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NOTES ON THE ARTICLE: A CURRENT APPROACH TO MORSE AND NOVIKOV THEORIES

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The following article, *A current approach to Morse and Novikov Theories*, is the doctoral thesis of Giulio Minervini, which was defended in 2003 and archived at the Università degli Studi di Roma "La Sapineza". Giulio, a good friend and colleague, was in the process of preparing an expanded version of this thesis for publication when he met an untimely death. Since his results are somewhat difficult to access, and there is current interest in the subject, we have decided to submit his thesis for publication.

As a guidepost to the reader, we present in this prefatory note an outline of some of the highlights of Giulio's results and a brief discussion of how they relate to the work of others.

The thesis is concerned with complete vector fields with generalized hyperbolic singularities on a smooth manifold X. The first main result provides very general sufficient conditions under which the stable and unstable manifolds of the flow in X, and the graph of the flow in $X \times X$, have locally finite volume. In fact he shows that these sets have a structure, which he introduces, called *horned stratified*. This then provides a direct connection between the geometry of the flow and other more topological quantities associated with the manifold.

Starting earlier with [14], and certainly since 2003, there have been many expositions geometrizing the gradient flow of a Morse-Bott function. (For example , see [1] – [5], [9], [11], [14] – [16] and the references therein.) However, all of these, in one form or the other (including the most recent, [5]), assume a strong "tameness" condition on the vector field $V = \nabla f$ in a neighborhood of each critical set. For example, for isolated critical points, they assume that in certain local coordinates $z = (x, y)$, V is linear, i.e., $V = x \cdot (\partial/\partial x) - y \cdot (\partial/\partial y)$. By contrast, Minervini's results apply to quite general vector fields, and could be considered a part of the theory of dynamical systems, utilizing the boundary value technique of Shilnikov [18].

Recently Daniel Cibotaru [5] has proved a strong desingularization theorem for certain gradients of Morse-Bott functions with some very nice applications. This provided the impetus for our endeavor here.

The general set-up of Minervini's work starts with a complete vector field which is gradient-like with respect to a smooth function f, i.e., $Vf > 0$ outside the zeros of V. (Later Minervini noted that all he used was the existence of a continuous function f which is strictly increasing in time on any flow line through a non-critical point.)

The simplest case of his main theorem occurs when the manifold X is compact and the singularities of the vector field are isolated points. The replacement for the linear Morse condition is that each singularity be hyperbolic, i.e., that the linear term for the vector field at each critical point have no purely imaginary eigenvalue. In this case the stable and unstable manifolds S_x and U_x of a critical point x are well defined, and one assumes the standard Smale condition that $S_x \pitchfork U_y$, i.e., each pair is transversal. As a consequence of his finite volume theorem, a fundamental current equation holds on $X \times X$:

$$\partial T \;=\; \Delta - \sum_{x \in \mathrm{Cr}} S_x \times U_x \tag{1}$$

connecting the graph T of the flow, the diagonal Δ in $X \times X$, and the stable manifold S_x and unstable manifold U_x at each critical point $x \in \mathrm{Cr}$. This equation is very geometric and powerful. It gives rise to an operator equation (cf. [13]):

$$d \circ \mathbf{T} + \mathbf{T} \circ d \;=\; I - \mathbf{P} \tag{2}$$

which is a chain homotopy between the identity and the operator $\mathbf{P}$ from forms to currents given by $\mathbf{P}(\alpha) = \lim_{t \to \infty} \varphi_t^* \alpha$, that is, by pulling back under the flow and taking $t \to \infty$. The operator $\mathbf{P}$ is also defined directly as the following linear combination

$$\mathbf{P}(\alpha) \;=\; \sum_{x \in \mathrm{Cr}} \left(\int_{U_x} \alpha \right) S_x$$

of stable manifolds with "residue" coefficients. It is also straightforward to deduce the formula (cf. [14], [9], [11])

$$\partial S_x \;=\; \sum_{x \in \mathrm{Cr}} N_{x,y} S_y \tag{3}$$

where $N_{x,y}$ is the algebraic number of one-dimensional components of $S_x \cap U_y$. This is an equation of currents for these general flows.

