Homework 1

Due Monday Jan 29 at the beginning of class

1. Prove that

$$\left|\frac{z-a}{1-\overline{a}z}\right| \;<\; 1 \qquad \text{if } |a|<1 \;\text{ and }\; |z|<1.$$

Fix a with $|a|<1$. Show that

$$f(z) \;=\; \frac{z-a}{1-\overline{a}z} \quad \text{gives a holomorphic diffeomorphism of } \;\{z : |z|<1\}$$

2. Show that there are complex numbers z satisfying

$$|z-a|+|z+a| \;=\; 2|c|$$

if and only if $|a| \leq |c|$.

3. Let $\omega = e^{\frac{2\pi i}{n}}$. Show that for any integer p which is not a multiple of n,

$$1+\omega^p+\omega^{2p}+\cdots+\omega^{(n-1)p} \;=\; 0.$$

4. Show that if $z, w \in \mathbf{C}$ correspond to antipodal points on the sphere S^2 under stereographic projection, then $z\overline{w} = -1$.

5. Show that a holomorphic function cannot have a constant absolute value without being (locally) constant.

6. Prove that if $f(z)$ is holomorphic on $U^{\text{open}} \subset \mathbf{C}$, then $\overline{f(\overline{z})}$ is holomorphic on $\overline{U}$ (= the image of U under complex conjugation).

7. If $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence R_1 and R_2, then $\sum a_n b_n z^n$ has radius of convergence $\geq R_1 R_2$.

8. (Do not do this one this week). Let $f : S^2 \to S^2$ be a mapping given by the rational function

$$\frac{a_0+a_1 z+\cdots+a_n z^n}{b_0+b_1 z+\cdots+b_m z^m}$$

where the numerator and denominator have no common roots and $a_n b_m \neq 0$. What is the degree of f? Take any definition of degree that you want. One possibility is

$$deg(f) \;=\; \int_{S^2} f^*\omega$$

where ω is a volume form of integral 1 on S^2.