REMARKS ON THE ALEXANDER-WERMER THEOREM FOR CURVES

F. Reese Harvey and H. Blaine Lawson, Jr.*

Abstract

We give a new proof of the Alexander-Wermer Theorem that characterizes the oriented curves in $\mathbf{C}^n$ which bound positive holomorphic chains, in terms of the linking numbers of the curve with algebraic cycles in the complement. In fact we establish a slightly stronger version which applies to a wider class of boundary 1-cycles. Arguments here are based on the Hahn-Banach Theorem and some geometric measure theory. Several ingredients in the original proof have been eliminated.

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*Partially supported by the N.S.F.

1. The Alexander-Wermer Theorem

We present a different proof of the Alexander-Wermer Theorem [AW], [W_2] for curves which uses the Hahn-Banach Theorem and techniques of geometric measure theory. Several ingredients of the original proof are eliminated, such as the reliance on the result in [HL_1] that if a curve satisfies the moment condition, then it bounds a holomorphic 1-chain. The arguments given here have been adapted to study the analogous problem in general projective manifolds (cf. [$HL_{3,4,5}$]).

Our arguments will also apply to a more general class of curves which we now introduce.

Definition 1.1. Let X be a complex manifold and suppose there exists a closed subset $\Sigma(\Gamma)$ of Hausdorff 1-measure zero and an oriented, properly embedded C^1-submanifold of $X - \Sigma(\Gamma)$ with connected components $\Gamma_1, \Gamma_2, ...$ If, for given integers $n_1, n_2, ...$, the sum $\Gamma = \sum_{k=1}^{\infty} n_k \Gamma_k$ defines a current of locally finite mass in X which is d-closed (i.e., without boundary), and if sptΓ has only a finite number of connected components[1] , then Γ will be called a **scarred 1-cycle (of class C^1) in X**. By a unique choice of orientation on each Γ_k we may assume each $n_k > 0$.

Example 1.2. Any real analytic 1-cycle is automatically a scarred 1-cycle (of class C^∞) – see [F, p. 433].

Definition 1.3. By a **positive holomorphic 1-chain with boundary** Γ we mean a sum $V = \sum_{k=1}^{\infty} m_k [V_k]$ with $m_k \in \mathbf{Z}^+$ and V_k an irreducible 1-dimensional complex analytic subvariety of $X - \text{spt}\Gamma$ such that V has locally finite mass in X and, as currents in X,

$$dV = \Gamma$$

Remark 1.4. Standard projection techniques (cf. [Sh], [H]) show that any 1-dimensional complex subvariety W of $X - \text{spt}\Gamma$ automatically has locally finite 2-measure at points of Γ, and furthermore, its current boundary is of the form $dW = \sum \epsilon_k \Gamma_k$ where $\epsilon_k = -1, 0$ or 1 for all k. See [H] and the "added in proof" for the more general case where T is a positive d-closed current on $\mathbf{C}^2 - \text{spt}\Gamma$.

Definition 1.5. A scarred 1-cycle Γ in $\mathbf{C}^n$ satisfies the **(positive) winding condition** if

$$\frac{1}{2\pi i} \int_\Gamma \frac{dP}{P} \; > \; 0$$

for all polynomials $P \in \mathbf{C}[z]$ with $P \neq 0$ on Γ.

There are many equivalent formulations of this condition. We mention three.

Proposition 1.6. *Γ satisfies the (positive) winding condition if and only if any of the following equivalent conditions holds:*

[1] More generally we need only assume that sptΓ is contained in a compact connected set of finite linear measure

1) $\int_\Gamma d^c\varphi \ \geq \ 0$ *for all smooth plurisubharmonic functions* φ *on* $\mathbf{C}^n$.

2) For each polynomial $P \in \mathbf{C}[z]$, *the unique compactly supported solution* $W_P(\Gamma)$ *to the equation* $dW_P(\Gamma) = P_*(\Gamma)$ *satisfies* $W_P(\Gamma) \geq 0$.

