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A generalization of pde's from a Krylov point of view ☆

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ABSTRACT

We introduce and investigate the notion of a "generalized equation", which extends nonlinear elliptic equations, and which is based on the notions of subequations and Dirichlet duality. Precisely, a subset $\mathbb{H} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a **generalized equation** if it is an intersection $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ where $\mathbb{E}$ and $\mathbb{G}$ are subequations and $\widetilde{\mathbb{G}}$ is the subequation dual to $\mathbb{G}$. We utilize a viscosity definition of "solution" to $\mathbb{H}$. The **mirror** of $\mathbb{H}$ is defined by $\mathbb{H}^* \equiv \mathbb{G} \cap (-\widetilde{\mathbb{E}})$. One of the main results (Theorem 2.6) concerns the Dirichlet problem on arbitrary bounded domains $\Omega \subset \mathbb{R}^n$ for solutions to $\mathbb{H}$ with prescribed boundary function $\varphi \in C(\partial\Omega)$. We prove that:
(A) Uniqueness holds $\iff$ $\mathbb{H}$ has no interior, and
(B) Existence holds $\iff$ $\mathbb{H}^*$ has no interior.
For (B) the appropriate boundary convexity of $\partial\Omega$ must be assumed. Many examples of generalized equations are discussed, including the constrained Laplacian, the twisted Monge-Ampère equation, and the $C^{1,1}$-equation.
The closed sets $\mathbb{H} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ which can be written as generalized equations are intrinsically characterized. For such an $\mathbb{H}$ the set of subequation pairs $(\mathbb{E}, \mathbb{G})$ with $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ is partially ordered (see (1.10)). If $(\mathbb{E}, \mathbb{G}) \prec (\mathbb{E}', \mathbb{G}')$, then any solution for the first is also a solution for the second. Furthermore, in this ordered set there is a canonical least element, contained in all others.

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A general form of the main theorem, which holds on any manifold, is also established.

Contents

1. Introduction

For some time we have been studying fully nonlinear pde's from a perspective of generalized potential theory. This was initiated by the discovery [6,7] of a robust potential theory attached to every calibrated manifold – a fact which generalized the classical pluripotential theory in Kähler geometry. Our point of view had some reflections in early work of Krylov [18], and eventually went far outside calibrated geometries. Good references for these techniques and results are [8], [9] and [10]

The purpose of this paper is to examine, to the fullest extent, when these viscosity and Dirichlet duality techniques can be employed to study nonlinear differential relations. Our fundamental Definition 2.2 is completely natural in the context of duality, but may seem to be too general or abstract. However, it turns out that there are lots of interesting examples: the constrained Laplace equation, the twisted Monge-Ampère equation, the relation $\det(D^2u) \leq 0$ in dimension 2, the $C^{1,1}$-equation, and many, many more. Furthermore, our fundamental Theorem 2.6 gives a simple and somewhat surprising relationship with the two questions of existence and uniqueness.

For clarity and simplicity we restrict attention to the constant coefficient case until the last Section 6.

Before making more detailed comments about examples and results, we recall the fundamental background ideas.

Short preliminaries

We adopt the subequation point of view from [8], [9], where a differential operator f is replaced by the constraint set $\mathbb{F} \equiv \{A \in \mathrm{Sym}^2(\mathbb{R}^n); f(A) \geq 0\}$. (Here $\mathrm{Sym}^2(\mathbb{R}^n)$ denotes the space of quadratic forms on $\mathbb{R}^n$.) The equation $f(D^2u) = 0$ is replaced by the constraint condition $D^2u \in \partial\mathbb{F}$. For that reason we will refer to $\partial\mathbb{F}$ as the **equation**

associated to the subequation $\mathbb{F}$. The ellipticity hypothesis is assumed in the weakest possible form:

$$\mathbb{F}+\mathcal{P} \ \subset\ \mathbb{F}. \tag{1.1}$$

Here $\mathcal{P}=\{P\in \mathrm{Sym}^2(\mathbb{R}^n) : P\geq 0\}$. Any closed subset $\mathbb{F}\subset \mathrm{Sym}^2(\mathbb{R}^n)$ satisfying this *positivity*, or $\mathcal{P}$*-monotonicty, condition* (1.1), is called a **subequation.**[1]

The viscosity definition of a subsolution takes the following form. Consider an upper semi-continuous function u defined on an open set $X\subset\mathbb{R}^n$ and taking values in $[-\infty,\infty)$. An *upper test function* for u at a point $x\in X$ is a C^2-function φ defined near x with $u\leq\varphi$ and $u(x)=\varphi(x)$. The function u is said to be $\mathbb{F}$**-subharmonic** or an $\mathbb{F}$**-subsolution** on X if for every upper test function φ at any point $x\in X$ we have $D^2_x\varphi\in\mathbb{F}$. For C^2-functions u, the **consistency** of this definition with the classical definition that $D^2_x u\in\mathbb{F}$ for all x, follows from the positivity condition (1.1). (In fact, consistency mandates positivity.) We will denote the space of $\mathbb{F}$-subharmonics functions on X by $\mathbb{F}(X)$. (For a complete introduction to viscosity theory see [2,3].)

The **Dirichlet dual** $\widetilde{\mathbb{F}}$ of a subequation $\mathbb{F}$ is defined to be

$$\widetilde{\mathbb{F}} \ = \sim(-\mathrm{Int}\,\mathbb{F}) \ =\ -(\sim \mathrm{Int}\mathbb{F}) \tag{1.2}$$

It is also a subequation and provides a true duality $\widetilde{\widetilde{\mathbb{F}}}=\mathbb{F}$. Moreover, one has the key relationship

$$\partial\mathbb{F}\ =\ \mathbb{F}\cap(-\widetilde{\mathbb{F}}) \tag{1.3}$$

which enables one to replace $f(D^2u)=0$ by $\partial\mathbb{F}$ via the viscosity definitions (see Definition 2.1). It is easy to see that

$$\mathrm{Int}\widetilde{\mathbb{F}}\ =\ -(\sim\mathbb{F})\ =\sim(-\mathbb{F}), \tag{1.4}$$

and to see that (1.1), together with $\mathbb{F}$ being closed, implies the **topological property**

$$\mathbb{F}\ =\overline{\mathrm{Int}\,\mathbb{F}}. \tag{1.5}$$

(Also $\mathrm{Int}\mathbb{F},\partial\mathbb{F}$ and $\mathbb{F}$ are all path-connected.)

By (1.3) applied to $\widetilde{\mathbb{F}}$, instead of $\mathbb{F}$, we have

$$\partial\widetilde{\mathbb{F}}\ =\ \widetilde{\mathbb{F}}\cap(-\mathbb{F})\ =\ -\partial\mathbb{F}. \tag{1.6}$$

It is easy to see that, when $\mathbb{F}\neq\emptyset$ or $\mathrm{Sym}^2(\mathbb{R}^n)$,

[1] It is convenient occasionally in this paper to allow $\mathbb{F}=\emptyset$ or $\mathbb{F}=\mathrm{Sym}^2(\mathbb{R}^n)$.

$$\mathbb{F} \;=\; \partial\mathbb{F} + \mathcal{P} \tag{1.7}$$

and hence, by (1.6) we have

$$\widetilde{\mathbb{F}} \;=\; -\partial\mathbb{F} + \mathcal{P} \tag{1.8}$$

This formula for the dual subequation $\widetilde{\mathbb{F}}$ could have replaced (1.2) as the definition of the dual subequation. It is frequently the easiest way to compute $\widetilde{\mathbb{F}}$ for examples, as in Section 4.

Generalized equations

A **generalized equation** is simply a pair $(\mathbb{E}, \mathbb{G})$ of subequations on $\mathrm{Sym}^2(\mathbb{R}^n)$. We associate to this pair the constraint set

$$\mathbb{H} \;=\; \mathbb{H}_{\mathbb{E},\mathbb{G}} \;\stackrel{\text{def}}{=}\; \mathbb{E} \cap (-\widetilde{\mathbb{G}})$$

and define an $\mathbb{H}$**-harmonic** on an open $X \subset \mathbb{R}^n$ to be a function h on X with

$$h \in \mathbb{E}(X) \qquad \text{and} \qquad -h \in \widetilde{\mathbb{G}}(X),$$

i.e., h is $\mathbb{E}$-subharmonic and $-h$ is $\widetilde{\mathbb{G}}$-subharmonic on X. An $\mathbb{H}$-harmonic on X which is C^2 satisfies the differential relation

$$D^2_x h \in \mathbb{H} \qquad \text{for all} \;\; x \in X \tag{1.9}$$

(by the C^2 consistency referred to above). One example is the **constrained Laplacian** (see Example 2.3 below)

$$\mathbb{H} = \{\mathrm{tr}A = 0 \text{ and } -r\mathrm{Id} \le A \le r\mathrm{Id}\} \qquad \text{for} \;\; r \ge 0,$$

where $\mathbb{E} = \mathbb{H} + \mathcal{P}$ and $-\widetilde{\mathbb{G}} = \mathbb{H} - \mathcal{P}$, i.e., $\mathbb{G} = \widetilde{\mathbb{E}}$. Here the $\mathbb{H}$-harmonics are classical harmonic functions h with $-r\mathrm{Id} \le D^2 h \le r\mathrm{Id}$. This example can be generalized with $\mathbb{H}$ being any closed subset of $\{\mathrm{tr}A = 0\}$ (see Example 4.3).

Another Example is the twisted Monge-Ampère Equation (Example 4.8) which has been studied by Streets, Tian and Warren, [22,23]. This equation requires a splitting of space, and uses a mixture of a convex and concave Monge Ampere operator on the two pieces. The authors proved an Evans-Krylov type estimate despite the nonconvexity of the operator. We are grateful for Jeff Streets communicating their work on this equation and asking us if any of our methods could apply.

A very nice example $\mathbb{H}_\lambda$ is given by taking $\mathbb{E} = \mathcal{P} - \lambda\mathrm{Id}$ and $-\widetilde{\mathbb{G}} = -\mathbb{E} = -\mathcal{P} + \lambda\mathrm{Id}$ for $\lambda \ge 0$. This is a special case of the quasi-convex/quasi-concave equation in Example 4.1. By a result of Hiriart-Urruty and Plazanet (which we present in Appendix A) we have

$$h \;\text{ is } \mathbb{H}_\lambda\text{-harmonic} \qquad \iff \qquad h \text{ is } C^{1,1} \text{ with Lipschitz coefficient } \lambda.$$

Related to this is the quasi-subaffine/quasi-superaffine equation in Example 4.2. This equation is the mirror $\mathbb{H}_\lambda^*$ of the $\mathbb{H}_\lambda$ above. Let's let $\lambda = 1$ and set $\mathbb{H} = \mathbb{H}_1$. Then the intersection $\mathbb{H} \cap \mathbb{H}^*$ is another generalized equation (see Remark before (2.4)). The $\mathbb{H} \cap \mathbb{H}^*$-harmonics h of class C^2 satisfy

$$D^2h + \mathrm{Id} \geq 0, \quad \mathrm{Id} - D^2h \geq 0 \quad \text{and} \quad \det(D^2h + \mathrm{Id}) = -\det(\mathrm{Id} - D^2h).$$

A basic example is given by taking $\mathbb{E} = \widetilde{\mathcal{P}}$ and $\mathbb{G} = \mathcal{P}$, i.e., $-\widetilde{\mathbb{G}} = -\widetilde{\mathcal{P}}$ so that $\mathbb{H} = \widetilde{\mathcal{P}} \cap (-\widetilde{\mathcal{P}}) = \mathrm{Sym}^2(\mathbb{R}^n) - \mathrm{Int}\{\mathcal{P} \cup (-\mathcal{P})\}$. This is a special case of the quasi-subaffine/quasi-superaffine equation in Example 4.2 (see also Example 4.16(d)). When $n = 2$, we see that $\mathbb{H} = \{A : \lambda_1(A)\lambda_2(A) = \det(A) \leq 0\}$.

When $\mathbb{G} = \mathbb{E}$, we have $\mathbb{H} = \partial\mathbb{E}$ and we are in the case discussed in the preliminaries above. Here there is no other pair giving the same $\mathbb{H}$.

Note however that a generalized equation $\mathbb{H}$ can possibly be written as $\mathbb{E} \cap (-\widetilde{\mathbb{G}})$ for many pairs of subequations $(\mathbb{E}, \mathbb{G})$. These pairs are partially ordered by saying

$$(\mathbb{E}, \mathbb{G}) \prec (\mathbb{E}', \mathbb{G}') \quad \text{if} \quad \mathbb{E} \subset \mathbb{E}' \ \text{ and } \ \mathbb{G}' \subset \mathbb{G}. \tag{1.10}$$

If this holds, then any $\mathbb{H}_{\mathbb{E},\mathbb{G}}$-harmonic on X is automatically $\mathbb{H}_{\mathbb{E}',\mathbb{G}'}$-harmonic on X.

We shall see in Chapter 3 that for any generalized equation $\mathbb{H}$, there exits a unique **canonical pair** $(\mathbb{E}_{\min}, \mathbb{G}_{\max})$ defining $\mathbb{H}$, which is $\prec$ all other pairs defining the set $\mathbb{H}$. Thus an $\mathbb{H}_{\mathbb{E}_{\min},\mathbb{G}_{\max}}$-harmonic on X, is harmonic for all pairs $(\mathbb{E}, \mathbb{G})$ defining $\mathbb{H}$. It will be called a **canonical $\mathbb{H}$-harmonic** on X. There remain some very interesting questions concerning this story (see Chapter 5).

One may wonder whether the closed sets $\mathbb{H} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ which are generalized equations can be intrinsically characterized. They can be. They are exactly the sets satisfying:

$$\mathbb{H} \ = \ (\overline{\mathbb{H} + \mathcal{P}}) \cap (\overline{\mathbb{H} - \mathcal{P}}). \tag{1.11}$$

(See Theorem 3.5.)

Given this, it is natural to consider the canonical $\mathbb{H}$-harmonics on an open set $X \subset \mathbb{R}^n$. In Theorem 3.6 we show that if $\mathbb{H} \subset \mathbb{H}'$ are generalized equations,

$$\text{then any } \mathbb{H}_{\mathrm{can}}\text{-harmonic on } X \text{ is also a } \mathbb{H}'_{\mathrm{can}}\text{-harmonic on } X.$$

Now whether or not a closed subset $\mathbb{H} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ satisfies (1.11), the set

$$\mathbb{H}^\diamondsuit \equiv (\overline{\mathbb{H} + \mathcal{P}}) \cap (\overline{\mathbb{H} - \mathcal{P}}) \tag{1.12}$$

is a generalized equation, and it is the smallest generalized equation containing $\mathbb{H}$. That is, it is contained in every other generalized equation containing $\mathbb{H}$. (See Theorem 3.7 for all of these facts.)

Recall that if $\mathbb{H} = \partial\mathbb{F} \Rightarrow -\mathbb{H} = \partial\widetilde{\mathbb{F}}$. More generally, the negative of a generalized equation is also a generalized equation

$$-\mathbb{H}_{\mathbb{E},\mathbb{G}} \;=\; \mathbb{H}_{\widetilde{\mathbb{E}},\widetilde{\mathbb{G}}} \tag{1.13}$$

This might be viewed as the dual of a generalized equation.

More importantly, every generalized equation has a **mirror**, which is obtained by interchanging $\mathbb{E}$ and $\mathbb{G}$. A shortened form of our main result is the following.

Theorem 2.6. *Suppose $\mathbb{H} \equiv \mathbb{H}_{\mathbb{E},\mathbb{G}} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ is a generalized equation with mirror $\mathbb{H}^* = \mathbb{G} \cap (-\widetilde{\mathbb{E}})$, and that $\Omega \subset \mathbb{R}^n$ is a bounded domain. Then*

$$\textit{(A)} \quad \textit{Uniqueness for the (DP) for } \mathbb{H} \textit{holds on } \Omega \quad \iff \quad \mathrm{Int}\,\mathbb{H} = \emptyset$$

Suppose that $\partial\Omega$ is smooth and strictly $\mathbb{G}$ and $\widetilde{\mathbb{G}}$-convex. Then

$$\textit{(B)} \quad \textit{Existence for the (DP) for } \mathbb{H} \textit{ holds on } \Omega \quad \iff \quad \mathrm{Int}\,\mathbb{H}^* = \emptyset$$

This allows us to divide generalized equations into four types depending on whether the interiors of $\mathbb{H}$ and $\mathbb{H}^*$ are empty or not. These types are strictly tied to the uniqueness and existence questions.

In proving Theorem 2.6 we established some results of independent interest.

Proposition 2.16. *Fix boundary values $\varphi \in C(\partial\Omega)$. If there exist solutions h to the $\mathbb{H}$-(DP) and h^* to the $\mathbb{H}^*$-(DP) on Ω, then $h = h^*$. That is, $h = h^*$ is the common solution to the $\mathbb{H}$ and the $\mathbb{H}^*$ Dirichlet problems with boundary values φ.*

Proposition 2.20. *Assume uniqueness holds for $\mathbb{H}$, i.e., $\mathrm{Int}\,\mathbb{H} = \emptyset$, for the generalized equation $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$. Given a domain as above and $\varphi \in C(\partial\Omega)$, then:*

$$\begin{aligned} &\textit{There exists } h \in C(\overline{\Omega}) \textit{ with } h\big|_{\partial\Omega} = \varphi \textit{ and } h\big|_{\Omega}\ \mathbb{H}\text{-harmonic} \\ &\iff \qquad h_{\mathbb{E}} = h_{\mathbb{G}}, \quad \textit{in which case } h = h_{\mathbb{E}} = h_{\mathbb{G}}. \end{aligned}$$

We would like to thank the referee for some very helpful suggestions.

