Final Exam – MAT 536.

Instructions: Do 8 of the 10 problems.

1. Let $\Omega \subset\subset \mathbf{C}$ be a domain with a piecewise smooth boundary $\partial\Omega$.

(a) Let $f \in C(\overline{\Omega})$ be holomorphic on Ω. Please state the basic case of Cauchy's Integral Formula in this setting.

(b) Fix $z_0 \in \Omega$. Show that f has a convergent power series about z_0. For which radiii $r > 0$ do you know that this series is absolutely convergent?

2. Use the argument principle to prove the Open Mapping Theorem for holomorphic functions.

3. Let $\Delta = \{z : |z| < 1\}$ and $H = \{z : \mathrm{Im}(z) > 0\}$. Let $f : \Delta \to H$ be holomorphic with $f(0) = i$. Show that

$$|f(z) - i| \;\leq\; |z||f(z) + i|.$$

4. Compute

$$\frac{1}{2i}\int_{|z|=\frac{3}{2}} \frac{e^z \cos(\pi z)}{\sin(\pi z)}\, dz$$

(using any theorem proved in the course).

5. How many zeros (counted to multiplicity) does the function

$$f(z) = \frac{1}{2}e^z + 4z^6 + \frac{1}{2}z^8$$

have in the disk $|z| < 1$.

6. Let $\mathcal{F}$ be a family of holomorphic functions on a domain $\Omega \subset \mathbf{C}$.

(a) Show that if $\mathcal{F}$ is uniformly bounded on every closed disk $D \subset \Omega$, then $\mathcal{F}$ is equicontinuous on every closed disk $D \subset \Omega$.

(b) Show that if $\mathcal{F}$ is equicontinuous on every closed disk $D \subset \Omega$, then $\mathcal{F}$ is equicontinuous on every compact subset of Ω.

7. (a) State the Mittag-Leffler Theorem (about principle parts of meromorphic functions on $\mathbf{C}$).

(b) Give the proof.

8. Write down an infinite product of holomorphic functions on $\mathbf{C}$ which converges absolutely on compact subsets and has zeros, all of which are order one, at exactly the points 1, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, ...

9. (a) State the Riemann Mapping Theorem for a domain $\Omega \subset \mathbf{C}$ and a point $z_0 \in \Omega$. (Include all the neccesary hypotheses.)

(b) If z_0 is a real number and if $\Omega = \overline{\Omega}$, show that the Riemann map f satisfies

$$f(z) = \overline{f(\overline{z})}.$$

10. Let Ω be a simply-connected domain in $\mathbf{C}$, and $\{f_n\}_{n=1}^{\infty}$ a sequence of holomorphic functions on Ω. Suppose that for all n,

$$f_n(z) \neq 0 \text{ or } 1 \qquad \text{for any } z \in \Omega.$$

Show that $\{f_n\}_{n=1}^{\infty}$ has a subsequence that converges uniformly on compact subsets of Ω to a holomorphic function f on Ω. (You may use anything I have done in class.)