PROJECTIVE LINKING AND BOUNDARIES OF POSITIVE HOLOMORPHIC CHAINS IN PROJECTIVE MANIFOLDS, PART I

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Dedicated to Nigel Hitchin
in celebration of his 60th birthday

Abstract

We introduce the notion of the *projective linking number* $\text{Link}_{\mathbf{P}}(\Gamma, Z)$ of a compact oriented real submanifold Γ of dimension $2p-1$ in complex projective n-space $\mathbf{P}^n$ with an algebraic subvariety $Z \subset \mathbf{P}^n - \Gamma$ of codimension p. This notion is related to *projective winding numbers* and quasi-plurisubharmonic functions, and it generalizes directly from $\mathbf{P}^n$ to any projective manifold. Part 1 of this paper establishes the following result for the case $p = 1$. Let Γ be an oriented, stable, real analytic curve in $\mathbf{P}^n$. Then

Γ is the boundary of a positive holomorphic 1-chain of mass $\leq \Lambda$ in $\mathbf{P}^n$ if and only if $\widetilde{\text{Link}}_{\mathbf{P}}(\Gamma, Z) \geq -\Lambda$ for all algebraic hypersurfaces $Z \subset \mathbf{P}^n - \Gamma$.

where $\widetilde{\text{Link}}_{\mathbf{P}}(\Gamma, Z) = \text{Link}_{\mathbf{P}}(\Gamma, Z)/\deg(Z)$. An analogous theorem is implied in any projective manifold. Part 2 of this paper studies similar results for $p > 1$.

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1. Introduction

In 1998 Herb Alexander and John Wermer published the following result [AW_2].

THEOREM (Alexander-Wermer). *Let Γ be a compact oriented smooth submanifold of dimension $2p-1$ in $\mathbf{C}^n$. Then Γ bounds a positive holomorphic p-chain in $\mathbf{C}^n$ if and only if the linking number*

$$\mathrm{Link}(\Gamma, Z) \ \geq\ 0$$

for all canonically oriented algebraic subvarieties Z of codimension p in $\mathbf{C}^n - \Gamma$.

The **linking number** is an integer-valued topological invariant defined by the intersection $\mathrm{Link}(\Gamma, Z) \equiv N \bullet Z$ with any $2p$-chain N having $\partial N = \Gamma$ in $\mathbf{C}^n$. (See §3.) A **positive holomorphic p-chain** is a finite sum of canonically oriented complex subvarieties of dimension p and finite volume in $\mathbf{C}^n - \Gamma$. (See Definition 5.1). Here the notion of boundary is taken in the sense of currents, i.e., Stokes' Theorem is satisfied. However, for smooth Γ there is boundary regularity almost everywhere, and if Γ is real analytic, one has complete boundary regularity. (See HL_1].)

The main point of this paper is to formulate and prove an analogue of the Alexander-Wermer Theorem for oriented (not necessarily connected) curves in a projective manifold. In the sequel we shall study the corresponding result for submanifolds of any odd dimension.

Before stating the main result we remark that a key ingredient in the proof of the Alexander-Wermer Theorem is the following classical theorem and its generalizations [W_1].

THEOREM (Wermer) . *Let $\Gamma \subset \mathbf{C}^n$ be a compact real analytic curve and denote by*

$$\widehat{\Gamma}_{\mathrm{poly}} \ =\ \{z \in \mathbf{C}^n \ :\ |p(z)| \leq \sup_{\Gamma} |p| \ \ \textit{for all polynomials}\ \ p\}.$$

its polynomial hull. Then $\widehat{\Gamma}_{\mathrm{poly}} - \Gamma$ is a one-dimensional complex analytic subvariety of $\mathbf{C}^n - \Gamma$.

For compact subsets K of complex projective n-space $\mathbf{P}^n$, the authors recently introduced the notion of the **projective hull**

$$\widehat{K}_{\mathrm{proj}} \ =\ \{z \in \mathbf{P}^n : \exists C \text{ s.t. } \|\sigma_z\| \leq C^d \sup_K \|\sigma\| \ \ \forall \sigma \in H^0(\mathbf{P}^n, \mathcal{O}(d)), d > 0\}$$

and defined K to be **stable** if the constant C can be chosen independently of the point $z \in \widehat{K}_{\mathrm{proj}}$. A number of basic properties of $\widehat{K}_{\mathrm{proj}}$ were established in [HL_5]), and the following analogue of Wermer's Theorem was proved in [HLW].

THEOREM (Harvey-Lawson-Wermer). *Let $\Gamma \subset \mathbf{P}^n$ be a stable real analytic curve. Then $\widehat{\Gamma}_{\mathrm{proj}} - \Gamma$ is a one-dimensional complex analytic subvariety of $\mathbf{P}^n - \Gamma$.*

It is interesting to note that while Wermer's Theorem holds for curves with only weak differentiability properties (see [AW_1] or [DL] for an account), its projective analogue fails even for C^∞-curves. On the other hand there is much evidence for the following.

CONJECTURE A. *Every real analytic curve in $\mathbf{P}^n$ is stable.*

This brings us to the notion of projective linking numbers. Suppose that $\Gamma \subset \mathbf{P}^n$ is a compact oriented smooth curve, and let $Z \subset \mathbf{P}^n - \Gamma$ be an algebraic subvariety of codimension 1. The **projective linking number** of Γ with Z is defined to be

$$\mathrm{Link}_{\mathbf{P}}(\Gamma, V) \ \equiv\ N \bullet Z - \deg(Z) \int_N \omega$$

where ω is the standard Kähler form on $\mathbf{P}^n$ and N is any integral 2-chain with $\partial N = \Gamma$ in $\mathbf{P}^n$. Here Z is given the canonical orientation, and $\bullet : H_2(\mathbf{P}^n, \Gamma) \times H_{2n-2}(\mathbf{P}^n - \Gamma) \to \mathbf{Z}$ is the topologically defined intersection pairing. This definition is independent of the choice of N. (See §3.) The associated **reduced linking number** is defined to be

$$\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) \ \equiv\ \frac{1}{\deg(Z)} \mathrm{Link}_{\mathbf{P}}(\Gamma, Z)$$

The basic result proved here is the following.

THEOREM 6.1. *Let Γ be a oriented stable real analytic curve in $\mathbf{P}^n$ with a positive integer multiplicity on each component. Then the following are equivalent:*

(i) Γ *is the boundary of a positive holomorphic* 1*-chain of mass* $\leq \Lambda$ *in* $\mathbf{P}^n$.

(ii) $\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) \ \geq\ -\Lambda$ *for all algebraic hypersurfaces* Z *in* $\mathbf{P}^n - M$.

If Γ bounds any positive holomorphic 1-chain, then there is a unique such chain T_0 of least mass. (All others are obtained by adding algebraic 1-cycles to T_0.) Note that $\Lambda_0 \equiv M(T_0)$ is the smallest positive number such that (ii) holds.

COROLLARY 6.8. *Let Γ be as in Theorem 6.1 and suppose T is a positive holomorphic 1-chain with $dT = \Gamma$. Then T is the unique holomorphic chain of least mass with $dT = \Gamma$ if and only if*

$$\inf_Z \left\{ \frac{T \bullet Z}{\deg Z} \right\} \ =\ 0$$

where the infimum is taken over all algebraic hypersurfaces in the complement $\mathbf{P}^n - \Gamma$.