2. Main new definitions and the main theorem

To begin we discuss more completely the notion of a (determined) equation in the sense of [18], [8] and [9].

Definition 2.1 *(Equation).* A subset $\mathbb{H} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a **determined equation** or just an **equation** if $\mathbb{H} = \partial\mathbb{F}$ for some subequation $\mathbb{F}$. In this case a **solution to the equation** $\mathbb{H}$ (or an $\mathbb{H}$-**harmonic**) on X, is a function u such that $u \in \mathbb{F}(X)$ and $-u \in \widetilde{\mathbb{F}}(X)$.

Such functions are automatically continuous by definition. For C^2-functions u the consistency of this definition with the classical definition that $D_x^2 u \in \mathbb{H} = \partial\mathbb{F}$ for all x, follows from (1.3) above that $\partial\mathbb{F} = \mathbb{F} \cap (-\widetilde{\mathbb{F}})$, and the consistency property for subequations, mentioned in Section 1.

The main new concept in this paper is the following generalization.

Definition 2.2 *(Generalized Equation).* A subset $\mathbb{H} \equiv \mathbb{H}_{\mathbb{E},\mathbb{G}} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a **generalized equation** if

$$\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}}) = \mathbb{E} \cap (\sim \mathrm{Int}\mathbb{G}) \tag{2.1}$$

for some pair of subequations $\mathbb{E}, \mathbb{G}$. Just as in Definition 2.1 above, where $\mathbb{E} = \mathbb{G} = \mathbb{F}$, we define **a solution to the (generalized) equation** $\mathbb{H}_{\mathbb{E},\mathbb{G}}$ (or an $\mathbb{H}_{\mathbb{E},\mathbb{G}}$-**harmonic**) to be a function u with

$$u \in \mathbb{E}(X) \qquad \text{and} \qquad -u \in \widetilde{\mathbb{G}}(X), \tag{2.2}$$

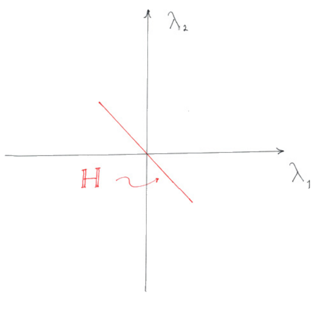
and let $\mathbb{H}(X)$ or $\mathbb{H}_{\mathbb{E},\mathbb{G}}(X)$ denote the space of $\mathbb{H}$-solutions on X.

As noted above, a generalized equation $\mathbb{H}$ can possibly be written as $\mathbb{E} \cap (-\widetilde{\mathbb{G}})$ for many pairs of subequations $(\mathbb{E}, \mathbb{G})$. These pairs are partially ordered by saying $(\mathbb{E}, \mathbb{G}) \prec (\mathbb{E}', \mathbb{G}')$ if $\mathbb{E} \subset \mathbb{E}'$ and $\mathbb{G}' \subset \mathbb{G}$. If this holds, then any $\mathbb{H}_{\mathbb{E},\mathbb{G}}$-harmonic on X is automatically $\mathbb{H}_{\mathbb{E}',\mathbb{G}'}$-harmonic on X.

We present a guiding elementary example at this point to help the reader assimilate the many definitions presented here. Examples are discussed more fully in Section 4.

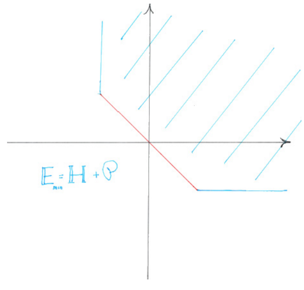
Example 2.3 *(The constrained Laplacian).* Fix $r \geq 0$ and let

$$\mathbb{H} \equiv \{A : \mathrm{tr}A = 0 \ \text{ and } \ -rI \leq A \leq rI\}. \tag{2.3}$$

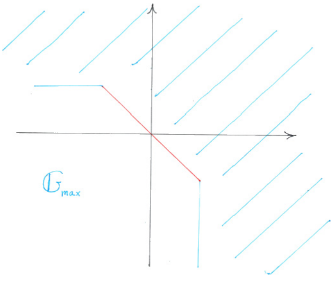


Define

$$\mathbb{E}_{\min} \stackrel{\text{def}}{=} \mathbb{H} + \mathcal{P} \;=\; \Delta \cap (\mathcal{P} - rI)$$



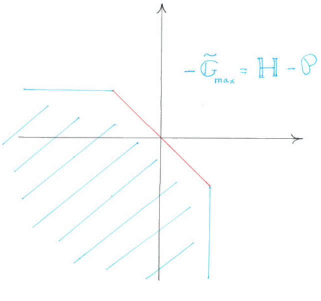
and note that $-\mathbb{E}_{\min} = \mathbb{H} - \mathcal{P}$ since $-\mathbb{H} = \mathbb{H}$. We then define $\mathbb{G}_{\max} \equiv \widetilde{\mathbb{E}}_{\min}$



Then we have

$$\mathbb{H} \;=\; \mathbb{E}_{\min} \cap (-\widetilde{\mathbb{G}}_{\max})$$

with $-\widetilde{\mathbb{G}}_{\max} = -\mathbb{E}_{\min} = \mathbb{H} - \mathcal{P}$



We note that, with $\mathbb{H}$ defined by $\mathbb{E}_{\min}, \mathbb{G}_{\max}$ as above, **$\mathbb{H}$-harmonic implies Δ-harmonic**, which is of course obvious for C^2-functions. This follows from Theorem 3.6 below.

Remark *(Intersections of GE's are GE's).* For general intersections of subequations and their negatives $\mathbb{H} \equiv \mathbb{E}_1 \cap \cdots \cap \mathbb{E}_k \cap (-\widetilde{\mathbb{G}}_1) \cap \cdots \cap (-\widetilde{\mathbb{G}}_\ell)$, we just get another generalized equation. The positivity condition (1.1) and the closure condition are both preserved under intersections. Hence $\mathbb{E} \equiv \mathbb{E}_1 \cap \cdots \cap \mathbb{E}_k$ and $\widetilde{\mathbb{G}} \equiv \widetilde{\mathbb{G}}_1 \cap \cdots \cap \widetilde{\mathbb{G}}_\ell$ are subequations, and $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$. (Also since $\widetilde{\widetilde{\mathbb{F}}} = \mathbb{F}$ and $\widetilde{\mathbb{F}_1 \cap \mathbb{F}_2} = \widetilde{\mathbb{F}}_1 \cup \widetilde{\mathbb{F}}_2$ for subequations, one can show that $\mathbb{G} = \mathbb{G}_1 \cup \cdots \cup \mathbb{G}_\ell$.)

As before by definition such functions are continuous with the coherence property that if u is C^2, then

$$u \text{ is } \mathbb{H}\text{-harmonic} \qquad \iff \qquad D^2_x u \in \mathbb{H} \text{ for all } x \tag{2.4}$$

For any generalized equation $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ it is easy to see by (1.4) that the interior satisfies

$$\mathrm{Int}\mathbb{H} \;=\; (\mathrm{Int}\mathbb{E}) \cap (-\mathrm{Int}\widetilde{\mathbb{G}}) \;=\; (\mathrm{Int}\mathbb{E}) \cap (\sim \mathbb{G}) \tag{2.5}$$

In particular,

$$\begin{aligned} &\text{If } \mathbb{H} = \partial\mathbb{F} = \mathbb{F} \cap (-\widetilde{\mathbb{F}}) \text{ is a determined equation,} \\ &\quad \text{then } \ \mathrm{Int}\,\mathbb{H} = (\mathrm{Int}\,\mathbb{F}) \cap (\sim \mathbb{F}) = \emptyset. \end{aligned} \tag{2.6}$$

The mirror

Each generalized equation has a mirror, which we now define.

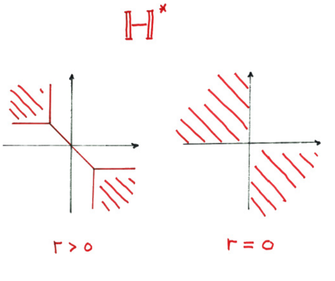
Definition 2.4 *(The mirror equation).* If

$$\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$$

is a generalized equation, its **mirror** is defined to be the generalized equation

$$\mathbb{H}^* = \mathbb{G} \cap (-\widetilde{\mathbb{E}}).$$

In Example 2.3 we have $\mathbb{H}^* = \mathbb{G}_{\max} \cap (-\widetilde{\mathbb{E}}_{\min}) = \mathbb{G}_{\max} \cap (-\mathbb{G}_{\max})$ since $\mathbb{E}_{\min}$ and $\mathbb{G}_{\max}$ are dual to one another.



This $\mathbb{H}^*$ is a somewhat surprising example of a generalized equation from the following point of view. Consider $n = 2$ and $r = 0$. By using viscosity theory in applying Definition 2.2 to $\mathbb{H}^*$, we have a notion of when D^2h has eigenvalues λ_1 and λ_2, which differ in sign, for *general continuous functions* h. Moreover, this is consistent with the classical definition if $h \in C^2$. On the other hand, there is nothing "elliptic" about a mixed sign constraint on D^2h. (The reader could also consider $(\mathbb{H}^\diamond)^*$ in Example 3.8 below where the eigenvalues also satisfy $|\lambda_1|, |\lambda_2| \geq r$.)

Existence and uniqueness

Examination of existence and uniqueness for the Dirichlet Problem for $\mathbb{H}$-harmonics leads to four distinct types of generalized equations, as follows.

Definition 2.5. Suppose Ω is a bounded domain in $\mathbb{R}^n$. We say that **existence for the (DP) for** $\mathbb{H}$ holds on Ω if for all prescribed boundary functions $\varphi \in C(\partial\Omega)$ there exists $h \in C(\overline{\Omega})$ satisfying

(a) $h\big|_{\Omega}$ is $\mathbb{H}$-harmonic, and
(b) $h\big|_{\partial\Omega} = \varphi$.

We say **uniqueness for the (DP) for** $\mathbb{H}$ holds on Ω if for all $\varphi \in C(\partial\Omega)$ there exists at most one $h \in C(\overline{\Omega})$ satisfying (a) and (b).

Now we can state our main result in this pure second-order constant coefficient case.

Theorem 2.6. *Suppose* $\mathbb{H} \equiv \mathbb{H}_{\mathbb{E},\mathbb{G}} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ *is a generalized equation with mirror* $\mathbb{H}^* = \mathbb{G} \cap (-\widetilde{\mathbb{E}})$, *and that* $\Omega \subset \mathbb{R}^n$ *is a bounded domain. Then*

$$(A) \quad \textit{Uniqueness for the (DP) for } \mathbb{H} \textit{ holds on} \Omega \qquad \iff \qquad \mathrm{Int}\, \mathbb{H} = \emptyset$$

Suppose that $\partial\Omega$ is smooth and strictly $\mathbb{G}$ and $\widetilde{\mathbb{G}}$-convex. Then

$$(B) \quad \textit{Existence for the (DP) for } \mathbb{H} \textit{ holds on} \Omega \qquad \iff \qquad \mathrm{Int}\, \mathbb{H}^* = \emptyset$$

In fact, the following are equivalent:

(1) $\mathrm{Int}\, \mathbb{H} = \emptyset$,
(2) Uniqueness for the (DP) for $\mathbb{H}$ holds on Ω,
(3) $\mathbb{E} \subset \mathbb{G}$,
(4) $\mathbb{H} = \partial\mathbb{E} \cap \partial\mathbb{G}$,

and assuming that $\partial\Omega$ is smooth and both strictly $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ convex, these conditions are equivalent to

(5) Existence for the (DP) for $\mathbb{H}^*$ holds on Ω.

Interchanging $\mathbb{E}$ with $\mathbb{G}$ and $\mathbb{H}$ with $\mathbb{H}^*$, we have the mirror list of equivalences:

*(1)** $\operatorname{Int}\mathbb{H}^* = \emptyset$,
*(2)** *Uniqueness for the (DP) for $\mathbb{H}^*$ holds on Ω,*
*(3)** $\mathbb{G} \subset \mathbb{E}$,
*(4)** $\mathbb{H}^* = \partial\mathbb{G} \cap \partial\mathbb{E}$,

and assuming that $\partial\Omega$ is smooth and both strictly $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ convex, these conditions are equivalent to

*(5)** *Existence for the (DP) for $\mathbb{H}$ holds on Ω.*

Proof. It suffices to prove that (1) through (5) are equivalent because: (A) is just the statement that (2) $\iff$ (1), the mirror equivalences $(1)^*$ through $(5)^*$ are immediate from (1) through (5), and (B) is just the statement $(1)^* \iff (5)^*$.

Before proving the equivalence of (1) through (5) we list some trivial equivalences for any sets $\mathbb{E}$ and $\mathbb{G}$.

Lemma 2.7. *Suppose that $\mathbb{E} = \overline{\operatorname{Int}\mathbb{E}}$ and $\mathbb{G} = \overline{\operatorname{Int}\mathbb{G}}$, then*

$$(i)\ \ \mathbb{E} \subset \mathbb{G} \quad (ii)\ \ \operatorname{Int}\mathbb{E} \subset \mathbb{G} \quad (iii)\ \ \operatorname{Int}\mathbb{E} \subset \operatorname{Int}\mathbb{G} \quad (iv)\ \ \widetilde{\mathbb{G}} \subset \widetilde{\mathbb{E}} \quad (v)\ \ -\widetilde{\mathbb{G}} \subset -\widetilde{\mathbb{E}}$$

are equivalent. Interchanging $\mathbb{E}$ and $\mathbb{G}$ yields that

$$(i)^*\ \ \mathbb{G} \subset \mathbb{E} \quad (ii)^*\ \ \operatorname{Int}\mathbb{G} \subset \mathbb{E} \quad (iii)^*\ \ \operatorname{Int}\mathbb{G} \subset \operatorname{Int}\mathbb{E} \quad (iv)^*\ \ \widetilde{\mathbb{E}} \subset \widetilde{\mathbb{G}} \quad (v)^*\ \ -\widetilde{\mathbb{E}} \subset -\widetilde{\mathbb{G}}$$

are equivalent.

Proof. Note that $(i) \Rightarrow (ii)$ obviously, $(ii) \Rightarrow (iii)$ since $\operatorname{Int}\mathbb{E}$ is an open subset contained in $\mathbb{G}$, $(iii) \Rightarrow (i)$ by the hypothesis, $(i) \iff (iv)$ follows from the definitions of the duals, and $(iv) \iff (v)$ is trivial. $\square$

Corollary 2.8. *If $\mathbb{H} \equiv \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ is a generalized equation with mirror $\mathbb{H}^* = \mathbb{G} \cap (-\widetilde{\mathbb{E}})$, then*

(1) $\operatorname{Int}\mathbb{H} = \emptyset$ *is equivalent to (i) through (v), and*
*(1)** $\operatorname{Int}\mathbb{H}^* = \emptyset$ *is equivalent to $(i)^*$ through $(v)^*$.*

Proof. Note that (2.5) says that

$$\operatorname{Int}\mathbb{H} \ = \ (\operatorname{Int}\mathbb{E}) \sim \mathbb{G}.$$

Hence, (1) $\iff$ (*ii*). Interchanging $\mathbb{E}$ and $\mathbb{G}$ yields $(1)^* \iff (ii)^*$. $\square$

Proof that (1) $\iff$ (3). By Corollary 2.8, (1) $\iff$ (i) which is (3). $\square$

Proof that (4) $\Rightarrow$ (1). If $\mathbb{H} = \partial\mathbb{E} \cap \partial\mathbb{G}$, then in particular $\mathbb{H} \subset \partial\mathbb{E}$ which has no interior.

Proof that (1) $\Rightarrow$ (4). Note that $\mathbb{E} = \partial\mathbb{E} \cup (\operatorname{Int}\mathbb{E})$ and $-\widetilde{\mathbb{G}} = \partial\mathbb{G} \cup (\sim\mathbb{G})$ are disjoint unions. Hence, $\mathbb{H} \equiv \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ is the disjoint union of the four sets: $\partial\mathbb{E} \cap \partial\mathbb{G}$, $\partial\mathbb{E} \cap (\sim\mathbb{G})$, $\operatorname{Int}\mathbb{E} \cap \partial\mathbb{G}$, and $\operatorname{Int}\mathbb{E} \cap (\sim\mathbb{G})$. By (2.1) and (2.5), the last set $\operatorname{Int}\mathbb{E} \cap (\sim\mathbb{G}) = \operatorname{Int}\mathbb{H} = \emptyset$, so that $\mathbb{H}$ is the disjoint union of the three remaining sets. However, $\operatorname{Int}\mathbb{H} = (\operatorname{Int}\mathbb{E}) \cap (\sim\mathbb{G}) = \emptyset$ implies (3) and hence $\partial\mathbb{E} \subset \mathbb{G}$ or $\partial\mathbb{E} \cap (\sim\mathbb{G}) = \emptyset$. By Lemma 2.7 (iii) $\operatorname{Int}\mathbb{E} \subset \operatorname{Int}\mathbb{G}$ so that $(\operatorname{Int}\mathbb{E}) \cap \partial\mathbb{G} = \emptyset$. Thus three of these four sets are empty leaving $\mathbb{H} = \partial\mathbb{E} \cap \partial\mathbb{G}$. $\square$

Proof that (1) $\Rightarrow$ (2). Recall from [9, Def. 8.1] the following form of comparison (C) for a subequation $\mathbb{F}$, which we will refer to as the **zero maximum principle for sums**, and abbreviate as either (ZMP for sums) or (C).

