HYPERBOLIC POLYNOMIALS AND THE DIRICHLET PROBLEM

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ABSTRACT

This paper presents a simple, self-contained account of Gårding's theory of hyperbolic polynomials, including a recent convexity result of Bauschke-Guler-Lewis-Sendov and an inequality of Gurvits. This account also contains new results, such as the existence of a real analytic arrangement of the eigenvalue functions.

In a second, independent part of the paper, the relationship of Gårding's theory to the authors' recent work on the Dirichlet problem for fully nonlinear partial differential equations is investigated. Each Gårding polynomial p of degree m on $\mathrm{Sym}^2(\mathbf{R}^n)$ (hyperbolic with respect to the identity) has an associated eigenvalue map $\lambda : \mathrm{Sym}^2(\mathbf{R}^n) \to \mathbf{R}^m$, defined modulo the permutation group acting on $\mathbf{R}^m$. Consequently, each closed symmetric set $E \subset \mathbf{R}^m$ induces a second-order p.d.e. by requiring, for a C^2-function u in n-variables, that

$$\lambda\left((D^2u)(x)\right) \ \in\ \partial E \quad \text{for all } x.$$

Assume that $A \geq 0 \Rightarrow \lambda(A) \geq 0$ and that $E + \mathbf{R}^m_+ \subset E$. A main result is that for smooth domains $\Omega \subset \mathbf{R}^n$ whose boundary is suitably (p, E)-pseudo-convex, the Dirichlet problem has a unique continuous solution for all continuous boundary data. This applies in particular to each of the m distinct branches of the equation $p\left(D^2u\right) = 0$

In the authors' recent extension of results from euclidean domains to domains in riemannian manifolds, a new global ingredient, called a monotonicity subequation, was introduced. It is shown in this paper that for every polynomial p as above, the associated Gårding cone is a monotonicity cone for all branches of the the equation $p(\mathrm{Hess}\, u) = 0$ where $\mathrm{Hess}\, u$ denotes the riemannian Hessian of u.

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1. Introduction

This paper is concerned with the Gårding theory of hyperbolic polynomials and its relation to the Dirichlet problem for fully nonlinear partial differentials equations.

We recall that a homogeneous polynomial of degree m on a finite dimensional real vector space V is called *hyperbolic with respect to a direction* $a \in V$ if (we assume $p(a) > 0$) the one-variable polynomial $t \mapsto p(ta + x)$ has m real roots for each $x \in V$. Thus we can write

$$p(ta + x) \;=\; p(a) \prod_{k=1}^{m} (t + \lambda_k(x)). \tag{1.1}$$

The functions $\lambda_k(x)$, called the *a-eigenvalues of* x, are well defined up to permutation. In 1959 Gårding developed a beautiful theory of these hyperbolic polynomials [G_2]. An important part of the theory concerned the set

$$\Gamma \;\equiv\; \{x \in V : \lambda_k(x) > 0, \text{ for all } k\} \tag{1.2}$$

which is proved to be a convex cone. Of crucial importance for the paper is the *monotonicity property*: that if one orders the eigenvalues $\lambda_1^{\uparrow}(x) \leq \cdots \leq \lambda_m^{\uparrow}(x)$, then

$$\lambda_k^{\uparrow}(x + b) \;>\; \lambda_k^{\uparrow}(x) \quad \text{for all } b \in \Gamma, x \in V. \tag{1.3}$$

Gårding's motivation came from the theory of hyperbolic partial differential equations [G_1]. (See also [H].) However, in subsequent years his results have attracted attention and found application in other areas. For example, Gårding theory was used to initiate hyperbolic programming by Guler [Gu]. (See also [R].)

This paper has two distinct and essentially independent objectives.

The first is to present a simple, self-contained account of Gårding's theory. Our account starts with a new result, Theorem 2.9 below, which states that

For $b \in \Gamma$ and $x \in V$ the a-eigenvalues can be arranged so that each of the functions $t \mapsto \lambda_k(x + tb)$ is a strictly increasing real-analytic mapping from $\mathbf{R}$ onto $\mathbf{R}$ with real analytic inverse $s \mapsto -\mu_k(x - sa)$ where μ_k is a b-eigenvalue.

The proof of this result uses only the classical elementary fact that each point on an algebraic curve admits a local uniformizing parameter. Gårding's theory, together with a recent convexity result of Bauschke, Güler, Lewis and Sendov [BGLS], is then presented as a series of elementary consequences of this theorem. This is done in Sections 2 and 3. Another recent result, due to Gurvits [$\text{Gur}_{1,2}$], is presented in Appendix B as an improvement of a basic inequality of Gårding. The parallels in the proofs are emphasized.

Historical Note: In 1958 P. Lax [L] conjectured that for homogeneous polynomials $p(x, y, z)$ in three variables which are hyperbolic with respect to a direction a, say $a = (0, 0, 1)$, there exist symmetric matrices A, B such that $p(x, y, z) = \det(xA + yB + zI)$. This conjecture was essentially established by Vinnikov [V] in 1993 (see [LPR]). Armed with this highly non-trivial result, one can quickly establish the full Gårding theory, since all arguments only involve three variables at a time. However, the presentation here emphasizes quite elementary proofs.

The second objective of this paper is to investigate the relationship of Gårding theory to the theory of fully nonlinear second-order (degenerate elliptic) equations, first discussed by Cafferelli, Nirenberg and Spruck [CNS] for O_n-invariant equations. This discussion can be easily ignored by readers interested only the first part of the paper on Gårding theory. Here the basic vector space in question is the space $V = \mathrm{Sym}^2(\mathbf{R}^n)$ of real symmetric $n \times n$ matrices. This is because a second-order equation can be viewed as a subset $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$. The *solutions* or *$F$-harmonic functions* are functions u satisfying (at least in the C^2-case)

$$(D^2u)(x) \ \in \ \partial F.$$

In [HL_1] the Dirichlet problem is studied and solved via the Perron method utilizing the notion of an F-subharmonic function. The only condition the closed subset F must satisfy in this theory is the positivity condition

$$F + P \ \subset \ F \qquad \text{for each} \quad P > 0 \quad \text{(positive definite)}. \tag{1.4}$$

In this case the closed set $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is referred to as a *subequation*. One of the main points of this paper is to show how Gårding's theory determines a vast array of interesting and natural subequations. His monotonicity result (1.3) is exactly what is needed to establish the crucial positivity condition (1.4).

The first and most basic example is $F = \overline{\Gamma}$, the Gårding cone itself. Here the positivity condition (1.4) must be assumed, not deduced. However, requiring (1.4) is equivalent to requiring that the polynomial p be hyperbolic in each positive definite direction $P > 0$ in $\mathrm{Sym}^2(\mathbf{R}^n)$. These polynomials will be called *Dirichlet-Gårding polynomials* for the purposes of this paper.

The main new construction of subequations in this paper can be described as follows. Suppose that p is a Dirichlet-Gårding polynomial of degree m, and choose a hyperbolic direction $a \in \Gamma$, for example any $P > 0$ or, more specifically, the identity $I \in \mathrm{Sym}^2(\mathbf{R}^n)$. Then the eigenvalue map $\lambda : \mathrm{Sym}^2(\mathbf{R}^n) \to \mathbf{R}^m$ is defined, as above, modulo coordinate permutations on $\mathbf{R}^m$. Consequently each closed symmetric subset $E \subset \mathbf{R}^n$ determines a second-order partial differential equation (at least for C^2-functions u) by requiring

$$\lambda\left((D^2u)(x)\right) \ \in \ \partial E. \tag{1.5}$$

One main result (Theorem 5.19) says that if a closed symmetric subset $E \subset \mathbf{R}^m$ satisfies the *universal positivity condition*:

$$E + \lambda \ \subset \ E \quad \text{for all } \lambda \in \mathbf{R}^m \text{ with } \lambda > 0 \quad \text{(i.e., each} \quad \lambda_k > 0), \tag{1.6}$$

then the subset F_E of $\mathrm{Sym}^2(\mathbf{R}^n)$ defined by

$$F_E \ \equiv \ \{A \ \in \ \mathrm{Sym}^2(\mathbf{R}^n) : \lambda(A) \ \in \ E\} \tag{1.7}$$

is a subequation, that is, F_E satisfies the positivity condition (1.4). The closed subsets E satisfying (1.6) can be regarded as *universal subequations on* $\mathbf{R}^m$ since they induce a subequation on $\mathbf{R}^n$ for each Dirichlet-Gårding polynomial of degree m on $\mathrm{Sym}^2(\mathbf{R}^n)$ (and each hyperbolic direction). For each such subequation on $\mathbf{R}^n$, Theorem 5.20 concludes that the Dirichlet problem has a unique continuous solution for all continuous boundary functions on any smooth domain Ω in $\mathbf{R}^n$ whose boundary is suitably "pseudo"-convex.

This applies in particular to each of the branches of the equation

$$p((D^2u) \ = \ 0,$$

where the k**th branch** F_k is defined by requiring that

$$\lambda_k^{\uparrow}(D^2u) \ \geq \ 0$$

where $\lambda_k^{\uparrow}$ is the kth ordered eigenvalue function of p. These branches F_k are independent of the choice of hyperbolic direction (Theorem 2.12). The universal kth branch $E_k \subset \mathbf{R}^m$ is defined by

$$E_k \ \equiv \ \{\lambda \in \mathbf{R}^m : \text{at least } m-k+1 \text{ of the } \lambda_j \text{ are } \geq 0\} \ = \ \{\lambda \in \mathbf{R}^m : \lambda_k^{\uparrow} \geq 0\}$$

In Appendix A we give an algebraic description of each of these branches in terms of polynomial inequalities.

We shall return to a discussion of examples of universal eigenvalue subequations $E \subset \mathbf{R}^m$, but first we present examples of Dirichlet-Gårding polynomials. Three basic irreducible classical examples are the three determinants: $\det_{\mathbf{R}}$ on $\mathrm{Sym}^2(\mathbf{R}^n)$, $\det_{\mathbf{C}}$ on $\mathrm{Sym}^2(\mathbf{R}^{2n})$ and $\det_{\mathbf{H}}$ on $\mathrm{Sym}^2(\mathbf{R}^{4n})$. They are discussed in Section 5. A new irreducible example of degree 2^n on $\mathrm{Sym}^2(\mathbf{R}^{2n})$, which yields a notion of Lagrangian harmonicity, is also discussed there. Our theory enables us to treat all branches of this new equation even though an explicit rendering of them is quite complicated.

Several methods of constructing new Dirichlet-Gårding polynomials from a given Dirichlet-Gårding polynomial are outlined in Section 5 as well. One method is to take a directional derivative in one of the directions in the Gårding cone. Equivalently, these polynomials are the elementary symmetric functions in the eigenvalue functions. A second method constructs a Dirichlet-Gårding polynomial whose eigenvalue functions are the $\binom{m}{p}$ possible p-fold sums of the given eigenvalue functions. A third method constructs (for each $\delta > 0$) a new Dirichlet-Gårding polynomial (of the same degree) whose open Gårding cone is uniformly elliptic, i.e., contains the closed set of all $A \geq 0$ in $\mathrm{Sym}^2(\mathbf{R}^n)$. Since implementing these methods is not a commutative process, taken together they yield a huge collection of Dirichlet-Gårding polynomials.

The second and third methods are special cases of a more general method. If Q is a symmetric polynomial on $\mathbf{R}^m$ of degree N which is hyperbolic in all directions $\lambda > 0$, and p is a Dirichlet-Gårding polynomial on $\mathrm{Sym}^2(\mathbf{R}^n)$, then $q(A) = Q(\lambda(A))$ is a Dirichlet-Gårding polynomial of degree N on $\mathrm{Sym}^2(\mathbf{R}^n)$.

Now we return to the other ingredient, the universal subequations E. A classification or "structure" theorem is given in Proposition 5.25 in terms of Lipschitz graphs.

However, there are examples which are obscured by this structure theorem, such as the universal Special Lagrangian subequation $E_c \subset \mathbf{R}^m$ defined by requiring that

$$\sum_{k=1}^{m} \arctan \lambda_k \ \geq \ c.$$

The structure theorem says that every universal subequation E has boundary which is a graph

$$\partial E \ = \ \{f(\lambda)e + \lambda : \ \lambda \cdot e = 0\}$$

over the hyperplane $\{e \cdot \lambda = 0\}$ where $e = (1, ..., 1)$. E is then the region above the graph in this picture. The graphing function f must be Lipschitz with Lipschitz constant one using a natural norm on $\{e \cdot \lambda = 0\}$.

The Dirichlet problem is solved in Theorem 5.20 without assuming that the subequation F_E defined by (1.7) is either convex or uniformly elliptic, but our construction includes many such examples because, if E is convex, then F_E is convex, and if E is uniformly ellpitic, then so is F_E (see Theorem 5.19 (a) and (b)).

The general Gårding theory has an important application to the analogous Dirichlet problem on riemannian manifolds. In a recent paper [HL$_2$] the authors have carried over the basic results of [HL$_1$] from constant coefficient purely second-order subequations to equations defined in a quite general setting on riemannian manifolds, and manifolds with reduced structure group – such as almost complex hermitian manifolds. A fundamental new feature of this theory is the introduction of a *monotonicity cone* M_F for a given subequation F. Existence and uniqueness results are established under the (essentially necessary) hypothesis that there exist some strictly M_F-subharmonic function defined on a neighborhood of the domain in question. The Gårding monotonicity result (1.3) exactly provides such monotonicity cones. For a given I-hyperbolic polynomial p on $\mathrm{Sym}^2(\mathbf{R}^n)$ the Gårding cone Γ is a monotonicity cone for each branch of the equation $\{p(D^2u) = 0\}$. More generally, the subequations F_E on $\mathbf{R}^n$, determined by p of degree m and a universal subequation E on $\mathbf{R}^m$, are not confined to domains in $\mathbf{R}^n$, but extend to all parallelizable riemannian n-manifolds, and when p has a non-trivial invariance group G, they extend to all riemannian n-manifolds with structure group reduced (topologically) to G. Details are discussed in Chapter 6.

2. Gårding's Theory of Hyperbolic Polynomials

This section gives an essentially self-contained account of the Gårding theory of hyperbolic polynomials. Some aspects of the presentation are new – for example Theorem 2.7. Its proof relies only on the classical elementary fact that each point on an algebraic curve admits local uniformizing parameters (cf. (2.12)). This Theorem 2.7 together with Remark 2.3 imply all the basic results in [G$_2$].

Preliminaries

Suppose that p is a homogeneous polynomial of degree m on a complex vector space $V_{\mathbf{C}}$. Given points $a, x \in V_{\mathbf{C}}$ with $p(a) \neq 0$, the one-variable polynomial $s \mapsto p(sa+x)$ factors as

$$p(sa+x) \;=\; p(a)\prod_{k=1}^{m}(s+\lambda_k(x)) \qquad \text{and} \;\; p(x) \;=\; p(a)\prod_{k=1}^{m}\lambda_k(x), \tag{2.1}$$

where, by definition, $\lambda_1(x), ..., \lambda_m(x)$ are the a**-eigenvalues of** x and $r_k(x) \equiv -\lambda_k(x)$ are the a**-roots of** x. Of course $\lambda(x) \equiv (\lambda_1(x), ..., \lambda_m(x))$ is defined only modulo the permutation group π_m. When necessary, the dependence of the eigenvalues of x on the choice of a will be denoted by $\lambda_a^k(x) \equiv \lambda_k(x)$ and $\lambda_a(x) = (\lambda_a^1(x), ..., \lambda_a^m(x))$, and similarly for the a-roots of x.

Everything proven in this section uses one or more of the elementary properties discussed next – and very little else.

Elementary Properties: Suppose $p(a) \neq 0$. It follows easily from (2.1) that for all $t \in \mathbf{C}$ and $x \in V_{\mathbf{C}}$:

(1) $\lambda_a^k(tx) \;=\; t\lambda_a^k(x) \bmod \pi_m$,

(2) $\lambda_a^k(ta+x) \;=\; t + \lambda_a^k(x) \bmod \pi_m$, and so (2a) $\lambda_a^k(a) \;=\; 1$ for all k.

Sometimes it is convenient to set $e = (1, ..., 1)$ and restate (2) in the form

(2)$'$ $\lambda_a(ta+x) \;=\; te + \lambda_a(x) \bmod \pi_m$.

Using (2.1) with $x = tb$, $t \in \mathbf{C}$, $b \in V_{\mathbf{C}}$ and applying (1) yields

$$p(sa+tb) \;=\; p(a)\prod_{k=1}^{m}(s+t\lambda_k(b)) \tag{2.1$'$}$$

Note that (2.1)$'$ gives an explicit factorization of p restricted to span $\{a, b\}$. This will be useful in establishing certain important properties.

We shall use the notation

$$p'_x(y) \;\equiv\; \frac{d}{dt}p(y+tx)\Big|_{t=0}$$

for the directional derivative of p at y in the direction x. Setting $s = 1$ in $(2.1)'$ and then taking the logarithmic derivative at $t = 0$ proves that

$$(3) \quad p(a+tx) \ = \ p(a)\prod_{k=1}^{r}\left(1+t\lambda_a^k(x)\right) \qquad \text{and} \qquad \frac{p_x'(a)}{p(a)} \ = \ \sum_{k=1}^{r}\lambda_a^k(x).$$

Here $r = r(x)$, called the **rank of** x, is the number of non-zero a-eigenvalues, which are here listed first. The **nullity of** x is the number $n(x) = m - r(x)$ of a-eigenvalues which are zero. The elementary formula (3) is particularly useful for studying degenerate cases where the rank of x is less than m, i.e., where $p(x) = 0$.

As a consequence of the second part of Property (3) we have

(4) The a-trace of x, $\text{trace}_a(x) \ = \ \sum_{k=1}^{r}\lambda_a^k(x)$ is a linear functional in x.