Condition (ii) in Theorem 6.1 has several equivalent formulations. The first is in terms of projective winding numbers. Given a holomorphic section $\sigma \in H^0(\mathbf{P}^n, \mathcal{O}(d))$, the **projective winding number** of σ on Γ is defined as the integral

$$\mathrm{Wind}_{\mathbf{P}}(\Gamma, \sigma) \ \equiv\ \int_\Gamma d^C \log\|\sigma\|,$$

and we set

$$\widetilde{\mathrm{Wind}}_{\mathbf{P}}(\Gamma, \sigma) \ \equiv\ \frac{1}{d} \mathrm{Wind}_{\mathbf{P}}(\Gamma, \sigma)$$

Another formulation involves the cone $PSH_\omega(\mathbf{P}^n)$ of **quasi-plurisubharmonic functions**. These are the upper semi-continuous functions $f : \mathbf{P}^n \to [-\infty, \infty)$ for which $dd^C f + \omega$ is a positive (1,1)-current on $\mathbf{P}^n$.

Proposition 5.2. *Let Γ be an oriented smooth curve in $\mathbf{P}^n$ with a positive integer multiplicity on each component. Then for any $\Lambda > 0$ the following are equivalent:*

(ii) $\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) \ \geq \ -\Lambda$ *for all algebraic hypersurfaces $Z \subset \mathbf{P}^n - \Gamma$.*

(iii) $\widetilde{\mathrm{Wind}}_{\mathbf{P}}(\Gamma, \sigma) \ \geq \ -\Lambda$ *for all holomorphic sections σ of $\mathcal{O}(d)$, and all $d > 0$.*

(iv) $\int_\Gamma d^C u \ \geq \ -\Lambda$ *for all $u \in PSH_\omega(\mathbf{P}^n)$.*

Any smooth curve Γ in $\mathbf{P}^n$ lies in some affine chart $\mathbf{C}^n \subset \mathbf{P}^n$, and it is natural to ask for a reformulation of condition (ii) in terms of the conventional linking numbers of Γ with algebraic hypersurfaces in that chart. This is done explicitly in Theorem 6.6.

The results above extend to any projective manifold X. Given a very ample hermitian line bundle λ on X there are intrinsically defined **λ-linking numbers** $\mathrm{Link}_\lambda(\Gamma, Z)$, **$\lambda$-winding numbers** $\mathrm{Wind}_\lambda(\Gamma, \sigma)$ for $s \in H^0(X, \lambda^d)$, and $\omega = c_1(\lambda)$-**quasi-plurisubharmonic functions** $PSH_\omega(X)$. With these notions, Proposition 5.2 and Theorem 6.1 carry over to X. This is done in section 7.

Theorem 6.1 leads to the following interesting result.

Theorem 8.1. *Let $\gamma \subset \mathbf{P}^n$ be a finite disjoint union of real analytic curves and assume γ is stable. Then a class $\tau \in H_2(\mathbf{P}^n, \gamma; \mathbf{Z})$ is represented by a positive holomorphic chain with boundary on γ if and only if*

$$\tau \bullet u \ \geq \ 0$$

for all $u \in H_{2n-2}(\mathbf{P}^n - \gamma; \mathbf{Z})$ represented by positive algebraic hypersurfaces in $\mathbf{P}^n - \gamma$.

This result expands to a duality between the cones of relative and absolute classes which are representable by positive holomorphic chains [HL_8].

The arguments given in §6 show that there is even more evidence for the following.

Conjecture B. *Every oriented real analytic curve in $\mathbf{P}^n$ which satisfies the equivalent conditions of Proposition 5.2 is stable.*

In Part II of this paper we prove that if Conjecture B holds for curves in $\mathbf{P}^2$, then all the results above continue to hold for real analytic Γ of any odd dimension $2p - 1$. (No stability hypothesis is needed.)

Remark . There are several quite different characterizations of the boundaries of general (i.e., not necessarily positive) holomorphic chains in projective and certain quasi-projective manifolds. See, for example, [Do], [$\mathrm{DH}_{1,2}$], and [$\mathrm{HL}_{2,4}$].

Note . To keep formulas simple throughout the paper we adopt the convention that

$$d^C \ = \ \frac{i}{2\pi}(\overline{\partial} - \partial)$$

2. Projective Hulls

In this section we recall the definition and basic properties of the projective hull introduced in [HL$_5$]. This material is not really necessary for reading the rest of the paper.

Let $\mathcal{O}(1) \to \mathbf{P}^n$ denote the holomorphic line bundle of Chern class 1 endowed with the standard unitary-invariant metric, and let $\mathcal{O}(d)$ be its dth tensor power with the induced tensor product metric.

Definition 2.1. Let $K \subset \mathbf{P}^n$ be a compact subset of complex projective n-space. A point $x \in \mathbf{P}^n$ belongs to the *projective hull of* K if there exists a constant $C = C(x)$ such that

$$\|\sigma(x)\| \ \leq\ C^d \sup_K \|\sigma\| \tag{2.1}$$

for all global sections $\sigma \in H^0(\mathbf{P}^n, \mathcal{O}(d))$ and all $d > 0$. This set of points is denoted $\widehat{K}$.

The set $\widehat{K}$ is independent of the choice of metric on $\mathcal{O}(1)$.

The projective hull possesses interesting properties. It and its generalizations function in projective and Káhler manifolds much as the polynomial hull and its generalizations function in affine and Stein manifolds. The following were established in [HL$_5$]:

(1) If $Y \subset \mathbf{P}^n$ is an algebraic subvariety and $K \subset Y$, then $\widehat{K} \subset Y$. That is, $\widehat{K}$ is contained in the Zariski closure of K. Furthermore, if $Y \subset \mathbf{P}^n$ is a projective manifold and $\lambda = \mathcal{O}(1)\big|_Y$, the λ-projective hull of $K \subset Y$, defined as in (2.1) with $\mathcal{O}(1)$ replaced by λ, agrees with $\widehat{K}$. The same is true of λ^k for any k.

(2) If K is contained in an affine open subset $\Omega \subset \mathbf{P}^n$, then $(\widehat{K})_{\mathrm{poly},\Omega} \subseteq \widehat{K}$.

(3) If $K = \partial C$, where C is a holomorphic curve with boundary in $\mathbf{P}^n$, then $C \subseteq \widehat{K}$.

(4) $\{\widehat{K}\}^- - K$ satisfies the *maximum modulus principle* for holomorphic functions on open subsets of $\mathbf{P}^n - K$. ($\{\widehat{K}\}^-$ denotes the closure of $\widehat{K}$.)

(5) $\{\widehat{K}\}^- - K$ is 1-pseudoconcave in the sense of [DL]. In particular, for any open subset $U \subset \mathbf{P}^n - K$, if the Hausdorff 2-measure $\mathcal{H}^2(\widehat{K} \cap U) < \infty$, then $\widehat{K} \cap U$ is a *complex analytic subvariety* of dimension 1 in U.

(6) If K is a real analytic curve, then the Hausdorff dimension of $\widehat{K}$ is 2.

(7) $\widehat{K}$ is pluripolar if and only if K is pluripolar. [1] Thus there exist smooth closed curves $\Gamma \subset \mathbf{P}^2$ with $\widehat{\Gamma} = \mathbf{P}^2$. However, real analytic curves are always pluripolar.

The projective hull has simple characterizations in both affine and homogeneous coordinates. For example, if $\pi : \mathbf{C}^{n+1} - \{0\} \longrightarrow \mathbf{P}^n$ is the standard projection, and we set $S(K) \equiv \pi^{-1}(K) \cap S^{2n+1}$, then

$$\widehat{K} \ =\ \pi\left\{\widehat{S(K)}_{\mathrm{poly}} - \{0\}\right\}.$$

[1] A set K is pluripolar if it is locally contained in the $-\infty$ set of a plurisubharmonic function.

where $\widehat{S(K)}_{\text{poly}}$ is the polynomial hull of $S(K)$ in $\mathbf{C}^{n+1}$.

