THE DE RHAM-FEDERER THEORY OF DIFFERENTIAL CHARACTERS AND CHARACTER DUALITY

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Abstract. The theory of differential characters is developed completely from a de Rham-Federer viewpoint. Characters are defined as equivalence classes of special currents, called sparks, which appear naturally in the theory of singular connections. As in de Rham-Federer cohomology, many different spaces of currents are shown to yield the character groups. The fundamental exact sequences are derived, and a multiplication of de Rham-Federer characters is defined using new transversality/intersection results for flat and rectifiable currents. An equivalence of de Rham-Federer characters with the classical Cheeger-Simons characters is given, as in de Rham cohomology, via integration. This discussion rounds out the approach to characters introduced by Gillet-Soulé and Harris. The groups of differential characters have a natural topology and *smooth Pontrjagin duals* (introduced here) which fit into fundamental exact sequences. A principal result is the formulation and proof of *duality* for characters on oriented manifolds. It is shown that the pairing $(a, b) \mapsto a * b([X])$ defined by multiplication and evaluation on the fundamental cycle, gives an isomorphism of the group of differential characters of degree k with the dual to characters in degree $n - k - 1$ where n = $\dim(X)$. Thom homomorphisms, which refine the usual ones in de Rham theory and integral cohmology, are established for differential characters. Gysin maps for characters are similarly established. New examples, namely Morse sparks and Hodge sparks, are introduced and examined in detail.

0. Introduction. In 1973 Jim Simons and Jeff Cheeger introduced the rings of differential characters and used them, among other things, to give an important generalization of the Chern-Weil homomorphism [S], [CS]. Their theory constituted a vast generalization of the Chern-Simons invariant, and was applied to prove striking global nonconformal-immersion theorems in Riemannian geometry. The theory also gave rise to new invariants for flat connections, and provided an insightful derivation of the whole family of foliation invariants due to Godbillon-Vey, Heitsch, Bott and others.

The first part of this paper presents a self-contained formulation of the theory of differential characters based on special currents, called *sparks*, which appear in the study of singular connections. This approach can be thought of as a *de Rham-Federer theory of differential characters* in analogy with de Rham-Federer cohomology theories. Such formulations are originally due to Gillet-Soulé [GS_1] and B. Harris [H]. Our discussion presents, rounds out, and unifies their work.

It is of course always useful to have distinct representations of the same theory, and there are many advantages of the de Rham-Federer viewpoint. One sees,

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for example, that the differential character groups carry a natural topology. One also finds that in many ways differential characters behave like a homology-cohomology theory. A striking instance is that differential characters obey a Poincaré-type *Duality* (a main result of this paper)—that is, on an oriented n-manifold, taking $*$-product and evaluating on the fundamental cycle identifies the compactly supported characters of degree k with the smooth Pontrjagin dual of the group of characters of degree $n-k-1$. That one should consider Pontrjagin duals when dealing with differential characters is a key point in this result. The theory of differential characters also carries Thom homomorphisms, Gysin mappings, and certain relative long exact sequences. There is also an *Alexander-Lefschetz-type Duality Theorem*. (This and the long exact sequences will be treated in a separate paper.)

A *spark* on a manifold X is a de Rham current a whose exterior derivative can be decomposed as

$$da = \phi - R \tag{0.1}$$

where ϕ is a smooth form and R is integrally flat. (A current is integrally flat if it can be written as $R + dS$ where R and S are rectifiable.) Such objects arise quite naturally. Suppose for example that $E \to X$ is a smooth oriented vector bundle with connection, and α is a section with nondegenerate zeros $\mathrm{Div}(\alpha)$. Pulling down the classical Chern transgression gives a canonical L^1_{loc}-form τ with $d\tau = \chi(\Omega) - \mathrm{Div}(\alpha)$ where $\chi(\Omega)$ is the Weil polynomial in the curvature which represents the Euler class of E. Quite general formulas of this type are derived systematically in [HL_{1-4}].