A property, not used here and left as an exercise, is that for $b \in V_{\mathbf{C}}$ with $p(b) \neq 0$,

$$\lambda_b^k(a) \ = \ \frac{1}{\lambda_a^k(b)} \mod \pi_m.$$

The standard algebraic fact that the roots of a polynomial depend continuously on the coefficients (if the degree does not drop) will be used in several guises. For instance,

(5) The $\lambda_a^k(x) \mod \pi_m$ are continuous in $x \in V_{\mathbf{C}}$.

Hyperbolic Polynomials

Now suppose that p is a homogeneous real polynomial of degree m on a real vector space V.

Definition 2.1. Given $a \in V$, the polynomial p is a-**hyperbolic** if $p(a) > 0$ and the eigenvalues $\lambda_k(x)$, $k = 1, ..., m$ are real for all $x \in V$. That is, the polynomial $t \mapsto p(ta + x)$ has exactly m real roots for each $x \in V$. The **Gårding cone** $\Gamma \subset V$ (also denoted by Γ_a) is defined to be the set of x such that $\lambda_k(x) > 0$ for $k = 1, ..., m$.

Of course, the elementary properties (1) through (5) hold for p on V.

A Basic Inequality

Here we prove an elementary version of Gårding's inequality.

Lemma 2.2. *Suppose p is a-hyperbolic of degree m. If $x \in \Gamma_a$, then*

$$\left(\frac{p(x)}{p(a)}\right)^{\frac{1}{m}} \ \leq \ \frac{1}{m}\frac{p_x'(a)}{p(a)}, \qquad \text{with equality} \iff \lambda_1(x) = \cdots \lambda_m(x) \tag{2.2}$$

Proof. Each of the a-eigenvalues λ_k of x is > 0. Using the second part of (2.1) and of property (3), one sees that (2.2) is the classical inequality between the geometric and the arithmetic mean:

$$\left(\prod_{k=1}^{m}\lambda_k\right)^{\frac{1}{m}} \ \leq \ \frac{1}{m}\sum_{k=1}^{m}\lambda_k, \qquad \text{with equality} \iff \lambda_1(x) = \cdots \lambda_m(x) \qquad \blacksquare \tag{2.2$'$}$$

Remark 2.3. The inequality (2.2) has a geometric formulation. The condition that the graph of a function $f(x)$ lies below its tangent plane at a point $(a, f(a))$ can be written as:

$$f(x) \ \leq \ f(a) + f'_{x-a}(a) \ = \ f(a) - f'_a(a) + f'_x(a) \qquad \text{for } x \text{ near } a. \tag{2.3}$$

If f is homogeneous of degree one, recall that $f'_a(a) = f(a)$, so that condition (2.3) simplifies to

$$f(x) \ \leq \ f'_x(a) \qquad \text{for } x \text{ near } a. \tag{2.3$'$}$$

Take $f(x) = p(x)^{\frac{1}{m}}$ with $x \in \Gamma_a$. Then $f'_x(a) = \frac{1}{m} p(a)^{\frac{1}{m}-1} p'_x(a)$ and we see that (2.3)$'$ and (2.2) are the same inequality. This proves that

$$\text{If } p \text{ is } b \text{ hyperbolic for all } b \in \Gamma_a, \text{ then } p(x)^{\frac{1}{m}} \text{ is a concave function on } \Gamma_a. \tag{2.4}$$

In particular, in this case Γ_a is convex, since $-p(x)^{\frac{1}{m}}$ is convex and negative on Γ_a with boundary values zero.

Ordered Eigenvalues

Suppose p is an a-hyperbolic polynomial. Given a vector $\lambda \in \mathbf{R}^m$, let $\lambda^{\uparrow} \in \mathbf{R}^m$ denote the non-decreasing reordering of λ, i.e., $\lambda_1^{\uparrow} \leq \lambda_2^{\uparrow} \leq \cdots \leq \lambda_m^{\uparrow}$. The **ordered eigenvalue functions** are then defined to be $\lambda^{\uparrow}(x) = (\lambda_1^{\uparrow}(x), ..., \lambda_m^{\uparrow}(x))$. Note that $\lambda_{\min}(x) = \lambda_1^{\uparrow}(x)$ and $\lambda_{\max}(x) = \lambda_m^{\uparrow}(x)$. It is useful to rewrite some of the properties listed above in terms of the ordered a-eigenvalues.

Elementary Properties: For each $k = 1, ..., m$, we have

(1a) $\lambda_k^{\uparrow}(tx) \ = \ t\lambda_k^{\uparrow}(x)$, for $t \geq 0$,

(1b) $\lambda_k^{\uparrow}(-x) = -\lambda_{m-k+1}^{\uparrow}(x)$, and in particular, $\lambda_{\max}(x) = -\lambda_{\min}(-x)$

(2) $\lambda_k^{\uparrow}(ta + x) \ = \ t + \lambda_k^{\uparrow}(x)$ for all $t \in \mathbf{R}$,

(5) $\lambda_k^{\uparrow}(x)$ is continuous.

Definition 2.4. The sets

$$F_k \equiv \{\lambda_k^{\uparrow} \geq 0\}$$

(rather than ∂F_k) will be referred to as the **branches** of $\{p = 0\}$. The **principal** or **smallest branch** $F_1 = \{\lambda_{\min} \geq 0\}$ is of particular interest.

Note that each branch F_k is a cone with vertex at the origin by (1a).

Remark 2.5. Given a continuous function $g : \mathbf{R}^{N-1} \to \mathbf{R}$, we define **the associated graphing function** $G : \mathbf{R}^N \to \mathbf{R}$ by $G(t, x) = t - g(x)$. The closed set $F \equiv \{G \geq 0\}$

above the graph of g has interior $\mathrm{Int}F = \{G > 0\}$ and boundary $\partial F = \{G = 0\}$ equal to the graph of g. Moreover, $\mathrm{Int}F$ is a connected open set with $F = \overline{\mathrm{Int}F}$. Finally note:

$$F \text{ is convex} \iff g \text{ is convex} \iff G \text{ is concave} \tag{2.5}$$

These obvious facts can be applied as follows. Suppose that p is an a-hyperbolic polynomial and consider the ordered eigenvalues. Take $\mathbf{R}^{N-1}$ to be a hyperplane $W \subset V$ transverse to a.

$$\text{If } g \text{ is the restriction of } -\lambda_k^{\uparrow} \text{ to } W, \text{ then } G = \lambda_k^{\uparrow}$$

since for $x \in W$, we have $G(ta+x) = t - g(x) = t + \lambda_k^{\uparrow}(x) = \lambda_k^{\uparrow}(ta+x)$ where the last equality follows from Property (2). Consequently, as a special case of Remark 2.5 we have:

Proposition 2.6. *Suppose that p is a-hyperbolic. Then $\partial F_k = \{\lambda_k^{\uparrow} = 0\}$, and $\mathrm{Int}F_k = \{\lambda_k^{\uparrow} > 0\}$. Furthermore, $\mathrm{Int}F_k$ is a connected set satisfying $F_k = \overline{\mathrm{Int}F_k}$. In particular*

$$\Gamma = \mathrm{Int}F_1, \qquad F_1 = \overline{\Gamma}, \qquad \text{and}$$

$$\Gamma \text{ is convex} \iff \lambda_{\min} \text{ is concave} \iff \lambda_{\max} \text{ is convex} \tag{2.6}$$

Thus the Gårding cone is the interior of the principal branch. The last equivalence in (2.6) follows since $\lambda_{\max}(x) = -\lambda_{\min}(-x)$ by (1b).

Corollary 2.7. *The Gårding cone Γ_a is the connected component of $\{p \neq 0\}$ which contains a.*

Proof. Let Γ denote the connected component of $\{p \neq 0\}$ containing a. By Property (2a) we have $a \in \Gamma$, and since Γ is open and connected, $\Gamma_a \subset \Gamma$. Since the a-eigenvalues $\lambda_k(x)$ never vanish on Γ and equal 1 at $a \in \Gamma$, they are strictly positive on Γ, i.e., $\Gamma \subset \Gamma_a$. ∎

Remark 2.8. Setting $k^* = m - k + 1$, we have

$$x \in \mathrm{Int}F_k \qquad \iff \qquad x \text{ has at least } k^* \text{ strictly positive eigenvalues.}$$

A Real Analytic Arrangement of the Restricted Eigenvalues

Throughout this subsection we assume that p is an a-hyperbolic polynomial of degree m on V. The eigenvalues $\lambda_1(x), ..., \lambda_m(x)$ of $x \in V$ are well defined mod π_m. The continuity of λ_k as a function of x (Property (5)) cannot be improved to differentiability. However, by restricting the λ_k to the line through x in a direction $b \in \Gamma$, the one-variable functions $\lambda_1(x+tb), \ ... \ , \lambda_m(x+tb)$, of t defined mod π_m, can be arranged so that each function is a strictly increasing bi-real-analytic mapping of $\mathbf{R}$ onto $\mathbf{R}$. The functions

$$\lambda_1(x-tb), \quad ... \quad , \lambda_m(x-tb)$$

of t defined mod π_m will be referred to as the **restricted eigenvalue functions**.

THEOREM 2.9. *Suppose that p is an a-hyperbolic polynomial of degree m on V with Gårding cone Γ. Then p is also b-hyperbolic for each $b \in \Gamma$, and $\Gamma_b = \Gamma_a$. Moreover, for each fixed $x \in V$ the restricted eigenvalue functions of t*

$$\lambda_a^1(x - tb), \quad \dots \quad , \lambda_a^m(x - tb) \tag{2.7}$$

can be arranged so that each one is a real analytic, strictly decreasing function from $\mathbf{R}$ onto $\mathbf{R}$, and inverses are given by

$$\lambda_b^1(x - sa), \quad \dots \quad , \lambda_b^m(x - sa). \tag{2.8}$$

Consequently,

$$\frac{d}{dt}\lambda_a^k(x + tb) \ > \ 0 \ \text{ for all } t \in \mathbf{R} \ \text{ and } \ k = 1, ..., m. \tag{2.9}$$

Remark. Note that by combining Corollary 2.7 and Theorem 2.9 we have that for each connected component Γ of the set $\{p \neq 0\}$,

p is a-hyperbolic for some $a \in \Gamma$ $\quad \iff \quad$ *p is a-hyperbolic for all $a \in \Gamma$.*

Consequently it is reasonable to use the phrase "p is Γ-hyperbolic" where Γ is a connected component of $\{p \neq 0\}$.

Theorem 2.9 can be deduced from a local result, which holds without assuming $b \in \Gamma$.

Lemma 2.10. *Suppose $x, b \in V$. Then near any point $t_0 \in \mathbf{R}$ the restricted root functions*

$$r_a^1(tb + x), \ ..., r_a^m(tb + x) \tag{2.10}$$

can be arranged to be real-analytic in t. Therefore, by uniqueness of real analytic continuation the local arrangements (2.10) extend to a global arrangement of the restricted roots, which we label as

$$s_k(t) \ = \ r_a^k(tb + x) \qquad k = 1, ..., m. \tag{2.11}$$

Proof. Let s_1 denote one of the root functions evaluated at t_0. That is, (s_1, t_0) is a point on the complex algebraic curve

$$C \ \equiv \ \{(s, t) : p(sa + tb + x) = 0\}$$

in $\mathbf{C}^2$. Choose a local irreducible component C_1 of C at (s_1, t_0), along with a local uniformizing parameter z. This means that we have holomorphic functions

$$s(z) \ = \ s_1 + z^p g(z) \qquad \text{and} \qquad t(z) \ = \ t_0 + z^q h(z), \quad \text{for } |z| < \delta, \tag{2.12}$$

with $g(0) \neq 0$, $h(0) \neq 0$, such that $z \mapsto (s(z), t(z))$ is a homeomorphism of a neighborhood of $0 \in \mathbf{C}$ onto a neighborhood of (s_1, t_0) in C_1. By extracting a qth root of $h(z)$ we may assume that $h(z) \equiv 1$, i.e.,

$$t(z) \ = \ t_0 + z^q.$$

It will suffice to show that $q = 1$, for then $s(z)$ is a root of $p(sa + (t_0 + z)b + x)$, and, as such, must be real if z is real. Setting

$$r_a^1(tb + x) \ \equiv \ s(t - t_0)$$

we obtain a local real analytic restricted root function. Repeating this construction for each of the m real points $(s_1, t_0), ..., (s_n, t_0) \in C$ (counting multiplicities) will complete the proof.

We now show that $q = 1$. Since $s(z)$ is a root of $s \mapsto p(sa + t(z)b + x)$ and p is a-hyperbolic,

$$z^q \ \text{ real } \quad \Rightarrow \quad s(z) \ \text{ real.} \tag{2.13}$$

In particular, z real $\quad \Rightarrow \quad s(z)$ real, so that the power series

$$s(z) = \sum_{k=0}^{\infty} a_k z^k$$

for $s(z)$ has real coefficients a_k. Set $\omega \equiv e^{\frac{\pi i}{q}}$ and $z = r\omega$ with r real. Then ω^k is not real unless k is a multiple of q. Now z^q is real and hence $\sum_{k=0}^{\infty} a_k \omega^k r^k$ is real by (2.13). It follows that $a_k = 0$ unless k is a multiple of q. Thus $s(z) = s_1 + \sum_{k=1}^{\infty} b_k z^{kq} = s_1 + f(z^q)$. Since $z \mapsto (f(z^q), z^q)) = (s(z) - s_1, t(z) - t_0)$ is one-to-one, this proves that $q = 1$. ■

Proof of Theorem 2.9. Now we will use the fact that $b \in \Gamma$. By Property (1)

$$r_a^k(tb + x) \ = \ t r_a^k \left(b + \frac{x}{t} \right) \quad \text{mod } \pi_m \qquad \text{for } t \neq 0. \tag{2.14}$$

For $|t| >> 0$ sufficiently large,

$$r_a^k \left(b + \frac{x}{t} \right) \ < \ 0. \tag{2.15}$$

This follows from the continuity of the eigenvalues (5) and the fact that $r_a^k(b) < 0$, i.e., $b \in \Gamma$. Consequently,

$$\lim_{t \to \pm\infty} r_a^k(tb + x) \ = \ \mp\infty. \tag{2.16}$$

In particular, none of the functions $s_k(t)$ are constant, and each maps onto all of $\mathbf{R}$.

Suppose that $\varphi(t)$ is one of the function $s_k(t)$ which is repeated ℓ-times. If s_0 is not a critical value of φ, then

$$\text{each point } t_j \in \varphi^{-1}(s_0) \text{ is a root of } t \mapsto p(s_0 a + tb + x) \text{ of multiplicity } \ell. \tag{2.17}$$

This follows from the fact that the functions $s - \varphi(t)$ and $t - \varphi^{-1}(s)$ vanish to first order on the same real curve in $\mathbf{R}^2$ near (s_0, t_0) and therefore differ by a smooth (in fact, real analytic) factor $\alpha(s,t)$ near (s_0,t_0) with $\alpha(s_0,t_0) \neq 0$.

Let S_0 denote the set of points which are not critical values for any of the functions $s_k(t)$. If $s_0 \in S_0$, then $s_k^{-1}(s_0)$ can have at most one point t_k. Otherwise by (2.17) the polynomial $t \mapsto p(s_0 a + tb + x)$ would have more than m roots. Hence each s_k is one-to-one outside its critical points. This is enough to conclude that each s_k is a homeomorphism from $\mathbf{R}$ onto $\mathbf{R}$. Therefore, each $s_k(t)$ defined by (2.11) is a strictly decreasing homeomorphism from $\mathbf{R}$ onto $\mathbf{R}$.

Moreover, for $s_0 \in S_0$ the points $t_1 = s_1^{-1}(s_0), ..., t_m = s_m^{-1}(s_0)$, listed to multiplicity, provide m (real) roots of $t \mapsto p(s_0 a + tb + x)$. This proves that

$$r_b^k(sa + x) \;=\; s_k^{-1}(s) \tag{2.18}$$

for $s \in S_0$. By continuity of the roots (2.18) remains true for critical values $s \in \mathbf{R}$.

In particular, the roots

$$r_b^1(x) \;=\; s_1^{-1}(0),\; \ldots\;, r_b^m(x) \;=\; s_m^{-1}(0)$$

of $p(tb + x)$ are all real, proving that the polynomial p is b-hyperbolic. By (2.16) the monotone functions $s_k(t) = r_a^k(tb + x)$ must be non-increasing homeomorphisms of $\mathbf{R}$ onto $\mathbf{R}$. Since p is b-hyperbolic, the results established for r_a apply to r_b. In particular, by (2.18) each of the inverses s_k^{-1} is real analytic. This proves that for some arrangement of the restricted roots

$$s \;=\; r_a^k(x + tb) \qquad \text{and} \qquad t \;=\; r_b^k(x + ta) \quad k = 1, ..., m \tag{2.19}$$

are inverses of each other. Also,

$$\begin{gathered}\text{Each } r_a^k(x+tb)\text{, as a function of } t\text{, is a}\\ \text{strictly decreasing bi-real-analytic map from } \mathbf{R} \text{ onto } \mathbf{R}.\end{gathered} \tag{2.20}$$

Finally note that (2.19) and (2.20) remain true with $r_a^k(x+tb)$ replaced by $\lambda_a^k(x-tb) = -r_a^k(x - tb)$, $k = 1, ..., m$. ∎

Example 2.11. (Comparing Real Analytic and Ordered Eigenvalues). The polynomial $p(x) = x_1^2 - x_2^2 - x_3^2$ on $\mathbf{R}^3$ is the prototype of all hyperbolic polynomials. It is $(1,0,0)$-hyperbolic with Gärding cone equal to the light cone $\Gamma = \{p(x) > 0\} \cap \{x_1 > 0\}$. The ordered eigenvalue functions are:

$$\lambda_{\min}(x) \;=\; x_1 - \sqrt{x_2^2 + x_3^2} \quad \leq \quad x_1 + \sqrt{x_2^2 + x_3^2} \;=\; \lambda_{\max}(x).$$

Choose $b \in \Gamma$ of the form $b_1 > b_2 > 0 = b_3$. The ordered arrangement of the restricted eigenvalue functions is

$$\begin{aligned}\lambda_{\min}(x + tb) \;&=\; x_1 + tb_1 - \sqrt{(x_2 + tb_2)^2 + x_3^2}\\ \lambda_{\max}(x + tb) \;&=\; x_1 + tb_1 + \sqrt{(x_2 + tb_2)^2 + x_3^2}\end{aligned}$$

If $x_3 \neq 0$, then both of these functions are real analytic in t, and hence the real analytic arrangement agrees with the ordered arrangement. If $x_3 = 0$, the ordered arrangement is:

$$\lambda_{\min}(x_t b) \ = \ x_1 + tb_1 - |x_2 + tb_2| \quad \leq \quad x_1 + tb_1 + |x_2 + tb_2| \ = \ \lambda_{\max}(x + tb).$$

while the real analytic arrangement is

$$\begin{aligned} \lambda^1(x+tb) &= x_1 + tb_1 - (x_2 + tb_2) \\ \lambda^2(x+tb) &= x_1 + tb_1 + (x_2 + tb_2). \end{aligned}$$

It is interesting to note the following asymptotics:

$$\text{for } \ t >> 0, \qquad \lambda_{\min}(x+tb) \ \sim \ t\lambda_{\min}(b) \qquad \text{and} \qquad \lambda_{\max}(x+tb) \ \sim \ t\lambda_{\max}(b)$$

$$\text{while for } \ t << 0, \qquad \lambda_{\min}(x+tb) \ \sim \ t\lambda_{\max}(b) \qquad \text{and} \qquad \lambda_{\max}(x+tb) \ \sim \ t\lambda_{\min}(b).$$

which shows that (2.14) only holds mod π_m.