There is also a *best constant function* $C : \widehat{K} \to \mathbf{R}^+$ defined at x to be the least $C = C(x)$ for which the defining property (2.1) holds. For $x \in \widehat{K}$, the set $\pi^{-1}(x) \cap \widehat{S(K)}_{\text{poly}}$ is a disk of radius $\rho(x) = 1/C(x)$. One deduces that $\widehat{K}$ is compact if C is bounded.

DEFINITION 2.2. A compact subset $K \subset \mathbf{P}^n$ is called *stable* if $C : \widehat{K} \to \mathbf{R}^+$ is bounded.

Combining a classical argument of E. Bishop with (5) and (6) above gives the following.

THEOREM 2.3. [HLW]. *Let $\Gamma \subset \mathbf{P}^n$ be a compact stable real analytic curve (not necessarily connected). Then $\widehat{\Gamma} - \Gamma$ is a one-dimensional complex analytic subvariety of $\mathbf{P}^n - \Gamma$.*

As mentioned in the introduction, there is much evidence for the conjecture that any compact real analytic curve in $\mathbf{P}^n$ is stable, and therefore the stability hypothesis could be removed from Theorem 2.3. Interestingly, the conclusion of Theorem 2.3 fails to hold in general for smooth curves (see [HL_5, §4]), but may hold for smooth curves which are pluripolar or, say, quasi-analytic.

REMARK 2.4. The close parallel between polynomial and projective hulls is signaled by the existence of a *projective Gelfand transformation*, whose relation to the classical Gelfand transform is analogous to the relation between Proj of a graded ring and Spec of a ring in modern algebraic geometry. To any Banach graded algebra $A_* = \bigoplus_{d \geq 0} A_d$ (a normed graded algebra which is a direct sum of Banach spaces) one can associate a topological space X_{A_*} and a hermitian line bundle $\lambda_{A_*} \to X_{A_*}$ with the property that A_* embeds as a closed subalgebra

$$A_* \subseteq \bigoplus_{d \geq 0} \Gamma_{\text{cont}}(X_{A_*}, \lambda^d_{A_*})$$

of the algebra of continuous sections of powers of λ with the sup-norm. When $K \subset \mathbf{P}^n$ is a compact subset and $A_d = H^0(\mathbf{P}^n, \mathcal{O}(d))\big|_K$ with the sup-norm on K, there is a natual homeomorphism

$$\widehat{K} \cong X_{A_*} \qquad \text{and} \qquad \mathcal{O}(1) \cong \lambda_{A_*}.$$

This parallels the affine case where, for a compact subset $K \subset \mathbf{C}^n$, the Gelfand spectrum of the closure of the polynomials on K in the sup-norm corresponds to the polynomial hull of K.

Furthermore, when A_* is finitely generated, the space X_{A_*} can be realized, essentially uniquely, by a subset $X_{A_*} \subset \mathbf{P}^n$ with $\lambda_{A_*} \cong \mathcal{O}(1)$ and

$$\widehat{X}_{A_*} = X_{A_*}.$$

This parallels the classical correspondence between finitely generated Banach algebras and polynomially convex subsets of $\mathbf{C}^n$. Details are given in [HL_5].

3. Projective Linking and Projective Winding Numbers

In this section we introduce the notion of projective linking numbers and projective winding numbers for oriented curves in $\mathbf{P}^n$.

Let $M = M^{2p-1} \subset \mathbf{P}^n$ be a compact oriented submanifold of dimension $2p-1$, and recall the *intersection pairing*

$$\bullet : H_{2p}(\mathbf{P}^n, M; \mathbf{Z}) \times H_{2(n-p)}(\mathbf{P}^n - M; \mathbf{Z}) \longrightarrow \mathbf{Z} \tag{3.1}$$

which under Alexander Duality corresponds to the Kronecker pairing:

$$\kappa : H^{2(n-p)}(\mathbf{P}^n - M; \mathbf{Z}) \times H_{2(n-p)}(\mathbf{P}^n - M; \mathbf{Z}) \longrightarrow \mathbf{Z}.$$

When homology classes are represented by cycles which intersect transversally in regular points, the map (3.1) is given by the usual algebraic intersection number.

Definition 3.1. Fix M as above and let $Z \subset \mathbf{P}^n - M$ be an algebraic subvariety of dimension $n-p$. Then the *projective linking number* of M and Z is defined as follows. Choose a $2p$-chain N in $\mathbf{P}^n$ with $dN = M$ and set:

$$\mathrm{Link}_{\mathbf{P}}(M, Z) \equiv N \bullet Z - \deg(Z) \int_N \omega^p \tag{3.2}$$

where ω is the standard Kähler form on $\mathbf{P}^n$. This definition extends by linearity to algebraic $(n-p)$-cycles $Z = \sum_j n_j Z_j$ supported in $\mathbf{P}^n - M$.

Lemma 3.2. *The linking number* $\mathrm{Link}_{\mathbf{P}}(M, Z)$ *is independent of the choice of the cobounding chain* N.

Proof. Let N' be another choice and set $W = N - N'$. Then $dW = 0$ and

$$W \bullet Z - \deg(Z) \int_W \omega^p = \deg(W) \cdot \deg(Z) - \deg(Z) \cdot \deg(W) = 0. \qquad \blacksquare$$

For now we shall be concerned with the case $p = 1$. Here there is a naturally related notion of projective winding number defined as follows.

Definition 3.3. Suppose $\Gamma \subset \mathbf{P}^n$ is a smooth closed oriented curve, and let σ be a holomorphic section of $\mathcal{O}(\ell)$ over $\mathbf{P}^n$ which does not vanish on Γ. Then the *projective winding number* of σ on Γ is defined to be

$$\mathrm{Wind}_{\mathbf{P}}(\Gamma, \sigma) \equiv \int_\Gamma d^C \log\|\sigma\| \tag{3.3}$$

Proposition 3.4. *Let Γ and σ be as in Definition 3.3, and let Z be the divisor of σ. Then*

$$\mathrm{Wind}_{\mathbf{P}}(\Gamma, \sigma) = \mathrm{Link}_{\mathbf{P}}(\Gamma, Z)$$

Proof. We recall the fundamental formula

$$dd^C \log\|\sigma\| \ = \ Z - \ell\omega \tag{3.4}$$

which follows by writing $\|\sigma\| = \rho|\sigma|$ in a local holomorphic trivialization of $\mathcal{O}(\ell)$ and applying the Chern and Poincaré-Lelong formulas: $dd^C \log\rho = -\ell\omega$ and $dd^C \log|\sigma| = \mathrm{Div}(\sigma) = Z$. We now write $\Gamma = \partial N$ for a rectifiable 2-current N in $\mathbf{P}^n$ and note that

$$\int_\Gamma d^C \log\|\sigma\| \ = \ \int_N dd^C \log\|\sigma\| \ = \ N \bullet Z - \deg(Z) \int_N \omega$$

by formula (3.4). ∎

It will be useful in what follows to normalize these linking numbers.

Definition 3.5. For Γ and Z as above, we define the *reduced projective linking number* to be

$$\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) \ \equiv \ \frac{1}{\deg(Z)} \mathrm{Link}_{\mathbf{P}}(\Gamma, Z)$$

and the *reduced projective winding number* to be $\widetilde{\mathrm{Wind}}_{\mathbf{P}}(\Gamma, Z) \equiv \mathrm{Wind}_{\mathbf{P}}(\Gamma, Z)/\deg(Z)$.

