Homework 3

Due Monday Feb. 12 at the beginning of class

Below all circular arcs are oriented in the counterclockwise direction.

1. Compute
$$\int_{|z|=1} \frac{e^z}{z}\,dz.$$

2. Compute
$$\int_{|z|=2} \frac{e^z}{z^n}\,dz.$$

3. Compute
$$\int_{|z|=2} \frac{1}{z^2+1}\,dz.$$

4. Compute
$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

under the assumption that $|a| \neq \rho$.

Hint: Make use of the equations $z\overline{z} = \rho^2$ and

$$|dz| \;=\; -i\rho\frac{dz}{z} \qquad \text{(prove this.)}$$

5. Let $F : \mathbf{C} \to \mathbf{C}$ be holomorphic in the entire complex plane, and suppose there exist constants M, c and an integer n such that

$$|f(z)| \;\leq\; M|z|^n$$

for all z with $|z| \geq c$. Show that $f(z)$ is a polynomial.

6. Show that the successive derivatives of a holomorphic function f at a point z can never satisfy $|f^{(n)}(z)| > n!n^n$.

7. Consider the function
$$f(z) \;=\; e^z\frac{1}{(z-1)^2}.$$

What is the radius of convergence of the power series for $f(z)$ at $a = 1 + 2i$?

8. Let $\Omega \subset \mathbf{C}$ be a domain, and fix a point $z_0 \in \Omega$. Using theorems proved in class this week, show that:

(i) If $f(z)$ is holomorphic in $\Omega - \{z_0\}$, and if

$$\lim_{z \to z_0} (z - z_0) f(z) \;=\; 0,$$

then there is an analytic extension of $f(z)$ across $z = z_0$.

(ii) If $f(z)$ is holomorphic in $\Omega - \{z_0\}$, and if

$$\lim_{z \to z_0} |f(z)| \;=\; \infty,$$

then there is an integer $n > 0$ and a function $g(z)$ analytic in a neighborhood of z_0, with $g(z_0) \neq 0$, such that

$$f(z) \;=\; \frac{1}{(z - z_0)^n} g(z)$$

in a neighborhood of z_0.