Since the a-eigenvalues are all real, the rank of $x \in V$ can be refined into the sum of the **plus rank**, denoted $r^+(x)$, and the **minus rank**, denoted $r^-(x)$.

THEOREM 2.12. *The quantities $r^\pm(x)$ and $n(x)$ are all independent of the direction $a \in \Gamma$ chosen to compute the eigenvalues of x. In particular, all of the branches $F_k \equiv \{\lambda_k^\uparrow \geq 0\}$ of $\{p = 0\}$ are independent of the point $a \in \Gamma$ chosen to compute the eigenvalues.*

Proof. Choose $a, b \in \Gamma$ and assume $r_a^+(x) = k^*$ (cf. Remark 2.8). Let $\gamma_1, ..., \gamma_m$ denote the m curves in the (t, s)-plane defined by (2.19). Exactly k^* of these curves cross the negative s-axis. By (2.20) this implies that at exactly k^* of these curves cross the negative t-axis, i.e., $r_b^+(x) = k^*$. The proof is similar for $r^-(x)$ and $n(x)$. ■

Convexity

Corollary 2.13.

(a) *The Gårding cone Γ is convex.*

(b) *$\lambda_{\min}(x)$ is concave.*

(c) *$\lambda_{\max}(x)$ is convex.*

(d) *$p(x)^{\frac{1}{m}}$ is concave on Γ.*

(e) *$-\log p(x)$ is convex on Γ.*

Proof. The conditions (a), (b) and (c) are equivalent by Proposition 2.6. Condition (d) follows from (2.4) in Remark 2.3. Since $-p(x)^{\frac{1}{m}}$ is convex on Γ, negative on Γ, and zero on $\partial\Gamma$, condition (a) follows. Finally, recall that if φ is increasing and convex, then $\varphi(f(x))$ is convex if f is convex. To prove (e) take $\varphi(t) = -\log(-t)$ for $t < 0$ and note that $\varphi(-p(x)^{\frac{1}{m}}) = -\frac{1}{m}\log p(x)$. ■

Second Proof of (a). Suppose $x, y \in \Gamma$, i.e., $\lambda_{\min}(x) > 0$ and $\lambda_{\min}(y) > 0$. We must show that $\lambda_{\min}(sx + ty) > 0$ for all $s, t > 0$. By Theorem 2.9, $\Gamma_x = \Gamma_a$ so that we can assume the eigenvalues are computed using $a = x$. Then by Properties (1a) and (2)

$$\lambda_{\min}(sx+ty) \ = \ s+t\lambda_{\min}(y) \ > \ 0. \qquad ■$$

Remark 2.14. For $x \in W$, a hyperplane transverse to a, we may apply Remark 2.5 to $g(x) \equiv -\lambda_{\min}(x)$, $x \in W$. Since g is convex, we prefer the following norm notation:

$$\|x\|^+ = \ -\lambda_{\min}(x) \qquad \text{and} \qquad \|x\|^- = \ \|-x\|^+ \ = \ \lambda_{\max}(x) \qquad \text{for } x \in W$$

where the last equality follows from Property (1b). By Remark 2.5 we have

$$\begin{aligned} \partial\Gamma \ &= \ \{\|x\|^+ a + x : x \in W\} \ \text{ is the graph of } \ \|\bullet\|^+, \ \text{ and} \\ -\partial\Gamma \ &= \ \{-\|x\|^+ a + x : x \in W\} \ \text{ is the graph of } \ -\|\bullet\|^- \end{aligned}$$

over the hyperplane W. The convexity of Γ and 1(a) imply that

$$\|x + y\|^\pm \ \le \ \|x\|^\pm + \|y\|^\pm \qquad \text{and} \qquad \|tx\|^\pm \ = \ t\|x\|^\pm \ \text{ for all } t > 0.$$

Monotonicity

THEOREM 2.15. *The ordered eigenvalues are strictly Γ-monotone, that is*

$$\lambda_k^\uparrow(x + b) \ > \ \lambda_k^\uparrow(x) \ \text{ for all } x \in V, b \in \Gamma.$$

Proof. Under the real analytic arrangement in Theorem 2.9 each of the restricted eigenvalue functions $\lambda_k(tb + x)$ is strictly increasing in t. Hence the ordered eigenfunctions

$$\lambda_1^\uparrow(tb + x) \ \le \ \cdots \ \le \ \lambda_m^\uparrow(tb + x)$$

are also strictly increasing. Now compare $t = 0$ with $t = 1$. ■

Definition 2.16. We say that a subset $F \subset V$ is **Γ-monotone** if

$$F + \Gamma \ \subset \ F \quad \text{or equivalently} \quad F + \Gamma \ \subset \ \mathrm{Int}F. \tag{2.21}$$

The equivalence follows since $F + \Gamma$ is open.

Corollary 2.17. *Each branch $F_k \equiv \{\lambda_k^\uparrow \ge 0\}$ of $\{p = 0\}$ is Γ-monotone.*

Constructing Hyperbolic Polynomials

In this subsection we describe some of the important ways of constructing new hyperbolic polynomials from a given hyperbolic polynomial.

I. Factors and Products. Suppose $p(x)$ and $q(x)$ be real homogeneous polynomials with $p(a), q(a) > 0$ for fixed $a \in V$. Then the following is obvious.

Proposition 2.18.

$$p(x)q(x) \text{ is } a\text{-hyperbolic} \quad \iff \quad p(x) \text{ and } q(x) \text{ are each } a\text{-hyperbolic.}$$

$$\text{in which case} \quad \Gamma(pq) = \Gamma(p) \cap \Gamma(q).$$

II. Restriction. Suppose that W is a vector subspace of V and that $a \in W$. Then the following is also obvious.

Proposition 2.19.

$$\text{If } p(x) \text{ is } a\text{-hyperbolic, then } p\big|_W \text{ is } b\text{-hyperbolic for each } b \in \Gamma(p) \cap W$$

$$\text{in which case} \quad \Gamma\left(p\big|_W\right) = \Gamma(p) \cap W.$$

III. Closure. Suppose $\{p_j\}$ is a sequence of a-hyperbolic polynomials of degree m converging to a polynomial p.

Proposition 2.20. *If $p(a) \neq 0$, then p is a-hyperbolic.*

Proof. Since $p(a) \neq 0$, the a-eigenvalues of $x \in V$ are defined as complex numbers by (2.1). By continuity of the eigenvalues mod π_m, the limiting a-eigenvalues of p must be real. Obviously $p(a) > 0$ and degree of p equals m. ■

IV. Derivatives. Let p' or p'_b denote the directional derivative of p in the direction b, i.e.,

$$p'(x) \;=\; \frac{d}{ds} p(x + sb)\big|_{s=0} \tag{2.22}$$

Proposition 2.21. *Suppose p is a-hyperbolic of degree > 1. If $b \in \Gamma(p)$, then p'_b is a-hyperbolic. That is, $\Gamma(p) \subset \Gamma(p'_b)$. Moreover, for any x the ordered b-eigenvalues for p'_b are interspersed between the ordered b-eigenvalues for p.*

Proof. Note that by homogeneity $p'_b(b) = mp(b) > 0$. Also $p'_b(tb + x) = \frac{d}{dt} p(tb + x)$, and therefore by Rolle's Theorem the b-eigenvalues of x for p' are interspersed between the b-eigenvalues of x for p. Namely, we have

$$\lambda_1(x) \le \lambda'_1(x) \le \lambda_2(x) \le \cdots \le \lambda_{m-1}(x) \le \lambda'_{m-1}(x) \le \lambda_m(x). \tag{2.23}$$

More sharply stated,

$$\begin{array}{lll} \text{(a)} & \lambda_k(x) < \lambda_{k+1}(x) \quad \Rightarrow \quad \lambda_k(x) < \lambda'_k(x) < \lambda_{k+1}(x), \quad \text{and} & (2.23)' \\ \text{(b)} & \lambda_k(x) \text{ has multiplicity } \ell \quad \Rightarrow \quad \lambda'_k(x) \text{ has multiplicity } \ell - 1. & \end{array}$$

Thus the b-eigenvalues of x for p'_b are all real, proving that p'_b is b-hyperbolic. In particular, by (2.23), if $x \in \Gamma(p)$ (i.e., $\lambda_k(x) > 0$, $1 \le k \le m$), then $\lambda'_k(x) > 0$, for $1 \le k \le m-1$ (i.e., $x \in \Gamma(p')$). Thus $\Gamma(p) \subset \Gamma(p'_b)$ proving that p'_b is a-hyperbolic. ∎

By induction we have

Corollary 2.22. *Suppose that p is Γ-hyperbolic and $b_1, ..., b_k \in \Gamma$. Then the k-fold directional derivative $p^{(k)}_{b_1,...,b_k}$ in the directions $b_1, ..., b_k$ is Γ-hyperbolic, i.e., $\Gamma(p) \subset \Gamma(p^{(k)})$.*

V. Elementary Symmetric Functions.

Let σ_k denote the kth elementary symmetric function of m variables. If p is a homogeneous polynomial of degree m with $p(a) \neq 0$, then by (2.1)

$$p(sa+x) \;=\; p(a)\left[s^m + \sigma_1(\lambda(x))\,s^{m-1} + \cdots + \sigma_m(\lambda(x))\right]. \tag{2.24}$$

Taking directional derivatives in the direction a we have

$$p^{(k)}(sa+x) \;=\; p(a)\left[\alpha_0 s^{m-k} + \alpha_1\sigma_1(\lambda(x))s^{m-k-1} + \cdots + \alpha_{m-k}\sigma_{m-k}(\lambda(x))\right] \tag{2.25}$$

with $\alpha_j = k!\binom{m-j}{k}$. Setting $s = 0$ proves that the kth directional derivative of p in the direction a is (up to a constant) the same as the $(m-k)$th elementary symmetric function of the a-eigenvalue functions, i.e.,

$$p^{(k)}(x) \;=\; k!\,p(a)\sigma_{m-k}(\lambda(x)). \tag{2.26}$$

In particular, this proves that

$$\sigma_j(x) \;\equiv\; \sigma_j(\lambda(x)) \quad \text{defines a homogeneous polynomial of degree } \; j \tag{2.27}$$

and that its derivative in the direction a is $(m-j+1)$ times $\sigma_{j-1}(\lambda(x))$, i.e.,

$$\sigma'_j(x) \;=\; (m-j+1)\sigma_{j-1}(x) \tag{2.28}$$

By Corollary 2.22 this proves

Corollary 2.23. *Suppose p is a-hyperbolic. Then $\sigma_k(x) = \sigma_k(\lambda(x))$ defines an a-hyperbolic polynomial of degree k, and $\Gamma(\sigma_{k+1}) \subset \Gamma(\sigma_k)$ for $k = 1, ..., m$.*

Let $\lambda^{(k)}_1(x), ..., \lambda^{(k)}_{m-k}(x) \bmod \pi_{m-k}$ denote the a-eigenvalue functions for the a-hyperbolic polynomial $p^{(k)}(x)$. The equation (2.25) for $p^{(k)}(sa+x)$ proves that the jth elementary symmetric functions

$$\sigma_j(\lambda^{(k)}(x)) \;\text{ and }\; \sigma_j(\lambda(x)) \;\text{ are equal modulo a positive scale.} \tag{2.29}$$

VI. Hyperbolic Polynomials Defined Universally. Set $e = (1, ..., 1) \in \mathbf{R}^m$. Suppose that Q is a symmetric homogeneous polynomial of degree N on $\mathbf{R}^m$. If p

is a homogeneous polynomial of degree m on V with a-eigenvalue functions $\lambda(x) = (\lambda_1(x), ..., \lambda_m(x))$, then

$$q(x) \;=\; Q(\lambda(x)) \tag{2.30}$$

defines a homogeneous polynomial of degree N on V (since Q is a polynomial in the σ_ℓ's and each $\sigma_\ell(\lambda(x))$ is a polynomial). If Q is e-hyperbolic on $\mathbf{R}^m$, let $\Lambda_1(\lambda), ..., \Lambda_N(\lambda)$ denote the e-eigenvalue functions for Q.

Proposition 2.24. *Suppose Q is a symmetric e-hyperbolic polynomial of degree N on $\mathbf{R}^m$. Then Q universally determines an a-hyperbolic polynomial $q(x)$ defined by (2.30) on any vector space V equipped with an a-hyperbolic polynomial $p(x)$ of degree m. Moreover, the a-eigenvalue functions for q are $\Lambda_1(\lambda(x)), ..., \Lambda_N(\lambda(x))$.*

Proof. First note that $q(a) = Q(\lambda(a)) = Q(e) > 0$. By (2.1) $Q(te+\lambda) = Q(e)\prod_{j=1}^N (t+\Lambda_j(\lambda))$. Property $(2)'$:

$$\lambda(ta + x) = \; te + \lambda(x)$$

implies that $q(ta + x) = Q(\lambda(ta + x)) = Q(te + \lambda(x)) = Q(e)\prod_{j=1}^N (t + \Lambda_j(\lambda(x)))$. This proves that $\Lambda_1(\lambda(x)), ..., \Lambda_N(\lambda(x))$, which are real, are the a-eigenvalues of the polynomial $q(x)$. ∎

Remark 2.25. Examples of e-hyperbolic polynomials Q of degree N on $\mathbf{R}^m$ abound. Take any a-hyperbolic polynomial R of degree N on a vector space W. Choose vectors $b_1, ..., b_m \in W$ with $b \equiv b_1 + \cdots + b_m \in \Gamma(p)$. Then $Q(\lambda) = R(\lambda_1 b_1 + \cdots + \lambda_m b_m)$ is e-hyperbolic of degree N on $\mathbf{R}^m$. To use Q in Proposition 2.24 it should first be symmetrized (thereby increasing its degree).

Special cases of this construction V are of particular importance. (The trivial case $Q(\lambda) = \lambda_1 \cdots \lambda_m$ induces the polynomial $q = p$.) To begin exploring other cases note the following obvious

Fact 2.26. *Given $w \in \mathbf{R}^m$ the linear polynomial $\ell(\lambda) \equiv w \cdot \lambda$ is e-hyperbolic if and only if $\ell(e) = w \cdot e = w_1 + \cdots + w_m > 0$, in which case the single e-eigenvalue function for ℓ is computed to be*

$$\Lambda(\lambda) \;=\; \frac{w \cdot \lambda}{w \cdot e} \tag{2.31}$$

This polynomial is not symmetric. Symmetrizing $\ell(\lambda)$ yields $Q(\lambda) = \prod_{\sigma\in\pi_m} (\sigma w) \cdot \lambda$ and we obtain the following important special case of Proposition 2.24.

Proposition 2.27. *Suppose p is a-hyperbolic of degree m on V with a-eigenvalue functions $\lambda(x) = (\lambda_1(x), ..., \lambda_m(x)) \bmod \pi_m$. Fix $w \in \mathbf{R}^m$ with $w \cdot e > 0$. Then*

$$q(x) \;=\; \prod_{\sigma\in\pi_m} (\sigma w) \cdot \lambda(x)$$

defines an a-hyperbolic polynomial on V of degree $m!$ with a-eigenvalue functions

$$\frac{(\sigma w) \cdot \lambda(x)}{w \cdot e}, \qquad \sigma \in \pi_m.$$

The convexity of the largest eigenvalue function generalizes as follows.

