A GENERALIZATION OF PDE’S FROM A KRYLOV POINT OF VIEW

F. REESE HARVEY AND H. BLAINE LAWSON, JR.

Abstract. We introduce and investigate the notion of a “generalized equation” of the form \( f(D^2u) = 0 \), based on the notions of subequations and Dirichlet duality. Precisely, a subset \( \mathcal{H} \subset \text{Sym}^2(\mathbb{R}^n) \) is a generalized equation if it is an intersection \( \mathcal{H} = E \cap (-\tilde{G}) \) where \( E \) and \( G \) are subequations and \( \tilde{G} \) is the subequation dual to \( G \). We utilize a viscosity definition of “solution” to \( \mathcal{H} \). The mirror of \( \mathcal{H} \) is defined by \( \mathcal{H}^* \equiv G \cap (-E) \). One of the main results here concerns the Dirichlet problem on arbitrary bounded domains \( \Omega \subset \mathbb{R}^n \) for solutions to \( \mathcal{H} \) with prescribed boundary function \( \varphi \in C(\partial \Omega) \). We prove that:

(A) Uniqueness holds \( \iff \) \( \mathcal{H} \) has no interior, and
(B) Existence holds \( \iff \) \( \mathcal{H}^* \) has no interior.

For (B) the appropriate boundary convexity of \( \partial \Omega \) must be assumed. Many examples of generalized equations are discussed.

A general form of the main theorem, which holds on any manifold, is also established.

Table of Contents

1. Introduction/Preliminaries
2. Main Definitions
3. The Canonical Pair Defining a Given \( \mathcal{H} \)
4. Examples
5. Unsettling Questions
6. A General Case of the Main Theorem

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1. Introduction/Preliminaries.

The purpose of this paper is to examine, to the fullest extent, when viscosity and Dirichlet duality techniques can be employed to study nonlinear differential relations. For clarity and simplicity we restrict attention to the constant coefficient case until the last Section 6. We adopt the subequation point of view from [13], [2] and [3], where a differential operator $f$ is replaced by the constraint set $\mathcal{F} \equiv \{ A \in \text{Sym}^2(\mathbb{R}^n); f(A) \geq 0 \}$ (Sym$^2(\mathbb{R}^n)$ denotes the space of quadratic forms on $\mathbb{R}^n$.) The equation $f(D^2u) = 0$ is replaced by the constraint condition $D^2u \in \partial \mathcal{F}$. The ellipticity hypothesis can be stated in the weakest possible form as

$$\mathcal{F} + \mathcal{P} \subset \mathcal{F}$$

(1.1)

where $\mathcal{P} = \{ A \in \text{Sym}^2(\mathbb{R}^n) : A \geq 0 \}$. Any closed subset $\mathcal{F} \subset \text{Sym}^2(\mathbb{R}^n)$ satisfying this positivity, or $\mathcal{P}$-monotonicity, condition (1.1), is called a subequation.$^1$

The viscosity definition of a subsolution takes the following form. Consider an upper semi-continuous function $u$ defined on an open set $X \subset \mathbb{R}^n$ and taking values in $[-\infty, \infty)$. An upper test function for $u$ at a point $x \in X$ is a $C^2$-function $\varphi$ defined near $x$ with $u \leq \varphi$ and $u(x) = \varphi(x)$. The function $u$ is said to be $\mathcal{F}$-subharmonic or an $\mathcal{F}$-subsolution if for every upper test function $\varphi$ at any point $x \in X$ we have $D_x^2 \varphi \in \mathcal{F}$. For $C^2$-functions $u$, the consistency of this definition with the classical definition that $D_x^2 u \in \partial \mathcal{F}$ for all $x$, follows from the positivity condition (1.1).

The Dirichlet dual $\overline{\mathcal{F}}$ of a subequation $\mathcal{F}$ is defined to be

$$\overline{\mathcal{F}} = \sim (-\text{Int} \mathcal{F}) = - (\sim \mathcal{F})$$

(1.2)

It is also a subequation and provides a true duality $\overline{\mathcal{F}} = \mathcal{F}$. Moreover, one has the key relationship

$$\partial \mathcal{F} = \mathcal{F} \cap (-\overline{\mathcal{F}})$$

(1.3)

which enables one to replace $f(D^2u) = 0$ by $\partial \mathcal{F}$ via the viscosity definitions (see Def. 2.1). It is easy to see that

$$\text{Int} \overline{\mathcal{F}} = - (\sim \mathcal{F}) = \sim (-\mathcal{F}),$$

(1.4)

and to see that (1.1), together with $\mathcal{F}$ being closed, implies the topological property

$$\mathcal{F} = \text{Int} \overline{\mathcal{F}}.$$ 

(1.5)

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$^1$It is convenient occasionally in this paper to allow $\mathcal{F} = \emptyset$ or $\mathcal{F} = \text{Sym}^2(\mathbb{R}^n)$. 

2. Main New Definitions.

To begin we recall the notion of an equation in the sense of [13], [2] and [3].

**Definition 2.1. (Equation).** A subset $\mathbb{H} \subset \text{Sym}^2(\mathbb{R}^n)$ is a **determined equation** or just an **equation** if $\mathbb{H} = \partial F$ for some subequation $F$. In this case a **solution to the equation** $\mathbb{H}$, or an **$\mathbb{H}$-harmonic**, is a function $u$ such that $u$ is $F$-subharmonic and $-u$ is $\tilde{F}$-subharmonic.

Such functions are automatically continuous by definition. For $C^2$-functions the consistency of this definition with the classical definition that $D^2_x u \in \mathbb{H} = \partial F$ for all $x$, follows from the fact that

$$\partial F = F \cap (-\tilde{F}).$$

(2.1)

and the consistency property for subequations.

This can be generalized as follows.

**Definition 2.2. (Generalized Equation).** A subset $\mathbb{H} \subset \text{Sym}^2(\mathbb{R}^n)$ is a **generalized equation** if $\mathbb{H} = E \cap (-\tilde{G}) = E \cap (\sim \text{Int} G)$ (2.2)

for some pair of subequations $E, G$. Just as in the case $E = G = F$ we define a **solution to the (generalized) equation** $\mathbb{H}$, or an **$\mathbb{H}$-harmonic** to be a function $u$ such that $u$ is $E$-subharmonic and $-u$ is $\tilde{G}$-subharmonic.

**Remark.** In fact, one could consider more general intersections of subequations and their negatives $\mathbb{H} = E_1 \cap \cdots \cap E_k \cap (-\tilde{G}_1) \cap \cdots \cap (-\tilde{G}_\ell)$, but this is not more general. The positivity condition (1.1) and the closure condition are both preserved under intersections. Hence $E = E_1 \cap \cdots \cap E_k$ and $\tilde{G} = \tilde{G}_1 \cap \cdots \cap \tilde{G}_\ell$ are subequations, and $\mathbb{H} = E \cap (-\tilde{G})$. (Also since $\tilde{\tilde{F}} = F$ and $\tilde{F}_1 \cap \tilde{F}_2 = \tilde{F}_1 \cup \tilde{F}_2$ for subequations, one can show that $G = G_1 \cup \cdots \cup G_\ell$.)

As before by definition such functions are continuous with the coherence property that if $u$ is $C^2$, then

$$u \text{ is } \mathbb{H}\text{-harmonic} \iff D^2_x u \in \mathbb{H} \text{ for all } x$$

Note that for any generalized equation $\mathbb{H} = E \cap (-\tilde{G})$, the interior satisfies

$$\text{Int} \mathbb{H} = (\text{Int} E) \cap (-\text{Int} \tilde{G}) = (\text{Int} E) \cap (\sim G)$$

(2.3)

by (1.4). In particular,

If $\mathbb{H} = \partial F = F \cap (-\tilde{F})$ is a determined equation,

then $\text{Int} \mathbb{H} = (\text{Int} F) \cap (\sim F) = \emptyset$.

(2.4)

Each generalized equation has a mirror.
Definition 2.3. (The Mirror Equation). If
\[ H = E \cap (-\tilde{G}) \]
is a generalized equation, its **mirror** is defined to be the generalized equation
\[ H^* = G \cap (-\tilde{E}). \]

Examination of the Dirichlet Problem for \( H \)-harmonics leads to four distinct types of generalized equations, as follows.

