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Topological structures in string theory

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In string theory space-time comes equipped with an additional geometric structure called a B-field or 'gerbe'. I describe this structure, mention its relationship with non-commutative geometry, and explain how to use the B-field to define a twisted version of the K-theory of space-time. String-theoretical space-time can contain topologically non-trivial dynamical structures called D-branes. These are simply accounted for in the framework of conformal field theory. In a highly simplified limiting case—topological field theory with a finite gauge group—the D-branes naturally represent elements of the twisted K-theory of space-time: the K-theory class is the 'charge' of the D-brane.

Keywords: *B*-fields; gerbes; string backgrounds; D-branes; twisted *K*-theory

1. *B*-fields

When quantum theory is ignored, gravitation is described by general relativity, which tells us that space-time is a smooth manifold X equipped with a pseudo-Riemannian metric g satisfying Einstein's field equation. This equation, in the absence of any other matter, states that (X, g) is a critical point for the Einstein action

$$S(X, g) = \int_X R \, \mathrm{d}\,\mathrm{vol}_g,$$

where R is the scalar curvature of the metric g.

String theory is, in principle, a quantum theory of gravity. It models space-time not by a Riemannian manifold, but by a much more sophisticated mathematical object, which for the moment I shall call a 'string background'. In its low-energy behaviour, a string background approximates something more classical, in fact, a manifold equipped with a 'B-field' as well as a Riemannian metric.† The geometrical significance of the B-field is the subject of this talk: the ideas I shall be describing come from joint work with Michael Atiyah and Greg Moore.

A B-field (also called a 'gerbe with connection' (Hitchin 2001)) on a manifold X is akin to an electromagnetic field. In the nineteenth century it was thought that an electromagnetic field on a four-dimensional space-time X was described by giving the six components $F_{ij}(x)$ of the field strength at each point x of X, or, better expressed, giving the 2-form

$$F = \sum F_{ij} \, \mathrm{d}x^i \mathrm{d}x^j$$

† For simplicity, I am thinking of bosonic string theory, and am ignoring the scalar 'dilaton' field. In superstring theory the manifold would be equipped with additional fields besides the B-field. For a mathematical reader, the obvious references for string theory are Green *et al.* (1988) and d'Hoker (1999), the relevant sections being 3.4 and 6.9, respectively.

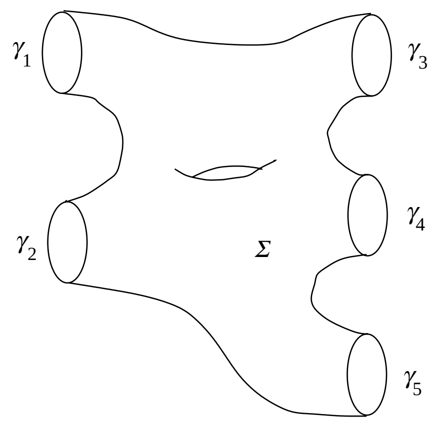

Figure 1.

on X, which had to be closed, as well as satisfying a field equation. Later, however, it was appreciated that the field strength does not describe the field completely, and that what one really has is a complex Hermitian line bundle L on X with a unitary connection, i.e. a complex line L_x associated to each point x, and a rule of parallel transport

$$A_\gamma : L_x \to L_y$$

associated to each path γ in X from x to y. The field strength F is the curvature of the connection: its value $F(\sigma)$ on an element σ of surface bounded by a very small closed curve γ is given by

$$\mathrm{e}^{2\pi \mathrm{i} F(\sigma)} = A_\gamma.$$

(Notice that transport from a line to itself is necessarily multiplication by a complex number of modulus 1, and we identify it with the number.) If the field strength vanishes, then the connection is flat, i.e. the transport A_γ depends only on the homotopy class of the path γ from x to y, but if X is not simply connected, the field can still have a physical effect, as was demonstrated in the Bohm–Aharonov experiment (Feynman *et al.* 1964, ch. 15). In all cases, the field is completely determined by giving the complex-valued function $\gamma \mapsto A_\gamma$ on all closed loops γ, though of course not all functions can arise.