Corollary 2.28. ([BGLS]). *Each non-decreaasing linear combination of the ordered eigenvalue functions is convex:*

$$w \in \mathbf{R}^m_{\uparrow} \qquad \Rightarrow \qquad w \cdot \lambda^{\uparrow}(x) \text{ is convex on } V.$$

That is, for $x, y \in V$ and $0 \le s \le 1$ one has

$$w \cdot \lambda^{\uparrow}(sx + (1-s)y) \ \le\ sw \cdot \lambda^{\uparrow}(x) + (1-s)w \cdot \lambda^{\uparrow}(y) \ =\ w \cdot \left(s\lambda^{\uparrow}(x) + (1-s)\lambda^{\uparrow}(y)\right).$$

Proof. Recall Property (4) that $e \cdot \lambda^{\uparrow}(x) = \lambda_1(x) + \cdots + \lambda_m(x)$ is a linear function of x. Therefore,

$$w \cdot \lambda^{\uparrow}(x) \ \text{ is convex} \qquad \Longleftrightarrow \qquad (w + te) \cdot \lambda^{\uparrow}(x) \ \text{ is convex},$$

and hence we may assume that $w \cdot e > 0$. Then it is easy to see that $w \cdot \lambda^{\uparrow}(x)/w \cdot e$ is the largest eigenvalue function for $q(x)$ defined in Proposition 2.27. Therefore, $w \cdot \lambda^{\uparrow}(x)$ is convex by Corollary 2.13c. ∎

Two particularly interesting choices for the $w \in \mathbf{R}^m$ in Proposition 2.27 are

$$w \ = \ (0, ..., 0, 1, ..., 1) \qquad \text{with } k \text{ ones}, \quad \text{and} \tag{2.32}$$

$$w \ = \ (\delta, ..., \delta, \delta + 1) \qquad \text{with } \delta > 0. \tag{2.33}$$

The first choice yields

Proposition 2.29. (k-Fold Sums). *Suppose p is a-hyperbolic of degree m on V. For each $1 \le k \le m$*

$$q(x) \ = \ {\prod_{|I|=k}}' \left(\lambda_{i_1}(x) + \cdots + \lambda_{i_k}(x)\right)$$

defines an a-hyperbolic polynomial of degree $\binom{m}{k}$ on V whose eigenvalue functions are the k-fold sums

$$\frac{1}{k}\left(\lambda_{i_1}(x) + \cdots + \lambda_{i_k}(x)\right) \qquad i_1 < \cdots < i_k.$$

The second choice yields

Proposition 2.30. ("δ-Uniformly Elliptic"). *Suppose p is a-hyperbolic of degree m on V. For each $\delta > 0$*

$$q_\delta(x) \ = \ \prod_{k=1}^{m} \left(\lambda_k(x) + \delta \mathrm{tr}\lambda(x)\right)$$

defines an a-hyperbolic polynomial of degree m on V whose eigenvalue functions are $\frac{1}{m\delta+1}$ times

$$\lambda_k(x) + \delta \mathrm{tr}\lambda(x) \qquad k = 1, ..., m.$$

The Gårding cones Γ_δ for q_δ form a conically-fundamental neighborhood system of the Gårding cone $\overline{\Gamma}$ for p.

The Edge and the Polar

To begin we review some standard results for any non-empty open convex cone $\Gamma \subset V$ with vertex at the origin. Set

$$\Gamma^+ \;=\; \overline{\Gamma} \qquad \text{and} \qquad \Gamma_+ \;=\; \overline{\Gamma}^0 \tag{2.34}$$

where the **polar** A^0 of a closed convex cone A with vertex at the origin is defined by

$$A^0 \;=\; \{y \in V : (x, y) \geq 0 \ \forall\, x \in A\}.$$

Recall the Bipolar Theorem: $(A^0)^0 = A$. Note that $\operatorname{span}\Gamma^+ = V$ since Γ is open and non-empty.

The **edge** of Γ^+ (or Γ) can be defined as

$$E \;=\; \{x \in V : x + \Gamma^+ = \Gamma^+\} \tag{2.35}$$

One sees easily that E is a convex cone contained in Γ^+, and since $x \in E \iff -x \in E$ ($x + \Gamma^+ = \Gamma^+ \iff \Gamma^+ = -x + \Gamma^+$), the edge E must be a vector subspace of V. Moreover,

$$E \text{ contains any linear subspace of } \Gamma^+. \tag{2.36}$$

The edge can also be defined by

$$E \;=\; \Gamma^+ \cap (-\Gamma^+). \tag{2.37}$$

To see this recall that with E defined by (2.35), $E \subset \Gamma^+ \cap (-\Gamma^+)$ was noted above. If $x \in \Gamma^+ \cap (-\Gamma^+)$, then the line through x is a subset of Γ^+. By (2.36) this line is a subset of the edge E defined by (2.35).

Again for a general open convex cone Γ we have:

$$E \text{ and } \operatorname{span}\Gamma_+ \text{ are polar cones.} \tag{2.38}$$

Proof. Note that $E \subset \Gamma^+ \;\Rightarrow\; \Gamma_+ \subset E^0$ and hence $\operatorname{span}\Gamma_+ \subset E^0$. Also we have $\Gamma_+ \subset \operatorname{span}\Gamma_+ \;\Rightarrow\; (\operatorname{span}\Gamma_+)^0 \subset \Gamma^+$. The polar of a vector space is a vector space, hence by (2.36) we have $(\operatorname{span}\Gamma_+)^0 \subset E$. Now apply the Bipolar Theorem. ∎

If $E = \{0\}$, then Γ^+ is said to be **complete**. This is equivalent to $V = \operatorname{span}\Gamma_+$ or $\Gamma_+ = \overline{\operatorname{Int}\Gamma_+}$ by (2.38).

Now we assume that Γ is the Gårding cone of an a-hyperbolic polynomial p. Then the edge E of $\Gamma(p)$ can be described in several ways.

THEOREM 2.31. *Suppose p is an a-hyperbolic polynomial with Gårding cone Γ.*

(a) The edge E of Γ equals the null set of p, i.e.,

$$E \;=\; \{x \in V : \lambda_1(x) = \cdots = \lambda_m(x) = 0\}. \tag{2.39}$$

(b) The edge E of Γ equals the linearity of p, i.e.,

$$E \;=\; \{x \in V : p(b+tx) = p(b) \;\; \forall\, t \in \mathbf{R}, b \in V\}. \tag{2.40}$$

(c) $$x \in E \quad \iff \quad p'_x \;\equiv\; 0.$$

Proof of (a). By 1(b) the negative of the Gårding cone, $-\Gamma^+$, is the set of points $x \in V$ whose a-eigenvalues are all ≤ 0. Thus $E = \Gamma^+ \cap (-\Gamma^+)$ is the null set of p.

Proof of (b) and (c). Apply Property (3), and then for (b) note that $p(b+tx) - p(b)$ vanishes for all $b \in \Gamma$ if and only if it vanishes for all $b \in V$. ∎

This has several important consequences. First, taking derivatives does not change the edge.

Corollary 2.32. *Suppose p is a-hyperbolic of degree $m \geq 2$. Then*

$$\mathrm{Edge}(p'_a) \;=\; \mathrm{Edge}(p).$$

Proof. By (2.23) $\mathrm{Edge}(p'_a) \supset \mathrm{Edge}(p)$. If $\lambda'_1(x) = \cdots = \lambda'_{m-1}(x) = 0$, then all $\lambda_k(x)$ vanish by the sharp application $(2.23)'$ of Rolle's Theorem given in the proof of Proposition 2.21. ∎

Second, the edge of the restriction is the restriction of the edge.

Corollary 2.33. *Suppose p is a-hyperbolic polynomial on V, and that W is a subspace of V. If $\Gamma \cap W \neq \emptyset$, then*

$$\mathrm{Edge}\left(p\big|_W\right) \;=\; \mathrm{Edge}(p) \cap W.$$

3. Γ-Monotone Sets

The Dirichlet problem will be studied (and solved) for equations determined by subsets F of $V = \mathrm{Sym}^2(\mathbf{R}^n)$ which are Γ-monotone. In this section we discuss these subsets. Recall that by Corollary 2.17 they include all branches of $\{p = 0\}$. Throughout this section we assume that p is a given a-hyperbolic polynomial, Γ is the associated Gårding cone, and $\lambda_1(x), ..., \lambda_m(x)$ are the a-eigenvalue functions.

The Structure Theorem

Recall (4) that

$$\mathrm{trace}(x) \;=\; \lambda_1(x) + \cdots + \lambda_m(x). \tag{3.1}$$

(i.e., the elementary symmetric function $\sigma_1(x)$) is a linear function of x. Define the hyperplane

$$V_0 \;\equiv\; \{x \in V : \mathrm{trace}(x) = 0\}.$$

We adopt the notation in Remark 2.14 with $W = V_0$.

Definition 3.1. A function $f : V_0 \to \mathbf{R}$ is **$\|\cdot\|^{\pm}$-Lipschitz** if

$$-\|y\|^{-} \ \leq \ f(x+y) - f(x) \ \leq \ \|y\|^{+} \ \text{ for all } x, y \in V_0. \tag{3.2}$$

THEOREM 3.2. (The Structure Theorem). *Suppose F is a closed subset of V with $\emptyset \neq F \neq V$. If F is Γ-monotone, then there exists a $\|\cdot\|^{\pm}$-Lipschitz function $f : V_0 \to \mathbf{R}$ such that*

$$\partial F \ = \ \{f(x)a + x : x \in V_0\} \ \text{ is the graph of } f, \tag{3.3}$$

and

$$F \ = \ \{ta + x : t \geq f(x) \text{ and } x \in V_0\} \ \text{ is the upper graph of } f. \tag{3.4}$$

Conversely, each such f defines a Γ-monotone set F via (3.4).

Corollary 3.3. *Suppose F is a closed subset of V which is Γ-monotone. Then* $\mathrm{Int}F$ *is a connected open set with $F = \overline{\mathrm{Int}F}$. Moreover, F is uniquely determined by its boundary since $F = \partial F + \Gamma$.*

Example 3.4. (Positive-Orphant-Monotone). Take $V \equiv \mathbf{R}^m$ and $p(\lambda) \equiv \lambda_1 \cdots \lambda_m$ where $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbf{R}^m$. Set $e = (1, ..., 1)$. Then p is e-hyperbolic with eigenvalues $\lambda_1, ..., \lambda_m$ at λ. The closure $\overline{\Gamma}$ of the Gårding cone is the closed positive orphant $\mathbf{R}^m_+$. The traceless hyperplane V_0 is the orthogonal complement of e while $\|\lambda\|^{+} = -\lambda_{\min}$ and $\|\lambda\|^{-} = \lambda_{\max}$. If E is a closed subset of $\mathbf{R}^m$ with $\emptyset \neq E \neq \mathbf{R}^m$, and E is $\mathbf{R}^m_+$-monotone, then there exists $f : V_0 \to \mathbf{R}$ which is $\|\cdot\|^{\pm}$-Lipschitz continuous such that ∂E is the graph of f over V_0 and E is the upper graph of f over V_0.

The next example is the key example for studying the Dirichlet problem.

Example 3.5. (The Structure of $\mathcal{P}$-Monotone Sets). Take $V = \mathrm{Sym}^2(\mathbf{R}^n)$ and $p(A) = \det A$. Then p is I-hyperbolic with eigenvalues exactly the standard eigenvalues of A. The closed Gårding cone $\overline{\Gamma} = \mathcal{P} \equiv \{A \geq 0\}$. Consider the hyperplane

$$\mathrm{Sym}^2_0(\mathbf{R}^n) \equiv \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \mathrm{tr}A = 0\}.$$

If F is a non-trivial closed subset of $\mathrm{Sym}^2(\mathbf{R}^n)$ ($\emptyset \neq F \neq \mathrm{Sym}^2(\mathbf{R}^n)$) which is $\mathcal{P}$-monotone, then there exists a function $f : \mathrm{Sym}^2_0(\mathbf{R}^n) \to \mathbf{R}$, which is $\|\cdot\|^{\pm}$-Lipschitz, such that

$$F \ = \ \{tI + A : A \in \mathrm{Sym}^2_0(\mathbf{R}^n) \text{ and } t \geq f(A)\} \ \text{ is the upper graph of } f. \tag{3.5}$$

Moreover, the properties of F enumerated in Corollary 3.3 hold.

Proof of Theorem 3.2. The monotonicity hypothesis $F + \Gamma \subset F$ implies that the set $\ell_x = \{t \in \mathbf{R} : ta + x \in F\}$ is either empty, all of $\mathbf{R}$, or of the form $[c, \infty)$ with $c \in \mathbf{R}$. If $\ell_x = \emptyset$, then $F = \emptyset$ since the existence of $y \in F$ implies that $x + ta = y + (x - y + ta) \in F$ if t is chosen large enough so that $x - y + ta \in \Gamma$. If $\ell_x = \mathbf{R}$, then $F = V$ since for each

$y \in V$, if $t << 0$, then $y - x - ta \in \Gamma$ which implies that $y = x + ta + (y - x - ta) \in F + \Gamma \subset F$.

Since V is neither empty nor all of V, there is a well-defined function $f : V_0 \to \mathbf{R}$ such that $\ell_x = [f(x), \infty)$. Note that ∂F is the graph of f over V_0. This proves (3.3) and (3.4).

We now claim that if F is the upper graph of f as in (3.4), then

$$F \text{ is } \Gamma\text{-monotone} \qquad \iff \qquad F \text{ is } \|\cdot\|^{\pm}\text{-Lipschitz.}$$

To see this suppose $\bar{x} = f(x)a + x$, $x \in V_0$, is a point on the graph of f. Then the cone $\bar{x} + \Gamma$ lies above the graph of F if and only if

$$f(x+y) \ \le\ f(x) + \|y\|^{+} \ \text{ for all } y \in V_0$$

by Remark 2.14. This inequality for all $y \in V_0$ is equivalent to

$$-\|y\|^{-} \ \le\ f(x+y) - f(x) \ \text{ for all } x, y \in V_0.$$

■

A Universal Construction of Γ-Monotone Subsets

Let $E \subset \mathbf{R}^m$ be a closed subset which is symmetric, i.e., invariant under the permutation of coordinates on $\mathbf{R}^m$. Then E universally determines a subset $F_E = F_E(p) \subset V$ for each a-hyperbolic polynomial p of degree m on V by setting

$$F_E \ =\ \{x \in V : \lambda(x) \in E\} \ =\ \lambda^{-1}(E) \tag{3.6}$$

where $\lambda(x)$ is the eigenvalue map (defined mod π_m). If E is a cone, then F_E is a cone. Note that the Gårding cone $\overline{\Gamma}$ for p is universally determined by the positive orphant $\mathbf{R}^m_+$. The next three theorems will be incorporated into one of our main results (Theorem 5.19) on the Dirichlet problem. The most basic for this application is the following.

THEOREM 3.6. *Suppose E is a closed symmetric subset of $\mathbf{R}^m$.*

If E is $\mathbf{R}^m_+$-monotone, then F_E is Γ-monotone.

Proof. Suppose $x \in F_E$, i.e., $\lambda_a(x) \in E$. Pick $b \in \Gamma$. Using the real analytic arrangement Theorem 2.9 states that the vector λ^+ with coordinates $\lambda^+_k \equiv \lambda^k_a(x+b) - \lambda^k_a(x)$ satisfies

$$\lambda^+ \ \in\ \mathbf{R}^m_+.$$

Hence $\lambda_a(x+b) = \lambda_a(x) + \lambda^+ \in E + \mathbf{R}^m_+ \subset E$ proving that $x + b \in F_E$. ■

Given an a-hyperbolic polynomial p of degree m with eigenvalue map $\lambda(x)$ and Gårding cone Γ, consider the conical neighborhoods $\overline{\Gamma_\delta}$ of $\overline{\Gamma}$ defined by

$$\lambda_k(x) + \delta \mathrm{tr}\lambda(x) \ \ge\ 0 \qquad k = 1, ..., m$$

as in Proposition 2.30. Let $\left(\mathbf{R}^m_+\right)_\delta \subset \mathbf{R}^m$ denote the conical neighborhood of the positive orphant $\mathbf{R}^m_+$ defined by

$$\lambda_k + \delta \mathrm{tr}\lambda \ \geq \ 0 \qquad k = 1, ..., m$$

THEOREM 3.7. *Suppose that E is a closed symmetric subset of $\mathbf{R}^m$ and $\delta > 0$.*

If E is $\left(\mathbf{R}^m_+\right)_\delta$-monotone, then F_E is $\overline{\Gamma_\delta}$-monotone.

Proof. Consider the a-hyperbolic polynomial q_δ defined in Proposition 2.30. It has eigenvalue map (up to scale) $\Lambda(x) \ = \ \lambda(x) + (\delta \mathrm{tr}\lambda(x))e$, and Gårding cone $\overline{\Gamma_\delta} = \Lambda^{-1}(\mathbf{R}^m_+)$. In order to apply Theorem 3.6 to q_δ consider the linear map $L : \mathbf{R}^m \to \mathbf{R}^m$ defined by $L(\lambda) = \lambda + (\delta \mathrm{tr}\lambda)e$. Note that $L^{-1}(\Lambda) = \Lambda - \left(\frac{\delta}{1+m\delta}\mathrm{tr}\Lambda\right)e$. It suffices to show that:

$$F_E \ = \ \Lambda^{-1}(LE), \ \ \text{and that}$$

$$E \ \text{ is } \ \left(\mathbf{R}^m_+\right)_\delta \ \text{monotone} \qquad \iff \qquad LE \ \text{ is } \ \mathbf{R}^m_+ \ \text{monotone} \tag{3.7}$$

to conclude by Theorem 3.6 that $F_E = \Lambda^{-1}(LE)$ is $\overline{\Gamma}_\delta = \Lambda^{-1}(\mathbf{R}^m_+)$-monotone. First, $\Lambda(x) = L\lambda(x)$ implies $F_E = \lambda^{-1}(E) = \Lambda^{-1}(LE)$. Second, $\left(\mathbf{R}^m_+\right)_\delta = L^{-1}\mathbf{R}^m_+$ implies (3.7). ∎

Finally we prove a recent convexity result of Bauschke, Gúler, Lewis and Sendov [BGLS] stated first with our applications in mind.

THEOREM 3.8. *Suppose that E is a symmetric subset of $\mathbf{R}^m$.*

If E is convex, then F_E is convex.

Proof. Suppose $x \notin F_E$, i.e., $\lambda^\uparrow(x) \notin E$. It will suffice to exhibit a convex set A which contains F_E but not x.

