1. Let $\Omega \subset \mathbb{C}$ be a domain with a piecewise smooth boundary $\partial \Omega$. Let $f \in C(\overline{\Omega})$ be holomorphic on $\Omega$. Please state the basic case of Cauchy’s Integral Formula in this setting.

**ANSWER:**

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad \text{for} \; z \in \Omega.$$ 

2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions in a domain $\Omega \subset \mathbb{C}$. Suppose that $f_n$ converges uniformly on compact subsets of $\Omega$ to a function $f : \Omega \rightarrow \mathbb{C}$. Show that $f$ is holomorphic. Show also that for every integer $k > 0$,

$$\frac{\partial^k f_n}{\partial z^k} \text{ converges uniformly on compact subsets of } \Omega \text{ to } \frac{\partial^k f}{\partial z^k}.$$

**ANSWER:** It will suffice to prove this on a small ball about each point in $\Omega$ by the Balzano-Weierstrauss Theorem. Fix $z_0 \in \Omega$ and $0 < \rho < r$ with $\{|z - z_0| \leq r\} \subset \Omega$.

$$f_n(z) - \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f_n(\zeta) - f(\zeta)}{\zeta - z} \, d\zeta$$

In $\{|z - z_0| \leq \rho\}$ the right hand side is bounded in absolute value by

$$\frac{1}{r - \rho} \sup_{|z - z_0| = r} |f_n - f| \rightarrow 0$$

and we see that

$$f(z) = \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

since $f_n$ converges to both sides uniformly in $\{|z - z_0| \leq \rho\}$. For the rest use the Cauchy formula for the derivatives of holomorphic functions and make the same estimate.
3. Let $f$ be holomorphic in \( \{ z \in \mathbb{C} : 0 < |z - z_0| < r \} \). Prove that:

(a) If \( \lim_{z \to z_0} (z - z_0)f(z) = 0 \), then $f$ extends holomorphically across $z_0$.

(b) If \( \lim_{z \to z_0} |f(z)| = \infty \), then $f(z) = (z - z_0)^{-k}g(z)$ where $k$ is a positive integer and $g(z)$ extends holomorphically across $z_0$ with $g(z_0) \geq 0$.

(Assume the existence of power series for holomorphic functions.)

(c) If neither (a) nor (b) happens, then the image of $f$ is dense in $\mathbb{C}$.

ANSWER: (a) I had in mind that you would use the Cauchy Formula on the annulus \( \{ \epsilon < |z| < r/2 \} \), and then take $\epsilon \to 0$ and show that the interior term vanishes. The resulting formula gives the extension. (On the other hand you could have quoted the version given in Ahlfors.)

(b) Take $g(z) = 1/f(z)$ and apply (a). The power series for $g$ gives $g(z) = (z - z_0)^kh(z)$ where $h(z_0) \neq 0$.

(c) If the range is not dense, then there exists $w_0 \in \mathbb{C}$ and $\epsilon > 0$ so that $|f(z) - w_0| \geq \epsilon$ for all $0 < |z - z_0| < r$. Then $1/(f(z) - w_0)$ is bounded by $1/\epsilon$ and (a) applies. This easily shows that $f(z)$ is either regular at $z_0$ or has a pole there.

4. Let $f$ be holomorphic in \( \{0 < |z| < 1\} \).

(a) What is the Laurent series for $f$?

(b) Prove that it exists, and show where and how it converges.

ANSWER: This should be in your lecture notes.

5. How many zeros (counted to multiplicity) does the function

\[ f(z) = z^6 + 13z^4 - 2z + 12 \]

have in the ball $|z| < 2$.

ANSWER: An easily calculation shows that on $|z| = 2$, we have $|z^6 - 2z + 12| < |13z^4|$ so by Rouché’s Theorem there are 4 zeros in \( \{|z| < 2\} \).

6. Let $f$ be holomorphic in \( \{|z| < 1\} \) with

\[ f(0) = f'(0) = \cdots f^{(n-1)}(0) = 0 \quad \text{and} \quad |f| \leq M. \]

Prove that

\[ |f(z)| \leq M|z|^n \]

ANSWER: We can factor $f(z) = z^n g(z)$ where $g$ is holomorphic across 0. Now $|g(z)| \leq M/r^n$ for $|z| = r$. By the maximum principle, $|g(z)| \leq M/r^n$ on \( \{|z| \leq r\} \). Let $r \to 1$.  

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7. Let $\gamma$ be a cycle in a domain $\Omega$ in $\mathbb{C}$.

   (a) What does it mean for $\gamma$ to be homologous to zero in $\Omega$?

   (b) Give an example of a domain $\Omega$ and two cycles $\gamma_1$ and $\gamma_2$ in $\Omega$ with one homologous to zero in $\Omega$ and the other not.

   (c) What is the general form of Cauchy’s Theorem?

ANSWER: This was straightforward.

8. What is the value of

$$\frac{1}{2\pi i} \int_{|z|=10} \frac{z^3}{z^4 + 1} \, dz$$

ANSWER: By applying the Argument Principle you can see (without any calculation) that the integral is equal to $\frac{1}{4} N$ where $N$ is the number of zeros of $z^4 + 1$ in the disk $\{ |z| < 10 \}$. Namely, we have $N = 4$ and the integral is 1.