3) The linking number $\mathrm{Link}(\Gamma, Z) \geq 0$ *for all algebraic hypersurfaces* Z *contained in* $\mathbf{C}^n - \mathrm{spt}\Gamma$.

PROPOSITION 1.7. *If* Γ *is the boundary of a positive holomorphic 1-chain* V *in* $\mathbf{C}^n$, *then* Γ *satisfies the positive winding condition.*

Proof. We have $\int_\Gamma d^c\varphi = \int_{dV} d^c\varphi = \int_V dd^c\varphi \geq 0$ since $dd^c\varphi \geq 0$. ∎

The following converse of Proposition 1.7 is due to Alexander and Wermer [AW], [W_2].

MAIN THEOREM 1.8. *Let* Γ *be a scarred 1-cycle in* $\mathbf{C}^n$. *If* Γ *satisfies the (positive) winding condition, then* Γ *bounds a positive holomorphic 1-chain in* $\mathbf{C}^n$.

This slightly generalizes the theorem in [AW] which applies only to smooth oriented curves. However, the essential point of this paper is to provide a conceptually different proof of the result which has other applications. This proof has two distinct parts which constitute the following two sections.

NOTE . We adopt the following notation throughout the paper. The polynomial hull of a compact subset $K \subset \mathbf{C}^n$ is denoted by $\widehat{K}$. The mass of a current T with compact support in $\mathbf{C}^n$ is denoted by $M(T)$.

2. A Dual Interpretation

In this section we shall use the Hahn-Banach Theorem to establish a dual interpretation of the positive winding condition. The main result is the following. Recall that if C is a convex cone in a topological vector space V, its **polar** is the set $C^0 = \{v' \in V' : v'(v) \geq 0$ for all $v \in C\}$.

THEOREM 2.1. (The Duality Theorem). *The cone A in the space $\mathcal{E}^1_{\mathbf{R}}(\mathbf{C}^n)$ of smooth 1-forms on $\mathbf{C}^n$, defined by*

$$A \;\equiv\; \{d\psi + d^c\varphi : \psi \in C^\infty(\mathbf{C}^n) \text{ and } \varphi \in \mathcal{PSH}(\mathbf{C}^n)\}$$

and the cone B in the dual space $\mathcal{E}'_1(\mathbf{C}^n)_{\mathbf{R}}$ of compactly supported one-dimensional currents in $\mathbf{C}^n$, defined by

$$B \;\equiv\; \{S : S = d(T+R) \text{ with } T \geq 0 \text{ and } R \text{ of bidimension } 2,0+0,2\}$$

are each the polar of the other.

Moreover,

(i) The cone B coincides with the cone

$$B' \;\equiv\; \{S : dS = 0 \text{ and } \exists T \geq 0 \text{ with compact support and } dd^cT = -d^cS\}, \quad \text{and}$$

(ii) If $S = d(T+R) \in B$ with T and R as above, then

$$\mathrm{sptT} \;\subseteq \widehat{\mathrm{sptS}} \quad (\textit{the polynomial hull of } \mathrm{sptS}).$$

This result can be restated as follows.

THEOREM 2.1′. *A real 1-dimensional current S with $dS = 0$ and compact support in $\mathbf{C}^n$ satisfies the (positive) winding condition if and only if*

$$S \;=\; d(T+R) \tag{2.1}$$

where T is a positive 1,1 current and R has bidimension 2,0 + 0,2, or equivalently,

$$dd^cT \;=\; -d^cS \tag{2.2}$$

for some compactly supported $T \geq 0$. Moreover, for each such T,

$$\mathrm{sptT} \;\subseteq \widehat{\mathrm{sptS}} \tag{2.3}$$

Proof. We will show that $B^0 \;=\; A$ and that B is closed. This is enough to conclude that A and B are each the polar of the other because of the bipolar theorem: $(C^0)^0 = \overline{C}$.