Note that by Corollary 2.28, for each $w^\uparrow \in \mathbf{R}^m_\uparrow$, the set

$$A \ \equiv \ \{y \in V : w^\uparrow(\lambda^\uparrow(y)) < 1\}$$

is convex. By the Hahn-Banach Theorem there exists a $w \in (\mathbf{R}^m)^*$ such that

$$\sup_E w \ < \ 1 \ = \ w\left(\lambda^\uparrow(x)\right).$$

Since $\sup_E w^\uparrow = \sup_E w$ and $w\left(\lambda^\uparrow(x)\right) \leq w^\uparrow\left(\lambda^\uparrow(x)\right)$,

$$\sup_E w^\uparrow \ < \ 1 \ \leq w^\uparrow\left(\lambda^\uparrow(x)\right). \qquad ∎$$

Remark 3.9. Adopt the notation in Corollary 2.28 and set $u \equiv \lambda^\uparrow(sx + (1-s)y), v = s\lambda^\uparrow(x) + (1-s)\lambda^\uparrow(y)$. Then for $w^\uparrow \in \mathbf{R}^m_\uparrow$ the Corollary concludes that $w^\uparrow(u) \leq w^\uparrow(v)$. Now

$$u \ \in \ E \ \equiv \ \text{the convex hull of the orbit } \pi_m v. \tag{3.8}$$

Otherwise, as in the proof of Theorem 3.8, there would exist $w^{\uparrow} \in \mathbf{R}^m_{\uparrow}$ with

$$\sup_E w^{\uparrow} \ < \ 1 \ \le \ w^{\uparrow}(u)$$

which contradicts $w^{\uparrow}(u) \le w^{\uparrow}(v)$.

THEOREM 3.10. *Suppose f is a symmetric function on $\mathbf{R}^m$.*

$$\text{If } f(\lambda) \text{ is convex, then } f(\lambda(x)) \text{ is convex} \tag{3.9}$$

Proof. Note that by (3.8) there exist $v_1, ..., v_N$ in the orbit $\pi_m v$ and $0 \le t_j \le 1$, $j = 1, ..., N$ with $\sum_j t_j = 1$ such that $u = \sum_j t_j v_j$. By the convexity and symmetry of f,

$$f(u) \ = \ f\left(\sum_j t_j v_j\right) \ \le \ \sum_j t_j f(v_j) \ = \ \sum_j t_j f(v) \ = \ f(v). \qquad \blacksquare$$

4. The Dirichlet Problem

In this section we summarize without proof the existence and uniqueness theorem in [HL_1] for the Dirichlet problem. It can be read independently of Sections 2 and 3. The equations considered are on euclidean space. They are purely second order and have constant coefficients; but are fully nonlinear and degenerate elliptic.

Subequations and Positivity

Each subset $F \subset \text{Sym}^2(\mathbf{R}^n)$ determines a class of C^2-functions u on an open subset $X \subset \mathbf{R}^n$ by the requirement that its hessian (second derivative) $D^2 u(X)$ lie in F at each $x \in X$. This class in $C^2(X)$ can be extended to a class in $\text{USC}(X)$, the upper semi-continuous $[-\infty, \infty)$-valued functions on X, provided that F satisfies monotonicity with respect to the set

$$\mathcal{P} \ \equiv \ \{A \in \text{Sym}^2(\mathbf{R}^n) : A \ge 0\} \tag{4.1}$$

Two justifications for imposing this condition are given in Remark 4.3 and Remark 4.10.

Definition 4.1. A closed subset $F \subset \text{Sym}^2(\mathbf{R}^n)$ with $\emptyset \ne F \ne \text{Sym}^2(\mathbf{R}^n)$ will be called a **subequation** or **Dirichlet set** if F satisfies the *positivity condition*

$$F + \mathcal{P} \ \subset \ F. \tag{4.2}$$

The Structure Theorem 3.2 for $\mathcal{P}$-monotone sets, as explained in Example 3.5, is now the "Structure Theorem for Subequations". In particular, Corollary 3.3 states that $\text{Int}F$ is connected, $F = \overline{\text{Int}F}$, and F is completely determined by its boundary. In fact, it is the upper graph of a Lipschitz function on $\text{Sym}^2_0(\mathbf{R}^n) = \{\text{tr}A = 0\}$.

Definition 4.2. Suppose that $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is a subequation and $u \in C^2(X)$ (where X is an open subset of $\mathbf{R}^n$). If $D^2u(x) \in F$ for all $x \in X$, then u is called **F-subharmonic on X**. If $D^2u(x) \in \mathrm{Int}F$ for all $x \in X$, then u is called **strictly F-subharmonic on X**. If $D^2u(x) \in \partial F$ for all $x \in X$, then u is called **F-harmonic on X**.

Remark 4.3. (Motivation). This definition of F-subharmonicity will be extended to non-differentiable functions. Perhaps the most fundamental property is that the maximum $\max\{u, v\}$ of two F-subharmonic functions u and v is again F-subharmonic. Computing the distributional hessian of $\max\{u, v\}$ in the case where u and v are C^2 provides strong support for assuming positivity if one wants $\max\{u, v\}$ to be F-subharmonic. (See Remark 3.3 in [HL$_1$]).

Dirichlet Duality

Each subequation F has a natural "dual" subequation.

Definition 4.4. The **Dirichlet dual** of a subequation $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is the set

$$\widetilde{F} \ = \sim(-\mathrm{Int}F) \ = \ -(\sim \mathrm{Int}F). \tag{4.3}$$

Proposition 4.5. *The Dirichlet dual $\widetilde{F}$ of a subequation F is also a subequation. Furthermore, duality holds, that is*

$$\widetilde{\widetilde{F}} \ = \ F. \tag{4.4}$$

Proof. Assertion (4.4) follows straightforwardly from the fact that $F = \overline{\mathrm{Int}F}$. To prove that $\widetilde{F}$ satisfies (4.2) a lemma is required.

Lemma 4.6. *Suppose $F = \overline{\mathrm{Int}F}$. Then*

$$\widetilde{F + A} \ = \ \widetilde{F} - A \quad \text{for each } A \in \mathrm{Sym}^2(\mathbf{R}^n). \tag{4.5}$$

Proof. Note that $B \in \widetilde{F + A} \iff -B \notin \mathrm{Int}(F + A) = \mathrm{Int}F + A \iff -(B + A) \notin \mathrm{Int}F \iff B + A \in \widetilde{F}$. ∎

Proof of Proposition 4.5. Suppose $P \in \mathcal{P}$. Then $F + P \subset F$, or equivalently $F \subset F - P$. Taking duals reverses this inclusion and by Lemma 4.6, $\widetilde{F - P} = \widetilde{F} + P$. Thus $\widetilde{F} + P \subset \widetilde{F}$ as desired. ∎

Proposition 4.7. *Suppose F is a subequation and $u \in C^2(X)$. Then*

$$u \text{ is } F\text{-harmonic} \iff u \text{ is } F\text{-subharmonic and } -u \text{ is } \widetilde{F}\text{-subharmonic.}$$

Proof. Note that $\partial F = F \cap (\sim \mathrm{Int}F) = F \cap (-\widetilde{F})$. ∎

F-Subharmonic and F-Harmonic Functions

It is necessary and quite useful to extend the definition of F-subharmonic to non-differentiable functions u. Let $\mathrm{USC}(X)$ denote the set of $[-\infty, \infty)$-valued, upper semi-continuous functions on X.

Definition 4.8. A function $u \in \mathrm{USC}(X)$ is said to be F-**subharmonic** if for each $x \in X$ and each function φ which is C^2 near x, one has that

$$\left\{\begin{array}{lll} u - \varphi & \leq 0 & \text{near } x_0 \text{ and} \\ & = 0 & \text{at } x_0 \end{array}\right\} \quad \Rightarrow \quad D^2_x \varphi \ \in \ F. \tag{4.6}$$

Note that if $u \in C^2(X)$, then

$$u \in F(X) \quad \Rightarrow \quad D^2_x u \in F \qquad \forall\, x \in X$$

since the test function φ may be chosen equal to u in (4.10). The converse is not true for general subsets F. However, we have the following.

Proposition 4.9. *Suppose F satisfies the Positivity Condition (4.2) and $u \in C^2(X)$. Then*

$$D^2_x u \ \in \ F \ \ \text{for all } x \in X \qquad \Rightarrow \qquad u \ \in \ F(X).$$

Proof. Assume $D^2_{x_0} u \in F$ and and φ is a C^2-function such that

$$\left\{\begin{array}{lll} u - \varphi & \leq 0 & \text{near } x_0 \\ & = 0 & \text{at } x_0 \end{array}\right\}$$

Since $(\varphi - u)(x_0) = 0$, $Dx_0(\varphi - u)_=0$, and $\varphi - u \geq 0$ near x_0, we have $D^2_{x_0}(\varphi - u) \in \mathcal{P}$. Now the Positivity Condition implies that $D^2_{x_0}\varphi \in D^2_{x_0} u + \mathcal{P} \subset F$. This proves that $u \in F(X)$. ■

Remark 4.10. Because of Proposition 4.9 *we must assume that F satisfies the Positivity Condition*. Otherwise the definition of F-subharmonicity would not extend the natural one for smooth functions (cf. Remark 4.3). The positivity condition is rarely used in proofs. This is because without it $F(X)$ is empty and the results are trivial. For example, the positivity condition is not required in the following theorem. (F need only be closed.)

It is remarkable, at this level of generality, that F-subharmonic functions share many of the important properties of classical subharmonic functions.

THEOREM 4.11. Elementary Properties of F-Subharmonic Functions. *Let F be an arbitrary closed subset of $J^2(X)$.*

(Maximum Property) If $u, v \in F(X)$, then $w = \max\{u, v\} \in F(X)$.

(Coherence Property) If $u \in F(X)$ is twice differentiable at $x \in X$, then $J^2_x u \in F_x$.

(Decreasing Sequence Property) If $\{u_j\}$ is a decreasing ($u_j \geq u_{j+1}$) sequence of functions with all $u_j \in F(X)$, then the limit $u = \lim_{j\to\infty} u_j \in F(X)$.

(Uniform Limit Property) Suppose $\{u_j\} \subset F(X)$ is a sequence which converges to u uniformly on compact subsets to X, then $u \in F(X)$.

(Families Locally Bounded Above) Suppose $\mathcal{F} \subset F(X)$ is a family of functions which are locally uniformly bounded above. Then the upper semicontinuous regularization v^* of the upper envelope

$$v(x) \ = \ \sup_{f \in \mathcal{F}} f(x)$$

belongs to $F(X)$.

Proof. See Appendix B in [HL$_2$].

Boundary Convexity

Associated to each subequation F is an open cone $\overrightarrow{F}$ which governs the geometry of those domains for which the Dirichlet problem is always solvable. If F is a cone with vertex at the origin, then $\overrightarrow{F}' = \mathrm{Int} F$. Otherwise, $\overrightarrow{F}'$ is constructed as in [HL$_1$, pp. 415-416]. Here is a summary.

Fix a vertex $B \in \mathrm{Sym}^2(\mathbf{R}^n)$ and consider the ray sets:

$$\overrightarrow{F_B} \ = \ \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : B + tA \in F \ \forall\, t \geq \text{ some } t_0\} \tag{4.7}$$

$$\overrightarrow{F_B}' \ = \ \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : B + tA \in \mathrm{Int} F \ \forall\, t \geq \text{ some } t_0\} \tag{4.8}$$

Examples show that $\overrightarrow{F_B}$ may not be closed, $\overrightarrow{F_B}'$ may not be open, and $\overrightarrow{F_B}, \overrightarrow{F_B}'$ may depend on B. However,

$$\overrightarrow{F} \ \equiv \ \mathrm{closure} \overrightarrow{F_B} \qquad \text{and} \qquad \overrightarrow{F}' \ \equiv \ \mathrm{Int} \overrightarrow{F_B}' \tag{4.9}$$

are independent of the choice of vertex B. In addition

$$\overrightarrow{F} \text{ is a subequation which is a cone with vertex at the origin.} \tag{4.10}$$

Moreover, by Lemma 5.8 and Elementary Property (3),

$$\overrightarrow{F}' \ = \ \mathrm{Int} \overrightarrow{F} \qquad \text{and} \qquad \overrightarrow{F} \ \equiv \ \mathrm{closure} \overrightarrow{F}' \tag{4.11}$$

Definition 4.12. We call $\overrightarrow{F}'$ the **asymptotic interior** of F, and $\overrightarrow{F}$ the **asymptotic subequation** of F. (In [HL$_1$], $\overrightarrow{F}$ is called the "asymptotic ray set associated to F".)

Note that (4.10) and (4.11) imply that

$$\mathrm{Int} \mathcal{P} \ \subset \ \overrightarrow{F}' \ = \ \mathrm{Int} \overrightarrow{F}. \tag{4.12}$$

Suppose now that $\Omega \subset\subset \mathbf{R}^n$ is a domain with smooth boundary $\partial\Omega$. Let ρ be a *local defining function* for $\partial\Omega$ at a point x, that is, a smooth function with $|\nabla\rho| > 0$, defined on a neighborhood U of x, such that $U \cap \Omega = \{\rho < 0\}$

Definition 4.13. The boundary $\partial\Omega$ is **strictly $\overrightarrow{F}$-convex at x if**

$$\mathrm{Hess}_x\rho\big|_{T_x\partial\Omega} \;=\; B\big|_{T_x\partial\Omega} \qquad \text{for some } B \in \overrightarrow{F}' \equiv \mathrm{Int}\overrightarrow{F}. \tag{4.13}$$

This is independent of the choice of ρ and is equivalent to the condition that:

$$\mathrm{Hess}_x\rho + tP_n \;\in\; \overrightarrow{F}' \equiv \mathrm{Int}\overrightarrow{F} \qquad \forall\, t \;\geq\; \text{some } t_0 \tag{4.14}$$

where P_n denotes orthogonal projection onto the normal line to $\partial\Omega$ at x.

Note that each classically strictly convex boundary is strictly $\overrightarrow{F}$-convex by (4.12). In general $\overrightarrow{F}$-convexity can be expressed purely in terms of the geometry of the boundary. Let II denote the second fundamental form of $\partial\Omega$ with respect to the interior normal at x. Then $\partial\Omega$ is strictly $\overrightarrow{F}$-convex at x if and only if either of the following equivalent conditions holds.

$$\begin{aligned} II \;&=\; B\big|_{T_x\partial\Omega} \qquad \text{for some } B \in \overrightarrow{F} \\ II + tP_n \;&\in\; \overrightarrow{F}' \equiv \mathrm{Int}\overrightarrow{F} \qquad \forall\, t \;\geq\; \text{some } t_0 \end{aligned}$$

The concept of $\overrightarrow{F}$-convexity is important because it leads to the existence of local barriers used in existence proofs. For domains $\Omega \subset\subset \mathbf{R}^n$ we have the following global result.

THEOREM 4.14. *Suppose $\partial\Omega$ is strictly $\overrightarrow{F}$-convex at each point. Then there exists a global defining function $\rho \in C^\infty(\overline{\Omega})$ and constants $\epsilon_0 > 0, R_0 > 0$ such that*

$C\left(\rho - \epsilon\frac{1}{2}|x|^2\right)$ is strictly F-subharmonic on Ω for all $C \geq C_0$ and $0 \leq \epsilon \leq \epsilon_0$.

See [HL_1, §5] for proofs of the above.

Existence and Uniqueness

THEOREM 4.15. *Suppose F is a subequation in* $\mathrm{Sym}^2(\mathbf{R}^n)$ *and Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. If $\partial\Omega$ is both $\overrightarrow{F}$ and $\overrightarrow{\widetilde{F}}$ strictly convex, then* the Dirichlet problem is uniquely solvable for all continuous boundary data. *That is, for each $\varphi \in C(\partial\Omega)$, these exists a unique $u \in C(\overline{\Omega})$ satisfying:*

(1) u is F-harmonic on Ω, and

(2) $u = \varphi$ on $\partial\Omega$.

Since $\mathcal{P} \subset \widetilde{F}$ and $\mathcal{P} \subset \overrightarrow{\widetilde{F}}$, if $\partial\Omega$ is strictly convex (in the standard way), then it is both $\overrightarrow{F}$ and $\overrightarrow{\widetilde{F}}$ strictly convex. In particular, the Dirichlet problem is uniquely solvable on all domains with strictly convex boundaries, such as a ball.

This Theorem 4.15 was first proved in [HL_1]. The proof involved the use of "subaffine" functions (equivalently $\widetilde{\mathcal{P}}$-subharmonic functions) and relied on a deep result of Slodkowski on quasi-convex functions [S]. The following definition of F-subharmonic functions was adopted there.

Definition 4.8$'$. A function $u \in \mathrm{USC}(X)$ is said to be F*-subharmonic* if for each $x \in X$ and each C^2-function v which is $\widetilde{F}$-subharmonic near x, the sum $u+v$ is subaffine near x,

See Remark 4.9 in [HL_1] for a proof that Definitions 4.9 and 4.9$'$ are equivalent.

A second proof of Theorem 4.15 was given in [HL_2] using standard viscosity methods, i.e., the Theorem on Sums and Definition 4.8 (see [C], [CIL]).

5. Subequations Determined by Hyperbolic Polynomials

Gårding's theory provides a unified approach to studying many basic subequations. For this application to subequations we only consider polynomials on the vector space $V \equiv \mathrm{Sym}^2(\mathbf{R}^n)$ which are hyperbolic in the direction I.

Dirichlet-Gårding Polynomials

Definition 5.1. A homogeneous real polynomial M of degree m on $\mathrm{Sym}^2(\mathbf{R}^n)$ is ***I*-hyperbolic** if $M(I) > 0$ and for all $A \in \mathrm{Sym}^2(\mathbf{R}^n)$ the polynomial $M_A(s) = M(sI+A)$ has m real roots.

We adopt the terminology and notation from the previous discussion, with Γ denoting the Gårding cone associated to the polynomial M. The principal branch (or **closed Gårding cone**) $F_1 = \overline{\Gamma}$ is defined by $\lambda_k(A) \geq 0, k = 1, ..., m$, and is a convex cone (Corollary 2.13) but may not be a subequation. We require that $\overline{\Gamma}$ be a subequation.

Definition 5.2. An I-hyperbolic polynomial M on $\mathrm{Sym}^2(\mathbf{R}^n)$ is said to be a **Dirichlet-Gårding** polynomial if $\overline{\Gamma}$ is a subequation, i.e., if positivity holds:

(1) $\overline{\Gamma} + \mathcal{P} \ \subset\ \overline{\Gamma}$.