**Definition 2.4.** Suppose \( \Omega \) is a bounded domain in \( \mathbb{R}^n \). We say that **existence for the (DP) for** \( H \) **holds on** \( \Omega \) if for all prescribed boundary functions \( \varphi \in C(\partial \Omega) \) there exists \( h \in C(\overline{\Omega}) \) satisfying
(a) \( h|_\Omega \) is \( H \)-harmonic, and
(b) \( h|_{\partial \Omega} = \varphi. \)

We say **uniqueness for the (DP) for** \( H \) **holds on** \( \Omega \) if for all \( \varphi \in C(\partial \Omega) \) there exists at most one \( h \in C(\overline{\Omega}) \) satisfying (a) and (b).

Now we can state our main result in this pure second-order constant coefficient case.

**Theorem 2.5.** Suppose \( H = E \cap (-\tilde{G}) \) is a generalized equation with mirror \( H^* = G \cap (-\tilde{E}) \), and that \( \Omega \subset \mathbb{R}^n \) is a bounded domain. Then

(A) Uniqueness for the (DP) for \( H \) holds on \( \Omega \) \( \iff \) \( \text{Int} \ H = \emptyset \)

Suppose that \( \partial \Omega \) is smooth and strictly \( G \) and \( \tilde{G} \)-convex. Then

(B) Existence for the (DP) for \( H \) holds on \( \Omega \) \( \iff \) \( \text{Int} \ H^* = \emptyset \)

In fact, the following are equivalent:

1. \( \text{Int} \ H = \emptyset \),
2. Uniqueness for the (DP) for \( H \) holds on \( \Omega \),
3. \( E \subset G \),
4. \( H = \partial E \cap \partial G \),

and assuming that \( \partial \Omega \) is smooth and both strictly \( E \) and \( \tilde{E} \) convex,

5. Existence for the (DP) for \( H^* \) holds on \( \Omega \).

Interchanging \( E \) with \( G \) and \( H \) with \( H^* \), we have the mirror list of equivalences:

1. * \( \text{Int} \ H^* = \emptyset \),
2. * Uniqueness for the (DP) for \( H^* \) holds on \( \Omega \),
A GENERALIZATION OF PDE’S FROM A KRYLOV POINT OF VIEW

(3)* $G \subset E,
(4)* \quad H^* = \partial G \cap \partial E,$

and assuming that $\partial \Omega$ is smooth and both strictly $G$ and $\tilde{G}$ convex,

(5)* Existence for the (DP) for $H$ holds on $\Omega.$

Proof. It suffices to prove that (1) through (5) are equivalent because: (A) is just the statement that (2) $\iff$ (1), the mirror equivalences (1)* through (5)* are immediate from (1) through (5), and (B) is just the statement (1)* $\iff$ (5)*.

Before proving the equivalence of (1) through (5) we list some trivial equivalences for any sets $E$ and $G.$

Lemma 2.6. Suppose that $E = \text{Int} E$ and $G = \text{Int} G,$ then

(i) $E \subset G$  (ii) $\text{Int} E \subset G$  (iii) $\text{Int} E \subset \text{Int} G$  (iv) $\tilde{G} \subset E$  (v) $-\tilde{G} \subset -E$

are equivalent. Interchanging $E$ and $G$ yields that

(i)* $G \subset E$  (ii)* $\text{Int} G \subset E$  (iii)* $\text{Int} G \subset \text{Int} E$  (iv)* $\tilde{E} \subset \tilde{G}$  (v)* $-\tilde{E} \subset -\tilde{G}$

are equivalent.

Proof. Note that (i) $\implies$ (ii) obviously, (ii) $\implies$ (iii) since $\text{Int} E$ is an open subset contained in $G,$ (iii) $\implies$ (i) by the hypothesis, (i) $\iff$ (iv) follows from the definitions of the duals, and (iv) $\iff$ (v) is trivial.

Corollary 2.7. If $H \equiv E \cap (-\tilde{G})$ is a generalized equation with mirror $H^* = G \cap (-E),$ then

(1) $\text{Int} H = \emptyset$ is equivalent to (i) through (v), and

(1)* $\text{Int} H^* = \emptyset$ is equivalent to (i)* through (v)*.

Proof. Note that (2.3) says that

\[ \text{Int} H = (\text{Int} E) \sim G. \]  \tag{2.5}

Hence, (1) $\iff$ (ii). Interchanging $E$ and $G$ yields (1)* $\iff$ (ii)*.

Proof that (1) $\iff$ (3): By Corollary 2.7, (1) $\iff$ (i) which is (3).

Proof that (4) $\implies$ (1): If $H = \partial E \cap \partial G,$ then in particular $H \subset \partial E$ which has no interior.

Proof that (1) $\implies$ (4): Note that $E = \partial E \cup (\text{Int} E)$ and $-\tilde{G} = \partial G \cup (\sim G)$ are disjoint unions. Hence, $H \equiv E \cap (-\tilde{G})$ is the disjoint union of the four sets: $\partial E \cap \partial G,$ $\partial E \cap (\sim G),$ $\text{Int} E \cap \partial G,$ and $\text{Int} E \cap (\sim G).$ By (2.2) and (2.3), the last set $\text{Int} E \cap (\sim G) = \text{Int} H = \emptyset,$ so that $H$ is the disjoint union of the three remaining sets. However, $\text{Int} H = (\text{Int} E) \cap (\sim G) = \emptyset$ implies
(3) and hence $\partial E \subset G$ or $\partial E \cap (\sim G) = \emptyset$. By Lemma 2.6 (iii) Int $E \subset \text{Int} G$ so that $(\text{Int} E) \cap \partial G = \emptyset$. This leaves $H = \partial E \cap \partial G$.

**Proof that (1) $\Rightarrow$ (2):** Recall from [3, Def. 8.1] the following form of comparison (C) for a subequation $F$, which we will refer to as the **zero maximum principle for sums**, and abbreviate as either (ZMP for sums) or (C).

**Definition 2.8.** Given a relatively compact domain $\Omega$ we say that comparison holds for $F$ on $\Omega$ if for all upper semi-continuous functions $u, v$ on $\Omega$, with $u_{\Omega} F$-subharmonic and $v_{\Omega} \tilde{F}$-subharmonic, one has

$$u + v \leq 0 \text{ on } \partial \Omega \Rightarrow u + v \leq 0 \text{ on } \overline{\Omega} \text{ (ZMP for sums)}$$

Comparison (C) “always” holds for subequations $F \subset \text{Sym}^2(\mathbb{R}^n)$ and domains $\Omega \subset \subset \mathbb{R}^n$. This was first established in [2, Rmk. 4.9 and Thm. 6.4]. (There are also proofs in [3] with extensions to simply-connected, non-positively curved manifolds.) More precisely we have:

**THEOREM 2.9.** Suppose $F \subset \text{Sym}^2(\mathbb{R}^n)$ is a subequation and $\Omega \subset \subset \mathbb{R}^n$ is a bounded domain. Then comparison (C) holds for $F$ on $\Omega$.

Now we can prove that (1) $\Rightarrow$ (2)

**Proposition 2.10.** Comparison (C) for both $E$ and $G$ on a domain $\Omega$ implies that:

$$\text{Int } H = \emptyset \Rightarrow \text{uniqueness for the } H-(DP) \text{ on } \Omega$$

**Proof.** By Corollary 2.7, $\text{Int } H = \emptyset \Rightarrow \tilde{G} \subset \tilde{E}$. Therefore (C) for $E$ implies the (ZMP for sums) if $u$ is $E$-subharmonic and $v$ is $\tilde{G}$-subharmonic. If $h_1, h_2$ are two solutions to the $H-(DP)$ on $\Omega$ with the same boundary values, then $u = h_1$ is $E$-subharmonic and $v = -h_2$ is $\tilde{G}$–subharmonic on $\Omega$. Since $u + v = 0$ on $\partial \Omega$, the (ZMP) $\Rightarrow h_1 \leq h_2$ on $\overline{\Omega}$. Interchanging $h_1$ and $h_2$ is possible since we are also assuming (C) for $G$. This proves $h_1 = h_2$.

**Proof that (2) $\Rightarrow$ (1):**

**Proposition 2.11.** If there exists a function $h \in C^2(\Omega) \cap C(\overline{\Omega})$ with $D_x^2 h \in \text{Int } H$ for all $x \in \Omega$, then uniqueness for the $H-(DP)$ on $\Omega$ fails.