(*a*) *The definition*

The string theory associated with a space-time X should have something to do with the loop space of X, so it is natural to define a B-field on X as a complex Hermitian line bundle L on the space $\mathcal{L}X$ of smooth closed loops in X, equipped with what I shall call a *string connection*. This is a rule that associates a transport operator

$$B_\Sigma : L_{\gamma_1} \otimes \cdots \otimes L_{\gamma_p} \to L_{\gamma_{p+1}} \otimes \cdots \otimes L_{\gamma_{p+q}}$$

to each smooth surface, or 'worldsheet', Σ in X, which has p incoming parametrized boundary circles $\gamma_1, \dots, \gamma_p$ and q outgoing parametrized boundary circles $\gamma_{p+1}, \dots, \gamma_{p+q}$ (see figure 1). As for an ordinary connection $\gamma \mapsto A_\gamma$, the essential properties of the assignment $\Sigma \mapsto B_\Sigma$ are that it is transitive with respect to concatenating the worldsheets Σ, and that it is parametrization independent in the sense that it does

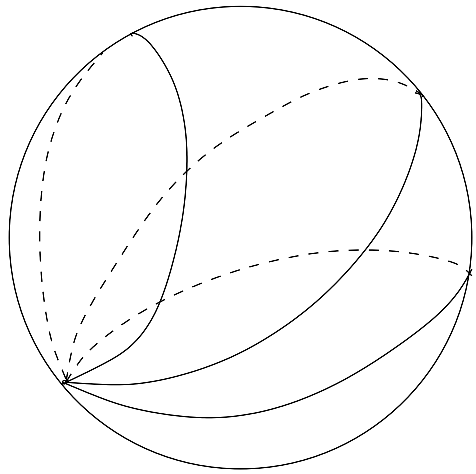

Figure 2.

not change if $\Sigma \to X$ is replaced by the composite $\Sigma' \to \Sigma \to X$, where $\Sigma' \to \Sigma$ is a diffeomorphism.

The *field strength*, or *curvature*, of a B-field is defined as the closed 3-form H on X whose value $H(v)$ on an element v of 3-volume at x bounded by a surface Σ is given by

$$e^{2\pi i H(v)} = B_\Sigma,$$

where Σ is regarded as a path in the loop space from a point loop at x to itself (see figure 2). Just as for the electromagnetic field, the field strength H nearly, but not quite, determines the B-field. It is, however, determined completely by giving the complex-valued function $\Sigma \mapsto B_\Sigma$ defined for all closed surfaces Σ in X.

The field equation satisfied by a Riemannian manifold (X, g, B), with a B-field B with curvature H, states that the action

$$S(X, g, B) = \int_X \{R \,\mathrm{d}\,\mathrm{vol}_g + H \wedge *H\}$$

is stationary, where $*$ is the Hodge star operator determined by the metric g. (If H is replaced by the field strength F of an electromagnetic field, the same formula gives the usual Einstein–Maxwell equations.) Explicitly, the field equation is

$$R_{ij} = \frac{1}{4} \sum H_{ipq} H_j^{pq}$$

in standard notation (see Green *et al.* 1988, eqn (3.4.56); d'Hoker 1999, eqn (6.69)).

(*b*) *Examples*

There are two basic examples of Riemannian manifolds with B-fields satisfying the field equations.

(i) (X, g) is a flat Riemannian torus, and the B-field is given by a constant 2-form B on X, in the sense that the line bundle on the loop space is trivial, and the transport B_Σ is multiplication by the complex number

$$\exp 2\pi i \int_X B.$$

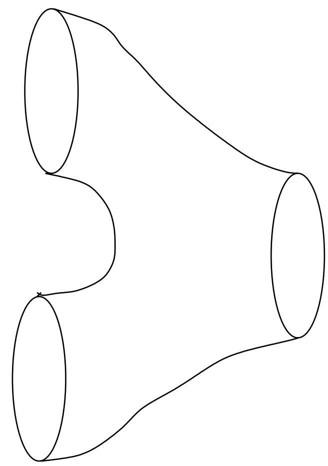

Figure 3.

(ii) (X, g) is a compact simply connected Lie group with a bi-invariant metric coming from an invariant inner product on its Lie algebra, and the curvature of the B-field is the bi-invariant 3-form H defined by

$$H(\xi, \eta, \zeta) = \langle \xi, [\eta, \zeta] \rangle$$

in terms of the invariant inner product, which must be chosen so that the 3-form represents an integral cohomology class. The transport B_Σ for a closed surface is given by

$$\exp 2\pi\mathrm{i} \int_v H,$$

where v is a three-dimensional region in X bounded by Σ.