Remark 5.3. There are other useful ways of stating this positivity condition for $\overline{\Gamma}$. Since $\overline{\Gamma}$ is a convex cone, (1) is equivalent to

(2) $\mathcal{P} \ \subset\ \overline{\Gamma}$, that is, $A \geq 0 \ \Rightarrow\ \lambda_k(A) \geq 0, \quad k = 1, ..., m.$

In other words, for $A \geq 0$, all the roots of the polynomial $t \mapsto M(tI + A)$ are ≤ 0. Equivalently,

(2)$'$ $M(tI + A) \ \neq\ 0$ if $A \geq 0$ and $t > 0$.

Since the extreme rays in $\mathcal{P}$ are generated by orthogonal projections P_e with $e \in \mathbf{R}^n$, it is enough to verify (2) for $A = P_e$.

(3) $\lambda_k(P_e) \ \geq\ 0$ for all unit vectors $e \in \mathbf{R}^n$ and $k = 1, ..., m$.

(3)$'$ $M(tI + P_e) \ >\ 0$ for all $t > 0$ and all unit vectors $e \in \mathbf{R}^n$.

By definition the branches F_k are subequations if positivity holds. That is,

(4) $F_k + \mathcal{P} \ \subset\ F_k$ for all $k = 1, ..., m$.

By Corolloary 2.17 and the convexity of the principal branch $\overline{\Gamma} = F_1$, condition (4) is equivalent to condition (2). Finally, by the Structure Theorem 3.2 the condition that:

(5) The ordered eigenvalue function $\lambda_k^{\uparrow}(x)$ is $\|\cdot\|^{\pm}$-Lipschitz $(k = 1, ..., m)$,

is equivalent to (4).

Branches Considered as Subequations

Some of Gårding's results from Section 2 can be summarized as follows. The ordered eigenvalues determine m branches $F_k \equiv \{\lambda_k^{\uparrow} \geq 0\}$ of $\{M = 0\}$. Each $\partial F_k = \{\lambda_k^{\uparrow} = 0\}$

is contained in $\{M = 0\}$, and $F_k = \overline{\mathrm{Int} F_k}$ with $\mathrm{Int} F_k = \{\lambda_k > 0\}$. Each F_k is a cone, and the principal branch $\overline{\Gamma} = F_1$ is a convex cone, which is a monotonicity cone for each of the other branches, i.e.,

$$F_k + \overline{\Gamma} \ \subset \ F_k \qquad k = 1, ..., m. \tag{5.1}$$

THEOREM 5.4. *Suppose that M is a Dirichlet-Gårding polynomial of degree m on* $\mathrm{Sym}^2(\mathbf{R}^n)$. *Then each branch F_k, $k = 1, ..., m$, is a subequation with dual subequation $\widetilde{F}_k = F_{m-k+1}$. Moreover, each branch is a cone.*

Proof. Property (1b), that $\lambda_k^{\uparrow}(-A) = -\lambda_{m-k+1}^{\uparrow}(A)$, for the ordered eigenfunctions, implies that $\widetilde{F}_k = F_{m-k+1}$. Property (1a), that $\lambda_k(tA) = t\lambda_k(A)$ for $t \geq 0$, implies F_k is a cone. ■

Thus, as a special case of Theorem 4.15 we can solve the Dirichlet problem for each branch of the equation $\{M = 0\}$.

THEOREM 5.5. *Suppose that M is a Dirichlet-Gårding polynomial on* $\mathrm{Sym}^2(\mathbf{R}^n)$. *Let $i \equiv \max\{k, n-k+1\}$. Then for each bounded domain Ω with smooth strictly $\overrightarrow{F}_i$-convex boundary, the Dirichlet problem for F_k-harmonic functions is uniquely solvable for all continuous boundary data.*

Remark 5.6. (a) Fix $c \in \mathbf{R}$ and replace $F_k = \{\lambda_k^{\uparrow} \geq 0\}$ by $F_k^c = \{\lambda_k^{\uparrow} \geq c\}$. Both Theorem 5.4 and Theorem 5.5 remain true except for the fact that F_k^c is no longer a cone. Note that F_1^c remains a convex subequation.

(b) The principal branch $F_1 = \overline{\Gamma}$ can be perturbed to the convex cone subequation ($c > 0$)

$$\overline{\Gamma_c} \ = \ \{A \in \Gamma : M(A) \geq c\}.$$

Convexity follows from Corollary 2.13 (d) or (e).

Examples of Dirichlet-Gårding Polynomials

There are a number of geometrically interesting nonlinear equations that arise from the construction in Theorem 5.4, and for which Theorem 5.5 solves the Dirichlet problem.

Example 5.7. (The Real Determinant). Of course the fundamental example of a Dirichlet-Gårding polynomial is the polynomial on $\mathrm{Sym}^2(\mathbf{R}^n)$ given by

$$M(A) \ = \ \det A \tag{5.2}$$

where the M-eigenvalues of A are just the usual eigenvalues of A as a symmetric matrix. The corresponding F_1-harmonic functions are convex solutions u of the homogeneous Monge-Ampère equation $\det \mathrm{Hess}\, u = 0$ (in the viscosity sense (cf. [HL$_1$], [HL$_2$])). The other branches F_k, $k = 2, ..., n$ lead to other branches of this equation. Theorem 5.5 says that the Dirichlet problem is uniquely solvable for F_k-harmonic functions on any domain which is strictly $\overrightarrow{F_i}$-convex where $i = \max\{k, n-k+1\}$.

Example 5.8. (The Complex and Quaternionic Determinant). Consider $\mathbf{C}^n = (\mathbf{R}^{2n}, J)$ where $J : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ with $J^2 = -I$ is the standard almost complex structure. Then any $A \in \mathrm{Sym}^2(\mathbf{R}^{2n})$ has a *hermitian symmetric component*

$$A_{\mathbf{C}} \ \equiv\ \tfrac{1}{2}(A - JAJ)$$

which commutes with J. The eigenvalues of $A_{\mathbf{C}}$, considered as a real $2n \times 2n$ symmetric matrix, occur in pairs, $\lambda_1, \lambda_1, \lambda_2, \lambda_2, ..., \lambda_n, \lambda_n$ with complex lines as eigenspaces, while $A_{\mathbf{C}}$, considered as an $n \times n$ complex hermitian symmetric matrix, has n real eigenvalues $\lambda_1(A_{\mathbf{C}}) = \lambda_1, ..., \lambda_n(A_{\mathbf{C}}) = \lambda_n$.

The complex determinant

$$\det_{\mathbf{C}}(A_{\mathbf{C}}) \ =\ \lambda_1(A_{\mathbf{C}}) \cdots \lambda_n(A_{\mathbf{C}}) \tag{5.3}$$

is a Dirichlet-Gårding polynomial in $A \in \mathrm{Sym}^2(\mathbf{R}^{2n})$.

Similarly, one can consider quaternion n-space $\mathbf{H}^n = (\mathbf{R}^{4n}, I, J, K)$, where I, J, K satisfy the usual quaternion relations. Then any $A \in \mathrm{Sym}^2(\mathbf{R}^{4n})$ has a *quaternionic hermitian symmetric component*

$$A_{\mathbf{H}} \ \equiv\ \tfrac{1}{4}(A - IAI - JAJ - KAK)$$

which commutes with I, J and K. The eigenvalues of A, considered as a real $4n \times 4n$ symmetric matrix, occur in multiples of four $\lambda_1, \lambda_1, \lambda_1, \lambda_1, \lambda_2, ...$ with quaternion lines as eigenspaces, while $A_{\mathbf{H}}$, considered as a quaternionic hermitian symmetric $n \times n$ matrix can be diagonalized under the action of $\mathrm{Sp}_n \cdot \mathrm{Sp}_1$ with n real eigenvalues $\lambda_1(A_{\mathbf{H}}) = \lambda_1, ..., \lambda_n(A_{\mathbf{H}}) = \lambda_n$. The quaternion determinant

$$M(A) \ =\ \det_{\mathbf{H}}(A_{\mathbf{H}}) \ =\ \lambda_1(A_{\mathbf{H}}) \cdots \lambda_n(A_{\mathbf{H}}) \tag{5.4}$$

is a Dirichlet-Gårding polynomial in $A \in \mathrm{Sym}^2(\mathbf{R}^{4n})$.

Example 5.9. (Lagrangian Harmonicity). This is a non-classical case. Consider $\mathbf{C}^n = (\mathbf{R}^{2n}, J)$ as above, and for $A \in \mathrm{Sym}^2(\mathbf{R}^{2n})$ define the *skew hermitian component*

$$A_{\mathrm{skew}} \ \equiv\ \tfrac{1}{2}(A + JAJ)$$

which anti-commutes with J. The eigenvalues occur in opposite pairs $\pm\mu_1, ..., \pm\mu_n$ where each $\pm\mu_k$-eigenspace generates a complex line. Let $\tau = \frac{1}{2}\mathrm{trace}_{\mathbf{R}} A$.

Proposition 5.10. *The product*

$$M_{\mathrm{LAG}}(A) \ \equiv \prod_{2^n \text{ times}} (\tau \pm \mu_1 \pm \cdots \pm \mu_n) \tag{5.5}$$

taken over all sequences $\pm \cdots \pm$*, is a Dirichlet-Gårding polynomial on* $\mathrm{Sym}^2(\mathbf{R}^{2n})$.

More generally, for any $1 \leq p \leq n$, the product

$$M(A) \;=\; {\prod_{|I|=p}}' \left(\tau \pm \mu_{i_1} \pm \cdots \pm \mu_{i_p}\right) \tag{5.6}$$

is a Dirichlet-Gårding polynomial. (See [HL$_3$] for the proof.)

We shall refer to these four basic examples as the *real, complex, quaternionic, and Lagrangian/Isotropic Monge-Ampère polynomials*. Now we describe three methods of constructing a new Dirichlet-Gårding polynomial from a given one.

The first method decreases the degree of the Dirichlet-Gårding polynomial.

Method 5.11. (Derivatives – The Elementary Symmetric Functions). Given a Dirichlet-Gårding polynomial M with Gårding cone Γ, for each $A_0 \in \Gamma$, the kth derivative $M^{(k)}$ in the direction A_0 is also a Dirichlet-Gårding polynomial. This is because $\Gamma_{M^{(k+1)}} \supset \Gamma_{M^{(k)}} \supset \cdots \supset \Gamma_M$, by Proposition 2.21, and so $\mathcal{P} \subset \overline{\Gamma}_M$ implies $\mathcal{P} \subset \overline{\Gamma}_{M^{(k)}}$. By (2.26) this construction is equivalent to taking the elementary symmetric functions of the A_0-eigenvalues (as functions on $\mathrm{Sym}^2(\mathbf{R}^n)$).

This method can be applied, of course, to the real, complex, quaternionic, and Lagrangian/Isotropic Monge-Ampère polynomials.

Note. Different choices of $A \in \Gamma$ produce different Dirichlet-Gårding polynomials.

The second method increases the degree of the Dirichlet-Gårding polynomial.

Method 5.12. (k-Fold Sums of Eigenvalues). Suppose we are given a Dirichlet-Gårding polynomial M on $\mathrm{Sym}^2(\mathbf{R}^n)$ with Gårding cone Γ. Fix $A_0 \in \Gamma$ and consider the A_0-eigenvalue functions $\lambda_1(A), ..., \lambda_m(A)$. Then

$$M_k(A) \;=\; {\prod_{|I|=k}}' \left(\lambda_{i_1}(A) + \cdots + \lambda_{i_k}(A)\right) \qquad A \in \mathrm{Sym}^2(\mathbf{R}^n) \tag{5.7}$$

defines a Dirichlet-Gårding polynomial of degree $\binom{m}{k}$ on $\mathrm{Sym}^2(\mathbf{R}^n)$ with Gårding cone $\Gamma_k \supset \Gamma$ and A_0-eigenvalues functions

$$\Lambda_I(A) \;=\; \frac{1}{k}\left((\lambda_{i_1}(A) + \cdots + \lambda_{i_k}(A)\right). \tag{5.8}$$

This follows from Proposition 2.29 and Theorem 3.6.

Note 5.13.

(a) It is easy to see that $\Gamma = \Gamma_1 \subset \Gamma_2 \subset \;\cdots\; \subset \Gamma_n$.

(b) Any vector $w \in \mathrm{Int}\mathbf{R}^m_+$ can replace $e = (1, ..., 1)$. We would then have:

$$M_k(A) \;=\; {\prod_{|I|=k}}' \left(w_{i_1}\lambda_{i_1}(A) + \cdots + w_{i_k}\lambda_{i_k}(A)\right) \qquad A \in \mathrm{Sym}^2(\mathbf{R}^n)$$

The third method leaves the degree of the Dirichlet-Gårding polynomial fixed.

Method 5.14. (δ-Uniformity). Define for $\delta > 0$

$$\mathcal{P}_\delta \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : A + (\delta \mathrm{tr} A)\cdot I \geq 0\}. \tag{5.9}$$

Each $\mathcal{P}_\delta$ is a convex cone subequation whose interior contains $\mathcal{P}$. the family is *conically-fundamental* for $\mathcal{P}$ in the following sense.

$$\begin{array}{c}\text{If } F \text{ is any convex cone subequation with } \mathcal{P} - \{0\} \subset \mathrm{Int}F,\\ \text{then there exists } \delta > 0 \text{ such that } \mathcal{P} \subset \mathcal{P}_\delta \subset F\end{array} \tag{5.10}$$

Definition 5.15. A subequation $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is **uniformly elliptic** if for some $\delta > 0$, F is $\mathcal{P}_\delta$-monotone, i.e.,

$$F + \mathcal{P}_\delta \;\subset\; F. \tag{5.11}$$

Using (5.10) it is easy to show that our definition of uniform ellipticity is equivalent to any of the usual definitions.

Now for the third method. Suppose M is a Dirichlet-Gårding polynomial with Gårding cone Γ. Choose $A_0 \in \Gamma$. Let $\lambda_1(A), ..., \lambda_m(A)$ denote the A_0-eigenvalue functions, and let $\mathrm{tr}\lambda(A) = \lambda_1(A) + \cdots + \lambda_m(A)$ denote the trace.

THEOREM 5.16. *The function*

$$M_\delta(A) \;=\; \prod_{j=1}^m (\lambda_j(A) + \delta \mathrm{tr}(A)) \tag{5.12}$$

is a Dirichlet-Gårding polynomial with uniformly elliptic branches F_k^δ containing the branches F_k of $\{M = 0\}$.

Proof. By Proposition 2.30, M_δ is A_0-hyperbolic, and up to scale the A_0-eigenvalue functions are $\Lambda_k(A) = \lambda_k(A) + \delta \mathrm{tr}(A)$. Each branch F_k^δ of $\{M_\delta = 0\}$ is Γ_δ-monotone by (5.1). It remains to show that $\mathrm{Int}\mathcal{P}_{\delta'} \subset \Gamma_\delta$ for some $\delta' > 0$. This follows from (5.10) with $F = \overline{\Gamma_\delta}$ since $\mathcal{P} - \{0\} \subset \overline{\Gamma} - \{0\} \subset \mathrm{Int}\overline{\Gamma_\delta} = \Gamma_\delta$. (Note that $\mathrm{tr}\lambda(A) > 0$ if $A \in \Gamma$.)∎

Remark 5.17. The three methods of generating new DGPs, and hence new subequations for which Theorem 5.5 solves the Dirichlet problem, do not commute with each other. Thus combining these results creates a large number of new Dirichlet-Gårding polynomials.

Additional methods for generating subequations and solving the Dirichlet problem are investigated next.

Universal Eigenvalue Subequations

The branches F_k of $\{M = 0\}$ are universally determined by requiring that:

$$\lambda(A) \;\in\; E_k$$

where $E_k \equiv \{\lambda_k^\uparrow \geq 0\} = \{\lambda \in \mathbf{R}^m : \text{At least } m{-}k{+}1 \text{ of the } \lambda_j \text{ are} \geq 0\}$. Thus the subset $E_k \subset \mathbf{R}^m$ universally determines the kth branch of $\{M = 0\}$ for any Dirichlet-Gårding polynomial of degree m on $\mathrm{Sym}^2(\mathbf{R}^n)$. For example, E defined by $\lambda_1 \geq 0, ..., \lambda_m \geq 0$ is the **universal Monge-Ampère subequation**, and it induces an *M-Monge-Ampère subequation* for each Dirichlet-Gårding polynomial M of degree m.

Definition 5.18. A closed symmetric subset E of $\mathbf{R}^m$, with $\emptyset \neq E \neq \mathbf{R}^m$, which is positive-orphant monotone, will be called a **universal eigenvalue subequation**.

If ,in addition,

(a) E is convex, then E will be called a **convex (universal) eigenvalue subequation**

(b) E is $(\mathbf{R}^m_+)_\delta$-monotone for some $\delta > 0$, then E will be called a **uniformly elliptic (universal) eigenvalue subequation**.

The structure of universal eigenvalue subequations is described in Proposition 5.25 below. We leave to the reader a similar result for uniformly elliptic universal eigenvalue subequations.

THEOREM 5.19. *A universal eigenvalue subequation E on $\mathbf{R}^m$ universally determines a subequation F_E on $\mathbf{R}^n$ for each Dirichlet-Gårding polynomial M on $\mathrm{Sym}^2(\mathbf{R}^n)$ of degree m by setting*

$$F_E \;\equiv\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \lambda(A) \in E\} \;=\; \lambda^{-1}(E).$$

Moreover,

$$\widetilde{F}_E \;=\; F_{\widetilde{E}} \qquad \text{and} \qquad \overrightarrow{F}_E = F_{\overrightarrow{E}} \tag{5.13}$$

Finally,

(a) If E is convex, then F_E is a convex subequation.

,

(b) If E is uniformly elliptic, then F_E is a uniformly elliptic subequation.

Proof. The important monotonicity result Theorem 3.6 implies that F_E is a subequation (utilizing Definitions 5.2 and 5.18). We leave (5.13) to the reader. Part (a) is Theorem 3.8 while for (b) Theorem 3.7 says that F_E is $\overline{\Gamma}_\delta$-monotone. As noted in the proof of Theorem 5.16, this implies that F_E is uniformly elliptic. ∎

Because of the results proved in [HL$_1$] (stated as Theorem 4.15 here) for each of the subequations F_E constructed in Theorem 5.18, the Dirichlet problem can be solved.