**Proof.** Take $\varphi = h|_{\partial \Omega}$. For any function $\psi \in C^\infty_{\text{cpt}}(\Omega)$, if $\epsilon > 0$ is sufficiently small, we have $D_x^2(h + \epsilon \psi) \in H$ for all $x \in \Omega$. Thus the functions $h + \epsilon \psi$ give many solutions to the Dirichlet problem with the same boundary values $\varphi$. 


The following trivial fact is peculiar to the pure second-order, constant coefficient case (and the pure first-order case).

**Lemma 2.12.** Given any non-empty subset $S \subset \text{Sym}^2(\mathbb{R}^n)$, there exists a function $h \in C^2(\mathbb{R}^n)$ with $D^2_x h \in S$ for all $x \in \mathbb{R}^n$.

**Proof.** Pick $A \in S$ and take $h(x) \equiv \frac{1}{2} \langle Ax, x \rangle$ so that $D^2_x h = A$ for all $x \in \mathbb{R}^n$.

Combining this Lemma with the previous Proposition proves the implication (2) $\Rightarrow$ (1) in the form:

$$\text{Int } H \neq \emptyset \Rightarrow \text{uniqueness for the } H-(DP) \text{ fails on all domains } \Omega \subset \mathbb{R}^n. \quad (2.6)$$

Next we treat the implication (1) $\text{Int } H = \emptyset \Rightarrow (5)$ existence for the $H^*-(DP)$.

**Proposition 2.13.** If existence for the $\partial \mathbb{E}-(DP)$ holds on $\Omega$ (Definition 2.5), then

$$\text{Int } H = \emptyset \Rightarrow \text{existence for the } H^*-(DP) \text{ on } \Omega. \quad (2.7)$$

**Proof.** By Corollary 2.7 $\text{Int } H = \emptyset \Rightarrow \mathbb{E} \subset \mathbb{G}$. Let $h$ denote the $\partial \mathbb{E}$-harmonic function solving the (DP) with boundary values $\varphi$. Since $h$ is $\mathbb{E}$-subharmonic and $\mathbb{E} \subset \mathbb{G}$, it is also $\mathbb{G}$-subharmonic. Since $-h$ is $\tilde{\mathbb{E}}$-subharmonic, this proves that $h$ is $H^* = \mathbb{G} \cap (-\tilde{\mathbb{E}})$-harmonic.

Recall the following from [2]:

**THEOREM 2.14. (Existence).** Suppose $\Omega \subset \mathbb{R}^n$ has a smooth boundary which is both $\mathbb{E}$ and $\tilde{\mathbb{E}}$ strictly convex. Given $\varphi \in C(\partial \Omega)$, the Perron function $h(x) \equiv \sup \{ u \in \text{USC}(\overline{\Omega}) : u \in \mathbb{E}(\Omega) \text{ and } u|_{\partial \Omega} \leq \varphi \}$ solves the $\partial \mathbb{E}-(DP)$ on $\Omega$ for boundary values $\varphi$.

Combining Proposition 2.13 with Theorem 2.14 yields

$$(1) \text{ Int } H = \emptyset \Rightarrow (5) \text{ existence for the } H^*-(DP) \quad (2.8)$$

on domains $\Omega$ with strictly $\mathbb{E}$ and $\tilde{\mathbb{E}}$ convex smooth boundaries.

Before proving that (5) $\Rightarrow$ (1), or that $\text{Int } H \neq \emptyset$ implies non-existence for the $H^*-(DP)$, we need to establish some preliminary facts, which are also of independent interest.

**Proposition 2.15.** Fix boundary values $\varphi \in C(\partial \Omega)$. If there exist solutions $h$ to the $H-(DP)$ and $h^*$ to the $H^*-(DP)$ on $\Omega$, then $h = h^*$. That is, $h = h^*$ is
the common solution to the \( H \) and the \( H^* \) Dirichlet problems with boundary values \( \varphi \).

**Proof.** By definition \( h \) is \( E \)-subharmonic and \( -h \) is \( \tilde{G} \)-subharmonic on \( \Omega \). Also, \( h^* \) is \( G \)-subharmonic and \( -h^* \) is \( \tilde{E} \)-subharmonic on \( \Omega \). Therefore,

\[
    h - h^* = 0 \quad \text{on } \partial \Omega \quad \Rightarrow \quad h - h^* \leq 0 \quad \text{on } \overline{\Omega} \quad \text{by } E\text{-comparison},
\]

\[
    h^* - h = 0 \quad \text{on } \partial \Omega \quad \Rightarrow \quad h^* - h \leq 0 \quad \text{on } \overline{\Omega} \quad \text{by } G\text{-comparison}.
\]

Thus \( h - h^* = 0 \) on \( \Omega \).

**Note:** Then \( h = h^* \) solves the \( E \cap G \cap (\neg \tilde{E} \cup \tilde{G}) \) generalized equation.

(One can show that \( \tilde{E} \cap \tilde{G} = \tilde{E} \cup \tilde{G} \). See [3, Property (2) after Def. 3.1] for arbitrary subsets of \( J^2(X) \).)

**Proposition 2.16.** Recall again that comparison holds for \( E \) and \( G \) on \( \Omega \). From this we conclude the following. If there exists a function \( f \in C^2(\Omega) \cap C(\Omega) \) with \( D^2f \in \text{Int } H \) for all \( x \in \Omega \), then there is no solution \( h^* \) to the \( H^* \)-\((DP) \) on \( \Omega \) with boundary values \( \varphi \equiv f \mid_{\partial \Omega} \).

**Proof.** If \( h^* \) exists, then since \( f \) is an \( H \)-solution, by Proposition 2.15 we have \( h^* = f \), and hence \( h^* \) is \( C^2 \). Thus \( D^2f \in (\text{Int } H) \cap H^* = (\text{Int } E \sim G) \cap (G \sim \text{Int } E) = \emptyset \). So this is impossible.

**Proof that (5) \( \Rightarrow \) (1) or that \( \text{Int } H \neq \emptyset \Rightarrow \) Non-Existence for \( H^* \).**

The fact that \( \text{Int } H \neq \emptyset \) guarantees the existence of such a function \( f \) by Lemma 2.12, and hence the non-existence for the \( H^* \) Dirichlet problem.

This completes the proof of Theorem 2.5.

In light of Theorem 2.5, if one is given a generalized equation \( H = E \cap (\neg \tilde{G}) \) with mirror \( H^* = G \cap (\neg \tilde{E}) \), there are four distinct types possible which we label as follows.

**Type I:** \( \text{Int } H = \emptyset \) and \( \text{Int } H^* = \emptyset \)

**Type II:** \( \text{Int } H = \emptyset \) and \( \text{Int } H^* \neq \emptyset \)

**Type III:** \( \text{Int } H \neq \emptyset \) and \( \text{Int } H^* = \emptyset \)

**Type IV:** \( \text{Int } H \neq \emptyset \) and \( \text{Int } H^* \neq \emptyset \)

We shall now discuss each type.

**Type I:** \( \text{Int } H = \text{Int } H^* = \emptyset \). This type is a “determined equation” \( \partial F \) as defined in Definition 2.1, because by (1) \( \iff \) (3) and (1)* \( \iff \) (3)*, this is just the case where \( E \) and \( G \) are equal. We will call this subequation \( F \).

Thus \( H \) and \( H^* \) are \( F \cap (\neg F) = \partial F \). Theorems 2.9 and 2.14 apply directly.
Comparison holds for all bounded domains, and existence holds if $\partial \Omega$ is smooth and strictly $F$ and $\tilde{F}$ convex using results from [2].

**Type II:** $\text{Int } H = \emptyset$ and $\text{Int } H^* \neq \emptyset$. Collecting together (1)–(5) and the negations of (1)*–(5)* we have that

$$E$$ is a proper subset of $G$ and $H = \partial E \cap \partial G \neq H^*$

Uniqueness but not existence holds for $H$ on any bounded domain $\Omega$. The opposite is true for $H^*$, namely uniqueness for $H^*$ fails on all bounded domains $\Omega$, but if $\partial \Omega$ is smooth and both strictly $E$ and $\tilde{E}$ convex, then existence holds for $H^*$ on $\Omega$. In addition $H$ is a proper subset of both $\partial E$ and $\partial G$.