A basic fact is that the decomposition (0.1) is *unique*. Consequently, every spark has two differentials: $d_1 a = \phi$ and $d_2 a = R$. Two sparks a_1 and a_2 are defined to be equivalent if $a_1 = a_2 + db + S$ where b is any current and S is integrally flat. The spark equivalence classes define the space of characters $\widehat{\mathbb{H}}^*(X)$, and the differentials d_1 and d_2 yield the two fundamental exact sequences

$$0 \longrightarrow \widehat{\mathbb{H}}^k_\infty(X) \longrightarrow \widehat{\mathbb{H}}^k(X) \xrightarrow{\delta_2} H^{k+1}(X, \mathbf{Z}) \longrightarrow 0.$$

$$0 \longrightarrow H^k(X, S^1) \longrightarrow \widehat{\mathbb{H}}^k(X) \xrightarrow{\delta_1} Z_0^{k+1}(X) \longrightarrow 0.$$

in the theory. (See Propositions 1.11 and 1.12.)

In the classical case of de Rham-Federer cohomology there are many spaces of currents which represent the same functor. (For example, $H^*(X; \mathbf{R})$ can be represented as the cohomology of all currents, or of normal currents, or of flat currents, or of smooth currents. Similarly, $H^*(X; \mathbf{Z})$ can be represented by integral currents, or by integrally flat currents, or by the currents induced by smooth singular chains, etc..) Analogously we show in §2 that there are many spaces of sparks which under suitable equivalence yield differential characters. For example, one can restrict a to be flat and R to be rectifiable, or one can restrict a to be a form with L^1_{loc}-coefficients and R to be a smooth singular chain. This flexibility is quite useful in practice.

In §3 we introduce a product, denoted $*$, on differential characters using a basic "exterior/intersection" product on currents. The generic existence of this exterior/intersection product itself constitutes a new result which is essentially a *transversality theorem for flat and rectifiable currents*. The proof, which combines certain geometric arguments with Federer slicing theory, is given in the appendix.

We remark that defining products at the chain/cochain-level in topology is always problematical, and the intersection theory for currents in Appendix B should have considerable independent interest.

In §4 we exhibit an equivalence of functors from deRham-Federer characters to the classical Cheeger-Simons theory. As in the cohomology analogue, this equivalence is essentially given by integration over cycles (or "spark holonomy"). It is then shown that our intersection $*$-product agrees with the product introduced by Cheeger on differential characters [C]. Thus the equivalence is an equivalence of ring functors.

From our point of view it is natural to consider the *smooth characters* namely those which are representable by sparks which are smooth forms. The space of smooth characters is exactly the kernel of the differential δ_2. The characters also naturally inherit a topology, and this ideal of smooth characters is exactly the connected component of 0 in this topology.

The form ϕ in equation (0.1) could be considered the "curvature" of the spark a, and in this sense the kernel of δ_1 consists of the "curvature-flat" sparks.

One of the main new results here is the establishment of a *Poincaré-Pontrjagin duality* for differential characters. We preface this result by showing that taking Pontrjagin duals carries the two fundamental exact sequences above into strictly analogous sequences with roles reversed. We then prove that if X is compact and oriented, the pairing

$$\widehat{\mathbb{H}}^{n-k-1}(X) \times \widehat{\mathbb{H}}^k(X) \longrightarrow S^1$$

given by

$$(a, b) \mapsto a * b([X])$$

is nondegenerate. In fact, the resulting mapping

$$\widehat{\mathbb{H}}^{n-k-1}(X) \longrightarrow \text{Hom}(\widehat{\mathbb{H}}^k(X), S^1)$$

into continuous group homomorphisms has dense range consisting exactly of the *smooth* homomorphisms (See §6). In §7 we exhibit a number of examples of this duality on standard spaces.

In §8 we introduce characters with compact support. This allows us to extend duality to noncompact manifolds. There are in fact further extensions. On manifolds with boundary there is a Lefschetz-Pontrjagin Duality Theorem [HL_5]. There is also a general result of Alexander duality type.

The sparks studied here appear often in topology and geometry. They play a fundamental role in Morse Theory (see §11). They also appear in the Hodge decomposition of rectifiable cycles on a Riemannian manifold. From this we introduce a notion of linear equivalence on rectifiable cycles and define riemannian "Abel-Jacobi mappings" which strictly generalize the well-known maps from Kähler geometry. (See §12). On 3-manifolds these maps appear in the work of Chatterjee [Cha] and in Hitchin's discussion of special Lagrangian cycles and the mirror symmetry conjecture [Hi].

Sparks are also central in the theory of singular connections developed in [HL_{1-4}]. In this theory they appear as canonical transgression forms T with $dT = \Phi - \Sigma$ where Φ is a smooth "geometric" form (typically a Chern-Weil characteristic form) and Σ is an integrally flat cycle (typically defined by the singularities of a mapping).