Note that if $Z = \mathrm{Div}(\sigma)$ for $\sigma \in H^0(\mathbf{P}^n, \mathcal{O}(\ell))$, then by Proposition 3.4 we have

$$\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma, Z) \ = \ \int_\Gamma d^C \log\|\sigma\|^{\frac{1}{\ell}} \tag{3.5}$$

Remark 3.6. **(The relation to affine linking numbers.)** Note that a smooth curve $\Gamma \subset \mathbf{P}^n$ does not meet the generic hyperplane $\mathbf{P}^{n-1}$ and is therefore contained in the affine chart $\mathbf{C}^n = \mathbf{P}^n - \mathbf{P}^{n-1}$. Let N be an integral 2-chain with support in $\mathbf{C}^n$ and $dN = \Gamma$. Then

$$\mathrm{Link}_{\mathbf{P}}(\Gamma, \mathbf{P}^{n-1}) \ = \ N \bullet \mathbf{P}^{n-1} - \int_N \omega \ = \ -\int_N \omega, \tag{3.6}$$

and for any divisor Z of degree ℓ which does not meet Γ we have

$$\mathrm{Link}_{\mathbf{P}}(\Gamma, Z) - \mathrm{Link}_{\mathbf{P}}(\Gamma, \ell\mathbf{P}^{n-1}) \ = \ N \bullet Z \ = \ \mathrm{Link}_{\mathbf{C}^n}(\Gamma, Z),$$

the classical linking number of Γ and Z.

Remark 3.7. **(The relation to sparks and differential characters.)** The deRham-Federer approach to differential characters (cf. [GS], [Ha], [HLZ]) is built on objects called *sparks*. These are generalized differential forms (or currents) α which satisfy the equation

$$d\alpha \ = \ R - \phi$$

where R is rectifiable and ϕ is smooth. By (3.4) the generalized 1-form $d^C \log\|\sigma\|$ is such a creature. Its associated Cheeger-Simons character $[\alpha] : Z_1(\mathbf{P}^n) \longrightarrow \mathbf{R}/\mathbf{Z}$ on smooth 1-cycles is defined by the formula

$$[\alpha](\Gamma) \ \equiv \ \int_N \ell\omega \pmod{\mathbf{Z}}$$

where N is a 2-chain with $dN = \Gamma$. Thus we see that the "correction term" in the projective linking number is the value of a secondary invariant related to differential characters.

REMARK 3.8. Note that by (3.4) the function $u \equiv \log\|\sigma\|^{\frac{1}{\ell}}$ which appears in (3.5) has the property that

$$dd^C u + \omega \ = \frac{1}{\ell} Z \ \geq \ 0 \tag{3.7}$$

by the fundamental equation (3.4). This leads us naturally to the next section.

4. Quasi-plurisubharmonic Functions

The following concept, due to Demailly [D_1] and developed systematically by Guedj and Zeriahi [GZ], is central to the study of projective hulls, and it is intimately related to projective linking numbers.

DEFINITION 4.1. An upper semi-continuous function $u : X \longrightarrow [-\infty, \infty)$ on a Kähler manifold (X, ω) is *quasi-plurisubharmonic* (or *ω-quasi-plurisubharmonic*) if $u \not\equiv -\infty$ and

$$dd^C u + \omega \ \geq 0 \qquad \text{on} \ \ X. \tag{4.1}$$

The convex set of all such functions on X will be denoted by $PSH_\omega(X)$.

These functions enjoy many of the properties of classical plurisubharmonic functions and play an important role in understanding various capacities in projective space (cf. [GZ]). One of the appealing geometric properties of this class is the following. Suppose ω is the curvature form of a holomorphic line bundle $\lambda \to X$ with hermitian metric g. Then a smooth function $u : X \to \mathbf{R}$ is quasi-plurisubharmonic iff the hermitian metric $e^u g$ has curvature ≥ 0 on X.

The quasi-plurisubharmonic functions are directly relevant to projective hulls, as the next result shows (cf. [GZ, Pf. of Thm. 4.2] and Theorem 5.3 below).

THEOREM 4.2. *Let ω denote the standard Kähler form on $\mathbf{P}^n$. Then the projective hull $\widehat{K}$ of a compact subset $K \subset \mathbf{P}^n$, is exactly the subset of points $x \in \mathbf{P}^n$ for which there exists a constant $\Lambda = \Lambda(x)$ with*

$$u(x) \ \leq \sup_K u + \Lambda \qquad \text{for all} \ \ u \in PSH_\omega(X) \tag{4.2}$$

This enables one to generalize the notion of projective hull from projective algebraic manifolds to general Kähler manifolds.

Note that the least constant $\Lambda(x)$ for which (4.2) holds is exactly $\Lambda(x) = \log C(x)$ where $C(x)$ is the best constant function discussed in §2.

By considering functions of the form $u = \log\{\|\sigma\|^{\frac{1}{d}}\}$ for sections $\sigma \in H^0(\mathbf{P}^n, \mathcal{O}(d))$, one immediatly sees the necessity of the condition in Theorem 4.2. Sufficiency follows from the fact that such functions are the extreme points of the cone $PSH_\omega(\mathbf{P}^n)$.

One importance of quasi-plurisubharmonic functions is that they enable us to establish Poisson-Jensen measures for points in $\widehat{K}$, [HL_5, Thm. 11.1].

5. Boundaries of Positive Holomorphic Chains.

Let Γ be a smooth oriented closed curve in $\mathbf{P}^n$. We recall that (even if Γ is only class C^1) any irreducible complex analytic subvariety $V \subset \mathbf{P}^n - \Gamma$ of dimension one has finite Hausdorff 2-measure and defines a current $[V]$ of dimension 2 in $\mathbf{P}^n$ by integration on the canonically oriented manifold of regular points. This current satisfies $d[V] = 0$ in $\mathbf{P} - \Gamma$. (See [H] for example.)

Definition 5.1. By a *positive holomorphic 1-chain with boundary* Γ we mean a finite sum $T = \sum_k n_k[V_k]$ where each $n_k \in \mathbf{Z}^+$ and each $V_k \subset \mathbf{P}^n - \Gamma$ is an irreducible subvariety of dimension 1, so that

$$dT \;=\; \Gamma \qquad \text{(as currents on } \mathbf{P}^n\text{)}$$

We shall be interested in conditions on Γ which are necessary and sufficient for it to be such a boundary.

Proposition 5.2. *Let $\Gamma \subset \mathbf{P}^n$ be a smooth oriented closed curve (not necessarily connected) with a positive integer multiplicity on each component. Then the following are equivalent.*

(1) $\widetilde{\text{Link}}(\Gamma, Z) \;\geq\; -\Lambda$ *for all algebraic hypersurfaces $Z \subset \mathbf{P}^n - \Gamma$.*

(2) $\widetilde{\text{Wind}}(\Gamma, \sigma) \;\geq\; -\Lambda$ *for all $\sigma \in H^0(\mathbf{P}^n, \mathcal{O}(\ell)), \ell > 0,$ with no zeros on Γ.*

(3) $\int_\Gamma d^C u \;\geq\; -\Lambda$ *for all $u \in PSH_\omega(\mathbf{P}^n)$.*

Proof. The equivalence of (1) and (2) is follows immediately from Proposition 3.4. That (3) $\Rightarrow$ (1) follows from (3.5) and (3.7). To see that (1) $\Rightarrow$ (3) we use the following non-trivial fact due essentially to Demailly [D$_2$], [G].