This is proven in (2.10) below.

For Type III we interchange $E$ with $G$ and $H$ with $H^*$.

**Type III:** $\text{Int } H \neq \emptyset$ and $\text{Int } H^* = \emptyset$. Collecting together (1)*–(5)* and the negations of (1)–(5) we have that

$$G$$ is a proper subset of $E$ and $H^* = \partial G \cap \partial E \neq H$

Uniqueness but not existence holds for $H^*$ on any bounded domain $\Omega$. The opposite is true for $H$, namely uniqueness for $H$ fails on all bounded domains $\Omega$, but if $\partial \Omega$ is smooth and both strictly $G$ and $\tilde{G}$ convex, then existence holds for $H$ on $\Omega$. Also, $H^*$ is a proper subset of both $\partial E$ and $\partial G$ by (2.10).

**Type IV:** $\text{Int } H \neq \emptyset$ and $\text{Int } H^* \neq \emptyset$. By (2.3) this is equivalent to

$$(\text{Int } E) \cap (\sim G) \neq \emptyset \quad \text{and} \quad (\text{Int } G) \cap (\sim E) \neq \emptyset.$$  

Because of Lemma 2.6 (iii) and (iii)* this is equivalent to

$$\text{Int } E \subset \text{Int } G \quad \text{and} \quad \text{Int } G \subset \text{Int } E.$$  

The main point here is that both existence and uniqueness for the (DP) for both $H$ and $H^*$ fail.

Next we begin to examine to what extent a generalized equation $H = E \cap (\sim G)$ determines the subequations $E$ and $G$. The answer in the determined case is given as follows.

**Proposition 2.17. (The Determined Case).** If $H$ is a determined equation, then the subequation $F$ with $H = \partial F$ is uniquely determined by $H$. In fact, $\partial E \subset \partial G$ is enough to conclude that $E = G$ for any two subequations $E$ and $G$.

**Proof.** The first statement follows because

$$F = \partial F + \mathcal{P} \quad \text{for any subequation } F. \quad (2.9)$$

For the second statement note that one has $\partial E \subset \partial G \Rightarrow E = \partial E + \mathcal{P} \subset \partial G + \mathcal{P} = G$. However, $\partial E \subset \partial G \iff -\partial E \subset -\partial G$, but $-\partial E = \partial \tilde{E}$.
and $-\partial G = \partial \tilde{G}$. Hence, $\partial \tilde{E} \subset \partial \tilde{G}$, and this implies that $\tilde{E} \subset \tilde{G}$, which is equivalent to $G \subset E$.

It follows that:

If $\mathbb{H} \equiv E \cap (-\tilde{G})$ is Type II, then

$$\mathbb{H} = \partial E \cap \partial G$$

is a proper subset of both $\partial E$ and $\partial G$. (2.10)

**Proof.** If $\mathbb{H} \equiv \partial E = \partial \tilde{E} \cap \partial \tilde{G}$, then $\partial E \subset \partial G$, so that by Proposition 2.17, $E = G$ and $\mathbb{H}$ is Type I.

**Characterizing the Boundary Functions for Existence**

Here we turn to a natural question which arises in the Type II case, Non-Existence/Uniqueness for $H$:

For which boundary functions $\varphi \in C(\partial \Omega)$ does there exists a solution to the $H$-Dirichlet problem?

We make the assumption that $\Omega$ is a bounded domain with smooth boundary which is both strictly $E$ and $\tilde{G}$-convex. Using the equivalent versions (1)–(5) of the uniqueness hypothesis for $H$ in Theorem 2.5:

$$\text{Int } \mathbb{H} = \emptyset \iff E \subset G \iff \tilde{G} \subset \tilde{E},$$

this implies that $\partial \Omega$ is also strictly $G$ and $\tilde{E}$-convex. Let $h_E \in C(\partial \Omega)$ denote the (unique) $\partial E$-harmonic function on $\Omega$ with $h_E|_{\partial \Omega} = \varphi$, and $h_G \in C(\partial \Omega)$ denote the (unique) $\partial G$-harmonic function on $\Omega$ with $h_G|_{\partial \Omega} = \varphi$. One answer to the question is the following.

**Proposition 2.18.** Assume uniqueness holds for $\mathbb{H}$, i.e., $\text{Int } \mathbb{H} = \emptyset$, for the generalized equation $H = E \cap (-\tilde{G})$. Given a domain as above and $\varphi \in C(\partial \Omega)$, there exists $h \in C(\overline{\Omega})$ with $h|_{\partial \Omega} = \varphi$ and $h|_{\Omega} \mathbb{H}$-harmonic $\iff h_E = h_G$.

**Proof.** Suppose that $h_E = h_G$ and set $h = h_E = h_G$. Then $h|_{\partial \Omega} = \varphi$, $h$ is $E$-subharmonic, and $-h$ is $\tilde{G}$-subharmonic on $\Omega$, which proves that $h$ is a solution to the $\mathbb{H}$-Dirichlet Problem (DP) on $\Omega$ with boundary values $\varphi$.

Conversely, if these exists such an $h \in C(\overline{\Omega})$, then $h$ also solves the $\partial E$ (DP) since $h$ is $E$-subharmonic and $-h$ is $\tilde{E} \supset \tilde{G}$-subharmonic. Hence, $h_E = h$. Similarly, $h_G = h$ since $h$ is $G \supset E$-subharmonic and $-h$ is $\tilde{G}$-subharmonic.\[\square\]
3. The Canonical Pair Defining a Given $\mathbb{H}$.

Although it is only in the determined case that $\mathbb{H}$ uniquely determines $E = \mathbb{H} + \mathcal{P} = \mathbb{G}$, we always have the following.

**Proposition 3.1.** Suppose $\mathbb{H}$ is a generalized equation. Then there exists a canonical choice for the subequation pair $E, \mathbb{G}$ defining $\mathbb{H}$:

$$\mathbb{H} = E \cap (-\mathbb{G})$$

characterized by the following property:

If $E', G'$ is any other subequation pair with

$$\mathbb{H} = E' \cap (-\mathbb{G}')$$

then

$$E \subset E' \quad \text{and} \quad G' \subset G,$$

i.e., $E$ is minimal and $G$ is maximal.

In particular, if $h$ is $\mathbb{H} = E' \cap (-\mathbb{G}')$ harmonic, then $h$ is automatically also $\mathbb{H} = E \cap (-\mathbb{G})$ harmonic using the canonical min/max pair $E, \mathbb{G}$.

**Proof.** Define $E \equiv \mathbb{H} + \mathcal{P}$ and $\mathbb{G} \equiv -\mathbb{H} + \mathcal{P}$. Then $\mathbb{H} + \mathcal{P}$ and $-\mathbb{H} + \mathcal{P}$ satisfy positivity, and hence so do their closures. Since $E$ and $\mathbb{G}$ are closed, they are subequations.

Now assume that $\mathbb{H} = E' \cap (-\mathbb{G}')$. Then $\mathbb{H} \subset E'$ and so $E \equiv \mathbb{H} + \mathcal{P} \subset \mathbb{E} + \mathcal{P} = E' = E'$. We also have $-\mathbb{H} \subset \mathbb{G}'$ which implies that $\mathbb{G} \equiv -\mathbb{H} + \mathcal{P} \subset \mathbb{G}' + \mathcal{P} = \mathbb{G}' = \mathbb{G}'$. Thus we have $G' \subset G$.