Now one can proceed in either of two directions: allowing the critical set to be of higher dimension (where an abundance of applications occur, e.g. [7] – [10]), or dropping the compactness assumption on X (where Novikov Theory follows [17]) – or both.

First assume X is compact but the zero-set of V consists of the union of a family $\mathcal{C}r$ of connected disjoint submanifolds with possibly varying dimensions. In this case Giulio defines a notion of hyperbolicity for V along any $C \in \mathcal{C}r$ which extends the classical case where $\dim(C) = 0$ (see Def. 5.1.1). Under this assumption Giulio shows that each $C \in \mathcal{C}r$ has a stable and unstable manifold, S_C and U_C respectively, fibered by stable and unstable manifolds S_x, U_x of the points $x \in C$. He then makes a *generalized Smale hypothesis* that that for all critical points x, y one has $U_x \pitchfork S_{C_y}$ and $U_{C_x} \pitchfork S_y$ where $C_x \in \mathcal{C}r$ denotes the component of x. (This hypothesis goes back to Janko Latschev [15].) Finally, he assumes the existence of a continuous function $f : X \to \mathbb{R}$ which is strictly increasing on flow lines as above.

With these hypotheses Giulio establishes the structure of a horned stratified set for each S_C, U_C, $C \in \mathcal{C}r$, and for the graph of the flow. This then establishes the equation of currents which induces a chain homotopy of operators as in (1), (2) and (3) above. (See Theorem Thm. 5.2.11.) Such formulas together with their consequences were derived by Latchev under much stronger assumptions on the nature of the vector field in neighborhoods of the critical submanifolds.

In the second generalization Minervini considers the situation where V has isolated hyperbolic fixed points but the manifold X is non-compact. This is an important case because it leads to the establishment of a *current-theoretic approach to Novikov Theory*. In fact Minervini develops a quite general Morse Theory which yields a chain complex for the cohomology of X with *forward supports*. It pairs non-degenerately to the cohomology with *backward supports*, defined via the negative $-V$ of the given vector field V. The approach uses current formulas and chain homotopies of operators as above. The current approach to Novikov theory for circle valued Morse functions, or more generally for closed 1-forms with non-degenerate zeros, is an application of this general theory. So also is a certain duality in Novikov theory, which appears to be a new result.

In this second set-up one again assumes that the vector field V is gradient-like with respect to a smooth function $f : X \to \mathbb{R}$. One does *not* assume that f is a proper exhaustion. Instead Minervini introduces an important notion a condition of *weak properness* of V with respect to f (Def. 2.1.2). In the case we are now considering, where the zeros of V are isolated (or more generally for certain Bott-type singularities), weak properness is equivalent to the fact that each broken flow line contained in a slab $f^{-1}([a, b])$ is relatively compact. One also assumes the *Smale transversality condition* that $U_x \pitchfork S_y$ for each pair of critical points x and y.

Under these hypotheses Minervini establishes the local structure of a horned stratified set for each S_x and U_x and for the graph of the flow. This, in particular, proves locally finite volume for each such submanifold at any point of its closure. The result leads to an equation of currents (1). To obtain a resulting chain homotopy of operators as in (2) Minervini introduces the concept of a current with compact/forward support. He then proves that:

The complex of currents given by the stable manifolds computes the (co)homology of X with compact/forward supports .

A set $A ß X$ is a **compact/forward** set with respect to the function f if

(i) $A \cap f^{-1}([b, c])$ is compact for any $b \leq c$ in $\mathbb{R}$, and
(ii) $A ß f^{-1}([a, \infty))$ for some $a \in \mathbb{R}$.