Proposition 5.3. *The functions of the form $\log(\|\sigma\|^{\frac{1}{\ell}})$ for $\sigma \in H^0(\mathbf{P}^n, \mathcal{O}(\ell))$, $\ell > 0$, are weakly dense in $PSH_\omega(\mathbf{P}^n)$ modulo the constant functions.*

Proof. Fix $u \in PSH_\omega(\mathbf{P}^n)$ and consider the positive (1,1)-current $T \equiv dd^C u + \omega$. By [D$_2$], [G, Thm. 0.1], there exist sequences $\sigma_j \in H^0(\mathbf{P}^n, \mathcal{O}(\ell_j))$ and $N_j \in \mathbf{R}^+$, $j = 1, 2, 3, ...$ such that

$$T \;=\; \lim_{j\to\infty} \frac{1}{N_j}\text{Div}(\sigma_j) \;=\; \lim_{j\to\infty} \frac{\ell_j}{N_j}\left\{dd^C\log\left(\|\sigma_j\|^{\frac{1}{\ell_j}}\right) + \omega\right\}. \tag{5.1}$$

Since $T(\omega^{n-1}) = \int_{\mathbf{P}^n}(dd^C u \wedge \omega^{n-1} + \omega^n) = \int_{\mathbf{P}^n} \omega^n = 1 = \lim_{j\to\infty} \ell_j/N_j$ by (5.1), we may assume that $N_j = \ell_j$ for all j. Therefore, setting $u_j = dd^C\log\|\sigma_j\|^{\frac{1}{\ell_j}}$, we have by (5.1) that $\lim_j dd^C u_j = dd^C u$ and, in particular, $\lim_j \Delta u_j = \Delta u$. Renormalizing each u_j by an additive constant, we may also assume that $\lim_j \int u_j = \int u$.

Recall the formula $Id = H + G \circ \Delta$ on $C^\infty(X)$ where H is harmonic projection and G is the Green's function. This decomposition carries over to distributions $\mathcal{D}'(X)$ by adjoint. It follows that $u_j - \int u_j = G(\Delta u_j) \to G(\Delta u) = u - \int u$ and so $u_j \to u$ as claimed. ∎

Theorem 5.4. *Let Γ be as above. Suppose $\Gamma = dT$ where T is a positive holomorphic chain with mass $\mathbf{M}(T) \leq \Lambda$. Then Γ satisfies the equivalent conditions (1), (2), (3) of Proposition 5.2.*

Proof. Suppose $u \in PSH_\omega(\mathbf{P}^n)$. Since $dd^C u + \omega \geq 0$, we have

$$\int_\Gamma d^C u \;=\; \int_{dT} d^C u \;=\; T(dd^C u) \;=\; T(dd^C u + \omega) - T(\omega) \;\geq\; -T(\omega) \;=\; -\mathbf{M}(T) \;\geq -\Lambda$$

as asserted. ∎

6. The Projective Alexander-Wermer Theorem for Curves.

We now examine the question of whether the equivalent necessary conditions, given in Theorem 5.4 are in fact sufficient.

THEOREM 6.1. *Let $\Gamma \subset \mathbf{P}^n$ be an embedded, oriented, real analytic closed curve, not necessarily connected and with a positive integer multiplicity on each component. Suppose the underlying curve* supp (Γ) *is stable. If Γ satisfies the equivalent conditions (1), (2), (3) of Proposition 5.2, then there exists a positive holomorphic 1-chain T in $\mathbf{P}^n$ with $\mathbf{M}(T) \leq \Lambda$ such that*

$$dT \;=\; \Gamma.$$

NOTE. When it is clear from context, we also use Γ to denote the underlying curve supp Γ.

Proof. Fix a compact set $K \subset \mathbf{P}^n$ and consider the sets

$$\mathcal{C}_K \;\equiv\; \{dd^C T \;:\; T \in \mathcal{P}_{1,1}, \;\; \mathbf{M}(T) \leq 1 \;\text{ and }\; \text{supp}\, T \subset K\}$$
$$\mathcal{S}_K \;\equiv\; \{u \in C^\infty(\mathbf{P}^n) \;:\; dd^C u + \omega \geq 0 \;\text{ on } K\}$$

where $\mathcal{P}_{1,1} \subset \mathcal{E}_2(\mathbf{P}^n)$ denotes the convex cone of positive currents of bidimension (1,1) on $\mathbf{P}^n$. Note that $\mathcal{C}_K$ is a weakly closed, convex subset which contains 0 (since the set of $T \in \mathcal{P}_{1,1}$ with $\mathbf{M}(T) \leq 1$ and supp $T \subset K$ is weakly compact).

Suppose that $\mathcal{K}$ is a weakly closed convex subset containing 0 in a topological vector space V. Then the *polar* of $\mathcal{K}$ is defined to be the subset of the dual space given by

$$\mathcal{K}^0 \;\equiv\; \{L \in V' : (L, v) \;\geq\; -1 \;\text{ for }\; v \in \mathcal{K}\}.$$

Similarly given a subset $\mathcal{L} \subset V'$ we define

$$\mathcal{L}^0 \;\equiv\; \{v \in V : (L, v) \;\geq\; -1 \;\text{ for }\; L \in \mathcal{L}\}.$$

The Bipolar Theorem [S] states that

$$(\mathcal{K}^0)^0 \;=\; \mathcal{K}.$$

PROPOSITION 6.2.

$$\mathcal{S}_K \;=\; (\mathcal{C}_K)^0$$

Proof. Suppose that $u \in C^\infty(\mathbf{P}^n)$ satisfies

$$u(dd^C T) \;=\; T(dd^C u) \;\geq\; -1 \tag{6.1}$$

for all $T \in \mathcal{P}_{1,1}$ with $\mathbf{M}(T) \leq 1$ and supp $T \subset K$. Consider $T = \delta_x \xi$ where $x \in K$ and ξ is a positive (1,1)-vector at x with mass-norm $\|\xi\| = 1$. Then

$$T(dd^C u) \;=\; (dd^C u)(\xi) \;=\; (dd^C u + \omega)(\xi) - 1 \;\geq\; -1$$

by (6.1), and so $u \in \mathcal{S}_K$. This proves that $(\mathcal{C}_K)^0 \subset \mathcal{S}_K$.

For the converse, let $u \in \mathcal{S}_K$ and fix $T \in \mathcal{P}_{1,1}$ with $\mathbf{M}(T) \leq 1$ and supp $T \subset K$. Then

$$(dd^C T)(u) \;=\; T(dd^C u) \;=\; T(dd^C u + \omega) - T(\omega) \;\geq\; T(\omega) \;=\; -\mathbf{M}(T) \;\geq\; -1,$$

and so $u \in (\mathcal{C}_K)^0$. ∎

As an immediate consequence we have the following. If we set

$$\begin{aligned} \Lambda \cdot \mathcal{S}_K \;&\equiv\; \{dd^C T : T \in \mathcal{P}_{1,1}, \mathbf{M}(T) \leq \Lambda \text{ and supp } T \subset K\}, \qquad \text{then} \\ (\Lambda \cdot \mathcal{S}_K)^0 \;&=\; \{u \in C^\infty(\mathbf{P}^n) \;:\; dd^C u + \tfrac{1}{\Lambda}\omega \geq 0 \;\text{ on }\; K\} \end{aligned} \tag{6.2}$$

Recall from §4 that for each $\Lambda \in \mathbf{R}$ we have the compact set

$$\widehat{\Gamma}(\Lambda) \;=\; \{x \in \mathbf{P}^n \;:\; u(x) \leq \sup_\Gamma u + \Lambda \;\; \forall u \in PSH_\omega(\mathbf{P}^n)\}$$

and that $\widehat{\Gamma} = \bigcup_\Lambda \widehat{\Gamma}(\Lambda)$. The following lemma is established in [HL$_5$, 18.7].

Lemma 6.3. *Let u be a C^∞ function which is defined and quasi-plurisubharmonic on a neighborhood of $\{\widehat{\Gamma}\}^-$. Fix $\Lambda > 0$. Then there is a C^∞ function $\widetilde{u}$ which is defined and quasi-plurisubharmonic on all of $\mathbf{P}^n$ and agrees with u on a neighborhood of $\widehat{\Gamma}(\Lambda)$.*

Here $\{\widehat{\Gamma}\}^-$ denotes the closure of $\widehat{\Gamma}$. By our stability assumption, $\widehat{\Gamma} = \widehat{\Gamma}(\Lambda)$ for some Λ, and therefore $\widehat{\Gamma} = \{\widehat{\Gamma}\}^-$. However, we shall keep the notation $\{\widehat{\Gamma}\}^-$, when appropriate, in order to prove the more general result mentioned in Remark 6.5 below.