Note that $-\mathbb{G} = \mathbb{H} - \mathcal{P}$, so that $\mathbb{H} \subset -\mathbb{G}$. Also $H \subset \mathbb{H} + \mathcal{P} = E$ (since $0 \in \mathcal{P}$), which proves that $\mathbb{H} \subset E \cap (-\mathbb{G})$. By the above $E \subset E'$ and $-\mathbb{G} \subset -\mathbb{G}'$. Thus $E \cap (-\mathbb{G}) \subset E' \cap (-\mathbb{G}') = \mathbb{H}$. \hfill \blacksquare

**Proposition 3.2. (The Uniqueness Case).** Suppose $\mathbb{H}$ is a generalized equation. Let $E_{\min}, G_{\max}$ denote the canonical min/max pair with $\mathbb{H} = E \cap (-\mathbb{G})$ from Proposition 3.1. If $\mathbb{H} \subset \partial \mathbb{F}$ for any subequation $\mathbb{F}$ (thus, Int $\mathbb{H} = \emptyset$, i.e., uniqueness holds), then

$$E_{\min} \subset F \subset G_{\max}.$$ 

**Proof.** Note that $\mathbb{H} \subset \partial \mathbb{F} \Rightarrow E_{\min} \equiv \mathbb{H} + \mathcal{P} \subset \partial \mathbb{F} + \mathcal{P} = \mathbb{F} = F$. Now

$$\mathbb{H} \subset \partial \mathbb{F} \quad \iff \quad -\mathbb{H} \subset \partial \mathbb{F} = -\partial \mathbb{F}.$$ 

Hence, $\mathbb{H} \subset \partial \mathbb{F} \Rightarrow -\mathbb{H} \subset \partial \mathbb{F} \Rightarrow -\mathbb{H} + \mathcal{P} \subset \partial \mathbb{F} + \mathcal{P} = \mathbb{F} \Rightarrow G_{\max} \equiv -\mathbb{H} + \mathcal{P} \subset \mathbb{F} \iff F \subset G_{\max}. \quad \blacksquare
4. Examples of Generalized Equations \( H = E \cap (\tilde{G}) \).

We start with some classes of examples. First and foremost is the following.

**Class I. Type I Determined Equations (The \( G = E \) Case).** Here \( H = \partial F = F \cap (-\tilde{F}) \) is the boundary of a subequation \( F \). We refer to [2], [3] and [4] for an abundance of important specific examples.

Another important class of examples is

**Class II. (The \( G = \tilde{E} \) Case).** Here \( H = E \cap (\tilde{E}) \) and \( H^* = \tilde{E} \cap (-\tilde{E}) = \sim [(\text{Int } E) \cup (-\text{Int } E)] \). Note that the overlap between classes I and II consists of the boundaries \( H = \partial F \) of self-dual subequations (where the dual \( \tilde{F} \) equals \( F \)).

**Class IIa. (Edges).** In this case II, the most basic examples are when \( E \) is a convex cone subequation. Then \( H = E \cap (-E) \) is a vector subspace called the edge of the cone \( E \). Such edge harmonic include: (i) Affine functions, where \( \tilde{P} = P \), (ii) pluriharmonic functions in complex analysis, where \( \tilde{P} = P_C \), (iii) \( \Delta \)-harmonic functions, and many others. These particular generalized equations are the subject of [12].

Three specific non-edge examples are as follows.

**Some \( H \) Non-Uniqueness Examples**

**Example 4.1. (The Quasi-convex-Quasi-concave Equation).** Choose \( r_1, r_2 \geq 0 \) and let
\[
H \equiv (P - r_1 I) \cap (-P + r_2 I).
\]
Here \( E \equiv P - r_1 I \) is the subequation for \( r_1 \)-quasiconvex functions, and \( \tilde{G} \equiv P - r_2 I \) is the subequation for \( r_2 \)-quasiconvex functions. Thus \( H \equiv E \cap (-\tilde{G}) \) is the generalized equation for functions that are both \( r_1 \)-quasiconvex and \( r_2 \)-quasiconvex. Note that \( A \in H \iff -r_1 I \leq A \leq r_2 I \). A function \( u \) is \( H \)-harmonic \( \iff u + r_1 \frac{|x|^2}{2} \) is convex and \( u - r_2 \frac{|x|^2}{2} \) is concave. Note that if \( r_1 = r_2 \), then \( \tilde{G} = E \) and \( H = E \cap (-E) \) is class II above.

Here is a related example.

**Example 4.2. (The Quasi-subaffine-Quasi-superaffine Equation).** Choose \( r_1, r_2 \in \mathbb{R} \) and set
\[
H \equiv (\tilde{P} - r_1 I) \cap (-\tilde{P} + r_2 I).
\]
Here \( E \equiv \tilde{P} - r_1 I \) is the subequation for \( r_1 \)-quasi-subaffine functions, i.e., \( u(x) + \frac{r_1}{2} |x|^2 \) is subaffine, and \( \tilde{G} = \tilde{P} - r_2 I \) is again the subequation for \( r_2 \)-quasi-subaffine functions. Again if \( r_1 = r_2 \), then \( \tilde{G} = E \) and \( H = E \cap (-E) \) is a special case of Class II above.
Some Type II Examples of Non-Existence/Uniqueness for $\mathbb{H}$

Example 4.3. (Constrained Laplacian).

(a) Let $\mathbb{H} \equiv \{ A : \text{tr}A = 0 \text{ and } -rI \leq A \leq rI \}$. Then

$$E_{\text{min}} = \mathbb{H} + \mathcal{P} = \Delta \cap (\mathbb{P} - rI), \quad G_{\text{max}} = \sim (\mathbb{H} - \text{Int} \mathcal{P}), \quad \text{and} \quad \tilde{G}_{\text{max}} = E_{\text{min}}.$$  

In particular, note that $\mathbb{H}$-harmonic implies $\Delta$-harmonic, which is of course obvious for $C^2$-functions. This is because $\mathbb{H} = E_{\text{min}} \cap (-E_{\text{min}})$ and $E_{\text{min}} \subset \Delta$.

(b) Let $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^\ell$ and

$$\mathbb{H} \equiv \left\{ A \equiv \begin{pmatrix} a & c \\ c^t & b \end{pmatrix} : \text{tr}A = 0, \ a \geq 0 \text{ and } b \leq 0 \right\}$$

with $a \in \text{Sym}^2(\mathbb{R}^k)$, $b \in \text{Sym}^2(\mathbb{R}^\ell)$. Then

$$E_{\text{min}} = \{ A : a \geq 0 \text{ and } \text{tr}A \geq 0 \}, \quad G_{\text{max}} = \{ A : b \geq 0 \} \cup E_{\text{min}},$$

and $\tilde{G}_{\text{max}} = \{ A : b \geq 0 \text{ and } \text{tr}A \geq 0 \}$.

One could also look at this from the universal eigenvalue point of view (see the subsection of Section 5 in [5] and Remark 4.8 below), where it takes the form

$$H = \{ (x, y) : x \in Q^+(\mathbb{R}^k), y \in Q^-(\mathbb{R}^\ell), \text{tr}(x, y) = 0 \}$$

$$E_{\text{min}} = \{ (x, y) : x \in Q^+(\mathbb{R}^k), \text{tr}(x, y) \geq 0 \},$$

$$G_{\text{max}} = \{ (x, y) : y \in Q^+(\mathbb{R}^\ell) \} \cup E_{\text{min}}, \quad \tilde{G}_{\text{max}} = \{ (x, y) : y \geq 0 \text{ and } \text{tr}(x, y) \geq 0 \}.$$  

Here $Q^+(\mathbb{R}^k)$ denotes the positive orphant defined by $x_j \geq 0$ for all $j$. $Q^-(\mathbb{R}^\ell)$ is similar.

A great example of a non-existence/uniqueness $\mathbb{H}$ equation (Type II) has been introduced and studied in [14] and [15]. We wish to thank Jeff Streets for bringing it to our attention.

Example 4.4. (The Universal Version of the Twisted Monge-Ampère Equation). The real twisted Monge-Ampère equation is defined by $\mathbb{H} \subset \text{Sym}^2(\mathbb{R}^k \times \mathbb{R}^\ell)$ consisting of all

$$\begin{pmatrix} A & C \\ C^t & B \end{pmatrix} \text{ such that } A \geq 0, \ B \leq 0 \text{ and } \log \det A - \log \det(-B) = 0$$

i.e., $\det A = \det(-B)$, or $\det A = |\det B|$.

The universal version of this equation is defined on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^\ell$ by

$$H \equiv \{ (x, y) : x \in Q^+(\mathbb{R}^k), y \in Q^-(\mathbb{R}^\ell) \text{ and } x_1 \cdots x_k = |y_1 \cdots y_\ell| \}.$$
Lemma 4.5. Let $E \equiv H + Q^+(\mathbb{R}^n)$. Then $E$ is fibred over $Q^+(\mathbb{R}^k)$, where the fibre $E_x$ of $E$ at $x \in Q^+(\mathbb{R}^k)$ is the dual MA universal subequation:

$$
\tilde{P}_{x_1 \ldots x_k}(\mathbb{R}^\ell) = (\sim Q^-(\mathbb{R}^\ell)) \cup \{ y \in Q^{-}(\mathbb{R}^\ell) : |y_1 \cdots y_\ell| \leq x_1 \cdots x_k \}.
$$

Since it is easy to see that $E$ is closed, $E$ is equal to the minimal subequation defined above for this $H$.