There is a companion notion of compact/backward sets for f. They are just the compact/forward sets for $-f$. These two families of sets are *paracompactifying families* in the sense of Godement [6], and the Čech cohomology groups $H^*_\uparrow(X, \mathbb{R})$ and $H^*_\downarrow(X, \mathbb{R})$, with compact/forward and compact/backward supports respectively, are well defined. Furthermore, there is a **deRham Theorem** which asserts a natural isomorphism of $H^*_\uparrow(X, \mathbb{R})$ with the cohomology of the complex of smooth forms on X supported in compact/forward sets.

Minervini's theorem establishes a "Poincaré Duality" between the groups $H^k_\uparrow(X, \mathbb{R})$ and $H^{n-k}_\downarrow(X, \mathbb{R})$ where $n = \dim(X)$.

The application to Morse-Novikov Theory for a function $f_0 : X_0 \to S^1$ now follows by lifting f_0 to a function $f : X \to \mathbb{R}$ where $X \to X_0$ is the infinite cyclic covering induced by f_0 from the covering $\mathbb{R} \to S^1$.

The general case of a closed 1-form with non-degenerate singularities and possibly irrational periods is treated similarly with some additional work.

This current-geometric treatment of Novikov Theory is given in Chapter 4 of Minervini's thesis. It was actually published in [12] where certain technical aspects are discussed in more detail, otherwise this paper [12] is almost entirely due to Giulio Minervini.

REFERENCES

[1] D. M. Austin and P. J. Braam, *Morse-Bott theory and equivariant cohomology*. The Floer memorial volume, 123183, Progr. Math., **133**, Birkhuser, Basel, 1995.

[2] D. Burghelea, L. Friedlander and T. Kappeler, *On the space of trajectories of a generic gradient-like vector field*, An. Univ. Vest Timisoara, Ser. Mat.-Inform. **48** (2010), 45-126.

[3] D. Burghelea and S. Haller, *On the topology and analysis of a closed one form. I (Novikov's theory revisited).*Essays on geometry and related topics, Monogr. Enseign. Math. **38** (2001), 133175.

[4] —, *The geometric complex of a Morse-Bott-Smale pair and and extension of a theorem by Bismut-Zhang*, ArXiv: 0409166.

[5] D. Cibotaru, *Vertical flows and a general currential homotopy formula*, ArXiv: 1405.0952.

[6] R. Godement, *Topologie Algébrique et Théorie des Faisceaux*, Hermann, Paris, 1973

[7] R. Harvey and H. B. Lawson, A Theory of Characteristic Currents Associated with a Singular Connection, Astèrisque vol. 213, Soc. Math. de France, Montrouge, France, 1993.
[8] — , *Geometric residue theorems*, Amer. J. Math. **117** no. 4 (1995), 829-874.
[9] —-, *Morse Theory and Stokes' Theorem*, pp. 271-323 in Surveys in Differential Geometry, vol VII, International Press, Boston, 2000.
[10] —, *Singularities and Chern-Weil Theory I – The local MacPherson formula*, Asian J. Math **4** (2000), 71-96.
[11] —, *Finite volume flows and Morse theory*, Annals of Math. **153** no.1 (2001), 1-25.
[12] R. Harvey and G. Minervini, *Morse Novikov theory and cohomology with forward supports*, Math. Ann. **335** (2006), 787-818.
[13] R. Harvey and J. Polking, *Fundamental solutions in complex analysis, Part I*, Duke Math. J. **46** (1979), 253-300.
[14] F. Laudenbach, *On the Thom-Smale complex*, Appendix, Astérisque **205**, Soc. Math. de France, 1992.
[15] J. Latschev, *Gradient flows of Morse-Bott functions*, Math. Ann. **318** No. 4, 2000, 731-759.
[16] L. I. Nicolaescu, *Tame Flows*, Memoirs A.M.S. **208**, Amer. Math. Soc., Providence, 2010.
[17] S. P. Novikov, *Multi-valued functions and functionals. An analogue of Morse Theory*, Soviet Mathematics Dolklady **24** (1981), 222-226.
[18] L. Shilnikov, *Methods of non-qualitative theory in nonlinear dynamics Part I*, Nonlinear Science, World Scientific 1998.