For example, on the total space of an oriented bundle $E \to X$ with orthogonal connection, there is a canonical *Thom spark* γ with $d\gamma = \tau - [X]$ where τ is the canonical Thom form [HL_1, IV] and $[X]$ is the 0-section in E. Taking $*$-product with γ defines a Thom homomorphism for characters with the property that δ_1 and δ_2 transform it to the classical Thom mappings. (See §9.) This leads to general Gysin maps in the theory. (See §10.)

For a complex bundle $F \to X$ with unitary connection there are *Chern sparks* C_k with

$$dC_k = c_k(\Omega) - \mathbb{D}_k$$

on X where $c_k(\Omega)$ is the kth Chern characteristic form and $\mathbb{D}_k$ is the linear dependency current of a set of cross-sections of F. For a real bundle F with orthogonal connection there are similar *Pontrjagin-Ronga sparks* associated to rank-2 degeneracies, and if F is oriented, there are *Euler sparks* X_α associated to any atomic section α and satisfying: $dX_\alpha = \chi(\Omega) - \mathrm{Div}(\alpha)$. More generally to a bundle map $\alpha\colon\ E \to F$ there are *Thom-Porteous sparks* T_k with $dT_k = TP_k(\Omega^F, \Omega^E) - [\Sigma_k(\alpha)]$ where $\Sigma_k(\alpha)$ is the kth degeneracy locus of α and TP is the kth Thom-Porteous class expressed canonically in the curvatures of E and F. In each of these cases the character represented by the spark is independent of the sections (or bundle maps) involved. It is exactly the "characteristic character" defined by Cheeger-Simons in [CS]. This is elaborated in §13.

In low degrees differential characters have special interpretations, and so do the underlying sparks. A character of degree 0 corresponds to a smooth mapping $f\colon\ X \to S^1$ and its associated sparks are the functions $\tilde{f}\colon\ X \to \mathbf{R}$ in $L^1_{\mathrm{loc}}(X)$ with $\tilde{f} \equiv f$ (mod $\mathbf{Z}$). A character of degree 1 corresponds to a complex line bundle E with unitary connection up to gauge equivalence, and underlying sparks are given by pairs (E, σ) where σ is an atomic section of E. In degree 2 sparks are related to "gerbes with connection." This is discussed in §16 and will be treated in greater detail in a forthcoming paper [HL_6].

In this sequel [HL_6] we shall present an expanded approach to differential characters which involves a double Čech-deRhamCurrent complex of *hypersparks*. The theory is strikingly parallel to the one presented here (and, in fact, contains it). There is an equation analogous to the spark equation (0.1), and there is an equivalence relation, leading to characters, which generalizes the one given here. The sparks are contained in the hypersparks as a "full-subtheory" yielding an isomorphism at the level of characters. However, there are other interesting full-subtheories. For example, the subgroup of *smooth hypersparks* which are directly related to gerbes with connection ([Hi], [Br]).

Throughout this paper all manifolds are assumed to be oriented.

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1. De Rham-Federer characters. In this section we introduce and examine certain equivalence classes of special currents called sparks. We shall see later that our set of classes is naturally isomorphic to the Cheeger-Simons space of differential characters.

Let X be a smooth oriented manifold of dimension n (not necessarily compact). We adopt the following notation as in [deR]. Let $\mathcal{E}^k(X)$ denote the space of smooth differential k-forms on X with the C^∞-topology. By $\mathcal{D}^k(X)$ we mean the smooth k-forms with compact support on X. By the space of *currents of degree k (and dimension $n-k$)* on X we mean the topological dual space

$$\mathcal{D}'^k(X) \equiv \mathcal{D}'_{n-k}(X) \stackrel{\text{def}}{=} \{\mathcal{D}^{n-k}(X)\}'.$$

Note that $\mathcal{D}'^*(X)$ is a complex under d', the dual of d. The forms $\mathcal{E}^*(X)$ embed in the currents $\mathcal{D}'^*(X)$ by associating to a k-form φ the degree-k current, also denoted by φ, given by $\varphi(\psi) = \int \varphi \wedge \psi$ for $\psi \in \mathcal{D}^{n-k}(X)$. On this subspace $d' = (-1)^{n-k+1} d$, and thereby d extends to all of $\mathcal{D}'^k(X)$. The following subspaces of currents are defined in Federer [F].