Proof of Lemma 4.5. First note that $H$ is fibred over $Q^+(\mathbb{R}^k)$ with fibre $H_x$ at $x \in Q^+(\mathbb{R}^k)$ given by

$$
H_x = \{ y \in Q^{-}(\mathbb{R}^\ell) : |y_1 \cdots y_\ell| = x_1 \cdots x_k \}.
$$

Second note that this equals

$$
H_x = \partial \tilde{P}_{x_1 \ldots x_k}(\mathbb{R}^\ell)
$$

the boundary of the dual MA-subequation at level $c = x_1 \cdots x_k$. Third note that, since $\tilde{P}_{x_1 \ldots x_k}(\mathbb{R}^\ell)$ is a subequation,

$$
H_x = Q^+(\mathbb{R}^\ell) = \partial \tilde{P}_{x_1 \ldots x_k}(\mathbb{R}^\ell) + Q^+(\mathbb{R}^\ell) = \tilde{P}_{x_1 \ldots x_k}(\mathbb{R}^\ell)
$$

Defining $E'$ by its fibres $E'_{x_1 \ldots x_k} \equiv \tilde{P}_{x_1 \ldots x_k}(\mathbb{R}^\ell)$ over $x \in Q^+(\mathbb{R}^k)$, we have $E' \subset E$, and it remains to show $E \subset E'$. But $H \subset E'$, so it is enough to show $E'$ is $Q^+(\mathbb{R}^n)$-monotone. As noted above $E'$ is $Q^+(\mathbb{R}^\ell)$-monotone since $\tilde{P}_{x_1 \ldots x_k}(\mathbb{R}^\ell)$ is a subequation. Now increasing one of the $x$ coordinates with $x \in Q^+(\mathbb{R}^k)$ increases $\tilde{P}_{x_1 \ldots x_k}(\mathbb{R}^\ell)$ proving that $E'$ is $Q^+(\mathbb{R}^k)$-monotone. Finally the orphan $Q^+(\mathbb{R}^n)$ equals the product $Q^+(\mathbb{R}^k) \times Q^+(\mathbb{R}^\ell)$.

Lemma 4.6. Let $\tilde{G} \equiv -H + Q^+(\mathbb{R}^n)$. Then $\tilde{G}$ is fibred over $Q^+(\mathbb{R}^\ell)$, where the fibre $\tilde{G}_y$ of $\tilde{G}$ at $y \in Q^+(\mathbb{R}^\ell)$ is the dual MA universal subequation:

$$
\tilde{G}_y = \tilde{P}_{|y_1 \cdots y_\ell|}(\mathbb{R}^k).
$$

The proof of Lemma 4.6 is similar to the one for Lemma 4.5, and is skipped.

Proposition 4.7. $H = E \cap (-\tilde{G})$ is a universal version of a generalized equation with minimum subequation $E$ and maximum subequation $G$.

Proof. Note that $(x, y) \in E \iff x \in Q^+(\mathbb{R}^k)$ and $y \in \tilde{P}_{x_1 \ldots x_k}(\mathbb{R}^\ell)$ by Lemma 4.5. Note also that

$$
(x, y) \in -\tilde{G} \iff x \in -\tilde{P}_{|y_1 \cdots y_\ell|}(\mathbb{R}^k) \text{ and } y \in Q^-(\mathbb{R}^\ell).
$$

by Lemma 4.6.

Now assume $(x, y) \in E \cap (-\tilde{G})$. Then $x \in Q^+(\mathbb{R}^k) \cap (-\tilde{P}_{|y_1 \cdots y_\ell|}(\mathbb{R}^k))$ or otherwise said, $x \in Q^+(\mathbb{R}^k)$ and $|y_1 \cdots y_\ell| \leq x_1 \cdots x_k$. Also, $y \in Q^-(\mathbb{R}^\ell) \cap \tilde{P}_{x_1 \ldots x_k}(\mathbb{R}^\ell)$ or otherwise said, $x \in Q^-(\mathbb{R}^\ell)$ and $x_1 \cdots x_k \leq |y_1 \cdots y_\ell|$.
In summary, if \((x, y) \in E \cap (-\tilde{G})\), then

\[ x \in Q^+(\mathbb{R}^k), \quad y \in Q^-(\mathbb{R}^\ell), \quad \text{and} \quad x_1 \cdots x_k = |y_1 \cdots y_\ell| \]

that is, \((x, y) \in H\). It is easy to see that \(H \subset E \cap (-\tilde{G})\).

**Remark 4.8.** (Universal Equations and Gårding/Dirichlet Operators). A closed subset \(\Lambda \subset \mathbb{R}^n\) which is symmetric under permutations of the coordinates and satisfies \(\Lambda + \mathbb{R}^n \subset \Lambda\) is called a **universal eigenvalue subequation**. There is an obvious one-to-one correspondence between subequations \(\mathbb{F} \subset \text{Sym}^2(\mathbb{R}^m)\), which depend only on the eigenvalues of \(A \in \mathbb{F}\), and universal subequations \(\Lambda \subset \mathbb{R}^n\). However, this \(\mathbb{F}\) is only one of many subequations determined by \(\Lambda\) which are constructed by substituting Gårding eigenvalues for regular eigenvalues as follows.

Let \(g\) be a homogeneous polynomial of degree \(n > 0\), on some \(\text{Sym}^2(\mathbb{R}^m)\), which satisfies the conditions of being a **Gårding/Dirichlet**, or **GD**, operator (as defined in \([5, \S 5]\)). Then for each \(A \in \text{Sym}^2(\mathbb{R}^m)\) this operator gives \(n\) eigenvalues \(\lambda_g(A)\), and so \(\Lambda\) determines a subequation in \(\mathbb{R}^m\) by:

\[
\mathbb{F}^g_{\Lambda} = \{A \in \text{Sym}^2(\mathbb{R}^m) : \lambda_g(A) \in \Lambda\}.
\]

For example, the universal subequation \(\Lambda \equiv \{\lambda \in \mathbb{R}^n : \lambda_j \geq 0 \ \forall \ j\}\) determines the **Gårding Monge Ampère subequation** \(\mathbb{F}^g_{\Lambda} = \{A \in \text{Sym}^2(\mathbb{R}^m) : \lambda_{g,1}(A), \ldots, \lambda_{g,n}(A) \text{ are all } \geq 0\}\), which is just the closed Gårding cone for \(g\).

This carries over to generalized equations. A **generalized universal equation** is any closed \(\Lambda' \subset \mathbb{R}^n\) which is an intersection involving two universal subequations \(\Lambda' = \Lambda_1 \cap (-\tilde{\Lambda}_2)\). For example, the universal Laplacian \(\Lambda = \{\lambda \in \mathbb{R}^n : \lambda_1 + \cdots + \lambda_n \geq 0\}\) determines a Gårding Laplacian \(\mathbb{F}^g_{\Lambda}\) for each GD operator of degree \(n\). Moreover, given a pair of GD operators \(g_1, g_2\) of degrees \(n_1 + n_2 = n\), one has a constrained Laplacian generalized equation induced by the universal version of the constrained Laplacian given in Example 4.3. Namely, we have

\[
\mathbb{H} \equiv \{A \in \text{Sym}^2(\mathbb{R}^m) : \lambda_{g_1,j}(a) \geq 0, \lambda_{g_2,k}(b) \leq 0, \text{ and } \sum_j \lambda_{g_1,j}(a) + \sum_k \lambda_{g_2,k}(b) = 0\}.
\]

Similarly (we leave this to the reader) the universal twisted MA-equation (Example 4.4) spawns a huge family of generalized equations. For instance, in addition to the real version in \([14], [15]\), one has a complex version, a quaternionic version, a Lagrangian version, branched versions of these three, elementary symmetric versions of these three (the so-called “hessian equation” versions), just to name a few.

