Homework 6

Due Monday Mar. 5 at the beginning of class

1. Fix a domain $\Omega \subset \mathbf{C}$. Let $Z_1(\Omega)$ be the space of cycles in Ω (as defined in class). Note that $Z_1(\Omega)$ is a abelian group.

Definition 1. Two cycles γ_1, γ_2 in Ω are **homologous** (written $\gamma_1 \cong \gamma_2$) if $\gamma_1 - \gamma_2$ is homologous to zero in Ω. (The definition of "homologous to zero in Ω" is taken here from class or Ahlfors.)

Show that $\cong$ is an equivalence relation.

Definition 2. The **first homology group of** Ω is the set of equivalence classes:

$$H_1(\Omega) \ = \ Z_1(\Omega)/\cong$$

Show that $H_1(\Omega)$ is an abelian group.

2. (a) Calculate $H_1(\Omega)$ when

$$\Omega = \{z : 1 < |z| < 2\}.$$

(b) Calculate $H_1(\Omega)$ when

$$\Omega = \{z : |z| < 2\} - \{1\} - \{-1\}.$$

3. Let $\Omega \subset \mathbf{C}$ be a bounded domain whose complement has a finite number of connected components:

$$\mathbf{C} - \Omega \ = \ T_0 \cup T_1 \cup \cdots \cup T_N$$

where T_0 is the unbounded component.

Show using dyadic subdivisions that there exist domains $D_1, ..., D_N$ is $\mathbf{C}$ such that:

(i) $\overline{D}_i \cap \overline{D}_j = \emptyset$ if $i \neq j$.

(ii) ∂D_i is piece-wise linear (consisting of horizontal and vertical intervals).

(iii) $T_k \subset D_k$ for all k.

Let $C_k = \partial D_k$. Show that

$$H_1(\Omega) \ = \ \mathbf{Z}[C_1] \oplus \cdots \oplus \mathbf{Z}[C_N]$$

4. (a) Let $f(z) = z^k + 2$ for an integer $k > 0$. What is

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{f'(z)}{f(z)-a}\,dz \quad \text{for} \quad |a-2| \neq 1$$

(b) Same question for $f(z) = z^{-k} + 2$ for an integer $k > 0$.

(c) Let f be as in (a) and $|a-2| < 1$. What is

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{f'(z)\,z^k}{f(z)-a}\,dz$$

5. Using Rouché's Theorem find the number of zeros of

$$z^6 - 4z^4 + 1 \qquad \text{in} \qquad |z| < 3.$$

6. Using Rouché's Theorem find the number of zeros of

$$z^4 - 2z^3 + 9z^2 + z - 1 \qquad \text{in} \qquad |z| < 2.$$