Definition 2.9. Given a relatively compact domain Ω we say that **comparison holds for $\mathbb{F}$ on Ω** if for all upper semi-continuous functions u, v on $\overline{\Omega}$, with $u\big|_\Omega$ $\mathbb{F}$-subharmonic and $v\big|_\Omega$ $\widetilde{\mathbb{F}}$-subharmonic, one has

$$u + v \ \le\ 0 \quad \text{on} \quad \partial\Omega \qquad \Rightarrow \qquad u + v \ \le\ 0 \quad \text{on} \quad \overline{\Omega} \qquad\qquad (\textit{ZMP for sums})$$

Comparison (C) always holds for pure second-order subequations $\mathbb{F} \subset \operatorname{Sym}^2(\mathbb{R}^n)$ and domains $\Omega \subset\subset \mathbb{R}^n$. This was first established in [8, Rmk. 4.9 and Thm. 6.4]. (See [1] for many other constant coefficient situations where comparison always holds. There are also some extensions in [9] to simply-connected, non-positively curved manifolds.) More precisely we have:

Theorem 2.10. *Suppose $\mathbb{F} \subset \operatorname{Sym}^2(\mathbb{R}^n)$ is a subequation and $\Omega \subset \mathbb{R}^n$ is a bounded domain. Then comparison (C) holds for $\mathbb{F}$ on Ω.*

Now we can prove that (1) $\Rightarrow$ (2)

Proposition 2.11. *Comparison (C) for both $\mathbb{E}$ and $\mathbb{G}$ on a domain Ω implies that:*

$$\operatorname{Int}\mathbb{H} \ =\ \emptyset \qquad \Rightarrow \qquad \textit{uniqueness for the } \mathbb{H}\textit{-(DP) on } \Omega$$

Proof. By Corollary 2.8, $\operatorname{Int}\mathbb{H} = \emptyset \Rightarrow \widetilde{\mathbb{G}} \subset \widetilde{\mathbb{E}}$. Therefore (C) for $\mathbb{E}$ implies the (ZMP for sums) if u is $\mathbb{E}$-subharmonic and v is $\widetilde{\mathbb{G}}$-subharmonic. If h_1, h_2 are two solutions to the $\mathbb{H}$-(DP) on Ω with the same boundary values, then $u = h_1$ is $\mathbb{E}$–subharmonic and $v = -h_2$ is $\widetilde{\mathbb{G}}$–subharmonic on Ω. Since $u + v = 0$ on $\partial\Omega$, the (ZMP) $\Rightarrow h_1 \le h_2$ on $\overline{\Omega}$.

Interchanging h_1 and h_2 is possible since we are also assuming (C) for $\mathbb{G}$. This proves $h_1 = h_2$. $\square$

Proof that (2) $\Rightarrow$ (1):

Proposition 2.12. *If there exists a function $h \in C^2(\Omega) \cap C(\overline{\Omega})$ with $D_x^2 h \in \operatorname{Int} \mathbb{H}$ for all $x \in \Omega$, then uniqueness for the $\mathbb{H}$-(DP) on Ω fails.*

Proof. Take $\varphi = h\big|_{\partial\Omega}$. For any function $\psi \in C^\infty_{\text{cpt}}(\Omega)$, if $\epsilon > 0$ is sufficiently small, we have $D_x^2(h + \epsilon\psi) \in \mathbb{H}$ for all $x \in \Omega$. Thus the functions $h + \epsilon\psi$ give many solutions to the Dirichlet problem with the same boundary values φ. $\square$

The following trivial fact is peculiar to the pure second-order, constant coefficient case (and the pure first-order case).

Lemma 2.13. *Given any non-empty subset $S \subset \operatorname{Sym}^2(\mathbb{R}^n)$, there exists a function $h \in C^2(\mathbb{R}^n)$ with $D_x^2 h \in S$ for all $x \in \mathbb{R}^n$.*

Proof. Pick $A \in S$ and take $h(x) \equiv \frac{1}{2}\langle Ax, x\rangle$ so that $D_x^2 h = A$ for all $x \in \mathbb{R}^n$. $\square$

Combining this Lemma with the previous Proposition proves the implication (2) $\Rightarrow$ (1) in the form:

$$\begin{gathered}\operatorname{Int} \mathbb{H} \neq \emptyset \qquad \Rightarrow \\ \text{uniqueness for the } \mathbb{H}\text{-(DP) fails on all domains } \Omega \subset \mathbb{R}^n. \quad \square\end{gathered} \tag{2.7}$$

Next we treat the implication (1) $\operatorname{Int} \mathbb{H} = \emptyset \ \Rightarrow$ (5) existence for the $\mathbb{H}^*$-(DP).

Proposition 2.14. *If existence for the $\partial\mathbb{E}$-(DP) holds on Ω (Definition 2.5), then*

$$\operatorname{Int} \mathbb{H} = \emptyset \ \Rightarrow \ \textit{existence for the } \mathbb{H}^*\textit{-(DP) on } \Omega. \tag{2.8}$$

Proof. By Corollary 2.8 $\operatorname{Int} \mathbb{H} = \emptyset \ \Rightarrow \ \mathbb{E} \subset \mathbb{G}$. Let h denote the $\partial\mathbb{E}$-harmonic function solving the (DP) with boundary values φ. Since h is $\mathbb{E}$-subharmonic and $\mathbb{E} \subset \mathbb{G}$, it is also $\mathbb{G}$-submarmonic. Since $-h$ is $\widetilde{\mathbb{E}}$-subharmonic, this proves that h is $\mathbb{H}^* = \mathbb{G} \cap (-\widetilde{\mathbb{E}})$-harmonic. $\square$

Recall the following from [8]. (As mentioned in §1, $\mathbb{E}(\Omega)$ denotes the space of $\mathbb{E}$-subharmonic functions on Ω.)

Theorem 2.15 *(Existence). Suppose $\Omega \subset \mathbb{R}^n$ has a smooth boundary which is both $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ strictly convex. Given $\varphi \in C(\partial\Omega)$, the Perron function $h(x) \equiv \sup\{u \in USC(\overline{\Omega}) : u \in \mathbb{E}(\Omega)$ and $u\big|_{\partial\Omega} \leq \varphi\}$ solves the $\partial\mathbb{E}$-(DP) on Ω for boundary values φ.*

Combining Proposition 2.14 with Theorem 2.15 yields

$$(1)\ \ \mathrm{Int}\,\mathbb{H} = \emptyset \ \Rightarrow\ (5)\ \ \text{existence for the } \mathbb{H}^*\text{-(DP)} \tag{2.9}$$

on domains Ω with strictly $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ convex smooth boundaries.

Before proving that (5) $\Rightarrow$ (1), or that $\mathrm{Int}\,\mathbb{H} \neq \emptyset$ implies non-existence for the $\mathbb{H}^*$-(DP), we need to establish some preliminary facts, which are also of independent interest.

Proposition 2.16. *Fix boundary values $\varphi \in C(\partial\Omega)$. If there exist solutions h to the $\mathbb{H}$-(DP) and h^* to the $\mathbb{H}^*$-(DP) on Ω, then $h = h^*$. That is, $h = h^*$ is the common solution to the $\mathbb{H}$ and the $\mathbb{H}^*$ Dirichlet problems with boundary values φ.*

Proof. By definition h is $\mathbb{E}$-subharmonic and $-h$ is $\widetilde{\mathbb{G}}$-subharmonic on Ω. Also, h^* is $\mathbb{G}$-subharmonic and $-h^*$ is $\widetilde{\mathbb{E}}$-subharmonic on Ω. Therefore,

$$\begin{array}{lllll} h - h^* = 0 \text{ on } \partial\Omega & \Rightarrow & h - h^* \leq 0 \text{ on } \overline{\Omega} & \text{by } \mathbb{E}\text{-comparison,} \\ h^* - h = 0 \text{ on } \partial\Omega & \Rightarrow & h^* - h \leq 0 \text{ on } \overline{\Omega} & \text{by } \mathbb{G}\text{-comparison.} \end{array}$$

Thus $h - h^* = 0$ on $\overline{\Omega}$. $\square$

Note: Then $h = h^*$ solves the generalized equation

$$\mathbb{H} \cap \mathbb{H}^* \ = \ (\mathbb{E} \cap \mathbb{G}) \cap (-\widetilde{\mathbb{E} \cup \mathbb{G}}).$$

(One can show that $\widetilde{\mathbb{E}} \cap \widetilde{\mathbb{G}} = \widetilde{\mathbb{E} \cup \mathbb{G}}$. See [9, Property (2) after Def. 3.1] for arbitrary subsets of $J^2(X)$.)

Proposition 2.17. *Recall again that comparison holds for $\mathbb{E}$ and $\mathbb{G}$ on Ω. From this we conclude the following. If there exists a function $f \in C^2(\Omega) \cap C(\overline{\Omega})$ with $D^2_x f \in \mathrm{Int}\,\mathbb{H}$ for all $x \in \Omega$, then there is no solution h^* to the $\mathbb{H}^*$-(DP) on Ω with boundary values $\varphi \equiv f\big|_{\partial\Omega}$.*

Proof. If h^* exists, then since f is an $\mathbb{H}$-solution, by Proposition 2.16 we have $h^* = f$, and hence h^* is C^2. Thus $D^2 f \in (\mathrm{Int}\,\mathbb{H}) \cap \mathbb{H}^* = (\mathrm{Int}\,\mathbb{E} \sim \mathbb{G}) \cap (\mathbb{G} \sim \mathrm{Int}\,\mathbb{E}) = \emptyset$. So this is impossible. $\square$

Proof that (5) $\Rightarrow$ (1) or that $\mathrm{Int}\,\mathbb{H} \neq \emptyset$ $\Rightarrow$ **non-existence for** $\mathbb{H}^*$**.** The fact that $\mathrm{Int}\,\mathbb{H} \neq \emptyset$ guarantees the existence of such a function f by Lemma 2.13, and hence the non-existence for the $\mathbb{H}^*$ Dirichlet problem. $\square$

This completes the proof of Theorem 2.6. $\square$

Four types

In light of Theorem 2.6, if one is given a generalized equation $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ with mirror $\mathbb{H}^* = \mathbb{G} \cap (-\widetilde{\mathbb{E}})$, there are four distinct types possible which we label as follows.

Type I: $\mathrm{Int}\,\mathbb{H} = \emptyset$ and $\mathrm{Int}\,\mathbb{H}^* = \emptyset$.
Type II: $\mathrm{Int}\,\mathbb{H} = \emptyset$ and $\mathrm{Int}\,\mathbb{H}^* \neq \emptyset$.
Type III: $\mathrm{Int}\,\mathbb{H} \neq \emptyset$ and $\mathrm{Int}\,\mathbb{H}^* = \emptyset$.
Type IV: $\mathrm{Int}\,\mathbb{H} \neq \emptyset$ and $\mathrm{Int}\,\mathbb{H}^* \neq \emptyset$.

Note that Types I and II belong to part (A) of Theorem 2.6, and Types III and IV belong to part (B). We shall now discuss each type.

Type I: $\mathrm{Int}\,\mathbb{H} = \mathrm{Int}\,\mathbb{H}^* = \emptyset$. This type is a "determined equation" $\partial\mathbb{F}$ as defined in Definition 2.1, because by (1) $\iff$ (3), and $(1)^* \iff (3)^*$, this is just the case where $\mathbb{E}$ and $\mathbb{G}$ are equal. We will call this subequation $\mathbb{F}$. Thus $\mathbb{H}$ and $\mathbb{H}^*$ are $\mathbb{F} \cap (-\widetilde{\mathbb{F}}) = \partial\mathbb{F}$. Theorems 2.10 and 2.15 apply directly. Comparison holds for all bounded domains, and existence holds if $\partial\Omega$ is smooth and strictly $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ convex using results from [8].

Type II: $\mathrm{Int}\,\mathbb{H} = \emptyset$ and $\mathrm{Int}\,\mathbb{H}^* \neq \emptyset$. Collecting together (1)–(5) and the negations of $(1)^*$–$(5)^*$ we have that

$$\mathbb{E} \text{ is a proper subset of } \mathbb{G} \text{ and } \mathbb{H} = \partial\mathbb{E} \cap \partial\mathbb{G} \neq \mathbb{H}^*$$

Uniqueness but not existence holds for $\mathbb{H}$ on any bounded domain Ω. The opposite is true for $\mathbb{H}^*$, namely uniqueness for $\mathbb{H}^*$ fails on all bounded domains Ω, but if $\partial\Omega$ is smooth and both strictly $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ convex, then existence holds for $\mathbb{H}^*$ on Ω. In addition $\mathbb{H}$ is a proper subset of both $\partial\mathbb{E}$ and $\partial\mathbb{G}$. This is proven in (2.10) below.

For Type III we interchange $\mathbb{E}$ with $\mathbb{G}$ and $\mathbb{H}$ with $\mathbb{H}^*$.

Type III: $\mathrm{Int}\,\mathbb{H} \neq \emptyset$ and $\mathrm{Int}\,\mathbb{H}^* = \emptyset$. Collecting together $(1)^*$–$(5)^*$ and the negations of (1)–(5) we have that

$$\mathbb{G} \text{ is a proper subset of } \mathbb{E} \text{ and } \mathbb{H}^* = \partial\mathbb{G} \cap \partial\mathbb{E} \neq \mathbb{H}$$

Uniqueness but not existence holds for $\mathbb{H}^*$ on any bounded domain Ω. The opposite is true for $\mathbb{H}$, namely uniqueness for $\mathbb{H}$ fails on all bounded domains Ω, but if $\partial\Omega$ is smooth and both strictly $\mathbb{G}$ and $\widetilde{\mathbb{G}}$ convex, then existence holds for $\mathbb{H}$ on Ω. Also, $\mathbb{H}^*$ is a proper subset of both $\partial\mathbb{E}$ and $\partial\mathbb{G}$ by (2.10).

Type IV: $\mathrm{Int}\,\mathbb{H} \neq \emptyset$ and $\mathrm{Int}\,\mathbb{H}^* \neq \emptyset$. By (2.5) this is equivalent to

$$(\mathrm{Int}\,\mathbb{E}) \cap ({\sim}\,\mathbb{G}) \;\neq\; \emptyset \qquad \text{and} \qquad (\mathrm{Int}\,\mathbb{G}) \cap ({\sim}\,\mathbb{E}) \;\neq\; \emptyset.$$

Because of Lemma 2.7 (iii) and (iii)* this is equivalent to

$$\mathrm{Int}\,\mathbb{E} \not\subset \mathrm{Int}\,\mathbb{G} \qquad \text{and} \qquad \mathrm{Int}\,\mathbb{G} \not\subset \mathrm{Int}\,\mathbb{E}.$$

The main point here is that both existence and uniqueness for the (DP) for both $\mathbb{H}$ and $\mathbb{H}^*$ fail.

Next we consider the following.

Proposition 2.18 *(Boundaries of subequations). If $\mathbb{H}$ is a determined equation, then the subequation $\mathbb{F}$ with $\mathbb{H} = \partial\mathbb{F}$ is uniquely determined by $\mathbb{H}$. In fact, $\partial\mathbb{E} \subset \partial\mathbb{G}$ is enough to conclude that $\mathbb{E} = \mathbb{G}$ for any two subequations $\mathbb{E}$ and $\mathbb{G}$.*

Proof. The first statement follows from (1.7) which holds for any subequation $\mathbb{F}$.

For the second statement note that one has $\partial\mathbb{E} \subset \partial\mathbb{G} \Rightarrow \mathbb{E} = \partial\mathbb{E} + \mathcal{P} \subset \partial\mathbb{G} + \mathcal{P} = \mathbb{G}$. However, $\partial\mathbb{E} \subset \partial\mathbb{G} \iff -\partial\mathbb{E} \subset -\partial\mathbb{G}$, but $-\partial\mathbb{E} = \partial\widetilde{\mathbb{E}}$ and $-\partial\mathbb{G} = \partial\widetilde{\mathbb{G}}$. Hence, $\partial\widetilde{\mathbb{E}} \subset \partial\widetilde{\mathbb{G}}$, and this implies that $\widetilde{\mathbb{E}} \subset \widetilde{\mathbb{G}}$, which is equivalent to $\mathbb{G} \subset \mathbb{E}$. $\square$

It follows that:

$$\begin{aligned} &\text{If } \mathbb{H} \equiv \mathbb{E} \cap (-\widetilde{\mathbb{G}}) \text{ is Type II, then} \\ &\mathbb{H} = \partial\mathbb{E} \cap \partial\mathbb{G} \text{ is a proper subset of both } \partial\mathbb{E} \text{ and } \partial\mathbb{G}. \end{aligned} \tag{2.10}$$

Proof. If $\mathbb{H} \equiv \partial\mathbb{E} = \partial\mathbb{E} \cap \partial\mathbb{G}$, then $\partial\mathbb{E} \subset \partial\mathbb{G}$, so that by Proposition 2.18, $\mathbb{E} = \mathbb{G}$ and $\mathbb{H}$ is Type I. $\square$

Next we begin to examine to what extent a generalized equation $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ determines the subequations $\mathbb{E}$ and $\mathbb{G}$. The answer in the determined case is given by next proposition. However, open questions concerning more general cases can be found is Chapter 5.