The lemma above leads to the following.

Proposition 6.4. *If Γ satisfies condition (3) in Proposition 5.2, that is, if*

$$(-d^C\Gamma)(u) \;=\; \int_\Gamma d^C u \;\geq\; -\Lambda$$

for all $u \in PSH_\omega(\mathbf{P}^n)$, then there exists $T \in \mathcal{P}_{1,1}$ with $\mathbf{M}(T) \leq \Lambda$ and $\operatorname{supp} T \subset \{\widehat{\Gamma}\}^-$, such that

$$dd^C T \;=\; -d^C\Gamma \tag{6.3}$$

Proof. If this is not true, then it must fail on some compact neighborhood K of $\{\widehat{\Gamma}\}^-$. (Otherwise, there exists a sequence of positive currents $\{T_j\}_j$, satisfying (6.3), with $M(T_j) \leq \Lambda$ and $\operatorname{supp} T_j \subset K_j$ where K_j are compact neighborhoods squeezing down to $\{\widehat{\Gamma}\}^-$. By the standard compactness theorem for positive currents, there would then be a convergent subsequence whose limit T would satisfy the conclusion of Proposition 6.4.)

By (6.2) and the Bipolar Theorem, we then conclude that there is a smooth function u which is quasi-plurisubharmonic on K with $-d^C\Gamma(u) < -\Lambda$. Applying Lemma 6.3 contradicts the hypothesis. ■

Now let T be the current given by Proposition 6.4. Let V denote the projective hull of Γ and recall from [HLW, Theorem 4.1] that V has the following strong regularity. There exists a Riemann surface S with finitely many components, a compact region $W \subset S$ with real analytic boundary, and a holomorphic map $\rho : S \to \mathbf{P}^n$ which is generically injective and satisfies

(1) $\rho(W) \;=\; V$,

(2) ρ is an embedding on a tubular neighborhood of ∂W in S, and

(3) $\rho(\partial W)$ is a union of components of the support of Γ.

Let Σ denote an ϵ-tubular neighborhood of ∂W on S (with respect to some analytic metric), with ϵ chosen so that

(4) ρ is injective on Σ.

(5) $\rho(\partial^+\Sigma) \cap V \;=\; \emptyset$ where $\partial^+\Sigma$ denotes the "outer" boundary of Σ, i.e. the union of components of $\partial\Sigma$ not contained in W.

Write $\Sigma = \Sigma^+ \cup \partial W \cup \Sigma^-$ where Σ^+ denotes the union of components of $\Sigma - \partial W$ which are not contained in W. Then we have $d[\Sigma^+] = [\partial^+\Sigma] - [\partial W]$ (with standard orientations coming from the domains). Hence we have

$$dd^C[\Sigma^+] \;=\; -d^C[\partial^+\Sigma] + d^C[\partial W].$$

Let $\nu : \Sigma \to \mathbf{Z}$ denote the locally constant, integer-valued function with the property that

$$\rho_*\left(\nu[\partial W]\right) \;=\; \Gamma.$$

Then

$$dd^C\rho_*\left(\nu[\Sigma^+]\right) \;=\; -d^C\rho_*\left(\nu[\partial^+\Sigma]\right) + d^C\Gamma \;\stackrel{\text{def}}{=}\; -d^C\Gamma^+ + d^C\Gamma. \tag{6.4}$$

We now define

$$T^+ \;=\; T + \rho_*\left(\nu[\Sigma^+]\right)$$

and note that by (6.3) we have

$$dd^C T^+ \;=\; -d^C\Gamma^+. \tag{6.5}$$

Now in the open set $\mathbf{P}^n - \Gamma^+$, the current T^+ is a positive (1,1)-current which is supported in the analytic subvariety $V^+ \equiv \rho(W^+) \equiv \rho(\Sigma^+ \cup \overline{W})$ and satisfies $dd^C T^+ = 0$. We note that $\rho : W^+ \to V^+$ is just the normalization of V^+. It follows from [HL_3, Lemma 32] that there is a harmonic function $h : W \to \mathbf{R}^+$ so that

$$T^+ = \rho_*(h[W^+]) \qquad \text{in } \;\mathbf{P}^n - \Gamma^+.$$

This function is evidently constant ($=\nu$) in Σ^+, and hence h is constant on every component of W^+. Thus $T=\rho_*(h[W])$ is a positive holomorphic chain,

Now since $\operatorname{supp} dT \subset \operatorname{supp}\Gamma$ and $\operatorname{supp}\Gamma$ is a regular curve, it follows from the Federer Flat Support Theorem [F, 4.1.15] that dT is of the form $dT=\sum_k c_k[\Gamma_k]$ where the c_k's are constants and the Γ_k's are the oriented connected components of $\operatorname{supp}\Gamma$. The current Γ itself has the form $\Gamma=\sum_k \nu_k[\Gamma_k]$ for integers ν_k. Since $dd^C T = -d^C\Gamma$, we have $d^C(dT-\Gamma)=\sum(c_k-\nu_k)d^C[\Gamma_k]=0$. It follow that $c_k=\nu_k$ for all k, that is, $dT=\Gamma$ as claimed. ∎

Remark 6.5. Theorem 6.1 remains true if one replaces the second statement with the assumption that the closed projective hull $\{\widehat{\Gamma}\}^-$ of Γ is locally contained in a 1-dimensional complex subvariety at each point of $\{\widehat{\Gamma}\}^- - \Gamma$.

Theorem 6.1 can be restated in terms of the classical (affine) linking numbers.

Theorem 6.6. *Let $\Gamma\subset \mathbf{C}^n\subset\mathbf{P}^n$ be an oriented closed curve as in Theorem 6.1. Then there exists a constant Λ_0 so that the classical linking number*

$$\mathrm{Link}_{\mathbf{C}^n}(\Gamma, Z)\ \geq\ -\Lambda_0\ \mathrm{deg}Z$$

for all algebraic hypersurfaces in $\mathbf{C}^n-\Gamma$ if and only if Γ is the boundary of a positive holomorphic 1-chain T in $\mathbf{P}^n=\mathbf{C}^n\cup\mathbf{P}^{n-1}$ with (projective) mass

$$\mathbf{M}(T)\ \leq\ \Lambda_0+\frac{1}{2}\int_\Gamma d^C\log(1+\|z\|^2)$$

Proof. This follows from Remark 3.6 and the fact that the projective Kähler form $\omega=\frac{1}{2}dd^C\log(1+\|z\|^2)$ in $\mathbf{C}^n$. ∎

Note 6.7. The case $\Lambda_0=0$ corresponds to the theorem of Alexander and Wermer [AW$_2$]. A proof of the full Alexander-Wermer Theorem (for C^1-curves with no stability assumption) in the spirit of the arguments above is given in [HL$_6$].

Note that the current of least mass among those provided by Theorem 6.1 is uniquely determined by Γ. All others differ from this one by adding a positive algebraic cycle.

Corollary 6.8. *Let Γ be as in Theorem 6.1 and suppose T is a positive holomorphic 1-chain with $dT=\Gamma$. Then T is the unique holomorphic chain of least mass with $dT=\Gamma$ if and only if*

$$\inf_Z\left\{\frac{T\bullet Z}{\mathrm{deg}Z}\right\}\ =\ 0$$

where the infimum is taken over all algebraic hypersurfaces in the complement $\mathbf{P}^n-\Gamma$.

Proof. Suppose $\inf_Z\{T\bullet Z/\mathrm{deg}Z\}=c>0$. Then $\widetilde{\mathrm{Link}}_{\mathbf{P}}(\Gamma,Z)=T\bullet Z/\mathrm{deg}(Z)-T(\omega)\geq c-T(\omega)$ for all positive divisors in $\mathbf{P}^n-\Gamma$. Hence, by Theorem 6.1 there exists a positive holomorphic chain T' with $dT'=\Gamma$ and $M(T')\leq M(T)-c$.