2. String backgrounds

A string background is a conformal field theory with some additional structure. A conformal field theory, in turn, can be defined in language close to that just used for B-fields. Oversimplifying considerably, it consists of

(i) a complex vector space $\mathcal{H}$, and

(ii) a transport operator

$$\mathcal{U}_\Sigma : \mathcal{H}^{\otimes p} \to \mathcal{H}^{\otimes q}$$

associated to each surface Σ with a conformal structure, and with p incoming and q outgoing boundary circles each identified with the standard S^1. These operators should compose in the obvious way when the surfaces are concatenated.

In particular, $\mathcal{H}$ has a family of multiplications $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ associated to 'pairs of pants' (see figure 3), which make $\mathcal{H}$ approximately into a commutative algebra. The simplest case—not realistic as a description of space-time—is when $\mathcal{U}_\Sigma$ is independent of the conformal structure of Σ, and we have what is called a *topological field theory* (Atiyah 1988). In this case, there is just one multiplication, for all pairs of pants are isomorphic, and we have an exactly commutative algebra, with a trace $\theta : \mathcal{H} \to \mathbb{C}$

defined by taking Σ to be a disc. (It is well known in the topological case that to give the theory is completely equivalent to giving the commutative Frobenius algebra $(\mathcal{H}, \theta)$.†)

Replacing manifolds by conformal field theories has something in common with Connes's programme of non-commutative geometry, which begins with the observation that a manifold is completely encoded by its algebra A of smooth complex-valued functions, and then contemplates generalizing A to a non-commutative algebra. The string theory proposal, however, is very much more rigidly constrained.

The basic question at this point is why conformal field theories of a certain type should resemble—and provide an interesting generalization of—Riemannian manifolds with B-fields satisfying the variational equations mentioned above. There is no precise theorem to serve as a justification. However, from the classical data (X, g, B), one can, in good cases, construct a conformal field theory. Formally, one takes $\mathcal{H}$ to be the square-summable sections of the line bundle on the loop space $\mathcal{L}X$ defined by B, and one defines the transport $\mathcal{U}_\Sigma$ to be the integral operator with kernel

$$\mathcal{U}_\Sigma(\{\gamma_1, \ldots, \gamma_p\}, \{\gamma_{p+1}, \ldots, \gamma_{p+q}\}) = \int \mathrm{e}^{\mathrm{i}S(\phi)} \mathcal{D}\phi,$$

where the integral is over the space of all smooth maps $\phi : \Sigma \to X$, which restrict to the loops $\gamma_1, \ldots \gamma_{p+q}$ on the boundary of Σ, and the 'action' is given by

$$S(\phi) = \mathrm{energy}(\phi) + \mathrm{i}B_\phi,$$

in which the real part is the usual 'energy' of a map into a Riemannian manifold, and the imaginary part is the transport of the B-field along $\phi(\Sigma)$. In particular, the multiplication

$$L^2(\mathcal{L}T) \times L^2(\mathcal{L}T) \to L^2(\mathcal{L}T),$$

defined by a pair of pants, is a weighted compromise between ordinary pointwise multiplication and convolution with respect to concatenating the loops. To make precise sense of this schematic picture requires all the technology of two-dimensional quantum field theory, but the belief that underlies string theory is that the operator $\mathcal{U}_\Sigma$ can be constructed using only the conformal structure of Σ if and only if (X, g, B) satisfies the field equations.

Of course, one would still guess that the class of conformal field theories is vastly bigger than that of Riemannian manifolds with B-fields, but, surprisingly, that is not so. When one is given—by whatever means—a conformal field theory, then it is easy to find the moduli space of its infinitesimal deformations, and one finds that when the theory arises from some (X, g, B), there are no deformations other than the ones expected from deformations of the classical data. That is still not a very satisfactory situation, but to say any more would involve a much more elaborate discussion.