Proposition 2.19 *(Uniqueness of the defining pair $\mathbb{E}, \mathbb{G}$ in the determined case). Suppose that $\mathbb{H} = \partial\mathbb{F}$ where $\mathbb{F}$ is a subequation. If $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ for subequations $\mathbb{E}$ and $\mathbb{G}$, then $\mathbb{E} = \mathbb{G} = \mathbb{F}$.*

Proof. By (2.6) we have $\mathrm{Int}\mathbb{H} = \mathrm{Int}(\partial\mathbb{F}) = \emptyset$. By (2.5) we have $\mathrm{Int}\mathbb{H} = (\mathrm{Int}\mathbb{E}) \cap (\sim \mathbb{G})$. Therefore, $\mathrm{Int}\mathbb{E} \subset \mathbb{G}$, and so $\mathbb{E} = \overline{\mathrm{Int}\mathbb{E}} \subset \mathbb{G}$.

Now by (1.7) $\mathbb{F} = \partial\mathbb{F} + \mathcal{P} = \mathbb{H} + \mathcal{P}$, and by the hypothesis $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$, we have $\mathbb{H} \subset \mathbb{E}$. Thus, $\mathbb{F} = \mathbb{H} + \mathcal{P} \subset \mathbb{E} + \mathcal{P} = \mathbb{E}$. By the same hypothesis $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$, we have $-\mathbb{H} \subset \widetilde{\mathbb{G}}$. Hence by (1.8), $\widetilde{\mathbb{F}} = -\mathbb{H} + \mathcal{P} \subset \widetilde{\mathbb{G}} + \mathcal{P} = \widetilde{\mathbb{G}}$, so that $\mathbb{G} \subset \mathbb{F}$. This proves that $\mathbb{F} \subset \mathbb{E} \subset \mathbb{G} \subset \mathbb{F}$. $\square$

The question of characterizing the boundary functions for existence

Here we turn to a natural question which arises in the Non-Existence cases, Types II and IV.

$$\text{For which boundary functions } \varphi \in C(\partial\Omega)$$
$$\text{does there exists a solution to the } \mathbb{H}\text{-Dirichlet problem?}$$

First we discuss the Type II case: Non-Existence/Uniqueness for $\mathbb{H}$. This is interesting, for example, for the constrained Laplacian above.

We make the assumption that Ω is a bounded domain with smooth boundary which is both strictly $\mathbb{E}$ and $\widetilde{\mathbb{G}}$-convex. Using the equivalent versions (1)–(5) of the uniqueness hypothesis for $\mathbb{H}$ in Theorem 2.6:

$$\operatorname{Int}\mathbb{H} \;=\; \emptyset \qquad \Longleftrightarrow \qquad \mathbb{E} \;\subset\; \mathbb{G} \qquad \Longleftrightarrow \qquad \widetilde{\mathbb{G}} \;\subset\; \widetilde{\mathbb{E}},$$

this implies that $\partial\Omega$ is also strictly $\mathbb{G}$ and $\widetilde{\mathbb{E}}$-convex. Let $h_{\mathbb{E}} \in C(\overline{\Omega})$ denote the (unique) $\partial\mathbb{E}$-harmonic function on Ω with $h_{\mathbb{E}}\big|_{\partial\Omega} = \varphi$, and $h_{\mathbb{G}} \in C(\overline{\Omega})$ denote the (unique) $\partial\mathbb{G}$-harmonic function on Ω with $h_{\mathbb{G}}\big|_{\partial\Omega} = \varphi$. One answer to the question is the following.

Proposition 2.20. *Assume uniqueness holds for $\mathbb{H}$, i.e., $\operatorname{Int}\mathbb{H} = \emptyset$, for the generalized equation $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$. Given a domain as above and $\varphi \in C(\partial\Omega)$, then:*

$$\text{There exists } h \in C(\overline{\Omega}) \text{ with } h\big|_{\partial\Omega} = \varphi \text{ and } h\big|_{\Omega}\ \mathbb{H}\text{-harmonic}$$
$$\Longleftrightarrow \qquad h_{\mathbb{E}} = h_{\mathbb{G}}, \quad \text{in which case } h = h_{\mathbb{E}} = h_{\mathbb{G}}.$$

Proof. Suppose that $h_{\mathbb{E}} = h_{\mathbb{G}}$ and set $h = h_{\mathbb{E}} = h_{\mathbb{G}}$. Then $h\big|_{\partial\Omega} = \varphi$, h is $\mathbb{E}$-subharmonic, and $-h$ is $\widetilde{\mathbb{G}}$-subharmonic on Ω, which proves that h is a solution to the $\mathbb{H}$ Dirichlet Problem (DP) on Ω with boundary values φ.

Conversely, if there exists such an $h \in C(\overline{\Omega})$, then h also solves the $\partial\mathbb{E}$ (DP) since h is $\mathbb{E}$-subharmonic and $-h$ is $\widetilde{\mathbb{E}} \supset \widetilde{\mathbb{G}}$-subharmonic. Hence, $h_{\mathbb{E}} = h$. Similarly, $h_{\mathbb{G}} = h$ since h is $\mathbb{G} \supset \mathbb{E}$-subharmonic and $-h$ is $\widetilde{\mathbb{G}}$-subharmonic. $\square$

The question posed above can also be asked in the Type IV case. It is intriguing for the "$C^{1,1}$-equation" in Example 4.2 below. Here we take $r_1 = r_2 = \lambda > 0$, and ask the question: How do we characterize the functions on the boundary of a domain Ω which have a $C^{1,1}$-extension with Lipschitz constant λ to all of Ω? This is related to $C^{1,1}$ Glaeser-Whitney extensions, for which there is a very large literature. The interested reader could consult [20], [4] for results and some history.

We finish this section with a general comment.

Remark 2.21 *(Nice properties of generalized equations).* Recall that generalized equations are preserved under:

(1) Taking the mirror,
(2) Taking Intersections (see Remark before (2.4)),
(3) Taking the negative (see (1.13) $\mathbb{H}_{\mathbb{E},\mathbb{G}} = \mathbb{H}_{\widetilde{\mathbb{E}},\widetilde{\mathbb{G}}}$).

3. The canonical pair defining a given $\mathbb{H}$, and an intrinsic characterization of generalized equations

In this section we look at the question of which closed subsets of $\mathrm{Sym}^2(\mathbb{R}^n)$ are generalized equations, and we characterize them. We start with the following.

Lemma 3.1. *Suppose $\mathbb{H} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is any closed subset. Then*

$$\mathbb{E}_{\min} \stackrel{\text{def}}{=} \overline{\mathbb{H} + \mathcal{P}} \tag{3.1}$$

is a subequation containing $\mathbb{H}$, and it is minimal with respect to these properties, i.e., if $\mathbb{E}$ is any other subequation containing $\mathbb{H}$, then $\mathbb{E}_{\min} \subset \mathbb{E}$.

Proof. The set $\mathbb{H}+\mathcal{P}$ obviously satisfies positivity and so does its closure $\mathbb{E}_{\min} = \overline{\mathbb{H} + \mathcal{P}}$. To see this, let $B \in \mathbb{E}_{\min}$ and choose a sequence $B_k = A_k + P_k \to B$ with $A_k \in \mathbb{H}$ and $P_k \geq 0$. Now note that for all $P \geq 0$, $B_k + P \to B + P$. $\square$

Corollary 3.2. *The set $-(\overline{\mathbb{H} - \mathcal{P}}) = \overline{-\mathbb{H} + \mathcal{P}}$ is the minimal subequation containing $-\mathbb{H}$. Let*

$$\widetilde{\mathbb{G}}_{\max} \stackrel{\text{def}}{=} \overline{-\mathbb{H} + \mathcal{P}} \quad \textit{denote this subequation.} \tag{3.2}$$

Then $\mathbb{G}_{\max}$ denotes its dual $\widetilde{\widetilde{\mathbb{G}}}_{\max}$.

Example. Let $\mathbb{H} = \{A \in \mathrm{Sym}^2(\mathbb{R}^2) : \lambda_1 < 0, \lambda_2 > 0 \text{ and } \lambda_1\lambda_2 = -1\}$, where $\lambda_1 \leq \lambda_2$ are the ordered eigenvalues of A. Then $\mathbb{H} + \mathcal{P} = \{A : \lambda_2 > 0\}$ is not closed. Nor is $\mathbb{H} - \mathcal{P} = \{A : \lambda_1 < 0\}$ closed.

At the moment we do not have an example of this sort with $\mathbb{H}$ a generalized equation.

Although it is only in the determined case that $\mathbb{H}$ uniquely determines the defining pair $\mathbb{E}, \mathbb{G}$ (namely, $\mathbb{E} = \mathbb{H} + \mathcal{P} = \mathbb{G}$), we always have the following.

Proposition 3.3 *(The canonical pair). Suppose $\mathbb{H} \equiv \mathbb{E}_0 \cap (-\widetilde{\mathbb{G}}_0)$ is any generalized equation. Then there exists a canonical choice for the subequation pair defining $\mathbb{H}$, namely $\mathbb{E}_{\min}$ and $\mathbb{G}_{\max}$:*

$$\mathbb{H}_{\mathrm{can}} \equiv \mathbb{E}_{\min} \cap (-\widetilde{\mathbb{G}}_{\max}) = (\overline{\mathbb{H} + \mathcal{P}}) \cap (\overline{\mathbb{H} - \mathcal{P}}). \tag{3.3}$$

This canonical pair is characterized by the following property:

If $\mathbb{E}, \mathbb{G}$ is any other subequation pair yielding the same generalized equation $\mathbb{H}$, i.e., if $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$, then

$$\mathbb{E}_{\min} \subset \mathbb{E} \qquad \text{and} \qquad \mathbb{G} \subset \mathbb{G}_{\max},$$

i.e., $\mathbb{E}_{\min}$ is minimal and $\mathbb{G}_{\max}$ is maximal.
In particular, if h is $\mathbb{H}_{\text{can}} = \mathbb{E}_{\min} \cap (-\widetilde{\mathbb{G}}_{\max})$ harmonic for the canonical min/max pair $\mathbb{E}_{\min}, \mathbb{G}_{\max}$, then h is also $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ harmonic for all other pairs $\mathbb{E}, \mathbb{G}$.

Proof. Since $\mathbb{H} \subset \mathbb{E}_{\min} = \overline{\mathbb{H} + \mathcal{P}}$ and $\mathbb{H} \subset -\widetilde{\mathbb{G}}_{\max} = \overline{\mathbb{H} - \mathcal{P}}$, we have that

$$\mathbb{H} \ \subset \ \mathbb{E}_{\min} \cap (-\widetilde{\mathbb{G}}_{\max}).$$

Now assume that $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ for a subequation pair $(\mathbb{E}, \mathbb{G})$. Then $\mathbb{H} \subset \mathbb{E}$ and so $\mathbb{E}_{\min} \equiv \overline{\mathbb{H} + \mathcal{P}} \subset \overline{\mathbb{E} + \mathcal{P}} = \overline{\mathbb{E}} = \mathbb{E}$. We also have $-\mathbb{H} \subset \widetilde{\mathbb{G}}$ which implies that $\widetilde{\mathbb{G}}_{\max} \equiv \overline{-\mathbb{H} + \mathcal{P}} \subset \overline{\widetilde{\mathbb{G}} + \mathcal{P}} = \overline{\widetilde{\mathbb{G}}} = \widetilde{\mathbb{G}}$. Thus we have $\mathbb{E}_{\min} \subset \mathbb{E}$ and $\mathbb{G} \subset \mathbb{G}_{\max}$. Therefore, $\mathbb{E}_{\min} \cap (-\widetilde{\mathbb{G}}_{\max}) \subset \mathbb{E} \cap (-\widetilde{\mathbb{G}}) = \mathbb{H}$. With the display above, this implies that $\mathbb{H} = \mathbb{E}_{\min} \cap (-\widetilde{\mathbb{G}}_{\max})$. $\square$

The next result concerns a generalized equation $\mathbb{H}$ belonging to part (A) of Theorem 2.6 – the uniqueness case. This is the case where $\operatorname{Int} \mathbb{H} = \emptyset$, or equivalently by Theorem 2.6, this is the case of a generalized equation which is either of Type I or Type II. Given a generalized equation $\mathbb{H}$, another characterization is

$$\operatorname{Int} \mathbb{H} = \emptyset \qquad \iff \qquad \mathbb{H} \ \subset \ \partial \mathbb{F} \quad \text{for some subequation } \mathbb{F}.$$

(This follows easily from Theorem 2.6.)

Proposition 3.4. *Suppose $\mathbb{H}$ is a generalized equation with $\operatorname{Int}\mathbb{H} = \emptyset$ (that is, $\mathbb{H}$ belongs to part (A) of Theorem 2.6 – the uniqueness case). Let $\mathbb{E}_{\min}, \mathbb{G}_{\max}$ denote the canonical min/max pair with $\mathbb{H} = \mathbb{E}_{\min} \cap (-\widetilde{\mathbb{G}}_{\max})$. Any subequation $\mathbb{F}$ with $\mathbb{H} \subset \partial \mathbb{F}$ must satisfy*

$$\mathbb{E}_{\min} \ \subset \ \mathbb{F} \ \subset \ \mathbb{G}_{\max}.$$

Proof. Note that $\mathbb{H} \subset \partial \mathbb{F} \ \Rightarrow \ \mathbb{E}_{\min} \equiv \overline{\mathbb{H} + \mathcal{P}} \subset \overline{\partial \mathbb{F} + \mathcal{P}} = \overline{\mathbb{F}} = \mathbb{F}$. Now $\mathbb{H} \subset \partial \mathbb{F} \iff -\mathbb{H} \subset \partial \widetilde{\mathbb{F}} = -\partial \mathbb{F}$. Hence, $\mathbb{H} \subset \partial \mathbb{F} \ \Rightarrow \ -\mathbb{H} \subset \partial \widetilde{\mathbb{F}} \ \Rightarrow \ -\mathbb{H} + \mathcal{P} \subset \partial \widetilde{\mathbb{F}} + \mathcal{P} = \widetilde{\mathbb{F}} \ \Rightarrow \ \widetilde{\mathbb{G}}_{\max} \equiv \overline{-\mathbb{H} + \mathcal{P}} \subset \widetilde{\mathbb{F}} \iff \mathbb{F} \subset \mathbb{G}_{\max}$. $\square$

One may wonder whether the closed sets $\mathbb{H} \subset \operatorname{Sym}^2(\mathbb{R}^n)$, which are generalized equations, can be intrinsically characterized. They can be.

Theorem 3.5 *(The characterization of generalized equations). A closed subset $\mathbb{H} \subset \operatorname{Sym}^2(\mathbb{R}^n)$ is a generalized equation if and only if*

$$\mathbb{H} \;=\; (\overline{\mathbb{H}+\mathcal{P}}) \cap (\overline{\mathbb{H}-\mathcal{P}}). \tag{3.4}$$

Proof. First, by (3.3) this equation holds for any generalized equation $\mathbb{H}$.

For the converse, recall from Lemma 3.1 that the sets $\mathbb{E} \equiv \overline{\mathbb{H}+\mathcal{P}}$ and $\widetilde{\mathbb{G}} \equiv \overline{(-\mathbb{H}+\mathcal{P})}$ are always subequations, so that $\mathbb{E} \cap (-\widetilde{\mathbb{G}})$ is a generalized equation. $\square$

As we have seen, if $\mathbb{H}$ is a generalized equation then the set of pairs $(\mathbb{E}, \mathbb{G})$ with $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ is partially ordered with a unique minimal element $(\mathbb{E}_{\text{min}}, \mathbb{G}_{\text{max}})$. So associated to $\mathbb{H}$ we have this canonical subequation pair $(\mathbb{E}_{\text{min}}, \mathbb{G}_{\text{max}})$, and it is natural to consider the associated **canonical** $\mathbb{H}_{\text{can}}$**-harmonics.**

Theorem 3.6. *Let $\mathbb{H} \subset \mathbb{H}'$ be two generalized equations as in Theorem 3.5. Then any function h which is $\mathbb{H}_{\text{can}}$-harmonic on an open set $\Omega \subset \mathbb{R}^n$ is also $\mathbb{H}'_{\text{can}}$-harmonic on Ω.*

Proof. We shall use the second form of (3.3). Note that $\overline{\mathbb{H}+\mathcal{P}} \subset \overline{\mathbb{H}'+\mathcal{P}}$ and $\overline{\mathbb{H}-\mathcal{P}} \subset \overline{\mathbb{H}'-\mathcal{P}}$. Therefore, $\mathbb{E}_{\text{min}} \subset \mathbb{E}'_{\text{min}}$, and $-\widetilde{\mathbb{G}}_{\text{max}} \subset -\widetilde{\mathbb{G}}'_{\text{max}}$ which means that $\widetilde{\mathbb{G}}_{\text{max}} \subset \widetilde{\mathbb{G}}'_{\text{max}}$. Thus if h is $\mathbb{E}_{\text{min}}$-subharmonic and $-h$ is $\widetilde{\mathbb{G}}_{\text{max}}$-subharmonic, this also holds for the primed subequations. $\square$

This theorem says that the partial ordering by inclusion on the family of closed subsets $\mathbb{H} \subset \text{Sym}^2(\mathbb{R}^n)$ which are subequations, carries over to their canonical harmonics on any open $\Omega \subset \mathbb{R}^n$. The reader will also recall that a canonical $\mathbb{H}_{\text{can}}$-harmonic on Ω is also an $\mathbb{H}_{\mathbb{E},\mathbb{G}}$-harmonic for any other pair $(\mathbb{E}, \mathbb{G})$ with $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$. One could certainly wonder whether every $\mathbb{H}_{\mathbb{E},\mathbb{G}}$-harmonic is canonically $\mathbb{H}_{\text{can}}$-harmonic. (This is discussed as the “Broadened Equation Question” in Section 5.) If this were true, the theory would be very tidy.

