbf{F}_\theta$ for $\theta > (n-2)\frac{\pi}{2}$.

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for some k with $1 \leq k \leq n$.

Suppose $\partial\Omega$ is strictly $\min\{k, n-k+1\}$-convex.

Then there exists a unique solution $u \in C(\overline{\Omega})$

to the inhomogeneous Dirichlet problem

for all continuous boundary values $\varphi \in C(\partial\Omega)$.

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This phase function satisfies the equation

$$\nabla\psi = -\mathbf{JH} \qquad \text{on} \;\; \mathbf{L}$$

where H is the mean curvature vector of L.

Some Nice Results

Theorem. (Simon Brendle and Micah Warren). *Let $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$ be two domains with smooth strictly convex boundaries (2nd Fund Forms > 0). Then there exists a diffeomorphism*

$$F : \Omega \longrightarrow \widetilde{\Omega}$$

whose graph is Special Lagrangian.

A Bernstein Theorem

Theorem. (Yu Yuan also Jost-Xin and for $n = 2$ Lei Fu).

Let u be a smooth solution, over all of $\mathbb{R}^n$, to the equation

$$\mathrm{tr}\left\{\arctan(D^2 u)\right\} \ = \ \theta \qquad \text{with} \ \ |\theta| > (n-2)\frac{\pi}{2}$$

(in the critical interval). Then u is a quadratic polynomial.

Moment Conditions

Theorem. (Lei Fu).

Boundaries of SL submanifolds are **not** *characterized by a moment condition.*

The Degenerate Special Lagrangian Equation

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Much work has been done by Solomon, Yanir Rubinstein, Tamás Darvas, and Matt Dellatorre.

The SL Potential Equation and Mirror Symmetry

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Now an analogue of the SL potential equation plays a big role.

There are very good articles by T. C. Collins, A. Jacob, N. C. Leung, D. Xie and Y. Shi.

A Small Insight

(X, ω) an n-dimensional Kähler manfold

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This gives rise to a hermitian Yang-Mills equation

$$\Theta_\omega(\alpha) = \sum_k \arctan(\lambda_k) \;\equiv\; \theta \pmod{2\pi}$$

where the λ_k's are eigenvalues of an endomorphism $K : T^{1,0}X \to T^{1,0}X$ given by contracting by α and the dual of ω. Of course the elements in a all differ from a given one α_0 by $dd^c u$ for a function u on X.