Proof that $A = B^0$**.** The inclusion $A \subseteq B^0$ is essentially a restatement of Proposition 1.7 – the same proof applies. We need only show $B^0 \subseteq A$. Suppose $\alpha \in B^0$, i.e., $S(\alpha) \geq 0$ for all $S \in B$. Restricting to S of the form $S = dR$ where $R = R^{2,0} + R^{0,2}$ is of bidimension 2,0+0,2, we see that $S(\alpha) = dR(\alpha)$ must vanish (since $-dR$ is also in B). Hence, $\partial\alpha^{1,0} = 0$ and $\overline{\partial}\alpha^{0,1} = 0$. That is, $d\alpha = d^{1,1}\alpha$. In particular, $d^{1,1}\alpha$ is d-closed. Therefore, on $\mathbf{C}^n$ the equation $d\alpha = d^{1,1}\alpha = dd^c\varphi$ can be solved for some $\varphi \in C^\infty(\mathbf{C}^n)$.

Taking $S = dT$ where $T = \delta_p\xi \geq 0$ for $p \in \mathbf{C}^n$, yields $(d\alpha)(\delta_p\xi) = (d^{1,1}\alpha)(\delta_p\xi) \geq 0$. Hence, $dd^c\varphi = d^{1,1}\alpha \geq 0$, i.e., $\varphi \in \mathcal{PSH}(\mathbf{C}^n)$. Since $\alpha - d^c\varphi$ is d-closed, there exists $\psi \in C^\infty(\mathbf{C}^n)$ with $\alpha = d\psi + d^c\varphi$. ∎

To show that B is closed requires several preliminary results.

Proof of (i). If $S \in B$, then dd^cR is of bidegree $(n-1, n+1) + (n+1, n-1)$, and hence it must vanish. Therefore, $dd^cT = -d^cS$, i.e., $S \in B'$. Conversely, if $S \in B'$, then $S - dT$ is d^c-closed and of course also d-closed. Note that for $T \geq 0$ and R real and of bidimension $(2,0) + (0,2)$, the equations

$$d(T + R) \ = \ S \tag{2.4}$$

and

$$\partial T + \overline{\partial} R^{n,n-2} \ = \ S^{n,n-1} \tag{2.5}$$

are equivalent. Now the right hand side of the equation $\overline{\partial}R^{n,n-2} = S^{n,n-1} - \partial T$ is $\overline{\partial}$-closed. On $\mathbf{C}^n$, this implies that there exists a solution R with compact support. ∎

Proof of (ii). Since $T \geq 0$, we know from [DS] that $\mathrm{sptT} \subseteq \mathrm{spt}\,\widehat{\mathrm{dd^c}}\mathrm{T}$. Of course $\mathrm{spt\,dd^cT} = \mathrm{spt\,d^cS} \subseteq \mathrm{sptS}$. ∎

Lemma 2.2. *If* $S = d(T + R) \in B$*, then the mass* $M(T) = T(dd^c|z|^2) = S(d^c|z|^2)$*.*

Proof. Note that $T(dd^c|z|^2) = (T + R)(dd^c|z|^2) = (d(T + R), d^c|z|^2) = S(d^c|z|^2)$. ∎

Proposition 2.3. *The cone* B *is closed.*

Proof. Suppose $S_j = d(T_j + R_j) \in B$ and $S_j \to S$. Then by Lemma 2.2, $M(T_j) = S_j(d^c|z|^2) \to S(d^c|z|^2)$, and so the masses $M(T_j)$ are uniformly bounded in j. The convergence of $\{S_j\}$ means that all $\mathrm{sptS_j} \subset \mathrm{B(0,R)}$ for some R. Hence by part (ii) we have $\mathrm{sptT_j} \subset \mathrm{B(0,R)}$ for all j. By the basic compactness property of positive currents, there is a subsequence with $T_j \to T \geq 0$. Finally, since $dd^cT_j = -d^cS_j$ we have $dd^cT = -d^cS$. Hence, $S \in B' = B$. ∎

The Remainder of the Proof of the Main Theorem

Suppose now that Γ is a scarred 1-cycle in $\mathbf{C}^n$ which satisfies the positive winding condition. Applying Theorem 2.1 (in its second, "restated" form) with $S = \Gamma$, there exists a compactly supported, positive (1,1)-current T such

$$d(T + R) \ = \ \Gamma \tag{3.1}$$

where R is a current of compact support and bidimension (2,0)+(0,2). We shall show that $R = 0$ and T is a positive holomorphic chain. To proceed we utilize a fundamental result of Wermer [W_1] in a generalized form due to Alexander [A].