Theorem 3.7 *(The GE associated to a closed set). Let $\mathbb{H} \subset \text{Sym}^2(\mathbb{R}^n)$ be any closed subset. Then the pair of subequations*

$$(\mathbb{E}^\diamondsuit, \mathbb{G}^\diamondsuit) \;\stackrel{\text{def}}{=}\; (\overline{\mathbb{H}+\mathcal{P}}, (\overline{-\mathbb{H}+\mathcal{P}})^\sim) \tag{3.5}$$

gives a generalized equation

$$\mathbb{H}^\diamondsuit \;\stackrel{\text{def}}{=}\; \mathbb{E}^\diamondsuit \cap (-\widetilde{\mathbb{G}}^\diamondsuit) = (\overline{\mathbb{H}+\mathcal{P}}) \cap (\overline{\mathbb{H}-\mathcal{P}}) \tag{3.6}$$

containing $\mathbb{H}$, and it is the smallest such. That is, if $(\mathbb{E}, \mathbb{G})$ is a subequation pair with $\mathbb{H} \subset \mathbb{E} \cap (-\widetilde{\mathbb{G}})$, then $\mathbb{H}^\diamondsuit \subset \mathbb{E} \cap (-\widetilde{\mathbb{G}})$. Moreover, $(\mathbb{E}^\diamondsuit, \mathbb{G}^\diamondsuit)$ is the canonical min/max pair defining $\mathbb{H}^\diamondsuit$.

More succinctly, every closed subset $\mathbb{H} \subset \text{Sym}^2(\mathbb{R}^n)$ gives rise to a **minimal generalized equation** *containing $\mathbb{H}$, namely*

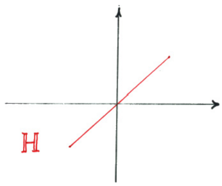
$$\mathbb{H}^\diamondsuit \;\equiv\; (\overline{\mathbb{H}+\mathcal{P}}) \cap (\overline{\mathbb{H}-\mathcal{P}}).$$

Proof. From Lemma 3.1 and Corollary 3.2 we know that $(\overline{\mathbb{H}+\mathcal{P}}, (\overline{-\mathbb{H}+\mathcal{P}})^{\sim})$ is a pair of subequations defining the generalized equation $\mathbb{H}^\diamondsuit$ given in (3.6), which clearly contains $\mathbb{H}$.

Suppose $\mathbb{E}, \mathbb{G}$ are subequations with $\mathbb{H} \subset \mathbb{E} \cap (-\widetilde{\mathbb{G}})$. Then $\mathbb{H} \subset \mathbb{E}$ and so $\overline{\mathbb{H}+\mathcal{P}} \subset \overline{\mathbb{E}+\mathcal{P}} = \overline{\mathbb{E}} = \mathbb{E}$. Also $\mathbb{H} \subset -\widetilde{\mathbb{G}}$, so $-\mathbb{H} \subset \widetilde{\mathbb{G}}$ and therefore $\overline{-\mathbb{H}+\mathcal{P}} \subset \overline{\widetilde{\mathbb{G}}+\mathcal{P}} \subset \widetilde{\mathbb{G}}+\mathcal{P} \subset \widetilde{\mathbb{G}}$. Hence, $\mathbb{H}^\diamondsuit = (\overline{\mathbb{H}+\mathcal{P}}) \cap (\overline{\mathbb{H}-\mathcal{P}}) \subset \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ as claimed. Finally, $\mathbb{E}_{\min} \overset{\text{def}}{=} (\overline{\mathbb{H}^\diamondsuit+\mathcal{P}}) \supset (\overline{\mathbb{H}+\mathcal{P}}) \overset{\text{def}}{=} \mathbb{E}^\diamondsuit$ (since $\mathbb{H}^\diamondsuit \supset \mathbb{H}$), and $\mathbb{H}^\diamondsuit \subset \overline{\mathbb{H}+\mathcal{P}} \overset{\text{def}}{=} \mathbb{E}^\diamondsuit$ implies $\overline{\mathbb{H}^\diamondsuit+\mathcal{P}} \subset \mathbb{E}^\diamondsuit$, which proves $\mathbb{E}_{\min} = \mathbb{E}^\diamondsuit$. The proof that $-\widetilde{\mathbb{G}}^\diamondsuit = \overline{\mathbb{H}-\mathcal{P}}$, so that $\mathbb{G}_{\max} = \mathbb{G}^\diamondsuit$, is similar. □

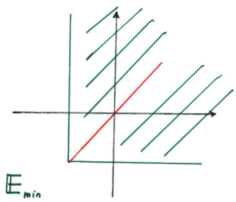
Example 3.8. Let

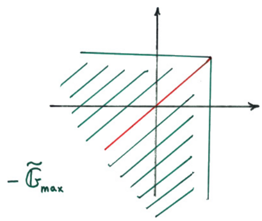
$$\mathbb{H} \equiv \{t \cdot \mathrm{Id} : -1 \leq t \leq 1\}.$$



Then

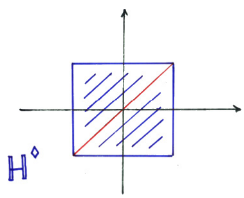
$$\mathbb{E}_{\min} \;=\; \mathbb{H}+\mathcal{P} \;=\; \{A : A \geq -\mathrm{Id}\} \qquad \text{and} \qquad -\widetilde{\mathbb{G}}_{\max} \;=\; \mathbb{H}-\mathcal{P} \;=\; \{A : A \leq \mathrm{Id}\}.$$





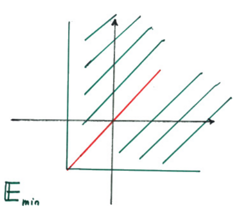
Therefore, the minimal generalized equation containing $\mathbb{H}$ is

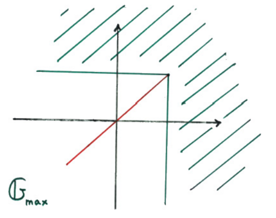
$$\mathbb{H}^\diamondsuit \;=\; (\mathbb{H}+\mathcal{P}) \cap (\mathbb{H}-\mathcal{P}) \;=\; \{A : -\mathrm{Id} \leq A \leq \mathrm{Id}\}.$$



Note that the canonical pair $\mathbb{E}_{\min}, \mathbb{G}_{\max}$ for this generalized equation $\mathbb{H}^\diamondsuit$ is:

$$\mathbb{E}_{\min} \;=\; \{A : A \geq -\mathrm{Id}\} \qquad \text{and} \qquad \mathbb{G}_{\max} \;=\; \{A : A - \mathrm{Id} \in \widetilde{\mathcal{P}}\}.$$





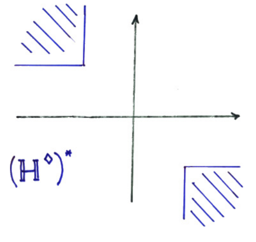
Some Facts: Here $\mathbb{G}_{\max}$ and $\mathbb{E}_{\min}$ are dual to one another, and so

$$\mathbb{H}^{\diamondsuit} \;\equiv\; \mathbb{E}_{\min} \cap (-\widetilde{\mathbb{G}}_{\max}) \;=\; \mathbb{E}_{\min} \cap (-\mathbb{E}_{\min})$$

is a Class II generalized equation (see Section 4).

The **mirror** of $\mathbb{H}^{\diamondsuit}$ is

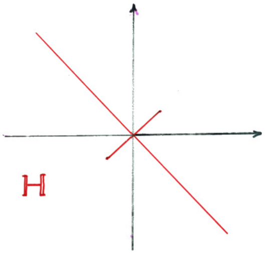
$$(\mathbb{H}^{\diamondsuit})^* \;=\; \mathbb{G}_{\max} \cap (-\mathbb{G}_{\max}) \;=\; \{A : A - \mathrm{Id} \in \widetilde{\mathcal{P}} \text{ and } -A - \mathrm{Id} \in \widetilde{\mathcal{P}}\}.$$



Both $\mathrm{Int}\mathbb{H}^{\diamondsuit} \neq \emptyset$ and $\mathrm{Int}(\mathbb{H}^{\diamondsuit})^* \neq \emptyset$, so that $\mathbb{H}^{\diamondsuit}$ is Type IV.

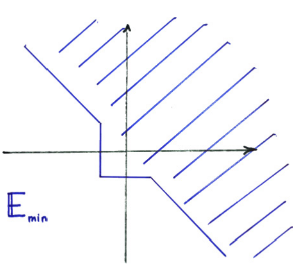
Example 3.9. Let

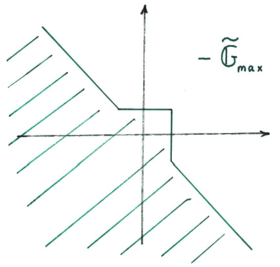
$$\mathbb{H} \equiv \{\mathrm{tr}(A) = 0\} \cup \{t \cdot \mathrm{Id} : -1 \leq t \leq 1\}.$$



Then we have

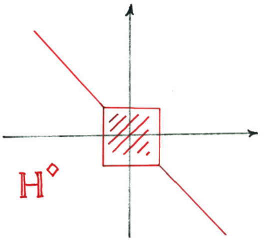
$$\mathbb{E}_{\min} \;=\; \mathbb{H} + \mathcal{P} \qquad\qquad \text{and} \qquad\qquad -\widetilde{\mathbb{G}}_{\max} \;=\; \mathbb{H} - \mathcal{P}$$





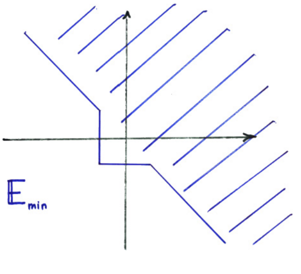
Therefore, the minimal generalized equation containing $\mathbb{H}$ is

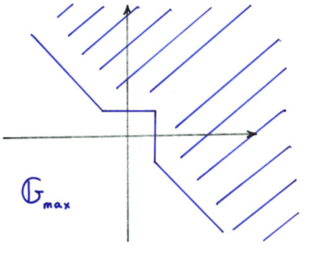
$$\mathbb{H}^\diamondsuit \;=\; (\mathbb{H}+\mathcal{P})\cap(\mathbb{H}-\mathcal{P}) \;=\; \{A : -\mathrm{Id} \le A \le \mathrm{Id}\} \cup \{A : \mathrm{tr}A = 0\}$$



Note that the two subequations are:

$$\mathbb{E}_{\min} \;=\; \{A \ge -\mathrm{Id}\} \cup \{\mathrm{tr}A \ge 0\} \qquad \text{and} \qquad \mathbb{G}_{\max} \;=\; \{A - \mathrm{Id} \in \widetilde{\mathcal{P}}\} \cap \{\mathrm{tr}A \ge 0\}.$$





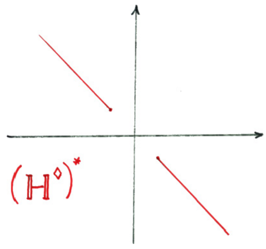
Some Facts: Here $\mathbb{G}_{\max}$ and $\mathbb{E}_{\min}$ are dual to one another, and so

$$\mathbb{H}^\diamondsuit \;\equiv\; \mathbb{E}_{\min} \cap (-\widetilde{\mathbb{G}}_{\max}) \;=\; \mathbb{E}_{\min} \cap (-\mathbb{E}_{\min})$$

is a Class II generalized equation (see Section 4).

The **mirror** of $\mathbb{H}^\diamondsuit$ is

$$(\mathbb{H}^\diamondsuit)^* \;=\; \mathbb{G}_{\max} \cap (-\mathbb{G}_{\max}) \;=\; \{\mathrm{tr}A = 0\} \sim \{-\mathrm{Id} < A < \mathrm{Id}\}.$$



Here we have $\mathrm{Int}\mathbb{H}^{\diamondsuit} \neq \emptyset$ and $\mathrm{Int}(\mathbb{H}^{\diamondsuit})^* = \emptyset$, so that $\mathbb{H}^{\diamondsuit}$ is Type III.

4. Examples of generalized equations $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$

We start with some classes of examples. First and foremost is the following.

Class I. Type I Determined Equations (the $\mathbb{G} = \mathbb{E}$ Case). Here $\mathbb{H} = \partial\mathbb{F} = \mathbb{F} \cap (-\widetilde{\mathbb{F}})$ is the boundary of a subequation $\mathbb{F}$. We refer to [8], [9] and [10] for an abundance of important specific examples.

Another important class of examples is

Class II. (The $\mathbb{G} = \widetilde{\mathbb{E}}$ Case.) Here $\mathbb{H} = \mathbb{E} \cap (-\mathbb{E})$ and $\mathbb{H}^* = \widetilde{\mathbb{E}} \cap (-\widetilde{\mathbb{E}}) = \sim [(\mathrm{Int}\,\mathbb{E}) \cup (-\mathrm{Int}\,\mathbb{E})]$. Note that the overlap between Classes I and II consists of the boundaries $\mathbb{H} = \partial\mathbb{F}$ of self-dual subequations (where the dual $\widetilde{\mathbb{F}}$ equals $\mathbb{F}$).

Class IIa. (Edges). In this Class II, the most basic examples are when $\mathbb{E}$ is a convex cone subequation. Then $\mathbb{H} = \mathbb{E} \cap (-\mathbb{E})$ is a vector subspace called the **edge** of the cone $\mathbb{E}$. Since $\mathbb{E}$ is a proper subspace, $\mathbb{H}$ is also a proper subspace, and hence $\mathrm{Int}\mathbb{H} = \emptyset$. Now note that $\sim \mathbb{E} \subset -\mathrm{Int}\mathbb{E}$ is false for a proper convex cone $\mathbb{E} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ because $-\mathrm{Int}\mathbb{E}$ is an open convex cone. Equivalently, $\mathbb{G} \equiv \sim (-\mathrm{Int}\mathbb{E}) \subset \mathbb{E}$ is false. By the equivalence of $(1)^*$ and $(3)^*$ in Theorem 2.6, this proves that $\mathrm{Int}\mathbb{H}^* \neq \emptyset$. Thus $\mathbb{H}$ is Type II. Such edge harmonics include: (i) Affine functions, where $\mathbb{H} = \{0\}$ and $\mathbb{E} = \mathcal{P}$, (ii) pluriharmonic functions in complex analysis, where $\mathbb{E} = \mathcal{P}_{\mathbb{C}}$ and the edge $\mathbb{H}$ is $\mathrm{SkewHerm}(\mathbb{C}^n)$, (iii) Δ-harmonic functions, and many others. The "edge" generalized equations are the subject of [15].

Three specific non-edge Class II examples are as follows.

Some $\mathbb{H}$ non-uniqueness examples

Example 4.1 *(The quasi-convex/quasi-concave equation).* Choose $r_1, r_2 \in \mathbb{R}$ with $-r_1 \leq r_2$, and let

$$\mathbb{H} \;\equiv\; (\mathcal{P} - r_1 I) \cap (-\mathcal{P} + r_2 I).$$

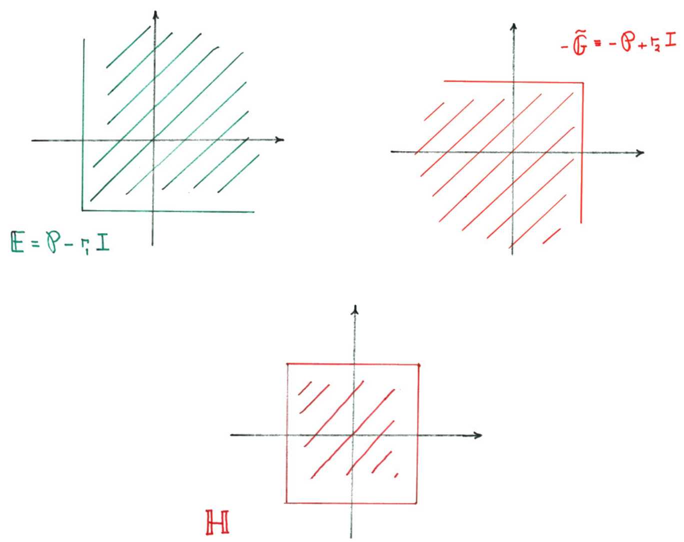
Here $\mathbb{E} \equiv \mathcal{P} - r_1 I$ is the subequation for r_1-quasiconvex functions, and $\widetilde{\mathbb{G}} \equiv \mathcal{P} - r_2 I$ is the subequation for r_2-quasiconvex functions. Thus $\mathbb{H} \equiv \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ is the generalized equation for functions that are both r_1-quasiconvex and r_2-quasiconcave. Note that $A \in \mathbb{H} \iff -r_1 I \leq A \leq r_2 I$. **A function u is $\mathbb{H}$-harmonic $\iff$ $u + r_1 \frac{|x|^2}{2}$ is convex and $u - r_2 \frac{|x|^2}{2}$ is concave.**

Observe now that if u satisfies this generalized (r_1, r_2) equation, then the function $u(x) + \frac{\rho}{2}|x|^2$ satisfies the generalized $(r_1 - \rho, r_2 - \rho)$ equation. Thus, by simply adding multiples of $|x|^2$ we translate the equation up and down the line $\{t\mathrm{Id} : t \in \mathbb{R}\}$. So we can assume that $r_1, r_2 \geq 0$, in fact let's assume $r_1 = r_2 = \lambda \geq 0$. In this case $\widetilde{\mathbb{G}} = \mathbb{E}$ and $\mathbb{H} = \mathbb{E} \cap (-\mathbb{E})$ is class II above, and Type IV. Furthermore, there is the following result, which has been known for some time. (See page 265, line 15-16 in [19], and also [16], [5], for example.) For the benefit of the reader we include a proof in Appendix A.