On the other hand suppose that T is not the unique positive holomorphic chain T_0 of least mass. Then $T=T_0+W$ where W is a positive algebraic cycle, and one has $T\bullet Z/\mathrm{deg}Z=T_0\bullet Z/\mathrm{deg}Z+W\bullet Z/\mathrm{deg}Z\geq W\bullet Z/\mathrm{deg}Z=\mathrm{deg}W$. ∎

7. Theorems for General Projective Manifolds.

The results established above generalize from $\mathbf{P}^n$ to any projective manifold. Let X be a compact complex manifold with a positive holomorphic line bundle λ. Fix a hermitian metric on λ with curvature form $\omega > 0$, and give X the Kähler metric associated to ω. Let Γ be a closed curve with integral weights as in Theorem 6.1, and assume $[\Gamma] = 0$ in $H_1(X; \mathbf{Z})$.

Definition 7.1. Let $Z = \mathrm{Div}(\sigma)$ be the divisor of a holomorphic section $\sigma \in H^0(X, \mathcal{O}(\lambda^\ell))$ for some $\ell \geq 1$. If Z does not meet Γ, we can define the **linking number** and the **reduced linking number** by

$$\mathrm{Link}_\lambda(\Gamma, Z) \;\equiv\; N \bullet Z - \ell \int_N \omega \qquad \text{and} \qquad \widetilde{\mathrm{Link}}_\lambda(\Gamma, Z) \;\equiv\; \frac{1}{\ell}\,\mathrm{Link}(\Gamma, Z)$$

respectively, where N is any 2-chain in Z with $dN = \Gamma$ and where the intersection pairing $\bullet$ is defined as in (3.1) with $\mathbf{P}^n$ replaced by X.

To see that this is well-defined suppose that N' is another 2-chain with $dN' = \Gamma$. Then $(N - N') \bullet Z - \ell \int_{N-N'} \omega = (N - N') \bullet (Z - \ell[\omega]) = 0$ because $Z - \ell\omega$ is cohomologous to zero in X.

Definition 7.2. The **winding number** of a section $\sigma \in H^0(X, \mathcal{O}(\lambda^\ell))$ with $\|\sigma\| > 0$ on Γ, is defined to be

$$\mathrm{Wind}_\lambda(\Gamma, \sigma) \;\equiv\; \int_\Gamma d^C \log\|\sigma\|.$$

The **reduced winding number** is

$$\widetilde{\mathrm{Wind}}_\lambda(\Gamma, \sigma) \;\equiv\; \frac{1}{\ell}\,\mathrm{Wind}_\lambda(\Gamma, \sigma) \;=\; \int_\Gamma d^C \log\|\sigma\|^{\frac{1}{\ell}}$$

From the Poincaré-Lelong equation

$$dd^C \log\|\sigma\| \;=\; \mathrm{Div}(\sigma) - \ell\omega \tag{7.1}$$

we see that

$$\mathrm{Wind}_\lambda(\Gamma, \sigma) \;=\; \mathrm{Link}_\lambda(\Gamma, \mathrm{Div}(\sigma)). \tag{7.2}$$

From (7.1) we also see that $\log\|\sigma\|^{\frac{1}{\ell}}$ belongs to the class $PSH_\omega(X)$ of quasi-plurisubharmonic functions on X defined in 4.1.

Proposition 7.3. *The following are equivalent:*

(a) $\widetilde{\mathrm{Link}}_\lambda(\Gamma, Z) \;\geq\; -\Lambda$ *for all divisors Z of holomorphic sections of λ^ℓ, and all $\ell > 0$.*

(b) $\widetilde{\mathrm{Wind}}_\lambda(\Gamma, \sigma) \;\geq\; -\Lambda$ *for all holomorphic sections σ of λ^ℓ, and all $\ell > 0$.*

(c) $\int_\Gamma d^C u \;\;\geq\;\; -\Lambda$ *for all $u \in PSH_\omega(X)$.*

Proof. That (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) is clear. To see that (c) $\Rightarrow$ (b) we use results of Demailly. Fix $u \in PSH_\omega(X)$ and consider the positive closed (1,1)-current $T \equiv dd^C u + \omega$. Note that $[T] = [\omega] = c_1(\lambda) \in H^2(X; \mathbf{Z})$. It follows from [$\mathrm{D}_2$] that T is the weak limit

$$T = \lim_{j\to\infty} \frac{1}{N_j} \mathrm{Div}(\sigma_j)$$

where $\sigma_j \in H^0(X, \mathcal{O}(\lambda^{\ell_j}))$ and $N_j > 0$. We can normalize this sequence by scalars so that $\mathbf{M}(\frac{1}{N_j}\mathrm{Div}(\sigma_j)) = \mathbf{M}(T)$ for all j. Set $\Omega = \frac{1}{(n-1)!}\omega^{n-1}$. Then $\mathbf{M}(T) = \mathbf{M}(\frac{1}{N_j}\mathrm{Div}(\sigma_j)) = (\frac{1}{N_j}\mathrm{Div}(\sigma_j), \Omega) = \frac{1}{N_j}(\ell_j\omega, \Omega) = \frac{\ell_j}{N_j}(\omega, \Omega) = \frac{\ell_j}{N_j}\mathbf{M}(T)$. Therefore, $\ell_j = N_j$ for all j. Since $\mathrm{Div}(\sigma_j) = dd^C \log\|\sigma_j\| + \ell_j\omega$, we conclude that

$$dd^C u = T - \omega = \lim_{j\to\infty} \left\{ dd^C \log\|\sigma_j\|^{\frac{1}{\ell_j}} + \omega \right\} - \omega \ = \ \lim_{j\to\infty} dd^C \log\|\sigma_j\|^{\frac{1}{\ell_j}}.$$

The remainder of the argument replicates the one given for Proposition 5.3. ■

For a compact subset $K \subset X$ the authors introduced the notion of the λ-hull $\widehat{K}_\lambda$ of K and showed that $\widehat{K}_\lambda = \widehat{K}$ for any embedding $X \hookrightarrow \mathbf{P}^N$ by sections of some power of λ. We shall say that K is **λ-stable** if $\widehat{K}_\lambda$ is compact.

THEOREM 7.4. *Let $\Gamma = \sum_{a=1}^M m_\alpha \Gamma_\alpha$ be an embedded, oriented, real analytic closed curve with integer multiplicities in X, and assume Γ is λ-stable. Then there exists a positive holomorphic 1-chain T in X with $dT = \Gamma$ and $\mathbf{M}(T) \le \Lambda$ if and only if any of the equivalent conditions of Proposition 7.3 is satisfied.*

Proof. If $dT = \Gamma$ and $\mathbf{M}(T) \le \Lambda$, then (a) follows as in the proof of 5.4 above.

For the converse we recall from [HL_5,(4.4)] that the hull $\widehat{K}_\lambda$ of a compact subset $K \subset X$ consists exactly of the points x where the extremal function

$$\Lambda_K(x) \ \equiv \ \sup\{u(x) : u \in PSH_\omega(X) \text{ and } \ u \le 0 \text{ on } X\}$$

is finite. This enables one to directly carry through the arguments for Theorem 6.1 in this case. ■

8. Relative holomorphic cycles.

Our main theorem 6.1 has a nice reinterpretation in terms of the Alexander-Lefschetz duality pairing discussed in section 3.