THEOREM 5.20. *Suppose that E is a universal eigenvalue subequation on $\mathbf{R}^m$, and M is a Dirichlet-Gårding polynomial on $\mathbf{R}^n$ of degree m. Then the Dirichlet problem for the subequation F_E induced by E and M can be solved uniquely for all continuous boundary data on any domain with smooth boundary which is both $\overrightarrow{F}_E$ and $\overrightarrow{F}_{\widetilde{E}}$ strictly convex.*

Examples of Universal Eigenvalue Subequations

The first four examples of universal subequations E have already been discussed along with their induced subequations F_E. In these examples E is a branch of an e-hyperbolic polynomial Q on $\mathbf{R}^m$ whose Gårding cone $\overline{\Gamma}$ contains $\mathbf{R}^m_+$. Taking M to be any Dirichlet-Gårding polynomial of degree m and composing Q with the eigenvalue map of M, one sees that Methods 5.10, 5.11 and 5.13 can now be considered special cases of Theorem 5.18.

Example 5.21. (The Branches of the Universal Monge-Ampère Equation). These are the subsets

$$E_k \;=\; \{\lambda \in \mathbf{R}^m : \lambda_k^{\uparrow} \geq 0\} \qquad k = 1, ..., m.$$

which are the branches of the universal Monge-Ampère equation $Q(\lambda) = \lambda_1 \cdots \lambda_m$.

Example 5.22. (The Branches of the Universal p$^{\text{th}}$ Elementary Symmetric Function). These subsets $E \subset \mathbf{R}^m$ are defined by requiring that at least $k^* = p-k+1$ of the e -eigenvalues of

$$Q(\lambda) \;=\; \sum_{|I|=p}{}' \lambda_{i_1} \cdots \lambda_{i_p}$$

are ≥ 0.

Example 5.23. (The Universal Geometrically p-Convex Equation and its Branches). Here the subsets E of $\mathbf{R}^m$ are the branches of

$$Q(\lambda) \;=\; \prod_{|I|=p}{}' \left(\lambda_{i_1} + \cdots + \lambda_{i_p}\right)$$

defined by requiring at least r of the p-fold sums $\lambda_{i_1} + \cdots + \lambda_{i_p}$ (these are p-times the e-eigenvalues of Q) to be ≥ 0.

Example 5.24. (The Universal δ-Uniformly Elliptic Subequation). Fix $\delta > 0$ and let $\text{tr}\lambda = \lambda \cdot e = \lambda_1 + \cdots + \lambda_m$. Set

$$Q_\delta(\lambda) \;=\; \prod_{k=1}^{m} (\lambda_k + \delta \text{tr}\lambda)$$

The branches of $\{Q_\delta = 0\}$ defined by requiring at least $k^* = m-k+1$ of the eigenvalues $\Lambda_j = \lambda_j + \delta\text{tr}\lambda$, $j = 1, ..., m$ to be ≥ 0, are universal uniformly elliptic subequations. Note that as $\delta > 0$ varies, the principal branches $\left(\mathbf{R}^m_+\right)_\delta$ form a conically-fundamental neighborhood system for $\mathbf{R}^m_+ = \mathbf{R}^m_+$.

The following provides a classification of universal subequations.

Proposition 5.25. *A subset $E \subset \mathbf{R}^m$ is a universal eigenvalue subequation if and only if there exists a $\|\cdot\|^{\pm}$-Lipschitz symmetric function f on the hyperplane normal to e with*

$$E \;=\; \{\lambda + te : \lambda \cdot e = 0 \;\;\text{and}\;\; t \geq f(\lambda)\}$$

Proof. See Example 3.4.

Example 5.26. (Using Functions to Construct Subequations). For simplicity we first consider the universal case. Suppose

$$E_{c_1} \;=\; \{\lambda : f_1(\lambda) \geq c_1\} \qquad \text{and} \qquad E_{c_2} \;=\; \{\lambda : f_2(\lambda) \geq c_2\}.$$

The description of the set $E_c \;=\; \{\lambda : f_1(\lambda) + f_2(\lambda) \geq c_2\}$ as $\bigcup_{c_1+c_2 \geq c}(E_{c_1} \cap E_{c_2})$ is awkward. Similarly, if $f_1 \geq 0$ and $f_2 \geq 0$, the set $E_c = \{\lambda : f_1(\lambda) f_2(\lambda) \geq c\}$ is awkward to describe. Consequently in describing examples of universal eigenvalue subequations it is useful to state an obvious result.

Lemma 5.27. *Suppose that $f(\lambda)$ is a continuous symmetric function defined on an open symmetric subset of $\mathbf{R}^m$. Choose $c \in \mathbf{R}$, set $E_c = \{\lambda : f(\lambda) \geq c\}$, and assume $\emptyset \neq E_c \neq \mathbf{R}^m$.*

If f is non-decreasing in each variable, then $\overline{E_c}$ is a universal subequation.

If, in addition, f is concave, then $\overline{E_c}$ is a convex subequation.

Example 5.28. (Special Lagrangian Type). Suppose $\varphi : \mathbf{R} \to \mathbf{R}$ is strictly increasing. For each $c \in \mathbf{R}$

$$E_c \;=\; \left\{\lambda \in \mathbf{R}^m : \sum_{k=1}^{m} \varphi(\lambda_k) \geq c\right\}$$

is a universal eigenvalue subequation. Taking $\varphi : \mathbf{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ to be $\varphi(t) = \arctan(t)$, we obtain the **universal special Lagrangian subequation**:

$$E_c \;\equiv\; \left\{\lambda \in \mathbf{R}^m : \sum_{k=1}^{m} \arctan \lambda_k \geq c\right\}, \qquad c \in \left(-\tfrac{m\pi}{2}, \tfrac{m\pi}{2}\right).$$

The arctangent function is concave on $t > 0$, but convex on $t < 0$. Consequently, in order for E_c to be convex, the condition $\sum_{k=1}^{m} \arctan \lambda_k \geq c$ must imply that each $\lambda_k \geq 0$. This proves easily that E_c is convex if $c \geq \frac{(m-1)\pi}{2}$. A detailed calculation due to Yuan [Y, Lemma 2.1] shows that

$$E_c \ \text{ is convex} \qquad \iff \qquad c \;\geq\; \frac{(m-2)\pi}{2} \tag{5.14}$$

Note that the function f in the structure Theorem 5.25 is difficult to describe for this set E_c.

Example 5.29. (Krylov Type). First note that by Theorem 2.9 for any a-hyperbolic polynomial p on a vector space V

$$-\frac{p'_a(x)}{p(x)} \;=\; -\sum_{k=1}^{m} \frac{1}{\lambda_a^k(x)} \quad \text{is } \Gamma - \text{monotone on } \Gamma, \tag{5.15}$$

i.e., strictly increasing in t on lines $x+tb$ (with $x,b\in\Gamma$ and $t>0$). Equivalently,

$$\frac{p(x)}{p'(x)} \quad \text{is } \Gamma - \text{monotone on } \Gamma.$$

Since each $p^{(k)}(x)$ is positive on Γ, this implies that

$$\frac{p(x)}{p^{(k)}(x)} \;=\; \frac{p(x)}{p^{(1)}(x)}\frac{p^{(1)}(x)}{p^{(2)}(x)}\cdots\frac{p^{(k-1)}(x)}{p^{(k)}(x)}$$

is also Γ-monotone on Γ. Equivalently, each function $-p^{(k)}(x)/p(x)$ is Γ-monotone on Γ. Consequently, we have that for all $c_0,...,c_{m-1}\geq 0$, the set

$$F \;=\; \left\{A\in\overline{\Gamma(p)} : p(A)-\sum_{k=0}^{m-1} c_k r p^{(k)}(A) \;\geq\; 0\right\} \tag{5.16}$$

is a subequation for any Dirichlet-Gårding polynomial of degree m on $\mathrm{Sym}^2(\mathbf{R}^n)$.

Obviously there are many other ways to use (5.15) to construct suequations. In Krylov [K], the Dirichlet problem was solved for F defined by (5.16) when $p(A)=\det A$ is the determinant.

Using the following lemma, one can show that F is a convex subequation.

Lemma 5.30. *Suppose that p is a-hyperbolic on V. Then on the Gårding cone Γ the function $p(x)/p'_a(x)$ is concave.*

Proof. Set $f(x)=p(x)/p'_a(x)$ and note that f is homogeneous of degree one, so that $(2.3)'$ can be used. By Theorem 3.6 it suffices to consider the universal case

$$p(x) \;=\; x_1\cdots x_m \qquad \text{with } \; a \;=\; (1,...,1).$$

Then $f(x)=p(x)/p'_a(x)=1/(\sum\frac{1}{x_k})$, and $f'_x(y)=(\sum\frac{x_k}{y_k})/(\sum\frac{1}{y_k})^2$ so that $(2.3)'$ becomes

$$\left(\sum_k\frac{1}{y_k}\right)^2 \;\leq\; \left(\sum_k\frac{1}{x_k}\right)\left(\sum_k\frac{x_k}{y_k^2}\right)$$

which is the Cauchy-Schwartz inequality where $a_k=\frac{1}{\sqrt{x_k}}$ and $b_k=\frac{\sqrt{x_k}}{y_k}$. ■

The fact that F is a convex subequation is an immediate consequence of the next result.

Corollary 5.31.

$$\log\frac{p^{(k)}(x)}{p(x)}, \quad \textit{and therefore also} \quad \frac{p^{(k)}(x)}{p(x)}, \quad \textit{is convex on} \;\; \Gamma.$$

Proof. Applying the convex increasing function $\varphi(t) = -\log(-t)$, $t < 0$, to the convex function $-\frac{p(x)}{p'(x)}$ shows that $\log \frac{p'(x)}{p(x)}$ is convex. Hence $\log \frac{p^{(k)}}{p(x)} = \sum_{j=1}^{k} \log \frac{p^{(j)}}{p^{(j-1)}(x)}$ is convex. ∎

6. The Dirichlet Problem on Riemannian Manifolds

Many of the equations discussed above can be carried over to riemannian manifolds, and existence and uniqueness results for the Dirichlet problem have been established in this context in [HL$_2$]. An essential ingredient for obtaining results in this general setting centers on the concept of a monotonicity cone. The discussion above fits neatly into this theory since for any I-hyperbolic polynomial on $\mathrm{Sym}^2(\mathbf{R}^n)$, the closed Gårding cone is a monotonicity cone for each of the branches $F_k = \{\lambda_k \geq 0\}$. We give here a brief sketch of these ideas.

Let X be a riemannian manifold and recall that any $u \in C^2(X)$ has a well-defined riemannian Hessian given by

$$(\mathrm{Hess}\, u)(V, W) \;\equiv\; VWu - (\nabla_V W)u$$

for vector fields V and W, where ∇ is the Levi-Civita connection. This defines a symmetric 2-form on TX, that is, a section $\mathrm{Hess}\, u$ of $\mathrm{Sym}^2(T^*X)$.

Suppose now that $F \subset \mathrm{Sym}^2(T^*X)$ is a closed subset satisfying the positivity condition $F + \mathcal{P} \subset F$ where $\mathcal{P} \equiv \{A \in \mathrm{Sym}^2(T^*X) : A \geq 0\}$. Then a function $u \in C^2(X)$ is said to be F-subharmonic if $\mathrm{Hess}_x u \in F$ for all $x \in X$. This notion extends to any $u \in \mathrm{USC}(X)$ essentially as in Definition 4.8 above. With this understood, the formulation given in Section 4 of the Dirichlet problem for a domain $\Omega \subset\subset X$ and the formulation of strict F-convexity for $\partial\Omega$ carry over to this setting.

However, the global nature of the problem requires a further global hypothesis before analogues of Theorem 4.15 can be established. Let $M \subset \mathrm{Sym}^2(T^*X)$ be a closed subset such that $M + \mathcal{P} \subset M$ and each fibre M_x is a convex cone with vertex at the origin. Then M is called a **monotonicity cone for** F if under point-wise sum,

$$F + M \;\subset\; F.$$

The equations of interest here are those which are,in a sense, “universal” in geometry. We now make this precise. Fix a closed subset $\mathbf{F} \subset \mathrm{Sym}^2(\mathbf{R}^n)$ with $\mathbf{F} + \mathcal{P} \subset \mathbf{F}$ and suppose that

$$g(\mathbf{F}) = \mathbf{F} \qquad \text{for all } g \in \mathrm{O}_n \tag{6.2}$$

where O_n acts on $\mathrm{Sym}^2(\mathbf{R}^n)$ in the standard way. Then $\mathbf{F}$ naturally determines a subequation F on any riemannian manifold X as follows. Let $e_1, ..., e_n$ be an orthonormal tangent frame field defined on an open set $U \subset X$. This determines a trivialization

$$\mathrm{Sym}^2(T^*U) \;\xrightarrow[\cong]{\varphi_U}\; U \times \mathrm{Sym}^2(\mathbf{R}^n), \tag{6.3}$$

and we define

$$F \cap \mathrm{Sym}^2(T^*U) \ \equiv\ \varphi_U^{-1}(U \times \mathbf{F}). \tag{6.4}$$

By the invariance (6.2) this set is independent of the choice of orthonormal frame field. The resulting subequation

$$F \ \subset\ \mathrm{Sym}^2(T^*X)$$

is said to be **universally determined** by the constant coefficient subequation $\mathbf{F}$.

This construction can be generalized. For $\mathbf{F} \subset \mathrm{Sym}^2(\mathbf{R}^n)$ as above, let

$$G \ =\ G(\mathbf{F}) \ \equiv\ \{g \in \mathrm{O}_n : g(\mathbf{F}) = \mathbf{F}\},$$

and suppose X is provided with a **topological G structure**. This means that there is given a distinguished covering of X by open sets $\{U_\alpha\}$ with an orthonormal frame field $e^\alpha = (e_1^\alpha, ..., e_n^\alpha)$ on each U_α so that the each change of framing has values in G, i.e.,

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \ \to\ G \subset\ \mathrm{O}_n.$$

Then in the manner above, $\mathbf{F}$ *universally determines a subequation $F \subset J^2(X)$ on any riemannian manifold with a topological G-structure.*

For example any almost complex manifold with a compatible riemannian metric (i.e., for which J is point-wise orthogonal) carries a topological U_n-structure.

The following is one of the results proved in [HL$_2$].

THEOREM 6.1. *Let X be a riemannian manifold with a topological G-structure. Let $\mathbf{F} \subset \mathrm{Sym}^2(\mathbf{R}^n)$ be a subequation with monotonicity cone $\mathbf{M} \subset \mathrm{Sym}^2(\mathbf{R}^n)$ and suppose both are G-invariant. Let $F, M \subset J^2(X)$ be the subequations on X universally determined by $\mathbf{F}$ and $\mathbf{M}$.*

Suppose X carries a C^2 strictly M-subharmonic function.

Then for each bounded domain $\Omega \subset\subset X$ with smooth boundary $\partial\Omega$ which is both $\overrightarrow{F}$ and $\overrightarrow{\widetilde{F}}$ strictly convex, the Dirichlet problem is uniquely solvable for all continuous boundary data. *That is, for each $\varphi \in C(\partial\Omega)$, these exists a unique $u \in C(\overline{\Omega})$ satisfying:*

(1) u is F-harmonic on Ω, and

(2) $u = \varphi$ on $\partial\Omega$.

The constructions above give many interesting examples of such universal equations in geometry. For any G-invariant Dirichlet-Gårding polynomial, the closed Gårding cone $\mathbf{F}_1$ is a monotonicity cone for every branch $\mathbf{F}_k$ and all branches are G-invariant. All further constructions can be carried over as well.

Appendix A. An Algebraic Description of the Branches

Suppose $p(x)$ is a-hyperbolic of degree m as in Section 2. Each of the branches

$$F_m \ \supset\ \cdots\ \supset\ F_1$$

of $\{p = 0\}$ can be described by polynomial inequalities involving the elementary symmetric functions. Normalize so that $p(a) = 1$ and recall from (2.20) that

$$p(ta + x) \;=\; \prod_{k=1}^{m}(t + \lambda_k(x)) \;=\; t^m + \sigma_1(x)t^{m-1} + \cdots + \sigma_m(x) \tag{A.1}$$

where

$$\sigma_k(x) \;=\; \sum_{i_1 < \cdots < i_k} \lambda_{i_1}(x) \cdots \lambda_{i_k}(x)$$

is the kth elementary symmetric function of the a-eigenvalues of x. Set $\sigma_0(x) \equiv 1$. Each $\sigma_k(x)$ is a homogeneous polynomial in x of degree k, which is a-hyperbolic as noted in (3) of Section 2.

Now suppose that

$$f(t) \;=\; \prod_{k=1}^{m}(t + \lambda_k) \;=\; t^m + \sigma_1 t^{m-1} + \cdots + \sigma_m$$

is any monic polynomial with real roots. Let E^+ denote the number of $\lambda_k \ge 0$. Let $\mathrm{Var}(\sigma)$ denote the total number of strict consecutive sign changes in the signs of $\sigma = (\sigma_0, \sigma_1, ..., \sigma_m)$ where $\sigma_0 = 1$. By definition *strict* means first drop all zeros in σ and then compute the number of consecutive sign changes in the remaining tuple of real numbers. The classical *Descartes rule of signs* is the statement that

$$E^+ + \mathrm{Var}(\sigma) \;=\; m \tag{A.2}$$

Lemma A.1. *The kth branch F_k of $\{p = 0\}$ is given by $F_k = \{x : \mathrm{Var}(\sigma(x)) \;\le\; k-1\}$.*

For example, $\mathbf{F}_1(M)$ is defined by

$$\sigma_1(x) \;\ge\; 0, \;\cdots,\; \sigma_m(x) \;\ge\; 0 \tag{A.3}$$

since $\mathrm{Var}(\sigma) \le 0$ if and only if $\sigma_1 \ge 0$,...,$\sigma_m \ge 0$.