We say that a function is $\lambda - C^{1,1}$ if it is C^1 and the first derivative is Lipschitz with Lipschitz coefficient λ.

Theorem A.1 *For a function u on a convex domain $\Omega \subset \mathbb{R}^n$,*

$$u \quad \lambda - C^{1,1} \qquad \iff \qquad \text{both} \quad \pm u \quad \text{are } \lambda\text{-quasi-convex}$$



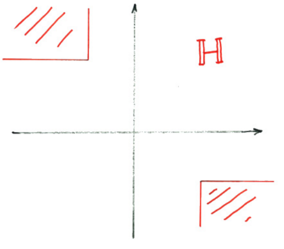
Here is a related example which produces the mirror equation to the one above.

Example 4.2 *(The quasi-subaffine/quasi-superaffine equation).* Choose $r_1, r_2 \in \mathbb{R}$ and set

$$\mathbb{H} \equiv (\widetilde{\mathcal{P}} - r_1 I) \cap (-\widetilde{\mathcal{P}} + r_2 I).$$

Here $\mathbb{E} \equiv \widetilde{\mathcal{P}} - r_1 I$ is the subequation for r_1-quasi-subaffine functions, i.e., $u(x) + \frac{r_1}{2}|x|^2$ is subaffine, and $\widetilde{\mathbb{G}} = \widetilde{\mathcal{P}} - r_2 I$ is again the subequation for r_2-quasi-subaffine functions. Again if $r_1 = r_2$, then $\widetilde{\mathbb{G}} = \mathbb{E}$ and $\mathbb{H} = \mathbb{E} \cap (-\mathbb{E})$ is a special case of Class II above.

If $r_1, r_2 \leq 0$, then consider $r_1' = -r_1, r_2' = -r_2$ and $\mathbb{E}' \equiv \mathcal{P} - r_1'\mathrm{Id}$ in Example 4.1. Then here in Example 4.2, the subequation $\mathbb{E} \equiv \widetilde{\mathcal{P}} - r_1\mathrm{Id}$ is equal to $\widetilde{\mathbb{E}}' = \widetilde{\mathcal{P}} + r_1'\mathrm{Id}$. Thus the mirror of $\mathbb{H}'$ in Example 4.1 is $\mathbb{H}$ in Example 4.2.



The intersection of Examples 4.1 and 4.2. From Example 4.1 we have

$$\mathbb{E} = \{A \geq -\mathrm{Id}\} = \{\lambda_{\min}(A) \geq -1\} \quad \text{and} \quad -\widetilde{\mathbb{G}} = \{A \leq \mathrm{Id}\} = \{\lambda_{\max}(A) \leq 1\},$$
$$\mathbb{H} \;=\; \{-\mathrm{Id} \leq A \leq \mathrm{Id}\} \;=\; \{-1 \leq \lambda_{\min}(A) \text{ and } \lambda_{\max}(A) \leq 1\}.$$

Recall that the GE in Example 4.2 is just the mirror $\mathbb{H}^* = \mathbb{G} \cap (-\widetilde{\mathbb{E}})$ where

$$\mathbb{G} \;= \sim \{A < \mathrm{Id}\} = \{\lambda_{\max}(A) \geq 1\} \quad \text{and} \quad -\widetilde{\mathbb{E}} \;= \sim \{A > -\mathrm{Id}\} = \{\lambda_{\min}(A) \leq -1\}.$$

Hence we have

$$\mathbb{H} \cap \mathbb{H}^* \;=\; (\partial\mathbb{E}) \cap \partial(-\widetilde{\mathbb{G}}) \;=\; \{-1 = \lambda_{\min}(A) \text{ and } \lambda_{\max}(A) = 1\}$$

The C^2-harmonics for this GE are h's with $\lambda_{\min}(D^2h) = -1$ and $\lambda_{\max}(D^2h) = 1$. This can be expressed invariantly as

$$D^2h + \mathrm{Id} \geq 0, \quad \mathrm{Id} - D^2h \geq 0 \quad \text{and} \quad \det(D^2h + \mathrm{Id}) + \det(\mathrm{Id} - D^2h) \;=\; 0,$$

or in keeping with the Twisted Monge-Ampère Example 4.8

$$D^2h + \mathrm{Id} \geq 0, \quad \mathrm{Id} - D^2h \geq 0 \quad \text{and} \quad \det(D^2h + \mathrm{Id}) = -\det(\mathrm{Id} - D^2h).$$

Note: There are lots of such harmonics. For example,

$$h(x_1, x_2, y) \;=\; \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + u(y) \quad \text{where} \quad -\mathrm{Id} \;\leq\; D^2u \;\leq\; \mathrm{Id}.$$

Some Type II examples of non-existence and uniqueness for $\mathbb{H}$

Example 4.3 *(Generalized constrained Laplacians).*

Definition 4.4. A closed subset $\mathbb{H}$ of $\partial\Delta \equiv \{A : \mathrm{tr}A = 0\}$ (with $\emptyset \neq \mathbb{H} \neq \partial\Delta$) will be called a **(generalized) constrained Laplacian**.

Lemma 4.5. *If $\mathbb{H}$ is a (generalized) constrained Laplacian, then $\mathbb{E} \equiv \mathbb{H} + \mathcal{P}$ is closed and hence a subequation. Furthermore, $\mathbb{H} - \mathcal{P}$ is closed, and*

$$\mathbb{H} \;=\; (\mathbb{H} + \mathcal{P}) \cap (\mathbb{H} - \mathcal{P}),$$

so that $\mathbb{H}$ is a generalized equation with $\mathbb{E}_{\min} = \mathbb{H} + \mathcal{P}$ and $-\widetilde{\mathbb{G}}_{\max} = \mathbb{H} - \mathcal{P}$.

Proof. Suppose $B \in \overline{\mathbb{H} + \mathcal{P}}$, i.e., $B = \lim_j (A_j + P_j)$ with $A_j \in \mathbb{H}$ and $P_j \geq 0$. Set $B_j = A_j + P_j$. Since $\mathbb{H} \subset \{\mathrm{tr}A = 0\}$, we have $\mathrm{tr}B_j = \mathrm{tr}(A_j + P_j) = \mathrm{tr}P_j$. Thus $\mathrm{tr}P_j \to \mathrm{tr}B$. The set $\{P \in \mathcal{P} : \mathrm{tr}P \leq c\}$ with $c > 0$ is compact. Therefore we can extract a convergent subsequence of $\{P_j\}$, which we again call $\{P_j\}$, with $P_j \to P \geq 0$. Therefore, $A_j = B_j - P_j \to B - P \stackrel{\mathrm{def}}{=} A$. Since $\mathbb{H}$ is closed and each $A_j \in \mathbb{H}$, we have $A \in \mathbb{H}$. Thus $B = A + P$ with $A \in \mathbb{H}$ and $P \geq 0$, and so we have proved that $\mathbb{H} + \mathcal{P}$ is closed. Replacing $\mathbb{H}$ by $-\mathbb{H}$ we have that $-\mathbb{H} + \mathcal{P} = -(\mathbb{H} - \mathcal{P})$ is closed. Thus $\mathbb{H} - \mathcal{P}$ is closed.

Obviously $\mathbb{H} \subset (\mathbb{H} + \mathcal{P}) \cap (\mathbb{H} - \mathcal{P})$. Suppose now that $B \in (\mathbb{H} + \mathcal{P}) \cap (\mathbb{H} - \mathcal{P})$, i.e., $B = A + P = A' - P'$ for $A, A' \in \mathbb{H}$ and $P, P' \in \mathcal{P}$. Then $\mathrm{tr}B = \mathrm{tr}P = -\mathrm{tr}P'$. Since $P, P' \geq 0$, this implies that $P = P' = 0$, and hence $B = A = A' \in \mathbb{H}$. $\square$

Note that $\mathrm{Int}\mathbb{H} = \emptyset$, so $\mathbb{H}$ must be type I or type II. However, it cannot be type I without $\mathbb{H} = \partial\Delta$ (cf. Propositions 2.18 and 2.19). This proves

A generalized constrained Laplacian is of type II.

Question 4.6. Suppose $\mathbb{E}$ and $\mathbb{G}$ define $\mathbb{H}$ $(= \mathbb{E} \cap (-\widetilde{\mathbb{G}}))$ and $\mathbb{H} \subset \partial\Delta$ is a generalized constrained Laplacian. If h is $H_{\mathbb{E},\mathbb{G}}$-harmonic, then is h Δ-harmonic?

Example 4.7. Let $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^\ell$ and

$$\mathbb{H} \equiv \left\{ A \equiv \begin{pmatrix} a & c \\ c^t & b \end{pmatrix} : \mathrm{tr}A = 0,\ a \geq 0 \text{ and } b \leq 0 \right\}$$

with $a \in \mathrm{Sym}^2(\mathbb{R}^k),\ b \in \mathrm{Sym}^2(\mathbb{R}^\ell)$. Then

$$\mathbb{E}_{\min} \;=\; \{A :\ a \geq 0 \text{ and } \mathrm{tr}A \geq 0\}, \qquad \mathbb{G}_{\max} \;=\; \{A : b \geq 0\} \cup \mathbb{E}_{\min},$$
$$\text{and} \quad \widetilde{\mathbb{G}}_{\max} = \{A : b \geq 0 \ \text{ and } \ \mathrm{tr}A \geq 0\}.$$

One could also look at this from the universal eigenvalue point of view (see the subsection of Section 5 in [11] and Remark 4.12 below), by taking the eigenvalues of a and b. Let $Q^+(\mathbb{R}^k)$ denote the positive orphant defined by $x_j \geq 0$ for all j, and let $Q^-(\mathbb{R}^k)$ be similar with coordinates $y_i \leq 0$. Then $\mathbb{H}, \mathbb{E}_{\min}, \mathbb{G}_{\max}$ can be defined by

$$H = \{(x,y) : x \in Q^+(\mathbb{R}^k), y \in Q^-(\mathbb{R}^\ell), \mathrm{tr}(x,y) = 0\}$$

$$E_{\min} = \{(x,y) : x \in Q^+(\mathbb{R}^k), \mathrm{tr}(x,y) \geq 0\},$$

$$G_{\max} = \{(x,y) : y \in Q^+(\mathbb{R}^\ell)\} \cup E_{\min}, \qquad \widetilde{G}_{\max} = \{(x,y) : y \geq 0 \ \text{ and } \ \mathrm{tr}(x,y) \geq 0)\}.$$

A great example of a non-existence/uniqueness $\mathbb{H}$ equation (Type II) has been introduced and studied in [22] and [23]. This is discussed next.

Example 4.8 *(The universal version of the twisted Monge-Ampère equation).* The real twisted Monge-Ampère equation is defined by $\mathbb{H} \subset \mathrm{Sym}^2(\mathbb{R}^k \times \mathbb{R}^\ell)$ consisting of all

$$\begin{pmatrix} A & C \\ C^t & B \end{pmatrix} \quad \textit{such that} \ \ A \geq 0, \quad B \leq 0 \ \text{ and } \ \mathit{log}\,\det A - \mathit{log}\,\det(-B) = 0$$

i.e., $\det A = \det(-B)$, or $\det A = |\det B|$.

As in Example 4.7, the universal version of this equation is defined on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^\ell$ by

$$H \equiv \{(x,y) : x \in Q^+(\mathbb{R}^k), y \in Q^-(\mathbb{R}^\ell) \text{ and } x_1 \cdots x_k = |y_1 \cdots y_\ell|\}.$$

Lemma 4.9. *Let $E \equiv H + Q^+(\mathbb{R}^n)$. Then E is fibred over $Q^+(\mathbb{R}^k)$, where the fibre E_x of E at $x \in Q^+(\mathbb{R}^k)$ is the dual MA universal subequation:*

$$\widetilde{P}_{x_1 \cdots x_k}(\mathbb{R}^\ell) = (\sim Q^-(\mathbb{R}^\ell)) \cup \{y \in Q^-(\mathbb{R}^\ell) : |y_1 \cdots y_\ell| \leq x_1 \cdots x_k\}.$$

Since it is easy to see that E is closed, E is equal to the minimal subequation defined above for this H.

Proof of Lemma 4.9. First note that H is fibred over $Q^+(\mathbb{R}^k)$ with fibre H_x at $x \in Q^+(\mathbb{R}^k)$ given by

$$H_x = \{y \in Q^-(\mathbb{R}^\ell) : |y_1 \cdots y_\ell| = x_1 \cdots x_k\}.$$

Second note that this equals

$$H_x = \partial \widetilde{P}_{x_1 \cdots x_k}(\mathbb{R}^\ell)$$

the boundary of the dual MA-subequation at level $c = x_1 \cdots x_k$. Third note that, since $\widetilde{P}_{x_1 \cdots x_k}(\mathbb{R}^\ell)$ is a subequation,

$$H_x + Q^+(\mathbb{R}^\ell) = \partial \widetilde{P}_{x_1\cdots x_k}(\mathbb{R}^\ell) + Q^+(\mathbb{R}^\ell) = \widetilde{P}_{x_1\cdots x_k}(\mathbb{R}^\ell)$$

Defining E' by its fibres $E'_{x_1\cdots x_k} \equiv \widetilde{P}_{x_1\cdots x_k}(\mathbb{R}^\ell)$ over $x \in Q^+(\mathbb{R}^k)$, we have $E' \subset E$, and it remains to show $E \subset E'$. But $H \subset E'$, so it is enough to show E' is $Q^+(\mathbb{R}^n)$-monotone. As noted above E' is $Q^+(\mathbb{R}^\ell)$-monotone since $\widetilde{P}_{x_1\cdots x_k}(\mathbb{R}^\ell)$ is a subequation. Now increasing one of the x coordinates with $x \in Q^+(\mathbb{R}^k)$ increases $\widetilde{P}_{x_1\cdots x_k}(\mathbb{R}^\ell)$ proving that E' is $Q^+(\mathbb{R}^k)$-monotone. Finally the orphant $Q^+(\mathbb{R}^n)$ equals the product $Q^+(\mathbb{R}^k) \times Q^+(\mathbb{R}^\ell)$. $\square$

Lemma 4.10. *Let $\widetilde{G} \equiv -H + Q^+(\mathbb{R}^n)$. Then $\widetilde{G}$ is fibred over $Q^+(\mathbb{R}^\ell)$, where the fibre $\widetilde{G}_y$ of $\widetilde{G}$ at $y \in Q^+(\mathbb{R}^\ell)$ is the dual MA universal subequation:*

$$\widetilde{G}_y \;=\; \widetilde{P}_{|y_1\cdots y_\ell|}(\mathbb{R}^k).$$

The proof of Lemma 4.10 is similar to the one for Lemma 4.9, and is skipped.

Proposition 4.11. *$H = E \cap (-\widetilde{G})$ is a universal version of a generalized equation with minimum subequation E and maximum subequation G.*

Proof. Note that $(x, y) \in E \iff x \in Q^+(\mathbb{R}^k)$ and $y \in \widetilde{P}_{x_1\cdots x_k}(\mathbb{R}^\ell)$ by Lemma 4.9. Note also that

$$(x, y) \in -\widetilde{G} \quad\iff\quad x \in -\widetilde{P}_{|y_1\cdots y_\ell|}(\mathbb{R}^k) \;\text{ and }\; y \in Q^-(\mathbb{R}^\ell),$$

by Lemma 4.10.

Now assume $(x, y) \in E \cap (-\widetilde{G})$. Then $x \in Q^+(\mathbb{R}^k) \cap (-\widetilde{P}_{|y_1\cdots y_\ell|}(\mathbb{R}^k))$ or otherwise said, $x \in Q^+(\mathbb{R}^k)$ and $|y_1 \cdots y_\ell| \le x_1 \cdots x_k$. Also, $y \in Q^-(\mathbb{R}^\ell) \cap \widetilde{P}_{x_1\cdots x_k}(\mathbb{R}^\ell)$ or otherwise said, $x \in Q^-(\mathbb{R}^\ell)$ and $x_1 \cdots x_k \le |y_1 \cdots y_\ell|$.

In summary, if $(x, y) \in E \cap (-\widetilde{G})$, then

$$x \in Q^+(\mathbb{R}^k), \quad y \in Q^-(\mathbb{R}^\ell), \quad \text{and} \quad x_1 \cdots x_k = |y_1 \cdots y_\ell|$$

that is, $(x, y) \in \mathbb{H}$. It is easy to see that $H \subset E \cap (-\widetilde{G})$. $\square$

Remark 4.12 *(Universal equations and Gårding/Dirichlet operators).* A closed subset $\Lambda \subset \mathbb{R}^n$ which is symmetric under permutations of the coordinates and satisfies $\Lambda + \mathbb{R}^n_+ \subset \Lambda$ is called a **universal eigenvalue subequation**. There is an obvious one-to-one correspondence between subequations $\mathbb{F} \subset \mathrm{Sym}^2(\mathbb{R}^n)$, which depend only on the eigenvalues of $A \in \mathbb{F}$, and universal subequations $\Lambda \subset \mathbb{R}^n$. However, this $\mathbb{F}$ is only one of many subequations determined by Λ which are constructed by substituting Gårding eigenvalues for regular eigenvalues as follows.