A more important point to emphasize in this paper is that the same conformal field theory can arise from *different* classical data. The simplest and most famous example is given by 'T-duality'. The theory defined by a flat Riemannian torus T is exactly the same as that for the dual torus T^*. (If T is the quotient of a Euclidean vector space $\mathfrak{t}$ by a lattice, then T^* is the quotient of the dual Euclidean space by the dual lattice. Thus a circle of radius r becomes a circle of radius $1/r$.) Furthermore,

† An account of this subject from the present point of view can be found in my lectures on topological field theory available from www.cgtp.duke.edu/ITP99/segal.

if T has a constant B-field given by a skew operator $B : \mathfrak{t} \to \mathfrak{t}^*$, as well as its metric given by $g : \mathfrak{t} \to \mathfrak{t}^*$, then we get the same theory from (T, g, B) as from (T^*, g^*, B^*), where

$$g^* + \mathrm{i}B^* = (g + \mathrm{i}B)^{-1}.$$

This duality is rather stranger than one might hastily suppose. It is easy to see that the loop groups $\mathcal{L}T$ and $\mathcal{L}T^*$ are naturally Pontryagin dual, in the sense that there is a bimultiplicative map

$$\mathcal{L}T \times \mathcal{L}T^* \to \mathbb{T}$$

which makes each into the character group of the other, and hence identifies the L^2 functions on $\mathcal{L}T$ with those on $\mathcal{L}T^*$ by Fourier transformation. But the Fourier transform is far from being a ring-homomorphism, and the assertion of T-duality is that the isomorphism respects the conformal field theory multiplicative structure.

3. Twisted *K*-theory

A cruder way to pursue the relation between conformal field theories and the classical objects (X, g, B) is to look for invariants of the field theories that correspond to topological properties of (X, g, B). The obvious topological invariant of the manifold X is its integral cohomology $H^*(X; \mathbb{Z})$. Alongside that is the K-theory $K^*(X)$, which is easier to define (Atiyah 1990). The isomorphism classes of complex vector bundles E on X form an abelian semigroup under fibrewise direct sum. The abelian group of formal differences $E - E'$ of bundles is $K^0(X)$. Integral cohomology and K-theory resemble each other closely; in fact, they become canonically isomorphic when tensored with the rational numbers.

K-theory carries over straightforwardly from manifolds to non-commutative algebras. For the vector space $\Gamma(E)$ of sections of a vector bundle, E on X is a finitely generated projective module over the algebra $C^\infty(X)$ and $E \mapsto \Gamma(E)$ defines an equivalence between the category of vector bundles on X and the category of finitely generated projective modules over $C^\infty(X)$. Finitely generated projective modules make perfectly good sense for non-commutative algebras, so they extend the notion of vector bundle to the non-commutative context and enable us to define K-theory (Connes 1994). There is no way, however, to define integral cohomology for a non-commutative algebra.

Another hint that K-theory is more appropriate than cohomology for conformal field theories is that K-theory has a very natural modification for a space with a B-field, as I shall now explain.

The only deformation invariant of an electromagnetic field is the cohomology class $c \in H^2(X; \mathbb{Z})$, which in mathematical language is the first Chern class of the complex line bundle, and in physical terms is the total magnetic charge. Every element of $H^2(X; \mathbb{Z})$ can arise in this way. Similarly, the only deformation invariant of a B-field is a class $c \in H^3(X; \mathbb{Z})$, and all classes can arise. (The field strength H is a closed 3-form, which represents c as an element of $H^3(X; \mathbb{R})$. If H vanishes, then c is the torsion class in $H^3(X; \mathbb{Z})$ defined by the transport map $B : H_2(X; \mathbb{Z}) \to \mathbb{C}^\times$ for closed surfaces in X.) Just as an element of $H^2(X; \mathbb{Z})$ is an isomorphism class of circle bundles on X, so an element of $H^3(X; \mathbb{Z})$ is an isomorphism class of fibre bundles on X whose fibre is the projective space of a standard complex Hilbert space. The crucial thing about such a bundle P, with fibres P_x, is that for each x we have a well-defined Banach space $\mathrm{End}(P_x)$ of bounded linear operators in the Hilbert space $\mathcal{H}_x$ underlying P_x. For though $\mathcal{H}_x$ is unique only up to a scalar multiplication, the space of linear operators in it depends only on P_x. One consequence of this is that a projective bundle P on X gives us a line bundle on $\mathcal{L}X$: we choose a connection in P, i.e. a transport map $U_\gamma : P_x \to P_y$ for each path γ in X from x to y, and then the transport U_γ around a closed loop γ beginning at x defines a ray in $\mathrm{End}(P_x)$. (The connection in P is unique up to homotopy, and so the line bundle is unique up to isomorphism.)