THEOREM 8.1. *Let $\gamma \subset \mathbf{P}^n$ be a finite disjoint union of real analytic curves with γ stable. Then a class $\tau \in H_2(\mathbf{P}^n, \gamma; \mathbf{Z})$ is represented by a positive holomorphic chain with boundary on γ if and only if*

$$\tau \bullet u \ \ge \ 0 \tag{8.1}$$

for all $u \in H_{2n-2}(\mathbf{P}^n - \gamma; \mathbf{Z})$ represented by positive algebraic hypersurfaces in $\mathbf{P}^n - \gamma$.

Proof. The implication $\Rightarrow$ is clear from the positivity of complex intersections. For the converse, consider the short exact sequence

$$0 \longrightarrow H_2(\mathbf{P}^n;\mathbf{Z}) \longrightarrow H_2(\mathbf{P}^n,\gamma;\mathbf{Z}) \xrightarrow{\delta} H_1(\gamma;\mathbf{Z}) \longrightarrow 0.$$

Note that

$$\delta\tau = \sum_{k=1}^{\ell} m_k \overrightarrow{\gamma}_k \equiv \Gamma$$

where $\overrightarrow{\gamma}_1, ..., \overrightarrow{\gamma}_\ell$ are the connected components of γ with a chosen orientation and the n_k's are integers which we can assume to be positive. We are assuming that (8.1) holds for any positive algebraic class u. If $\Gamma = 0$, the desired conclusion is immediate, so we assume that $\Gamma \neq 0$. Now let $Z \subset \mathbf{P}^n - \gamma$ be any positive algebraic hypersurface, and note that

$$0 \leq \frac{\tau \bullet Z}{\deg Z} = \frac{\tau \bullet Z}{\deg Z} - \tau(\omega) + \tau(\omega) = \mathrm{Link}_{\mathbf{P}}(\Gamma, Z) + \tau(\omega).$$

Therefore, by Theorem 6.1, there exists a positive holomorphic 1-chain T with $dT = \Gamma$ and $M(T) \leq \tau(\omega)$. Note that $\delta([T] - \tau) = 0$ and $([T] - \tau)(\omega) \geq 0$. Hence, $[T] - \tau$ is a positive class in $H_2(\mathbf{P}^n;\mathbf{Z})$ and is represented by a positive algebraic 1-cycle, say W. Therefore, $\tau = [T + W]$ is represented by a positive holomorphic chain as claimed. ■

This result leads to a nice duality between the cone in $H_2(\mathbf{P}^n,\gamma;\mathbf{Z})$ of those classes which are represented by positive holomorphic chains, and the cone in $H_{2n-2}(\mathbf{P}^n - \gamma;\mathbf{Z})$ of classes represented by positive algebraic hypersurfaces. This duality, in fact, extends to more general projective manifolds. All this is discussed in detail in [HL$_8$]

References

[AW$_1$] H. Alexander and J. Wermer, Several Complex Variables and Banach Algebras, Springer-Verlag, New York, 1998.

[AW$_2$] H. Alexander and J. Wermer, *Linking numbers and boundaries of varieties*, Ann. of Math. **151** (2000), 125-150.

[D$_1$] J.-P. Demailly, *Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète*, Ann. Sci. E. N. S. **15** no. 4 (1982), 457-511.

[D$_2$] J.-P. Demailly, *Courants positifs extrêmaux et conjecture de Hodge*, Inventiones Math. **69** no. 4 (1982), 347-374.

[DL] T.-C. Dinh and M. Lawrence, *Polynomial hulls and positive currents*, Ann. Fac. Sci de Toulouse **12** (2003), 317-334.

[Do] P. Dolbeault, *On holomorphic chains with given boundary in* $\mathbf{CP}^n$, Springer Lecture Notes in Math., **1089** (1983), 1135-1140.

[DH_1] P. Dolbeault and G. Henkin , *Surfaces de Riemann de bord donné dans* $\mathbf{CP}^n$, pp. 163-187 in " Contributions to Complex Analysis and Analytic Geometry" , Aspects of Math. Vieweg**26** , 1994.

[DH_2] P. Dolbeault and G. Henkin, *Chaînes holomorphes de bord donné dans* $\mathbf{CP}^n$, Bull. Soc. Math. de France, **125** (1997), 383-445.

[DS] J. Duval and N. Sibony *Polynomial convexity, rational convexity and currents*, Duke Math. J. **79** (1995), 487-513.

[F] H. Federer, Geometric Measure Theory, Springer–Verlag, New York, 1969.

[GS] H. Gillet and C. Soulé, *Arithmetic chow groups and differential characters*, pp. 30-68 in "Algebraic K-theory; Connections with Geometry and Topology", Jardine and Snaith (eds.), Kluwer Academic Publishers, 1989.

[G] V. Guedj, *Approximation of currents on complex manifolds*, Math. Ann. **313** (1999), 437-474.

[GZ] V. Guedj and A. Zeriahi, *Intrinsic capacities on compact Kähler manifolds*, Preprint Univ. de Toulouse , 2003

[Ha] B. Harris, *Differential characters and the Abel-Jacobi map*, pp. 69-86 in "Algebraic K-theory; Connections with Geometry and Topology", Jardine and Snaith (eds.), Kluwer Academic Publishers, 1989.

[H] F.R. Harvey, Holomorphic chains and their boundaries, pp. 309-382 in "Several Complex Variables, Proc. of Symposia in Pure Mathematics XXX Part 1", A.M.S., Providence, RI, 1977.

[HL_1] F. R. Harvey and H. B. Lawson, Jr, *On boundaries of complex analytic varieties, I*, Annals of Mathematics **102** (1975), 223-290.

[HL_2] F. R. Harvey and H. B. Lawson, Jr, *On boundaries of complex analytic varieties, II*, Annals of Mathematics **106** (1977), 213-238.

[HL_3] F. R. Harvey and H. B. Lawson, Jr, *An intrinsic characterization of Kähler manifolds*, Inventiones Math., **74** (1983), 169-198.

[HL_4] F. R. Harvey and H. B. Lawson, Jr, *Boundaries of varieties in projective manifolds*, J. Geom. Analysis, **14** no. 4 (2005), 673-695. ArXiv:math.CV/0512490.

[HL_5] F. R. Harvey and H. B. Lawson, Jr, *Projective hulls and the projective Gelfand transformation*, Asian J. Math. **10**, no. 2 (2006), 279-318. ArXiv:math.CV/0510286.

[HL_6] F. R. Harvey and H. B. Lawson, Jr, *Remarks on the Alexander-Wermer Theorem for Curves*, Matemática Contemporânea, **31** (2006), 53-64. ArXiv: math.CV/0610611.

[HL_7] F. R. Harvey and H. B. Lawson, Jr, *Projective linking and boundaries of holomorphic chains in projective manifolds, Part II*, pp. 365-380 in "Inspired by S. S. Chern", P. Griffiths, Ed., Nankai Tracts in Math. **11**, World Scientific Publishing, Singapore, 2006.

[HL_8] F. R. Harvey and H. B. Lawson, Jr, *Relative holomorphic cycles and duality*, Preprint, Stony Brook, 2006.

[HLZ] F. R. Harvey, H. B. Lawson, Jr. and J. Zweck, *A deRham-Federer theory of differential characters and character duality*, Amer. J. of Math. **125** (2003), 791-847.

[HLW] F. R. Harvey, H. B. Lawson, Jr. and J. Wermer, *The projective hull of certain curves in* $\mathbf{C}^2$, ArXiv:math.CV/0611482.

[HP] F. R. Harvey, J. Polking, *Extending analytic objects*, Comm. Pure Appl. Math. **28** (1975), 701-727.

[S] H. H. Schaefer, Topological Vector Spaces, Springer Verlag, New York, 1999.

[W_1] J. Wermer *The hull of a curve in* $\mathbf{C}^n$, Ann. of Math., **68** (1958), 550-561.

[W_2] J. Wermer *The argument principle and boundaries of analytic varieties*, Operator Theory: Advances and Applications, **127** (2001), 639-659.