THEOREM 3.1. *Let Γ be a scarred 1-cycle of class C^1 in $\mathbf{C}^n$. Then $\widehat{\mathrm{spt}\Gamma} - \mathrm{spt}\Gamma$ is a 1-dimensional complex analytic subvariety of $\mathbf{C}^n - \mathrm{spt}\Gamma$.*

Proof. Alexander proves in [A] that if $K \subset \mathbf{C}^n$ is contained in a compact connected set of finite linear measure, then $\widehat{K} - K$ is a 1-dimensional complex analytic subvariety of $\mathbf{C}^n - K$. The set $\mathrm{spt}\Gamma$ has finite linear measure and only finitely many connected components. One sees from the definition that it is possible to make a connected set $K = \mathrm{spt}\Gamma \cup \tau$ of finite linear measure by adding a finite union of piecewise linear arcs τ contained in the complement of $\mathrm{spt}\Gamma$. Each irreducible component W of the complex analytic curve $\widehat{K} - K$ will have locally finite 2-measure at points of τ and will extend to $\mathbf{C}^n - \mathrm{spt}\Gamma$ as a variety with boundary of the form $\sum_k c_k\tau_k$, where the c_k's are constants and τ_k are the connected arcs comprising τ (cf. [HL_1], [H]). Suppose this boundary is non-zero. Then W must be contained in the union of the complex lines determined by the real line segments comprising $\partial W \cap \tau$. Since W is irreducible, it is contained in just one such complex line. Constructing τ so that each connected component of τ has at least two (complex independent) line segments, we have a contradiction. Thus, for generic choice of τ, the set $\widehat{K} - \mathrm{spt}\Gamma$ is a 1-dimensional subvariety of $\mathbf{C}^n - \mathrm{spt}\Gamma$. In particular, this proves that $\widehat{K} \subseteq \widehat{\mathrm{spt}\Gamma}$. Since $\widehat{\mathrm{spt}\Gamma} \subseteq \widehat{K}$, we are done. ∎

Let $V_1, V_2, ...$ denote the irreducible components of the complex curve given by Theorem 3.1. We are going to prove that $T = \sum_j n_j V_j$ for positive integers n_j. For this we first utilize a result from [HL_2, p. 182].

LEMMA 3.2. *Suppose T is a positive current of bidimension 1,1 with $dd^cT = 0$ on a complex manifold X. If T is supported in a complex analytic curve W in X, then T can be written as a sum $T = \sum_j h_j W_j$ where each W_j is an irreducible component of W and h_j is a non-negative harmonic function on W_j.*

The case needed here is the following.

COROLLARY 3.3. *If $T \geq 0$ satisfies $dd^cT = -d^c\Gamma$ on $\mathbf{C}^n$, then on $\mathbf{C}^n - \mathrm{spt}\Gamma$ one has $T = \sum_j h_j V_j$ with h_j harmonic on V_j.*

We first restrict attention to dimension $n = 2$, where the equation (2.5), namely

$$\overline{\partial} R^{2,0} \ = \ \Gamma^{2,1} - \partial T$$

implies that $R^{2,0}$ is a holomorphic 2-form outside the support of $\Gamma - dT$.

LEMMA 3.4. (n=2). *If* $d(T+R) = \Gamma$ *with* $T \geq 0$ *and* R *of bidimension* $(2,0)+(0,2)$, *then*

$$\mathrm{sptR} \ \subseteq \ \widehat{\mathrm{spt}\Gamma}$$

Proof. By Theorem 2.1(ii), $R^{2,0}$ is a holomorphic 2-form on $\mathbf{C}^2 - \widehat{\mathrm{spt}\Gamma}$, and $R^{2,0}$ vanishes outside of a compact subset of $\mathbf{C}^2$. The polynomially convex set $\widehat{\mathrm{spt}\Gamma}$ cannot have a bounded component in its complement. Therefore, $R^{2,0}$ must vanish on all of $\mathbf{C}^2 - \widehat{\mathrm{spt}\Gamma}$.
■

LEMMA 3.5. (n=2). *Each* $h_j \equiv c_j$ *is constant, and the current* $T = \sum_j c_j V_j$ *is d-closed on* $\mathbf{C}^2 - \mathrm{spt}\Gamma$.