(*a*) *First definition*

Before giving the definition of K-theory for a compact manifold X with a B-field, let us recall that elements of the ordinary K-theory of X can be defined by families $\{f_x\}$ of Fredholm operators parametrized by X. If f_x is a Fredholm operator in a fixed Hilbert space $\mathcal{H}$, the kernel $\ker(f_x)$ and cokernel $\mathrm{coker}(f_x)$ are finite-dimensional vector spaces varying with x. One can show that a map $f : X \to \mathrm{Fred}(\mathcal{H})$ can be deformed so that $\{\ker(f_x)\}$ and $\{\mathrm{coker}(f_x)\}$ form vector bundles $\ker(f)$ and $\mathrm{coker}(f)$ on X, and then $[\ker(f)]-[\mathrm{coker}(f)]$ is an element of $K(X)$. In fact, Atiyah and Jänich (Atiyah 1990, Appendix) proved that K-theory can be defined in this way.

Theorem 3.1. *Elements of $K(X)$ can be identified with homotopy classes of continuous maps $X \to \mathrm{Fred}(\mathcal{H})$. Addition in $K(X)$ corresponds to composition of Fredholm operators.*

To a bundle P of projective Hilbert spaces on X representing the class of a B-field, we can associate a bundle $\mathrm{Fred}(P)$ whose fibre at x consists of the Fredholm operators in P_x. (Thus the fibre is an open subspace of the Banach space $\mathrm{End}(P_x)$ described above.) We can now define $K_P(X)$, the K-theory of X twisted by P, as the group of homotopy classes of sections of the bundle $\mathrm{Fred}(P)$.

(*b*) *Another definition*

The definition just given is simple, but perhaps not illuminating. A more fundamental point of view on the B-field is probably that it deforms the manifold X into a 'non-commutative space'. For we can associate to the bundle P of projective spaces the bundle $\mathcal{K}_P$ of Banach algebras whose fibre at x is the (non-unital) algebra of compact operators in P_x. (It is well known that such a bundle of algebras is determined by its Dixmier class $c \in H^3(X;\mathbb{Z})$, which, for $\mathcal{K}_P$, is the class of the bundle P (Bouwknegt & Mathai 2000).) If X is compact, the space A_P of sections of $\mathcal{K}_P$ is a Banach algebra, which we can use to give an equivalent definition of twisted K-theory.

Theorem 3.2. *If X is compact, then $K_P(X)$ is the K-theory of the Banach algebra A_P.*

A twisting class c of finite order in $H^3(X;\mathbb{Z})$ corresponds to a bundle of finite-dimensional algebras on X, and in that case the twisted K-theory has been studied in Donovan & Karoubi (1970).

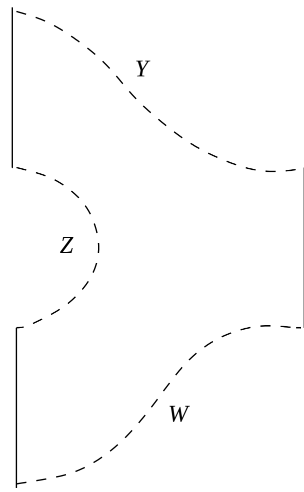

Figure 4.

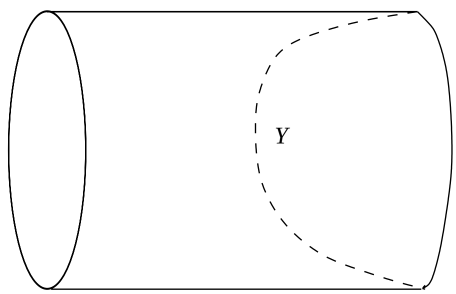

Figure 5.

