Homework 12

Due Monday April 23th at the beginning of class.

1. Let $\Omega \subset \mathbb{C}^2$ be a domain. A smooth function $f : \Omega \to \mathbb{C}$ is **holomorphic** if its tangent linear map $df_x : T_x \mathbb{C}^2 \cong \mathbb{C}^2 \to \mathbb{C}$ is complex linear at each point (i.e., it commutes with scalar multiplication by $i$). A smooth map from $\Omega$ to $\mathbb{C}^2$ is holomorphic if its two coordinate functions are holomorphic.

   (i) Set
   $$\Delta^2 = \{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < 1\} \quad \text{and} \quad I^4 = [-1, 1] \times \cdots \times [-1, 1]$$
   where the splitting of $I^4$ is according to the real and imaginary coordinates axes. Does there exist a homeomorphism $F : \Delta^2 \to I^4$ which is holomorphic on $\Delta^2$? Why?

   (ii) Let $f : \Delta^2 \to \mathbb{C}$ be a continuous map which is holomorphic in $\Delta^2$. Show that
   $$\sup_{\Delta^2} |f| \leq \sup_{|z|=|w|=1} |f(z, w)| \quad (\text{the max on the } 2 \text{-torus}).$$

   (iii) Show that there does not exist a homeomorphism $F : \Delta^2 \to B$ which is holomorphic on $\Delta^2$, where $B$ is the unit ball about the origin in $\mathbb{C}^2$.

2. Show that with proper choice of branches the function
   $$f(z) = \int_0^z \frac{1}{\sqrt{\zeta(1-\zeta^2)}} \, d\zeta$$
   maps the upper half plane onto a square.

3. Let $H$ be the upper half plane. Let $T$ be an equilateral triangle in $\mathbb{C}$ with vertices $A$, $B$ and $C$.

   (1) Why does there exist a holomorphic automorphism
   $$f : H \to T$$
   continuous on the closure, which sends $0, 1, \infty$ to $A, B, C$?

   (2) Why is $f$ unique?

   (3) Using Shwarz reflection, extend the mapping $f^{-1}$ to the entire complex plane.