Proof. Pick a regular point of one of the components V_j, let π denote a holomorphic projection (locally near the point) onto V_j, and let i denote the inclusion of V_j into $\mathbf{C}^2$. Note that T is locally supported in V_j by Theorem 2.1(ii) while R is locally supported in V_j by Lemma 3.4. Therefore, both of the push-forwards $\pi_* T$ and $\pi_* R$ are well defined. Now $\pi_* R$, being of bidimension $(2,0)+(0,2)$ in V_j must vanish. However, $T = h_j V_j$ satisfies $\pi_* T = h_j$. Since $d(T+R) = 0$, the push-forward $\pi_* d(T+R) = d\pi_*(T+R) = dh_j$ must also vanish, i.e., each $h_j = c_j$ is constant. This proves:

COROLLARY 3.6. (n=2). *The current* $T = \sum_j c_j V_j$ *on* $\mathbf{C}^2 - \mathrm{spt}\Gamma$ *has locally finite mass across* $\mathrm{spt}\Gamma$ *and its extension* T^0 *by zero across* $\mathrm{spt}\Gamma$ *satisfies*

$$dT^0 \ = \ \sum_j r_j \Gamma_j \qquad \text{on} \ \ \mathbf{C}^2$$

for real constants r_j.

Proof. See Remark 1.4. ■

Another corollary of Lemma 3.5 is the following.

COROLLARY 3.7. $\qquad \mathrm{sptR} \ \subseteq \ \mathrm{spt}\Gamma$

Proof. By (2.5) the current $R^{2,0}$ is a holomorphic 2-form on $\mathbf{C}^2 - \mathrm{spt}\Gamma$ since $dT = 0$ there. Since $R^{2,0}$ vanishes at infinity, this proves the result. ■

Completion of the case n=2. Now

$$T + R \ = \ T^0 + \chi T + R \tag{3.2}$$

where χ is the characteristic function of $\mathrm{spt}\Gamma$ and $\chi T + R$ has support in $\mathrm{spt}\Gamma$. We also have

$$d(\chi T + R) \ = \ \sum_j (n_j - r_j)\Gamma_j \qquad \text{on} \ \ \mathbf{C}^2 \tag{3.3}$$

Let ρ denote a local projection onto a regular point of Γ_j. Then $\rho_*(\chi T + R)$ is a well defined current on Γ_j, but of dimension 2. Hence it must vanish. Since ρ_* commutes with

d, this proves that $(n_j - r_j)\Gamma_j$ must vanish. Hence, $r_j = n_j$ for all j, and so $d(\chi T + R) = 0$ and $dT^0 = d(T + R)$ by equations (3.2) and (3.3). This proves that $dT^0 = \Gamma$ by (3.1) .

Proof for the case $n \geq 3$. The general case follow easily from the case where $n = 2$. Consider a generic linear projection $\pi : \mathbf{C}^n \to \mathbf{C}^2$ so that each mapping $V_j \to \pi V_j$ is one-to-one. Then the current $T = \sum_j h_j V_j$ in $\mathbf{C}^n - \mathrm{spt}\Gamma$ projects to the current $\pi_* T = \sum_j \widetilde{h}_j \pi(V_j)$ in $\mathbf{C}^2 - \pi(\mathrm{spt}\Gamma)$ where $\widetilde{h}_j \circ \pi = h_j$. Since each $\widetilde{h}_j = c_j$ is constant, so is each h_j. Now the current $T^0 = \sum_j c_j V_j$ satisfies $dT^0 = \sum_j r_j \Gamma_j$ and again by projecting we conclude that $r_j = n_j$. ∎

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