The Examples 4.1 and 4.2 can also be viewed as “universal subequations”, spawning many more examples of generalized equations as above.
Since we have no reason to rule out $F = \emptyset$ or $F = \text{Sym}^2(\mathbb{R}^n)$ as a subequation in this paper, we have that plus or minus a subequation is an example of a generalized equation of Type II or III respectively.

Example 4.9. For any subequation $E \neq \emptyset$, if we choose $G = \emptyset$, i.e., $-\tilde{G} = \text{Sym}^2(\mathbb{R}^n)$, then $H = E \cap \text{Sym}^2(\mathbb{R}^n) = E$ is a generalized equation. Now since $\text{Int} E = E$ and $E \neq \emptyset$, we have $\text{Int} H \neq \emptyset$. Also the mirror $H^* = \cap (\tilde{E}) = \emptyset$. Hence, $\text{Int} H^* = \emptyset$. In summary, if $E \neq \emptyset$ is any subequation, then with $G = \emptyset$, we have $H = E$ and $H^* = \emptyset$, so that $E$ itself (not $\partial E$) is a generalized equation which falls in the Existence/Non-Uniqueness case for $H$ (Type III).

Similarly, $-F = E \cap (-\tilde{G})$ with $E \equiv \text{Sym}^2(\mathbb{R}^n)$ and $G = \tilde{F}$ is Type II.

Some Type IV Examples of Non-Uniqueness/Non-Existence.

Example 4.10. With coordinates $z = (x,y) \in \mathbb{R}^a = \mathbb{R}^k \times \mathbb{R}^\ell$, define $E$ by $D_x^2u \geq 0$ and $\tilde{G}$ by $D_y^2u \geq 0$ (so that $G$ is the subaffine subequation on $\mathbb{R}^\ell$ considered as a subequation on $\mathbb{R}^n$). Then with $H \equiv E \cap (-\tilde{G})$ we have that the $H$-harmonics are continuous functions $h(x,y)$ that are separately convex in $x$ and concave in $y$. The mirror $H^*$-harmonics are continuous functions $h^*(x,y)$ that are separately subaffine in $x$ and superaffine in $y$.

5. Unsettling Questions.

In Section 8 of [11] we posed several such questions, starting with the single-valuedness of operators and the following equivalent restatement of that question.

(CCQ) Constant Coefficient Subequation Question: Can a pair of subequations $E, G$, with $E \subset \text{Int} G$, have a simultaneous harmonic $h$?

By definition such a function $h$ must be $E$ and $G$ subharmonic, and $-h$ must be $\tilde{E}$ and $\tilde{G}$ subharmonic. However, since $E \subset G$ and $G \subset \tilde{E}$ this reduces to $h$ being $E$-subharmonic and $-h$ being $G$-subharmonic, which is exactly our definition of $h$ satisfying the generalized equation

$$H = E \cap (-\tilde{G}) = E \cap (\sim \text{Int} G).$$ (5.1)

The assumption in (CCQ) that $E \subset \text{Int} G$ can be rephrased as

The generalized equation $H \equiv E \cap (-\tilde{G})$ is empty. (5.2)

In summary, the (CCQ) can be rephrased as:

(CCQ)': Does there exist a subequation pair $E, G$ defining the generalized empty equation $H = \emptyset$ with the property that $H \equiv E \cap (-\tilde{G})$ has a harmonic?
Note. There are lots of subequations pairs $E, G$ defining this generalized equation $H = \emptyset$. For some of these pairs we can prove that $E \cap (-\tilde{G})$ has no harmonics. For example, this holds if $E$-harmonics and $G$-harmonics are always $C^2$. At the other extreme for $H = \emptyset$, we have $E_{\text{min}} = \emptyset$, and so for the pair $E_{\text{min}}, G_{\text{max}}$ defining $H = \emptyset$ there are also no harmonics.

Now we can broaden our question as follows.

**Generalized Equation Question:** Given a generalized equation $H$ does there exists a subequation pair $E, G$ defining $H$ so that $H \equiv E \cap (-\tilde{G})$ has a harmonic which is not a harmonic for $H \equiv E_{\text{min}} \cap (-\tilde{G}_{\text{max}})$?

Note that by Proposition 3.1 $E_{\text{min}} \subset E$ and $-\tilde{G}_{\text{max}} \subset -\tilde{G}$ so that $H \equiv E_{\text{min}} \cap (-\tilde{G}_{\text{max}})$ harmonics are always $H \equiv E \cap (-\tilde{G})$ harmonics.

Any such harmonic $h$ in the Generalized Equation Question must be weird and pathological. In particular, it must be much worse than $C^2$ for sure.

6. The General Case of the Main Theorem.

For clarity and simplicity we have been restricting attention to pure second-order constant coefficient subequations $E$ and $G$ to define a generalized equation $H = E \cap (-\tilde{G})$ in $\mathbb{R}^n$. However, the main Theorem 2.5 holds for completely general subequations on manifolds, as defined in [3], and we give that result in this section. For general definitions we refer to [3]. However, there are many interesting cases which the reader could keep in mind (without consulting [3]), namely: constant coefficient subequations $E$ and $G$ (not necessarily pure second-order) in $\mathbb{R}^n$, variable coefficient subequations (constraint sets for subsolutions) on domains in $\mathbb{R}^n$, subequations on riemannian manifolds given canonically by $O(n)$-invariant equations in $\mathbb{R}^n$, subequations on hermitian manifolds given canonically by $U(n)$-invariant equations in $\mathbb{C}^n$, etc.

Let $J^2(X)$ be the 2-jet bundle on a manifold $X$. (When $X = \mathbb{R}^n$ this is just the bundle $\mathbb{R}^n \times (\mathbb{R} \oplus \mathbb{R}^n \oplus \text{Sym}^2(\mathbb{R}^n))$ over $\mathbb{R}^n$ of order-2 Taylor expansions.)

**THEOREM 6.1** Let $\Omega \subset \subset X$ be a domain in a manifold $X$, and suppose $E, G \subset J^2(X)$ are two subequations. Consider the generalized equation $H \equiv E \cap (-\tilde{G})$.

(a) $\text{Int } H = \emptyset \implies$ uniqueness for the $H$-(DP) on $\Omega$, assuming that comparison holds for $E$ and $G$ on $\Omega$.

(b) $\text{Int } H^* = \emptyset \implies$ existence holds for the $H$-(DP) on $\Omega$, assuming that existence for the $E$-(DP) holds on $\Omega$. 
(c) There exists $h \in C^2(\Omega) \cap C(\overline{\Omega})$ with $J^2 h \in \text{Int} \mathbb{H}$ for all $x \in \Omega \Rightarrow$ non-uniqueness for the $\mathbb{H}$-(DP) on $\Omega$ for the boundary values $\varphi \equiv h|_{\partial \Omega}$.

(d) There exists $f \in C^2(\Omega) \cap C(\overline{\Omega})$ with $J^2 f \in \text{Int} \mathbb{H}^*$ for all $x \in \Omega \Rightarrow$ non-existence for the $\mathbb{H}$-(DP) on $\Omega$ for the boundary values $\varphi \equiv f|_{\partial \Omega}$, assuming that comparison holds for $E$ and $G$ on $\Omega$.

**Proof of Assertion (a).** We begin by noting that assertions (1.2) – (1.5) hold for general subequations as defined in [3]. Our definition of $\mathbb{H}$ is the same as in Definition 2.2, and the assertion (2.3) carries over. As a result, Lemma 2.6 holds in this general case. We now look at the proof of Proposition 2.10, which says that under the assumption of comparison (C) for both $E$ and $G$ we have Part (a).

**Proof of Assertion (c).** This follows exactly the argument given for Proposition 2.11.

**Proof of Assertion (b).** This follows exactly the argument given for Proposition 2.13.

**Proof of Assertion (d).** We are assuming that comparison (C) holds on $\Omega$ for both $E$ and $G$. This means that Proposition 2.15 holds, and therefore also Proposition 2.16 is valid. This establishes Part (d).

**Example 6.2. (Generalized Constant Coefficient Equations in $\mathbb{R}^n$).** Here a subequation is, by definition (cf. [3], [4]), a closed subset

$$F \subset J^2 \equiv \mathbb{R} \oplus \mathbb{R}^n \oplus \text{Sym}^2(\mathbb{R}^n)$$

such that $F + (r, 0, P) \subset F$ for $r \leq 0$ and $P \geq 0$ and such that $\text{Int} F = \text{Int} \overline{F}$. The topological condition $F = \text{Int} F$ was not part of our definition of a subequation in Section 1 since in that case it follows easily from the positivity condition (1.1).

