Homework 3
Due Monday Feb. 12 at the beginning of class

Below all circular arcs are oriented in the counterclockwise direction.

1. Compute
\[ \int_{|z|=1} \frac{e^z}{z} \, dz. \]

2. Compute
\[ \int_{|z|=2} \frac{e^z}{z^n} \, dz. \]

3. Compute
\[ \int_{|z|=2} \frac{1}{z^2 + 1} \, dz. \]

4. Compute
\[ \int_{|z|=\rho} \frac{|dz|}{|z-a|^2} \]

under the assumption that $|a| \neq \rho$.

Hint: Make use of the equations $z\overline{z} = \rho^2$ and

\[ |dz| = -i\rho \frac{dz}{z} \quad \text{(prove this.)} \]

5. Let $F : \mathbb{C} \to \mathbb{C}$ be holomorphic in the entire complex plane, and suppose there exist constants $M, c$ and an integer $n$ such that

\[ |f(z)| \leq M|z|^n \]

for all $z$ with $|z| \geq c$. Show that $f(z)$ is a polynomial.

6. Show that the successive derivatives of a holomorphic function $f$ at a point $z$ can never satisfy $|f^{(n)}(z)| > n!n^n$.

7. Consider the function
\[ f(z) = e^z \frac{1}{(z-1)^2}. \]

What is the radius of convergence of the power series for $f(z)$ at $a = 1 + 2i$?
8. Let $\Omega \subset \mathbb{C}$ be a domain, and fix a point $z_0 \in \Omega$. Using theorems proved in class this week, show that:

(i) If $f(z)$ is holomorphic in $\Omega - \{z_0\}$, and if

$$\lim_{z \to z_0} (z - z_0)f(z) = 0,$$

then there is an analytic extension of $f(z)$ across $z = z_0$.

(ii) If $f(z)$ is holomorphic in $\Omega - \{z_0\}$, and if

$$\lim_{z \to z_0} |f(z)| = \infty,$$

then there is an integer $n > 0$ and a function $g(z)$ analytic in a neighborhood of $z_0$, with $g(z_0) \neq 0$, such that

$$f(z) = \frac{1}{(z - z_0)^n} g(z)$$

in a neighborhood of $z_0$. 
