Homework 7

Due Monday Mar. 19 at the beginning of class

1. Show that a C^2-function u on the annulus $\{R_1 < |z| < R_2\}$ is harmonic if and only if, in polar coordinates (r, θ), it satisfies

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

For u harmonic on the annulus $\{R_1 < |z| < R_2\}$, consider the function

$$\mu(r) \equiv \int_0^{2\pi} u(re^{i\theta})\, d\theta$$

Show that $\mu(r) = a \log r + b$ for constants a and b.

2. Suppose that $f(z)$ is holomorphic on the closed annulus $\{r_1 \leq |z| \leq r_2\}$ and consider the function

$$M(r) = \max_{|z|=r} |f(z)|$$

for $r_1 \leq r \leq r_2$. ($r_1 > 0$.) Prove that

$$M(r)^{\log \frac{r_2}{r_1}} = M(r_1)^{\log \frac{r_2}{r}} M(r_2)^{\log \frac{r}{r_1}}.$$

This is Hadamard's **Three Circle Theorem**.
Hint: Apply the maximum principle to a linear combination of $\log |f(z)|$ and $\log |z|$.

3. Let $U(t)$ be a bounded, piece-wise continuous (with a finite number of discontinuities), real-valued function on the real line $\mathbf{R}$. Show that

$$P_U(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\frac{y}{(x-t)^2 + y^2}\right\} U(t)\, dt$$

is a harmonic function in the upper half plane $\{y \geq 0\}$ with boundary values $U(t_0)$ at all points t_0 of continuity of U. (Do not worry about ∞.)

Hint: The integrand as a function of (x, y) is harmonic, in fact the imaginary part of a holomorphic function you might be able to see. Then use some basic analysis. After that, follow the argument given in the book for Schwarz's Theorem.

CONCERNING THE MIDTERM

For the midterm you should know the basic definitions and theorems, and the fundamental arguments for their proofs. I will not have any questions about the proof of Green's Theorem. However, you should know Green's Theorem in real and complex forms, and how it applies to basic results in complex analysis. Part of the exam will ask you to use the results we have proved to answer concrete problems.