4. D-branes

In recent years, string theorists have come to believe that the string backgrounds traditionally considered can acquire 'defects' or 'singularities' called *D-branes*. If we think of the initial background as corresponding to closed strings moving in a manifold X, then D-branes are submanifolds Y of X on which strings are allowed to begin and end: a theory with D-branes contains both open and closed strings. Furthermore, the submanifolds Y carry vector bundles with connections that provide boundary conditions at the ends of the strings. For a string background given by

$$(\mathcal{H}, \Sigma \mapsto \mathcal{U}_\Sigma),$$

it is in principle straightforward to determine the possible D-branes. The task is to find all ways of extending a conformal field theory to an 'open string' theory, by which I mean (roughly) the following structure. For any pair Y, Z of D-branes, there should be a vector space $\mathcal{H}_{YZ}$ of 'L^2 functions on the space of strings beginning on Y and ending on Z'. There should be a transport map

$$\mathcal{U}_\Sigma : \mathcal{H}_{YZ} \otimes \mathcal{H}_{ZW} \to \mathcal{H}_{YW}$$

for each surface Σ (with conformal structure) of the form shown in figure 4, where the dotted parts of the boundary are labelled by the D-branes. There should also be a map

$$\mathcal{U}_{\Sigma'} : \mathcal{H} \to \mathcal{H}_{YY}$$

for each surface Σ' (see figure 5) joining a closed to an open string. In general, there must be a map $\mathcal{U}_{\Sigma}$ for each surface Σ whose boundary is partitioned into three parts—an 'incoming' part consisting of a number of circles and intervals, an 'outgoing' part of the same type, and some additional 'free' boundary, which is itself a cobordism between the boundary of the incoming and the boundary of the outgoing parts—each connected component of the free boundary being labelled by a D-brane.

(*a*) *K-theory of a D-brane*

The D-branes for a given background form a moduli space analogous to those of holomorphic vector bundles or of coherent sheaves on a complex manifold. They should define classes—the 'D-brane charge'—in the K-theory of the background conformal field theory, if one can define such a thing (see Moore & Witten 2000). Greg Moore and I have analysed a baby version of this situation in which the string background is replaced by a topological field theory, i.e. by a commutative Frobenius algebra A. If A is semisimple, it can be identified with the ring of functions on the finite set $\mathrm{Spec}(A)$ of maximal ideals, and we could think of this finite set as a rudimentary space-time. But we might also think of A as a subalgebra of 'ground states' in a much larger conformal field theory, and then $\mathrm{Spec}(A)$ might be the set of connected components of the loop space of space-time, or perhaps even, if the larger theory were supersymmetric, A might be the cohomology algebra of space-time.

In any case, the definition of a D-brane is clear for a topological field theory, and we have shown that a D-brane corresponds precisely to a non-commutative Frobenius algebra B with a homomorphism $i_* : A \to B$ satisfying

(i) the image of A is contained in the centre of B, and

(ii) the 'Cardy condition' $i_\star i^* = \pi$ holds, where $i^* : B \to A$ is the adjoint of i_*, and $\pi : B \to B$ is defined by

$$\pi(x) = \sum b_i x b_i^*,$$

where $\{b_i\}$ is a vector space basis for B, and $\{b_i^*\}$ is the dual basis.

If A is semisimple, we can show that B must be of the form $\mathrm{End}_A(E)$ for some finitely generated A-module E. So, in this case, D-branes do indeed correspond to vector bundles on the 'space' $\mathrm{Spec}(A)$. This is too babyish an example to be very convincing, but one can attach a little more flesh to it in the light of an observation of Turaev (1999). When one has a topological field theory defined by a Frobenius algebra A, it is clear what it means for a finite group G to act as a group of automorphisms of the theory, and also what it means to 'gauge' the G-action. This richer concept of a gauged G-action—which I shall not define here—is what corresponds on the field theory side to an action of G on the classical space-time. It is what one needs in order to define an 'orbifold' theory. Turaev showed that when A is semisimple and a group G acts on A, then the possible G-gauged theories are classified precisely by the equivariant B-fields on the G-space $X = \mathrm{Spec}(A)$, i.e. by the elements of the equivariant cohomology $H^3_G(X; \mathbb{Z})$. In that situation, Moore and I can determine the D-branes in the gauged theory, and it turns out that they correspond to elements of the twisted G-equivariant K-theory of the G-space X.

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