Midterm Exam – MAT 536

March 19

1. Let $\Omega \subset\subset \mathbf{C}$ be a domain with a piecewise smooth boundary $\partial\Omega$.
Let $f \in C(\overline{\Omega})$ be holomorphic on Ω. Please state the basic case of Cauchy's Integral Formula in this setting.
ANSWER:

$$f(z) \;=\; \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad \text{for } z \in \Omega.$$

2. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions in a domain $\Omega \subset \mathbf{C}$. Suppose that f_n converges uniformly on compact subsets of Ω to a function $f : \Omega \to \mathbf{C}$. Show that f is holomorphic. Show also that for every integer $k > 0$,

$$\frac{\partial^k f_n}{\partial z^k} \text{ converges uniformly on compact subsets of } \Omega \text{ to } \frac{\partial^k f}{\partial z^k}.$$

ANSWER: It will suffice to prove this on a small ball about each point in Ω by the Balzano-Weierstrauss Theorem. Fix $z_0 \in \Omega$ and $0 < \rho < r$ with $\{|z - z_0| \le r\} \subset \Omega$.

$$f_n(z) - \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f_n(\zeta) - f(\zeta)}{\zeta - z}\, d\zeta$$

In $\{|z - z_0| \le \rho\}$ the right hand side is bounded in absolute value by

$$\frac{1}{r-\rho}\sup_{|z-z_0|=r} |f_n - f| \;\to\; 0$$

and we see that

$$f(z) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

since f_n converges to both sides uniformly in $\{|z - z_0| \le \rho\}$. For the rest use the Cauchy formula for the derivatives of holomorphic functions and make the same estimate.

3. Let f be holomorphic in $\{z \in \mathbf{C} : 0 < |z - z_0| < r\}$. Prove that:

(a) If $\lim_{z \to z_0} (z - z_0) f(z) = 0$, then f extends holomorphically across z_0.

(b) If $\lim_{z \to z_0} |f(z)| = \infty$, then $f(z) = (z - z_0)^{-k} g(z)$ where k is a positive integer and $g(z)$ extends holomorphically across z_0 with $g(z_0) \geq 0$. (Assume the existence of power series for holomorphic functions.)

(c) If neither (a) nor (b) happens, then the image of f is dense in $\mathbf{C}$.

ANSWER: (a) I had in mind that you would use the Cauchy Formula on the annulus $\{\epsilon < |z| < r/2\}$, and then take $\epsilon \to 0$ and show that the interior term vanishes. The resulting formula gives the extension. (On the other hand you could have quoted the version given in Ahlfors.)

(b) Take $g(z) = 1/f(z)$ and apply (a). The power series for g gives $g(z) = (z - z_0)^k h(z)$ where $h(z_0) \neq 0$.

(c) If the range is not dense, then there exists $w_0 \in \mathbf{C}$ and $\epsilon > 0$ so that $|f(z) - w_0| \geq \epsilon$ for all $0 < |z - z_0| < r$. Then $1/(f(z) - w_0)$ is bounded by $1/\epsilon$ and (a) applies. This easily shows that $f(z)$ is either regular at z_0 or has a pole there.

4. Let f be holomorphic in $\{0 < |z| < 1\}$.

(a) What is the Laurent series for f?

(b) Prove that it exists, and show where and how it converges.

ANSWER: This should be in your lecture notes.

5. How many zeros (counted to multiplicity) does the function

$$f(z) = z^6 + 13z^4 - 2z + 12$$

have in the ball $|z| < 2$.

ANSWER: An easly calculation shows that on $|z| = 2$, we have $|z^6 - 2z + 12| < |13z^4|$ so by Rouché's Theorem there are 4 zeros in $\{|z| < 2\}$.

6. Let f be holomorphic in $\{|z| < 1\}$ with

$$f(0) = f'(0) = \cdots f^{(n-1)}(0) = 0 \qquad \text{and} \qquad |f| \leq M.$$

Prove that

$$|f(z)| \leq M|z|^n$$

ANSWER: We can factor $f(z) = z^n g(z)$ where g is holomorphic across 0. Now $|g(z)| \leq M/r^n$ for $|z| = r$. By the maximum principle, $|g(z)| \leq M/r^n$ on $\{|z| \leq r\}$. Let $r \to 1$.

7. Let γ be a cycle in a domain Ω in $\mathbf{C}$.

(a) What does it mean for γ to be homologous to zero in Ω?

(b) Give an example of a domain Ω and two cycles γ_1 and γ_2 in Ω with one homologous to zero in Ω and the other not.

(c) What is the general form of Cauchy's Theorem?

ANSWER: This was straightforward.

8. What is the value of

$$\frac{1}{2\pi i}\int_{|z|=10}\frac{z^3}{z^4+1}\,dz?$$

ANSWER: By applying the Argument Principle you can see (without any calculation) that the integral is equal to $\frac{1}{4}N$ where N is the number of zeros of z^4+1 in the disk $\{|z|<10\}$. Namely, we have $N=4$ and the integral is 1.