Proof. Using the ordered a-eigenvalues of x, the kth branch F_k of $\{p = 0\}$ is defined by $\lambda_k(x) \ge 0$. or equivalently by the condition

$$E^+(x) \;\ge\; m - k + 1.$$

By Descarte's rule of signs (A.2), this is equivalent to $\mathrm{Var}(\sigma(x)) \le k - 1$. ■

Corollary A.2. (Algebraic description of the branches). *Set $\epsilon = (\epsilon_0, ..., \epsilon_m)$ with $\epsilon_0 = 1$ and each $\epsilon_j = \pm 1$. Define F^ϵ by the conditions*

$$\epsilon_1\sigma_1(x) \;\ge\; 0, \;\cdots\;,\; \epsilon_m\sigma_m(x) \;\ge\; 0. \tag{A.4}$$

Then

$$F_k \;=\; \bigcup_{\mathrm{Var}(\epsilon)\le k-1} F^{\epsilon}. \tag{A.5}$$

Example A.3. Consider the polynomial $p(x,y,z) = \frac{1}{3}(xy+xz+yz)$ in $\mathbf{R}^3$ and set $\mathbf{a} = (1,1,1)$. Then

$$p(t\mathbf{a}+\mathbf{x}) \;=\; t^2 + \tfrac{2}{3}(x+y+z)t + \tfrac{1}{3}(xy+xz+yz) \;=\; t^2+\sigma_1 t+\sigma_2.$$

One easily computes that $\{\sigma_2 \ge 0\} = C \cup (-C)$ where C is the convex circular cone with base $B = \{x+y+z=1\} \cap \{x^2+y^2+z^2 \le 1\}$. The hyperplane $\{\sigma_1 = 0\}$ divides the set $\{\sigma_2 \ge 0\}$ into the two pieces C and $-C$. The decompositions in Corollary A.2 are

$$F_1 \;=\; C \;=\; F^{(1,1,1)} \qquad \text{and} \qquad F_2 \;=\sim(-C) \;=\; F^{(1,1,1)} \cup F^{(1,-1,-1)} \cup F^{(1,1,-1)}$$

Appendix B. Gurvits' Inequality

Gårding's classical inequality and the recent improvement by Gurvits are derived in this appendix. We present the proofs in a way that emphasizes the parallels. Suppose that p is an a-hyperbolic polynomial of degree m, and let $\lambda_1(x),...,\lambda_m(x)$ denote the a-eigenvalues of $x \in V$. Elementary Property (3) says that

$$p(a+tb) \;=\; p(a)\prod_{k=1}^m \left(1+t\lambda_a^k(b)\right) \qquad \forall\, t\in\mathbf{R}, b\in V. \tag{B.1}$$

and

$$p_b'(a) \;=\; p(a)\sum_{k=1}^m \lambda_a^k(b). \tag{B.2}$$

If $\lambda_1,...,\lambda_m > 0$, then the classical inequality between the geometric and arithmetic means says that

$$\prod_{k=1}^m (1+t\lambda_k) \;\le\; \left(1+\frac{\sum\lambda_k}{m}t\right)^m \qquad \text{with equality} \iff \lambda_1=\cdots=\lambda_m. \tag{B.3}$$

Applying (B.3) to (B.1) and using (B.2) to substitute for $\sum\lambda_k(b)$, proves the following inequality.

Lemma B.1. *Suppose p is an a-hyperbolic polynomial of degree m and $b\in\Gamma(p)$. Then for $0<t<\infty$*

$$p(a+tb) \;\le\; p(a)\left(1+\frac{p_b'(a)}{m\,p(a)}t\right)^m \tag{B.4}$$

Equality holds if and only if one of the following equivalencies holds:

(i) $\lambda_1(b) \;=\; \cdots \;=\; \lambda_m(b)$.

(ii) a *and* b *are proportional module the edge* $E(p)$.

(iii) $p(sa+tb) \;=\; (\alpha s+\beta t)^m$ *for some* $\alpha>0, \beta>0$.

Proof that (i) $\iff$ (iii). Use formula $(2.1)'$ for $p(sa+tb)$.

Proof that (i) $\iff$ (ii). Recall (Theorem 2.3(a)) that the edge equals the null set of p. By Property (2), $\lambda_k(b-\mu a)=\lambda_k(b)-\mu$. This proves that

$$\lambda_k(b) \;=\; \mu \ \text{ for all } k \qquad \iff \qquad b-\mu a\in \mathrm{Edge}(p). \qquad \blacksquare \tag{B.5}$$

Gårding's inequality follows from (B.4) by dividing both sides by t^m and taking infimums, which occur at $t=\infty$ for both sides.

Proposition B.2. (Gårding). *For all* $a,b\in\Gamma(p)$ *one has*

$$p(b) \;=\; \inf_{t>0}\frac{1}{t^m}p(a+tb) \;\le\; p(a)^{1-m}\left(\frac{1}{m}p_b'(a)\right)^m \tag{B.6}$$

with equality as in Lemma B.1.

Gurvits' inequality follows from (B.4) by dividing both sides by t and taking infimums. If equality occurs, then the infimums must occur at the same point t, and at this point t, equality must also occur in (B.4). This establishes the equality statement in the next result.

Proposition B.3. (Gurvits). *Assume* $m\ge 2$. *For all* $a,b\in\Gamma(p)$

$$\frac{(m-1)^{m-1}}{m^m}\inf_{t>0}\frac{p(a+tb)}{t} \;\le\; \frac{1}{m}p_b'(a) \tag{B.7}$$

with equality the same as in Lemma B.1.

Proof. By (B.4) for each $t>0$

$$\frac{p(a+tb)}{t} \;\le\; p(a)\frac{\left(1+\frac{\alpha}{m}\right)^m}{t} \quad \text{with} \quad \alpha=\frac{p_b'(a)}{p(a)}. \tag{B.8}$$

The function $f(t)\equiv\frac{1}{t}\left(1+\frac{\alpha}{m}\right)^m$ on the interval $0<t<\infty$ blows up at the endpoints and has a single critical point at $t=\frac{m}{m-1}\frac{1}{\alpha}=\frac{m}{m-1}\frac{p(a)}{p_b'(a)}$ with critical value $(\frac{m}{m-1})^m\alpha=(\frac{m}{m-1})^m\frac{p_b'(a)}{p(a)}=\frac{m^m}{(m-1)^{m-1}}\frac{1}{p(a)}\frac{1}{m}p_b'(a)$. This proves the inequality (B.7). $\blacksquare$

Gårding's full inequality follows from the special case (B.6) by induction.

THEOREM B.4. (Gårding). *Suppose that* p *is* a*-hyperbolic of degree* m *and* $b_1,...,b_m\in\Gamma(p)$. *Then*

$$p(b_1)^{\frac{1}{m}}\cdots p(b_m)^{\frac{1}{m}} \;\le\; \frac{1}{m!}\,p^{(m)}_{b_1,\ldots,b_m}. \tag{B.9}$$

with equality if and only if one of the following equivalencies holds:

(i) $b_1, ..., b_m$ *are pairwise proportional modulo the edge* $E(p)$.

(ii) *Edge*$(p) \cap W$ *has codimension one in* $W \equiv \text{span}\{b_1, ..., b_m\}$.

(iii) $p(t_1 b_1 + \cdots t_m b_m) \;=\; (\alpha_1 t_1 + \cdots + \alpha_m t_m)^m$ *for some* $\alpha_1, ..., \alpha_m > 0$.

Proof. Induction can be employed because p'_b is also a-hyperbolic with $\Gamma(p'_b) \supset \Gamma(p)$ and $\text{Edge}(p'_b) = \text{Edge}(p)$. (See Propositions 2.21 and 2.32.) Employing (B.6)$'$ with $a = b_j$, $1 \le j \le m-1$ and with $b = b_m$ we have

$$p(b_j)^{\frac{m-1}{m}} p(b_m)^{\frac{1}{m}} \;\le \frac{1}{m} p'_{b_m}(b_j) \qquad \text{for} \;\; j = 1, ..., m-1.$$

Taking the product of both sides over $j = 1, ..., m-1$ yields

$$p(b_1)^{\frac{m-1}{m}} \cdots p(b_m)^{\frac{m-1}{m}} \;\le\; \frac{1}{m^{m-1}} p'_{b_m}(b_1) \cdots p'_{b_m}(b_{m-1}) \tag{B.10}$$

The induction hypothesis applied to the polynomial p'_{b_m} says that

$$p'_{b_m}(b_1)^{\frac{1}{m-1}} \cdots p'_{b_m}(b_{m-1})^{\frac{1}{m-1}} \;\le\; \frac{1}{(m-1)!} p^{(m)}_{b_1,...,b_m} \tag{B.11}$$

Taking the $(m-1)^{\text{st}}$ root of both sides of (B.10) and combining it with (B.11) proves (B.9). The equality statement follows easily from Lemma B.1. ■

Remark B.5. If $a, b_1, ..., b_m \in V$, then the m-fold directional derivative in the directions $b_1, ..., b_m$ is a constant independent of a. Let $\overline{p}$ denote the completely polarized form of p. Then

$$\frac{1}{m!} p^{(m)}_{b_1,...,b_m} \;=\; \frac{1}{m!} \frac{\partial^m}{\partial t_1 \cdots \partial t_m} p(a + t_1 b_1 + \cdots + t_m b_m)\bigg|_{t=0} \;=\; \overline{p}(b_1, ..., b_m) \tag{B.12}$$

If q is a homogeneous polynomial of degree k, then $q'_a(a) = kq(a)$ for any direction $a \in V$. Setting $b_{k+1} = \cdots = b_m = a \in \Gamma(p)$ in Gårding's inequality (B.9) yields

$$p(b_1)^{\frac{1}{m}} \cdots p(b_k)^{\frac{1}{m}} p(a)^{\frac{m-k}{m}} \;\le\; \frac{1}{k!}\, p^{(k)}_{b_1,...,b_k}(a). \tag{B.13}$$

Gurvits' full inequality follow from the special case (B.7) by induction.

THEOREM B.6. (Gurvits). *Suppose that* p *is* a*-hyperbolic and* $b_1, ..., b_m \in \Gamma(p)$. *Then*

$$\frac{1}{m^m} \text{Cap}(p) \;\stackrel{\text{def}}{=}\; \frac{1}{m^m} \inf_{t_1,...,t_m > 0} \frac{p\,(t_1 b_1 + \cdots + t_m b_m)}{t_1 \cdots t_m} \;\le\; \frac{1}{m!} p^{(m)}_{b_1,...,b_m} \tag{B.14}$$

with equality holding exactly as in Gårding's Theorem.

Proof. We use Lemma B.3 repeatedly.

$$\frac{1}{m^m}\inf_{t_m>0}\frac{p\left(t_1b_1+\cdots+t_mb_m\right)}{t_1\cdots t_m} \;\leq\; \frac{1}{m(m-1)^{m-1}}\frac{p^{(1)}_{b_m}(t_1b_1+\cdots+t_{m-1}b_{m-1})}{t_1\cdots t_{m-1}}$$

and

$$\frac{1}{m(m-1)^{m-1}}\inf_{t_{m-1}>0}\frac{p^{(1)}_{b_m}\left(t_1b_1+\cdots+t_{m-1}b_{m-1}\right)}{t_1\cdots t_{m-1}}$$
$$\leq\; \frac{1}{(m-1)(m-2)^{m-2}}\frac{p^{(2)}_{b_{m-1},b_m}(t_1b_1+\cdots+t_{m-1}b_{m-2})}{t_1\cdots t_{m-2}}$$

etc., proves that

$$\frac{1}{m^m}\inf_{t_2,\ldots,t_m>0}\frac{p\left(t_1b_1+\cdots+t_mb_m\right)}{t_1\cdots t_m} \;\leq\; \frac{1}{m!}\frac{p^{(m-1)}_{b_2,\ldots,b_m}(t_1b_1)}{t_1}$$

which equals $\frac{1}{m!}p^{(m)}_{b_1,\ldots,b_m}$ since $p^{(m-1)}_{b_2,\ldots,b_m}$ is homogeneous of degree 1. ■

Remark B.7. (Capacity). The **capacity** of p with respect to $b_1,...,b_m\in\Gamma(p)$ is defined by

$$\mathrm{Cap}_{b_1,\ldots,b_m}(p) \;=\; \inf_{t_1,\ldots,t_m>0}\frac{p(t_1b_1+\cdots+t_mb_m)}{t_1\cdots t_m}$$

The inductive step above can be written as

$$\frac{1}{k^k}\mathrm{Cap}_{b_1,\ldots,b_k}\left(p^{(m-k)}_{b_{k+1},\ldots,b_m}\right) \;\leq\; \frac{1}{k(k-1)^{k-1}}\mathrm{Cap}_{b_1,\ldots,b_{k-1}}\left(p^{(m-k+1)}_{b_k,\ldots,b_m}\right) \qquad (B.15)$$

Remark B.8 (Gurvits Refines/Improves Gårding). Suppose p is a-hyperbolic and $b_1,...,b_m\in\overline{\Gamma(p)}$.

$$p(b_1)^{\frac{1}{m}}\cdots p(b_m)^{\frac{1}{m}} \;\leq\; \frac{1}{m^m}\inf_{x_1,\ldots,x_m>0}\frac{p(x_1b_1+\cdots+x_mb_m)}{x_1\cdots x_m} \;\stackrel{\text{def}}{=}\; \frac{1}{m^m}\mathrm{Cap}(p) \qquad (B.16)$$

Proof. Using homogeneity and replacing b_j by x_jb_j the inequality becomes

$$p(b_1)^{\frac{1}{m}}\cdots p(b_m)^{\frac{1}{m}} \;\leq\; \frac{1}{m^m}p(b_1+\cdots+b_m). \qquad (B.16)'$$

We may assume that $p(b_j)>0$, $j=1,...,m$, i.e., $b_1,...,b_m\in\Gamma(p)$.

Now (B.16)$'$ follows from the classical geometric-arithmetic mean inequality:

$$p(b_1)^{\frac{1}{m}}\cdots p(b_m)^{\frac{1}{m}} \;\leq\; \left[\frac{p(b_1)^{\frac{1}{m}}+\cdots+p(b_m)^{\frac{1}{m}}}{m}\right]^m$$

and the concavity of $p^{\frac{1}{m}}$:

$$p(b_1)^{\frac{1}{m}}+\cdots+p(b_m)^{\frac{1}{m}} \;\leq\; p(b_1+\cdots+b_m)^{\frac{1}{m}}$$

.

Remark B.9. Note that the L.H.S. in the Gårding inequality vanishes if any one of $b_1,...,b_m$ are on the boundary of $\Gamma(p)$. However, the $(b_1,...,b_m)$-capacity of p occuring in the Gurvits inequality may not be zero. See [Gur$_1$] for a more general upper bound result for the $(b_1,...,b_m)$-capacity which allows for degeneracies.

References

[BGLS] H. Bauschke, O. Guler, A. Lewis and H. Sendov, *Hyperbolic polynomials and convex analysis*, Canad. J. Math. **53** no. 3, (2001), 470-488.

[CNS] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian*, Acta Math. **155** (1985), 261-301.

[C] M. G. Crandall, *Viscosity solutions: a primer*, pp. 1-43 in "Viscosity Solutions and Applications" Ed.'s Dolcetta and Lions, SLNM **1660**, Springer Press, New York, 1997.

[CIL] M. G. Crandall, H. Ishii and P. L. Lions *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N. S.) **27** (1992), 1-67.

[G_1] L. Gårding, *Linear hyperbolic differential equations with constant coefficients*, Acta. Math. **85** (1951), 2-62.

[G_2] L. Gårding, *An inequality for hyperbolic polynomials*, J. Math. Mech. **8** no. 2 (1959), 957-965.

[Gu] Guler, *Hyperbolic polynomials and interior point methods for convex programming*, Math. Oper. Res. **22** no. 2 (1997), 350-377.

[Gur_1] L. Gurvits, *Hyperbolic polynomials approach to Van der Waerden/Schrijver-Valiant like conjectures: sharper bounds, simpler proofs and algorithmic applications* , *Proc. 38 ACM Symp. on Theory of Computing (StOC-2006)*, 417-426, ACM, New York, 2006.

[Gur_2] L. Gurvits, *Van der Waerden/Schrijver-Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all*, ArXiv:0711.3496, May, 2008.

[HL_1] F. R. Harvey and H. B. Lawson, Jr., *Dirichlet duality and the non-linear Dirichlet problem*, Comm. on Pure and Applied Math. **62** (2009), 396-443.

[HL_2] F. R. Harvey and H. B. Lawson, Jr., *Dirichlet duality and the non-linear Dirichlet problem on riemannian manifolds*, ArXiv: 0907.1981.

[HL_3] F. R. Harvey and H. B. Lawson, Jr., *Lagrangian plurisubharmonicity and convexity*, Stony Brook Preprint (2010).

[H] L. Hörmander, *Lagrangian plurisubharmonicity and convexity*, Stony Brook Preprint (2010).

[K] N. V. Krylov, *Notions of Convexity*, Progress in Mathematics **127**, Birkháuser, Boston, 1994.

[L] P. Lax, *Differential equations, difference equations and matrix theory*, Comm. on Pure and Applied Math. **11** (1958), 175-194.

[LPR] A. S. Lewis, P. A. Parrilo, M. V. Ramana, *The Lax conjecture is true*, Proc. Amer. Math. Soc. **133** (2005), no. 9, 2495–2499.

[R] J. Renegar, *Hyperbolic programs, and their derivative relaxations*, Found. Comput. Math. **6** (2006) no. 1, 59-79.

[S] Z. Slodkowski, *The Bremermann-Dirichlet problem for q-plurisubharmonic functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **11** (1984), 303-326.

[V] V. Vinnikov, *Selfadjoint determinantal representations of real plane curves*, Math. Ann. **296** (1993), 453-479.

[Y] Y. Yuan, *Global solutions to special lagrangian equations*, Proc. A. M. S. **134** no. 5 (2005), 1355-1358.