Prop. 2. Fix \( x \in X \). Then
\[
\exists U \text{ open } \ni x \text{ and } \exists \epsilon > 0 \text{ s.t.}
\]
1. Any two points of \( U \) are endpoints of a unique geodesic of length < \( \epsilon \).

2. This geodesic \( Y(x_1, x_2, t) \) depends smoothly on the endpoints \( x_1, x_2 \) and on the parameter \( t \).
   In particular, if \( U_i = \frac{\partial Y}{\partial t} (x_1, x_2, 0) \)
   then \((x_1, U_i)\) is smooth in \((x_1, x_2)\).

3. \( \forall y \in U \)
   \[
   \exp_y : \{ v \in T_y X : \| v \| < \epsilon \} \xrightarrow{\sim} U_y
   \]
   is a diffeomorphism onto an open set \( U_y \subset U \).
Theorem 3. Let $y = y(x_1, x_2)$ be a geodesic from Prop. 2. Then $y$ is the unique shortest curve joining its endpoints.
Cor 1 \[ K^{\text{cpt}} \subset X \]

Then \( \exists \varepsilon > 0 \) so any two points \( x, y \in K \) with \( d(x, y) < \varepsilon \) are joined by a unique geodesic \( \gamma \) of length \( < \varepsilon \).

Furthermore, \( \gamma \) is minimal and \( \gamma \) depends \( C^\infty \) on its endpoints.

\[ Pf \]

For all \( x \in K \) choose \( \mathcal{U}_x, \varepsilon_x \) as in Prop. 2. Choose \( r_x \) so \( \mathcal{B}(x, 2r_x) \subset \mathcal{U}_x \).

Cover \( K \) by a finite no. of \( \mathcal{B}(x_j, r_j) \).

\( \varepsilon \equiv \min \{ r_j \} \).
If \( x, y \in K \) with
\[
d(x, y) < \varepsilon
\]

then \( \exists x_f \in K \) such that \( d(x, x_f) < \varepsilon \).

\[
d(x, y) < d(x, x_f) + d(x_f, y)
\]

\[
< \varepsilon + \varepsilon < 2\varepsilon
\]

\[
\therefore x, y \in W(x_f)
\]

and Prop 2 + Thm 3 apply.

\[qed\]
Corollary 2. Let $\sigma : [0,L] \to X$ be a piecewise $C^1$ curve parameterized by arc-length. If $\sigma$ is length-minimizing, then $\sigma$ is a geodesic.

If Cor. 1 with $K = V([0,L])$,

$$\forall t, \sigma \mid_{[t-\varepsilon, t+\varepsilon]}$$

is a geodesic.

Geodesic Convexity

Definition. A set $U \subset X$ is called geodesically convex if $\forall y_1, y_2 \in U$ there exists a unique minimal geodesic $\gamma$ in $X$ from $y_1$ to $y_2$ and $\gamma \in U$. 
Prop. 4. \textit{Fix }x \in X. \textit{Then}

1. $B(x, r)$ is geodesically convex if $r$ suff. small

2. \exists \ r_0 \text{ s.t. } B(y, r_0) \text{ is geodesically convex for } y \in B(x, \delta) \text{ and all } 0 < r < r_0.

For proof we need:

\textbf{Def} \ u \in C^2(X). \textit{The Riemannian Hessian of } u \textit{is the section}

$$\text{Hess}(u) \in \text{Sym}^2(T^*X)$$

\textit{defined by}

$$\{\text{Hess}(u)\}(V, W) = VWu - \nabla_V Wu$$

\textit{for } V, W \textit{- fields } V, W.
Note \( A \) tensor in \( V \)

\[
(\text{Hess} \ u)(v, w) = (\text{Hess} \ u)(W, V)
\]

Since

\[
vwu - \partial_v wu - wvwu + \partial_w vu = (\det (v, w) - \partial_v w - \partial_w v) u
\]

\[
= T_v w u = 0
\]

A tensor in \( W \) and symmetric.

Discuss: Only need \( \text{Tor} = 0 \)

However, in general \( \text{Hess} \ u \)

only makes sense at cr. points

Lemma

\[
(\text{Hess} \ u)(v, w) = \langle \nabla_v (\nabla u), w \rangle
\]

Pf

Fix \( v, w \in T_x X \). Extend to local \( v \)-fields \( V, W \).

\[
(\text{Hess} \ u)(v, w) = vw u - (\nabla_v w) u
\]

\[
= V \langle \langle w, \nabla u \rangle \rangle - \langle \nabla_v w, \nabla u \rangle
\]

\[
= \langle \nabla_v w, \nabla u \rangle + \langle w, \nabla_v (\nabla u) \rangle - \langle \nabla_v w, \nabla u \rangle
\]
\[
= \langle \nabla_v (\nabla u), w \rangle
\]

an eq of tensors. qed

**Lemma b** If \( y(s) \) is a geodesic,

\[
\frac{d^2}{ds^2} u(y(s)) = (\text{Hess } u)(\frac{dy}{ds}, \frac{dy}{ds})
\]