Let g be a homogeneous polynomial of degree $n > 0$, on some $\mathrm{Sym}^2(\mathbb{R}^m)$, which satisfies the conditions of being a **Gårding/Dirichlet**, or **GD**, operator (as defined in [11,

§5]). Then for each $A \in \mathrm{Sym}^2(\mathbb{R}^m)$ this operator gives n eigenvalues $\lambda_g(A)$, and so Λ determines a subequation in $\mathbb{R}^m$ by:

$$\mathbb{F}_\Lambda^g \;=\; \{A \in \mathrm{Sym}^2(\mathbb{R}^m) : \lambda_g(A) \in \Lambda\}.$$

For example, the universal subequation $\Lambda \equiv \{\lambda \in \mathbb{R}^n : \lambda_j \geq 0\, \forall\, j\}$ determines the Gårding Monge Ampère subequation $\mathbb{F}_\Lambda^g = \{A \in \mathrm{Sym}^2(\mathbb{R}^m) : \lambda_{g,1}(A), .., \lambda_{g,n}(A) \text{ are all } \geq 0\}$, which is just the closed Gårding cone for g.

This carries over to generalized equations. A **generalized universal equation** is any closed $\Lambda' \subset \mathbb{R}^n$ which is an intersection involving two universal subequations $\Lambda' = \Lambda_1 \cap (-\widetilde{\Lambda}_2)$. For example, the universal Laplacian $\Lambda = \{\lambda \in \mathbb{R}^n : \lambda_1 + \cdots + \lambda_n \geq 0\}$ determines a Gårding Laplacian $\mathbb{F}_\Lambda^g$ for each GD operator of degree n. Moreover, given a pair of GD operators g_1, g_2 of degrees $n_1 + n_2 = n$, one has a constrained Laplacian generalized equation induced by the universal version of the constrained Laplacian given in Example 2.3. Namely, we have

$$\mathbb{H} \;\equiv\; \{A \in \mathrm{Sym}^2(\mathbb{R}^m) : \lambda_{g_1,j}(a) \geq 0, \lambda_{g_2,k}(b) \leq 0, \text{ and } \sum_j \lambda_{g_1,j}(a) + \sum_k \lambda_{g_2,k}(b) = 0\}.$$

Example 4.13 *(Twisted Gårding MA generalized equations).* Similarly (we leave this to the reader) the universal twisted MA-equation (Example 4.8) spawns a huge family of generalized equations. For instance, in addition to the real version in [22], [23], one has a complex version, a quaternionic version, a Lagrangian version, branched versions of these three, elementary symmetric versions of these three (the so-called "hessian equation" versions), just to name a few.

The Examples 4.1 and 4.2 can also be viewed as "universal subequations", spawning many more examples of generalized equations as above.

Since we have no reason to rule out $\mathbb{F} = \emptyset$ or $\mathbb{F} = \mathrm{Sym}^2(\mathbb{R}^n)$ as a subequation in this paper, we have that $\mathbb{H}$ equal to plus or minus a subequation is an example of a generalized equation of Type III or II respectively.

Example 4.14 *(Subequations as generalized equations).* For any subequation $\mathbb{E} \neq \emptyset$, if we choose $\mathbb{G} = \emptyset$, i.e., $-\widetilde{\mathbb{G}} = \mathrm{Sym}^2(\mathbb{R}^n)$, then $\mathbb{H} = \mathbb{E} \cap \mathrm{Sym}^2(\mathbb{R}^n) = \mathbb{E}$ is a generalized equation. Now since $\overline{\mathrm{Int}\,\mathbb{E}} = \mathbb{E}$ and $\mathbb{E} \neq \emptyset$, we have $\mathrm{Int}\,\mathbb{H} \neq \emptyset$. Also the mirror $\mathbb{H}^* = \mathbb{G} \cap (-\widetilde{\mathbb{E}}) = \emptyset$. Hence, $\mathrm{Int}\,\mathbb{H}^* = \emptyset$. In summary, if $\mathbb{E} \neq \emptyset$ is any subequation, then with $\mathbb{G} = \emptyset$, we have $\mathbb{H} = \mathbb{E}$ and $\mathbb{H}^* = \emptyset$, so that $\mathbb{E}$ itself (not $\partial\mathbb{E}$) is a generalized equation which falls in the Existence/Non-Uniqueness case for $\mathbb{H}$ (Type III). Similarly, $-\mathbb{F} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ with $\mathbb{E} \equiv \mathrm{Sym}^2(\mathbb{R}^n)$ and $\mathbb{G} = \widetilde{\mathbb{F}}$ is Type II.

Some Type IV examples of non-uniqueness/non-existence

Example 4.15. With coordinates $z = (x, y) \in \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^\ell$, define $\mathbb{E}$ by $D_x^2 u \geq 0$ and $\widetilde{\mathbb{G}}$ by $D_y^2 u \geq 0$ (so that $\mathbb{G}$ is the subaffine subequation on $\mathbb{R}^\ell$ considered as a subequation on

$\mathbb{R}^n$). Then with $\mathbb{H} \equiv \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ we have that the $\mathbb{H}$-harmonics are continuous functions $h(x, y)$ that are separately convex in x and concave in y. The mirror $\mathbb{H}^*$-harmonics are continuous functions $h^*(x, y)$ that are separately subaffine in x and superaffine in y.

Elementary examples from $\mathcal{P}$ and $\widetilde{\mathcal{P}}$

Example 4.16.

(a) If we take $\mathbb{E} = \mathbb{G} = \mathcal{P}$, then $\mathbb{H} = \partial\mathcal{P}$ is the determined equation whose harmonics are solutions to the real Monge-Ampère equation.

(b) If we take $\mathbb{E} = \mathbb{G} = \widetilde{\mathcal{P}}$, then $\mathbb{H} = \partial\widetilde{\mathcal{P}} = -\partial\mathcal{P}$ is the determined subaffine equation, whose harmonics are the negatives of the harmonics for real Monge-Ampère equation.

(c) If we take $\mathbb{E} = \mathcal{P}$ and $\mathbb{G} = \widetilde{\mathcal{P}}$, then $\mathbb{H} = \mathcal{P} \cap (-\mathcal{P}) = \{0\}$, and the harmonics are functions which are both convex and concave, i.e., the affine functions. So far nothing is really new.

(d) If we take $\mathbb{E} = \widetilde{\mathcal{P}}$ and $\mathbb{G} = \mathcal{P}$, then $\mathbb{H} = \widetilde{\mathcal{P}} \cap (-\widetilde{\mathcal{P}}) = \mathrm{Sym}^2(\mathbb{R}^n) \sim [(\mathrm{Int}\mathcal{P}) \cup (-\mathrm{Int}\mathcal{P})]$. The solutions are functions u with D^2u non-definite, (i.e., never > 0 nor < 0) in the C^2 case.

The affine generalized equation

Example 4.17. Here we are interested in equations where $\mathbb{H}_{\mathbb{E},\mathbb{G}} = \{0\}$. Note that $\mathbb{H}_{\mathbb{E},\mathbb{G}} = \mathbb{E} \cap (-\widetilde{\mathbb{G}}) = \mathbb{E} \sim (\mathrm{Int}\mathbb{G}) = \{0\} \iff \mathbb{E} \sim \{0\} \subset \mathrm{Int}\mathbb{G}$. Therefore

$$\mathbb{H}_{\mathbb{E},\mathbb{G}} = \{0\} \qquad \iff \qquad \mathbb{E} \sim \{0\} \subset \mathrm{Int}\mathbb{G} \ \text{ and } \ 0 \in \partial\mathbb{E}, 0 \in \partial\mathbb{G}.$$

Proposition 4.18. *Suppose $\mathbb{E}$ is a convex cone and let $\mathbb{E}^0 \equiv \{B : \langle B, A\rangle \geq 0 \,\forall\, A \in \mathbb{E}\}$ be its polar cone.*

Suppose $\exists\, A \in \mathrm{Int}\mathbb{E}^0$ (equivalently $\mathbb{E} \subset \Delta_A = \{B : \langle A, B\rangle \geq 0\}$) such that $\Delta_A \subset \mathbb{G}$ (equivalently $\widetilde{\mathbb{G}} \subset \widetilde{\Delta}_A = \Delta_A$). Then h is $\mathbb{H}_{\mathbb{E},\mathbb{G}}$-harmonic $\Rightarrow$ h is Δ_A-harmonic $\Rightarrow$ h is affine.

Proof. h is $\mathbb{H}_{\mathbb{E},\mathbb{G}}$-harmonic $\iff$ h is $\mathbb{E}(\subset \Delta_A)$ subharmonic and $-h$ is $\widetilde{\mathbb{G}}(\subset \Delta_A)$ subharmonic. Therefore, h is $\mathbb{H}_{\mathbb{E},\mathbb{G}}$-harmonic $\Rightarrow$ h is Δ_A-harmonic $\Rightarrow$ h is smooth $\Rightarrow$ h is affine. $\square$

Proposition 4.19. *Any affine generalized equation $\mathbb{H}_{\mathbb{E},\mathbb{G}} = \{0\}$ is Type II, i.e.,* $\mathrm{Int}\mathbb{H} = \emptyset$ *and* $\mathrm{Int}\mathbb{H}^* \neq \emptyset$.

Proof. Of course $\mathrm{Int}\mathbb{H} = \mathrm{Int}\{0\} = \emptyset$. Now $\mathrm{Int}\mathbb{H}^* = (\mathrm{Int}\mathbb{G}) \sim \mathbb{E} = \emptyset \iff \mathrm{Int}\mathbb{G} \subset \mathbb{E} \iff \mathbb{G} \subset \mathbb{E}$, which proves that $\mathbb{E} = \mathbb{G}$ is Type I. Hence $\mathbb{H} = \partial\mathbb{E}$ and so $\partial\mathbb{E} = \{0\}$. This is impossible for a subequation $\mathbb{E}$, so $\mathrm{Int}\mathbb{H}^* \neq \emptyset$. $\square$

Recall for Type II that $\mathbb{H} = \partial\mathbb{E} \cap \partial\mathbb{G} = \{0\}$.

The **canonical pair** for $\mathbb{H} = \{0\}$ is given as follows.

$$\mathbb{E}_{\min} = \{0\} + \mathcal{P} = \mathcal{P}, \quad -\widetilde{\mathbb{G}}_{\max} = \{0\} - \mathcal{P} = -\mathcal{P}, \quad \widetilde{\mathbb{G}}_{\max} = \mathcal{P}, \quad \mathbb{G}_{\max} = \widetilde{\mathcal{P}}.$$

We have $\mathbb{H} = \{0\} = \mathcal{P} \cap (-\mathcal{P})$.

In this case an $\mathbb{H}_{\mathcal{P},\widetilde{\mathcal{P}}}$ harmonic h is affine since $\pm h$ are both convex.

5. Unsettling questions

In Section 8 of [14] we posed several such questions, starting with the single-valuedness of operators and the following equivalent restatement of that question.

(CCQuest) Constant coefficient subequation question: Can a pair of subequations $\mathbb{E}, \mathbb{G}$, with disjoint equations, i.e., $\partial\mathbb{E} \cap \partial\mathbb{G} = \emptyset$, have a simultaneous harmonic h?

Of course a simultaneous harmonic h cannot be C^2 since one would have $D_x^2 h \in \partial\mathbb{E} \cap \partial\mathbb{G} = \emptyset$. One reason for the success of viscosity theory is that the intuition gained from examining classical situations carries over to the viscosity approach. In fact, one can show that any simultaneous harmonic must be quite bizarre, but this is short of non-existence. (A known result which has some kinship to this open question is the fact that an arbitrary upper semi-continuous function has an upper test function at a dense set of points, cf. [13, Lemma 6.1′].)

In this section we first discuss some equivalent versions of the question above. We then examine an extension of this question to a natural one for any generalized equation.

It is easy to see from positivity that the boundary of a subequation can be expressed as the graph of a continuous function over the hyperplane $\{\mathrm{tr}A = 0\}$. Consequently, if $\partial\mathbb{E} \cap \partial\mathbb{G} = \emptyset$, we might as well assume $\mathbb{E} \subset \mathbb{G}$. Now the hypothesis of (CCQuest) can be reformulated as follows.

$$\partial\mathbb{E} \cap \partial\mathbb{G} = \emptyset \ \text{ and } \ \mathbb{E} \subset \mathbb{G} \quad \iff \quad \mathbb{E} \subset \mathrm{Int}\mathbb{G} \quad \iff \quad \partial\mathbb{E} \subset \mathrm{Int}\mathbb{G}. \tag{5.1a}$$

We leave the proof to the reader. The condition $\mathbb{E} \subset \mathrm{Int}\mathbb{G}$ in (5.1a) is obviously equivalent to:

$$\text{The generalized equation } \mathbb{H} \equiv \mathbb{E} \cap (-\widetilde{\mathbb{G}}) = \mathbb{E} \cap (\sim \mathrm{Int}\mathbb{G}) \text{ is empty.} \tag{5.1b}$$

Lemma 5.1. *Under the equivalent assumptions of (5.1)*

$$h \text{ is } \mathbb{H}_{\mathbb{E},\mathbb{G}}\text{-harmonic} \quad \iff \quad h \text{ is both } \partial\mathbb{E}\text{- and } \partial\mathbb{G}\text{-harmonic.} \tag{5.2}$$

Proof. First suppose h is $\mathbb{H}_{\mathbb{E},\mathbb{G}}$-harmonic, i.e., H is $\mathbb{E}$-subharmonic and $-h$ is $\widetilde{\mathbb{G}}$-subharmonic. By (5.1) $\mathbb{E} \subset \mathbb{G}$ and hence $\widetilde{\mathbb{G}} \subset \widetilde{\mathbb{E}}$ also. Thus h is both $\mathbb{E}$ and $\mathbb{G}$

subharmonic and $-h$ is both $\widetilde{\mathbb{G}}$ and $\widetilde{\mathbb{E}}$ subharmonic, which proves that h is both $\partial\mathbb{E}$ and $\partial\mathbb{G}$ harmonic.

For the converse suppose that h is both $\partial\mathbb{E}$ and $\partial\mathbb{G}$ harmonic. Then h is $\mathbb{E}$ subharmonic and $-h$ is $\widetilde{\mathbb{G}}$-subharmonic, so that h is $\mathbb{H}_{\mathbb{E},\mathbb{G}}$-harmonic. $\square$

This equivalence (5.2) means that we can reformulate the above (CCQuest) concerning simultaneous harmonics for $\partial\mathbb{E}$ and $\partial\mathbb{G}$ as follows.

(CCQuest)′: Does there exist a subequation pair $\mathbb{E}, \mathbb{G}$ defining the generalized empty equation $\mathbb{H} = \emptyset$ with the property that $\mathbb{H}_{\mathbb{E},\mathbb{G}}$ has a harmonic?

Summary. There are lots of subequations pairs $\mathbb{E}, \mathbb{G}$ defining this generalized empty equation $\mathbb{H} = \emptyset$. For some of these pairs we can prove that $\mathbb{E} \cap (-\widetilde{\mathbb{G}})$ has no harmonics. For example, this holds if $\mathbb{E}$-harmonics and $\mathbb{G}$-harmonics are always C^2 because of Lemma 5.1. To conclude we note that for $\mathbb{H} = \emptyset$ the canonical defining pair is $(\emptyset, \emptyset)$, so that for this pair $\mathbb{E}_{\min}, \mathbb{G}_{\max}$ defining $\mathbb{H} = \emptyset$ there are also no harmonics.

Now we can broaden our question as follows.

Broadened Equation Question: Given a generalized equation $\mathbb{H}$ does there exist a subequation pair $\mathbb{E}, \mathbb{G}$ defining $\mathbb{H}$ so that $\mathbb{H} \equiv \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ has a harmonic which is not a harmonic for $\mathbb{H} \equiv \mathbb{E}_{\min} \cap (-\widetilde{\mathbb{G}}_{\max})$?

Note that by Proposition 3.3 $\mathbb{E}_{\min} \subset \mathbb{E}$ and $-\widetilde{\mathbb{G}}_{\max} \subset -\widetilde{\mathbb{G}}$ so that $\mathbb{H} \equiv \mathbb{E}_{\min} \cap (-\widetilde{\mathbb{G}}_{\max})$ harmonics are always $\mathbb{H} \equiv \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ harmonics.

As for (CCQuest), any such harmonic h in this Broadened Equation Question must be weird and pathological, much worse than C^2 for sure.

6. The general case of the main theorem

For clarity and simplicity we have been restricting attention to pure second-order constant coefficient subequations $\mathbb{E}$ and $\mathbb{G}$ to define a generalized equation $\mathbb{H} = \mathbb{E} \cap (-\widetilde{\mathbb{G}})$ in $\mathbb{R}^n$. However, the main Theorem 2.6 holds for completely general subequations on manifolds, as defined in [9], and we give that result in this section. For general definitions we refer to [9]. However, there are many interesting cases which the reader could keep in mind (without consulting [9]), namely: constant coefficient subequations $\mathbb{E}$ and $\mathbb{G}$ (not necessarily pure second-order) in $\mathbb{R}^n$, variable coefficient subequations (constraint sets for subsolutions) on domains in $\mathbb{R}^n$, subequations on riemannian manifolds given canonically by $\mathrm{O}(n)$-invariant equations in $\mathbb{R}^n$, subequations on hermitian manifolds given canonically by $\mathrm{U}(n)$-invariant equations in $\mathbb{C}^n$, etc.