With regards to Assertion (a) comparison does not hold for all such equations. However it does hold for many interesting classes, for instance, all gradient free ones. Other such classes can be found in [1].

On the other hand, existence does hold for all these equations $F \subset J^2$, under the hypothesis that the domain $\Omega \subset \mathbb{R}^n$ has a smooth strictly $F$ and $\overline{F}$ convex boundary. (See Theorem 12.7 in [3].) Now in Assertion (b) existence is only required for $E$. Therefore, Assertion (b) holds for $E, G \subset J^2$ provided $\partial \Omega$ is strictly $E$ and $\overline{E}$ convex.

**Example 6.3. (Generalized Equations on an open set $X \subset \mathbb{R}^n$).** The general subequation here is a closed subset of the 2-jet bundle

$$F \subset J^2(X) \equiv X \times (\mathbb{R} \oplus \mathbb{R}^n \oplus \text{Sym}^2(\mathbb{R}^n))$$
such that
\[ F + (x; r, 0, P) \subset F \text{ for } r \leq 0 \text{ and } P \geq 0 \text{ and for all } x \in X, \]
\[ F = \text{Int} F, \text{ and for the fibres } F_x \text{ we have} \]
\[ F_x = \text{Int}_x F_x \quad \text{and} \quad \text{Int}_x F_x = (\text{Int} F)_x. \]

These are barebones hypotheses needed for the constraint set for subsolutions of a nonlinear equation corresponding to \( \partial F \).

This is the general case for domains \( \Omega \subset\subset X \subset \mathbb{R}^n \), and so the comparison and existence hypotheses in Theorem 6.1 need to be verified, but, of course, the literature is enormous.

For subequations on manifolds given by “universal” equations, much has been done in [3]. We shall now look at some cases.

**Example 6.4. (Universal Subequations Defined on any Riemannian Manifold).** Let \( \mathcal{F} \subset J^2 \) be a subequation (as in Example 6.2) which is invariant under the natural action of the orthogonal group \( O(n) \) (or \( SO(n) \)).

Then \( \mathcal{F} \) determines an invariant subequation \( \mathcal{F}(X) \subset J^2(X) \) on any rie-
nmannian (or oriented rie mannian) manifold \( X \) as follows.

Every \( C^2 \)-function \( u \) on \( X \) has a re mian nian hessian \( \text{Hess} u \), which is a section of the bundle \( \text{Sym}^2(X) \) of symmetric 2-forms on \( X \) by
\[ \{\text{Hess}_x u\}(V, W) \equiv V_x W u - (\nabla_V W)_x u \]
where \( \nabla \) is the Levi-Civita connection on \( TX \). Note that \( \nabla_V W - \nabla_W V = VW - WV = [V, W] \), so the symmetry and the tensorial properties of \( \text{Hess} u \) follow.

Now this re mian nian Hessian gives a splitting of the 2-jet bundle
\[ J^2(X) \cong X \times (\mathbb{R} \oplus T^*X \oplus \text{Sym}^2(X)), \]
and the orthogonally invariant subequation \( \mathcal{F} \) canonically determines a subequa-
tion \( \mathcal{F}(X) \subset J^2(X) \) as follows. Any orthonormal frame field \( e_1, \ldots, e_n \) for \( TX \) on an open set \( U \subset X \) determines an orthonormal framing of \( J^2(U) \cong U \times (\mathbb{R} \oplus \mathbb{R}^n \oplus \text{Sym}^2(\mathbb{R}^n)) \). Via this framing, \( \mathcal{F} \) determines a subequation on \( U \). However, if we use a different frame field \( e'_1, \ldots, e'_n \), the two framings of \( J^2(U) \) differ pointwise by \( O(n) \)-transforms. By the \( O(n) \)-invariance of \( \mathcal{F} \) the subequation on \( U \) are the same. This also means that on two different open sets \( U, V \subset X \) the two subequations agree on \( U \cap V \). Hence, we have a well-defined global subequation \( \mathcal{F}(X) \subset J^2(X) \).

For example, if \( \mathcal{F} = \{(r, p, A) : \text{tr}(A) \geq 0\} \), we get the subequation
\[ \Delta u = \text{tr}\{\text{Hess} u\} \geq 0 \text{ for the riemannian Laplacian}. \]
If \( \mathcal{F} = \{(r, p, A) : \text{det}(A) \geq 0\} \), we get the real Monge-Ampère subequation \( \text{det}\{\text{Hess} u\} \geq 0 \). If \( \mathcal{F} = \{(r, p, A) : p^t A p \geq 0\} \), one gets the infinite Laplacian on \( X \).
The questions of comparison and of existence of solutions for the Dirichlet problem on manifolds are addressed in [3]. A cone subequation $M$ on $X$ is a cone monotonicity subequation for $F(X)$ if $F(X) + M \subset F(X)$. Then for such equations we have the following from Thm. 13.2 and Thm. 10.1 in [3]. (See section 14 of [3] for examples.)

**THEOREM ([3]).** Suppose $X$ admits a $C^2$ strictly $M$-subharmonic function. Then comparison for $F(X)$ holds on any domain $\Omega \subset X$, and if $\partial \Omega$ is smooth and strictly $F(X)$ and $\overline{F}(X)$ convex, then existence holds for the Dirichlet problem for all boundary functions $\varphi \in C(\partial \Omega)$.

This construction has important generalizations.

**Example 6.5. (Universal Subequations Defined on a Riemannian Manifold with $G$-Structure).** We now assume that the riemannian manifold $X$ can be covered by open sets $U$, with an orthogonal tangent frame field $e^U \equiv (e_1, ..., e_n)$ on $U$, such that on the intersection $U \cap V$ of two such, the change of frames from $e^U$ to $e^V$ always lies in a given compact subgroup $G \subset O(n)$.

For example, if $X^{2n}$ has an orthogonal almost complex structure $J$, then $X$ has an $U(n)$-structure.

If the euclidean subequation $F$ is $G$-invariant, then the above construction gives a canonical subequation on any riemannian manifold with $G$-structure. For example, for $(X^{2n}, J)$ above, we can define the complex Monge-Ampère operator.

The Theorem at the end of Example 6.4 extends to these cases.

**Example 6.6. (Geometric Cases).** Of particular importance are the geometric cases given by a closed subset $G \subset G(p, \mathbb{R}^n)$ of the Grassmannian of $p$-planes in $\mathbb{R}^n$. We assume that $G$ is invariant under a closed subgroup $G \subset O(n)$. Then we consider the universal euclidean subequation

$$F_G \equiv \{(r, p, A) : \text{tr} (A|_L) \geq 0 \text{ for all } L \in G\}.$$

This subequation now carries over to any riemannian manifold with $G$-structure. For instance, suppose $G$ is the set of special Lagrangian $n$-planes in $\mathbb{C}^n$. Then we get a subequation on any Calabi-Yau manifold $X$. If $G$ is the set of associative 3-planes in $\mathbb{R}^7$, then we get a subequation on any 7-manifold $X$ with holonomy $G_2$.

Theorems in [3] apply to these cases, but there is a better theorem in [6]. We define the $G$-core of $X$ to be the set

$$\text{Core}_G(X) \equiv \{x \in X : \text{no smooth strictly } F_G \text{-subharm. function is strict at } x\}.$$
THEOREM ([6, Thm. 7.6 and Thm. 7.7]). If Core_{\mathcal{G}}(X) = \emptyset, then comparison for \mathcal{F}_{\mathcal{G}}(X) holds on any domain \Omega \subset X, and if \partial \Omega is smooth and strictly \mathcal{F}_{\mathcal{G}} and \mathcal{\tilde{F}}_{\mathcal{G}} convex, then existence holds for the Dirichlet problem for all boundary functions \varphi \in C(\partial \Omega).

Remark 6.7. In Section 2 we made a remark which does not carry over to general subequations. Finite intersections of subequations are not always subequations. There are classes of subequations where this is true (see [1]). However in general this means that one could expand the definition of a generalized equation to cover many-fold intersections and unions of subequations. This will be done elsewhere.

SOME REFERENCES