**Pf**

\[
\frac{d}{ds} u(y(s)) = \langle \nabla u, \frac{dy}{ds} \rangle
\]

\[
\frac{d^2}{ds^2} u(y(s)) = \langle \nabla_{\frac{dy}{ds}} \nabla u, \frac{dy}{ds} \rangle
\]

\[
+ \langle \nabla u, \frac{D}{ds} \frac{dy}{ds} \rangle
\]

\[
= (\text{Hess } u)(\frac{dy}{ds}, \frac{dy}{ds})
\]

by Lemma a. qed
Lemma: Let $u(x) = \frac{1}{2} d^2(x, x_0)$. Then

$$(\text{Hess}_{x_0} u)(v, v) = ||v||^2$$

i.e. as a sym. map $TX \to TX$

$$\text{Hess}_{x_0} u \simeq \text{Id}.$$ 

\textbf{Pf} Fix $v \in T_{x_0} X$. Assume $||v|| = 1$. \textit{wlog}

Let $\gamma(s)$ be good with

$\gamma(0) = x_0, \quad \frac{d}{ds} \gamma(0) = v$

Now $u(\gamma(s)) = \frac{1}{2} s^2$.

.. by Lemma b,

$$(\text{Hess}_{x_0} u)(v, v) = \frac{d^2}{ds^2} u(\gamma(s)) \bigg|_{s=0} = 1.$$ 

\textit{q.e.d}
Lemma d  \( \Omega \) open

Suppose

1. \( \gamma: [a,b] \rightarrow \Omega \) a geodesic
2. \( u \in C^2(\Omega) \) satisfies
   \[ \text{Hess}_x u > 0 \quad \forall x \in \Omega \]

Then \( u(\gamma(s)) \) is strictly convex on \([a,b]\). In particular
- no interior maxima

Pf  Lemma b.
Proof of Prop 3

Fix $x_0$.

Let $U$ (and $\varepsilon > 0$) be from Prop 2,

Assume w.l.o.g.

$$ U = B(x_0, 4r) \quad \text{some} r > 0 $$

By shrinking $r$, assume by Lem. c

$$ \text{Hess} \left( \frac{1}{2} d(x, x_0^2) \right) > 0 \quad \text{on} \quad U. $$

We know for any $x, y \in B(x_0, r)$

$\exists! \text{ good } Y \text{ joining } x \to y$, and

1. $L(s) = d(x, y)$
2. $Y \subset B(x_0, 4r)$

(Thm 3)

Now $u = \frac{1}{2} d^2(x, x_0)$ is convex on $Y$ and $\partial u$ cannot have interior max. \[ \Rightarrow \quad U | Y \leq \max (u(x), u(y)). \]
\[ u_\epsilon \leq \max (u(x), u(y)) \]

\[
\text{dist} (x(t), x_0) \leq \\
\max \{ \text{dist} (x, x_0), \text{dist} (x, y_0) \}^2 < r
\]

\[ y \in B(x_0, r) \]

This proves part 1.

Proof of part 2

\[
\exists r > 0 \text{ s.t.} \\
\text{Hess}_x \left\{ \frac{1}{2} d^2(x, y) \right\} > 0
\]

\[ \forall x, y \in B(x_0, 2r) \]
Hence if \( y \in B(x_0, r) \)

Then \( B(y, r) \subset B(x, 2r) \)

\[
\text{Hess}_x \left\{ \frac{1}{2} d^2(x, y) \right\} > 0
\]

\[
\forall x \in B(y, r) \\
\forall y \in B(x_0, r)
\]

We assume \( r \leq \text{previous } r \) s.t. \( B(x_0, 8r) \subset U \)

\[
\forall \ B(y, 4r) \subset U
\]

Now every \( B(y, r) \) is good convex for \( y \in B(x_0, r) \) by same argument.  

\text{\textit{zed}}
Theorem 5: For each compact subset $K \subseteq X$, $\exists \varepsilon > 0$ s.t. 
\[ \forall x \in K, \; B(x, \varepsilon) \text{ is good convex.} \]

Proof (Exercise): A compactness argument like proof of Theorem Cor 1.

\[ \forall x \in K, \; \exists r_x > 0 \text{ s.t.} \]
\[ \forall y \in B(x, 2r_x), \; B(y, 2r_x) \text{ is $g$-convex.} \]

Take finite cover $B(x_t, r_t)$

\[ \varepsilon \equiv \min \{ r_t \} \]

Fix $x \in K$ and suppose $y \in B(x, \varepsilon)$

\[ \exists x_t \text{ s.t.} \; d(x, x_t) < r_t. \]

\[ d(y, x_t) < d(y, x) + d(x, x_t) < \varepsilon + r_t < 2r_t. \]

\[ \therefore B(y, \varepsilon) \subseteq B(y, 2r_t) \text{ is } g \text{-convex.} \]

$\qed$
For $x \in X$, the convexity radius $\rho$ of $X$ at $x$ is

$$\operatorname{conv-rad}(x) = \sup \{ r : B(x,r) \text{ is geod.-convex} \}$$

For $E \subset X$ a subset

$$\operatorname{conv-rad}(E) = \inf_{x \in E} \operatorname{conv-rad}(x)$$

Thm 5. If $E$ is cpt, then

$$\operatorname{conv-rad}(E) \geq 0$$