Let $J^2(X)$ be the 2-jet bundle on a manifold X. (When $X = \mathbb{R}^n$ this is just the bundle $\mathbb{R}^n \times (\mathbb{R} \oplus \mathbb{R}^n \oplus \mathrm{Sym}^2(\mathbb{R}^n))$ over $\mathbb{R}^n$ of order-2 Taylor expansions.)

Theorem 6.1. *Let $\Omega \subset\subset X$ be a domain in a manifold X, and suppose $\mathbb{E}, \mathbb{G} \subset J^2(X)$ are two subequations. Consider the generalized equation $\mathbb{H} \equiv \mathbb{E} \cap (-\widetilde{\mathbb{G}})$.*

(a) $\mathrm{Int}\,\mathbb{H} = \emptyset \;\Rightarrow\;$ *uniqueness for the $\mathbb{H}$-(DP) on Ω, assuming that comparison holds for $\mathbb{E}$ and $\mathbb{G}$ on Ω.*

(b) $\mathrm{Int}\,\mathbb{H}^* = \emptyset \;\Rightarrow\;$ *existence holds for the $\mathbb{H}$-(DP) on Ω, assuming that existence for the $\mathbb{E}$-(DP) holds on Ω.*

(c) There exists $h \in C^2(\Omega) \cap C(\overline{\Omega})$ with $J^2_x h \in \mathrm{Int}\,\mathbb{H}$ for all $x \in \Omega \;\Rightarrow\;$ non-uniqueness for the $\mathbb{H}$-(DP) on Ω for the boundary values $\varphi \equiv h\big|_{\partial\Omega}$.

(d) There exists $f \in C^2(\Omega) \cap C(\overline{\Omega})$ with $J^2_x f \in \mathrm{Int}\,\mathbb{H}^*$ for all $x \in \Omega \;\Rightarrow\;$ non-existence for the $\mathbb{H}$-(DP) on Ω for the boundary values $\varphi \equiv f\big|_{\partial\Omega}$, assuming that comparison holds for $\mathbb{E}$ and $\mathbb{G}$ on Ω.

Proof of Assertion (a). We begin by noting that assertions (1.2)–(1.5) hold for general subequations as defined in [9]. Our definition of $\mathbb{H}$ is the same as in Definition 2.2, and the assertion (2.5) carries over. As a result, Lemma 2.7 and Corollary 2.8 hold in this general case. We now look at the proof of Proposition 2.11, which carries over and says that under the assumption of comparison (C) for both $\mathbb{E}$ and $\mathbb{G}$ we have Part (a). □

Proof of Assertion (c). This follows exactly the argument given for Proposition 2.12. □

Proof of Assertion (b). This follows exactly the argument given for Proposition 2.14. □

Proof of Assertion (d). We are assuming that comparison (C) holds on Ω for both $\mathbb{E}$ and $\mathbb{G}$. This means that Proposition 2.16 holds, and therefore also Proposition 2.17 is valid. This establishes Part (d). □

Example 6.2 *(Generalized constant coefficient equations in $\mathbb{R}^n$).* Here a subequation is, by definition (cf. [9], [10]), a closed subset

$$\mathbb{F} \subset J^2 \equiv \mathbb{R} \oplus \mathbb{R}^n \oplus \mathrm{Sym}^2(\mathbb{R}^n)$$

such that $\mathbb{F} + (r, 0, P) \subset \mathbb{F}$ for $r \leq 0$ and $P \geq 0$ and such that $\mathbb{F} = \overline{\mathrm{Int}\,\mathbb{F}}$. The topological condition $\mathbb{F} = \overline{\mathrm{Int}\,\mathbb{F}}$ was not part of our definition of a subequation in Section 1, which was for pure second-order subequations. This allowed us to use the simpler definition that $\mathbb{F}$ is closed, since $\mathbb{F} = \overline{\mathrm{Int}\,\mathbb{F}}$ then follows easily from the positivity condition (1.1).

With regards to Assertion (a) comparison does not hold for all such equations. However it does hold for many interesting classes, for instance, all gradient free ones. Other such classes can be found in [1].

On the other hand, **existence does hold for all these equations** $\mathbb{F} \subset J^2$, under the hypothesis that the domain $\Omega \subset\subset \mathbb{R}^n$ has a smooth strictly $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ convex boundary. (See Theorem 12.7 in [9].) Now in Assertion (b) existence is only required for $\mathbb{E}$. Therefore, Assertion (b) holds for $\mathbb{E}, \mathbb{G} \subset J^2$ provided $\partial\Omega$ is strictly $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ convex.

Example 6.3 *(Generalized equations on an open set $X \subset \mathbb{R}^n$).* The general subequation here is a closed subset of the 2-jet bundle

$$\mathbb{F} \subset J^2(X) \equiv X \times (\mathbb{R} \oplus \mathbb{R}^n \oplus \mathrm{Sym}^2(\mathbb{R}^n))$$

such that

$$\mathbb{F} + (x; r, 0, P) \subset \mathbb{F} \text{ for } r \leq 0 \text{ and } P \geq 0 \text{ and for all } x \in X,$$

$\mathbb{F} = \overline{\mathrm{Int}\,\mathbb{F}}$, and for the fibres $\mathbb{F}_x$ we have

$$\mathbb{F}_x = \overline{\mathrm{Int}_x\,\mathbb{F}_x} \qquad \text{and} \qquad \mathrm{Int}_x\,\mathbb{F}_x = (\mathrm{Int}\,\mathbb{F})_x.$$

These are barebones hypotheses needed for the constraint set for subsolutions of a non-linear equation corresponding to $\partial\mathbb{F}$.

This is the general case for domains $\Omega \subset\subset X \subset \mathbb{R}^n$, and so the comparison and existence hypotheses in Theorem 6.1 need to be verified, but, of course, the literature is enormous.

For subequations on manifolds given by "universal" equations, much has been done in [9]. We shall now look at some cases.

Example 6.4 *(Universal subequations defined on any Riemannian manifold).* Let $\mathbb{F} \subset J^2$ be a subequation (as in Example 6.2) which is invariant under the natural action of the orthogonal group $\mathrm{O}(n)$ (or $\mathrm{SO}(n)$). Then $\mathbb{F}$ determines an invariant subequation $\mathbb{F}_X \subset J^2(X)$ on any riemannian (or oriented riemannian) manifold X as follows.

Every C^2-function u on X has a riemannian hessian $\mathrm{Hess}\,u$, which is a section of the bundle $\mathrm{Sym}^2(X)$ of symmetric 2-forms on X, given at $x \in X$ by

$$\{\mathrm{Hess}_x u\}\,(V, W) \;\equiv\; V_x W\,u - (\nabla_V W)_x\,u$$

where ∇ is the Levi-Civita connection on TX. Note that $\nabla_V W - \nabla_W V = VW - WV = [V, W]$, so the symmetry and the tensorial properties of $\mathrm{Hess}\,u$ follow.

Now this riemannian Hessian gives a splitting of the 2-jet bundle

$$J^2(X) \;\cong\; X \times (\mathbb{R} \oplus T^*X \oplus \mathrm{Sym}^2(X)),$$

and the orthogonally invariant subequation $\mathbb{F}$ canonically determines a subequation $\mathbb{F}_X \subset J^2(X)$ as follows. Any orthonormal frame field $e_1, ..., e_n$ for TX on an open set $U \subset X$ determines an orthonormal framing of $J^2(U) \cong U \times (\mathbb{R} \oplus \mathbb{R}^n \oplus \mathrm{Sym}^2(\mathbb{R}^n))$. Via this framing, $\mathbb{F}$ determines a subequation on U. However, if we use a different frame field $e_1', ..., e_n'$, the two framings of $J^2(U)$ differ pointwise by $\mathrm{O}(n)$-transforms. By the $\mathrm{O}(n)$-invariance of $\mathbb{F}$ the subequation on U are the same. This also means that on two different open sets $U, V \subset X$ the two subequations agree on $U \cap V$. Hence, we have a well-defined global subequation $\mathbb{F}_X \subset J^2(X)$.

For example, if $\mathbb{F} = \{(r, p, A) : \mathrm{tr}(A) \geq 0\}$, we get the subequation $\Delta u = \mathrm{tr}\{\mathrm{Hess}\, u\} \geq 0$ for the riemannian Laplacian. If $\mathbb{F} = \{(r, p, A) : \det(A) \geq 0\}$, we get the real Monge-Ampère subequation $\det\{\mathrm{Hess}\, u\} \geq 0$. If $\mathbb{F} = \{(r, p, A) : p^t Ap \geq 0\}$, one gets the infinite Laplacian on X.

The questions of comparison and of existence of solutions for the Dirichlet problem on manifolds are addressed in [9]. A cone subequation M on X is a cone monotonicity subequation for $\mathbb{F}_X$ if $\mathbb{F}_X + M \subset \mathbb{F}_X$. Then for such equations we have the following from Thm. 13.2 and Thm. 10.1 in [9]. (See section 14 of [9] for examples.)

Theorem. *([9]) Suppose X admits a C^2 strictly M-subharmonic function. Then comparison for $\mathbb{F}_X$ holds on any domain $\Omega \subset\subset X$, and if $\partial\Omega$ is smooth and strictly $\mathbb{F}_X$ and $\widetilde{\mathbb{F}}_X$ convex, then existence holds for the Dirichlet problem for all boundary functions $\varphi \in C(\partial\Omega)$.*

This construction has important generalizations.

Example 6.5 *(Universal subequations defined on a Riemannian manifold with G-structure).* We now assume that the riemannian manifold X can be covered by open sets U, with an orthogonal tangent frame field $e^U \equiv (e_1, ..., e_n)$ on U, such that on the intersection $U \cap V$ of two such, the change of frames from e^U to e^V always lies in a given compact subgroup $G \subset O(n)$.

For example, if X^{2n} has an orthogonal almost complex structure J, then X has an $\mathrm{U}(n)$-structure.

If the euclidean subequation $\mathbb{F}$ is G-invariant, then the above construction gives a canonical subequation on any riemannian manifold with G-structure. For example, for (X^{2n}, J) above, we can define the complex Monge-Ampère operator.

The Theorem at the end of Example 6.4 extends to these cases.

Example 6.6 *(Geometric cases).* Of particular importance are the geometric cases given by a closed subset $\mathbb{G} \subset G(p, \mathbb{R}^n)$ of the Grassmannian of p-planes in $\mathbb{R}^n$. We assume that $\mathbb{G}$ is invariant under a closed subgroup $G \subset O(n)$. Then we consider the universal euclidean subequation

$$\mathbb{F}_{\mathbb{G}} \equiv \left\{(r, p, A) : \mathrm{tr}\left(A\big|_L\right) \geq 0 \text{ for all } L \in \mathbb{G}\right\}.$$

This subequation now carries over to any riemannian manifold with G-structure. For instance, suppose $\mathbb{G}$ is the set of special Lagrangian n-planes in $\mathbb{C}^n$. Then we get a subequation on any Calabi-Yau manifold X. If $\mathbb{G}$ is the set of associative 3-planes in $\mathbb{R}^7$, then we get a subequation on any 7-manifold X with holonomy G_2.

Theorems in [9] apply to these cases, but there is a better theorem in [12]. We define the **$\mathbb{G}$-core** of X to be the set

$$\mathrm{Core}_{\mathbb{G}}(X) \;\equiv\; \{x \in X : \text{no smooth strictly } \mathbb{F}_{\mathbb{G}}\text{-subharm. function is strict at } x\}$$

Theorem. *([12, Thm. 7.6 and Thm. 7.7]) If $\mathrm{Core}_{\mathbb{G}}(X) = \emptyset$, then comparison for $\mathbb{F}_{\mathbb{G}}(X)$ holds on any domain $\Omega \subset\subset X$, and if $\partial\Omega$ is smooth and strictly $\mathbb{F}_{\mathbb{G}}$ and $\widetilde{\mathbb{F}_{\mathbb{G}}}$ convex, then existence holds for the Dirichlet problem for all boundary functions $\varphi \in C(\partial\Omega)$.*

Remark 6.7. In Section 2 we made a remark which does not carry over to general subequations. Finite intersections of subequations are not always subequations. There are classes of subequations where this is true (see [1]). However in general this means that one could expand the definition of a generalized equation to cover many-fold intersections and unions of subequations. This will be done elsewhere.

Appendix A. The quasi-convexity characterization of $\mathbf{C^{1,1}}$

Interestingly, the condition that a function u be locally $C^{1,1}$ is equivalent to u locally being simultaneously quasi-convex and quasi-concave. This was probably first observed by Hiriart-Urruty and Plazanet in [16]. An alternate proof appeared in Eberhard [5] and also in [17]. For the benefit of the reader we include a proof here.

We say that a function is $\lambda - C^{1,1}$ if it is C^1 and the first derivative is locally Lipschitz with Lipschitz coefficient λ.

Theorem A.1.

$$u \;\; \text{is } \lambda - C^{1,1} \qquad \iff \qquad \text{both} \quad \pm u \quad \text{are locally } \lambda - \text{quasi-convex}$$

Proof. We consider u on a convex set Ω.

($\Rightarrow$) Suppose that u is λ-$C^{1,1}$, i.e., $u \in C^1$ and $|D_x u - D_y u| \le \lambda|x - y|$ for all $x, y \in \Omega$. Set $f \equiv u + \frac{\lambda}{2}|x|^2$. Then

$$D_x f - D_y f \;=\; \lambda(x - y) + D_x u - D_y u,$$

and hence

$$\begin{aligned} \langle D_x f - D_y f, x - y\rangle \;&=\; \lambda|x - y|^2 + \langle D_x u - D_y u, x - y\rangle \\ &\ge\; \lambda|x - y|^2 - |D_x u - D_y u||x - y| \\ &=\; (\lambda|x - y| - |D_x u - D_y u|)|x - y| \;\ge\; 0. \end{aligned}$$

This form of monotonicity of Df is one of the standard definitions of f being convex. The same proof works for $-u$

($\Leftarrow$) We state the converse as a proposition.

Proposition A.2. *If u and $-u$ are λ-quasi-convex on a convex domain Ω, then $u \in C^1$ and*

$$|D_x u - D_y u| \le \lambda|x-y|, \quad i.e., \quad u \quad is \quad \lambda - C^{1,1}, \quad \text{on} \quad \Omega.$$

Proof. We first show that this is true if $u \in C^\infty$. Note that $\pm u$ are λ-quasi-convex $\iff$ $D_x^2 u + \lambda I \ge 0$ and $-D_x^2 u + \lambda I \ge 0$ for all $x \iff -\lambda I \le D_x^2 u \le \lambda I$ for all $x \in \Omega$. Fix $x, y \in \Omega$. By the Mean Value Inequality in [21], $|D_x u - D_y u| \le |D_\xi^2 u||x-y|$ for some $\xi \in [x, y]$, and hence $|D_x u - D_y u| \le \lambda|x-y|$. (Here $|A| \equiv \sup\{|A(e)| : |e| = 1\}$ is the operator norm of the symmetric transformation $A = D_\xi^2 u$, which is equal to the max of $|\langle Ae, e\rangle|$ over unit vectors e.)

In general, since the graph of $u + \frac{\lambda}{2}|x|^2$ has a supporting hyperplane from below and the graph of $u - \frac{\lambda}{2}|x|^2$ has a supporting hyperplane from above, at every point, the function u is differentiable everywhere. By partial continuity of the first derivative for quasi-convex functions (see for example Lemma 1.3 in [13]), we have $u \in C^1$.

Now standard convolution $u^\epsilon \equiv u * \varphi_\epsilon$ works just fine to complete the proof since $\pm u^\epsilon$ is λ-quasi-convex by the next lemma, and the fact that $u \in C^1 \Rightarrow Du^\epsilon \to Du$ locally uniformly. $\square$

Lemma A.3. *u is λ-quasi-convex $\Rightarrow$ $u^\epsilon \equiv u * \varphi_\epsilon$ is λ-quasi-convex.*

Proof. Suppose u is λ-quasi-convex, i.e., $f \equiv u + \frac{\lambda}{2}|x|^2$ is convex. Standard convolution of f with an approximate identity φ_ϵ based on φ (i.e., $\varphi_\epsilon(x) \equiv \frac{1}{\epsilon^n}\varphi(\frac{x}{\epsilon})$) yields $f^\epsilon \equiv f * \varphi_\epsilon$ smooth and convex. Note that $(|x|^2 * \varphi_\epsilon) = |x|^2 + \langle a, x\rangle + c$ preserves $|x|^2$ modulo an affine function, since $\int |x + \epsilon y|^2 \varphi(y)\, dy = |x|^2 + \epsilon\langle a, x\rangle + C\epsilon^2$ where $\langle a, x\rangle = 2\int \langle x, y\rangle \varphi(y)\, dy$ and $C = \int |y|^2 \varphi(y)\, dy$. Therefore, $D^2 f^\epsilon = D^2 u^\epsilon + \lambda I$, proving that each u^ϵ is λ-quasi-convex. $\square$

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