$$\begin{aligned}
\mathcal{E}^k_{L^1_{\text{loc}}}(X) &= \text{the } k\text{-forms with } L^1_{\text{loc}}\text{-coefficients on } X\\
\mathcal{R}^k(X) &= \text{the locally } \textit{rectifiable} \text{ currents of degree } k \text{ (dimension } (n-k)) \text{ on } X\\
\mathcal{F}^k(X) &= \text{the locally } \textit{flat} \text{ currents of degree } k \text{ on } X\\
\mathcal{IF}^k(X) &= \text{the locally } \textit{integrally flat} \text{ currents of degree } k \text{ on } X
\end{aligned}$$

In what follows the word *locally* will often be supressed. We recall that:

$$\begin{aligned}(1.1)\qquad \mathcal{F}^k(X) &= \{\phi + d\psi : \phi \in \mathcal{E}^k_{L^1_{\text{loc}}}(X) \text{ and } \psi \in \mathcal{E}^{k-1}_{L^1_{\text{loc}}}(X)\}\\ \mathcal{IF}^k(X) &= \{R + dS : R \in \mathcal{R}^k(X) \text{ and } S \in \mathcal{R}^{k-1}(X)\}.\end{aligned}$$

We now introduce a space of currents which appears naturally in geometry and analysis (cf. [$BC_{1,2}$], [GS_{1-4}], [BGS], [HL_{1-4}], [$HZ_{1,2}$]).

Definition 1.2. The space of *sparks* of degree k on X is defined to be

$$\mathcal{S}^k(X) \equiv \{a \in \mathcal{D}'^k(X) : da = \phi - R \text{ where } \phi \in \mathcal{E}^{k+1}(X) \text{ and } R \in \mathcal{IF}^{k+1}(X)\}$$

LEMMA 1.3. *For $a \in \mathcal{S}^k(X)$ the decomposition of da into smooth plus integrally flat is unique, i.e., if $da = \phi - R = \phi' - R'$ where ϕ, ϕ' are smooth and R, R' are integrally flat, then $\phi = \phi'$ and $R = R'$.*

Proof. It suffices to show that

$$(1.4)\qquad \mathcal{E}^{k+1}(X) \cap \mathcal{IF}^{k+1}(X) = \{0\},$$

i.e., if a current T is both smooth and integrally flat, then $T = 0$. If T is smooth, then it has finite mass. By [F, 4.2.16(3), p. 413] any integrally flat current of finite mass is rectifiable. Thus T is both smooth and rectifiable. Since T is smooth, its $(n-k-1)$-density $\Theta(\|T\|, x) = 0$ for all $x \in X$. However, for a rectifiable current T, one has $\Theta(\|T\|, x) \geq 1$ for $\|T\|$-almost all x (cf. [F, 4.1.28, p. 384]). Thus $\|T\|$ and hence T vanish. □

For any spark $a \in \mathcal{S}^k(X)$ with decomposition $da = \phi - R$ as in Lemma 1.3 we define

$$d_1 a = \phi \qquad \text{and} \qquad d_2 a = R.$$

LEMMA 1.5. *Let $a \in \mathcal{S}^k(X)$ be a spark. Then $d_1 a = \phi$ is a smooth d-closed differential form with integral periods and $d_2 a = R$ is an integrally flat cycle.*

Proof. Since $da = \phi - R$, the current $d\phi = dR$ is both smooth and integrally flat. Hence,

$$d\phi = dR = 0$$

by (1.4). Since R has integral periods and ϕ is cohomologous to R, ϕ also has integral periods. □

Motivated by the comparison formula for euler sparks [HZ_2] (and, of course, Cheeger-Simons) we define an equivalence relation on the space of sparks.

Definition 1.6. For each integer k, $0 \le k \le n$ we define the group of *de Rham-Federer characters* of degree k to be the quotient

$$\widehat{\mathbb{H}}^k(X) \stackrel{\text{def}}{=} \mathcal{S}^k(X)/\{d\mathcal{D}'^{k-1}(X) + \mathcal{IF}^k(X)\}.$$

The equivalence class in $\widehat{\mathbb{H}}^k(X)$ of a spark $a \in \mathcal{S}^k(X)$ will be denoted by $\langle a \rangle$ or $\widehat{a}$. We also set

$$\widehat{\mathbb{H}}^{-1}(X) \stackrel{\text{def}}{=} \mathbf{Z}$$

with generator $\mathbf{I}$ satisfying $d_1\mathbf{I} = 1$ and $d_2\mathbf{I} = [X] = 1$. Note that (1.4) fails for $k = -1$.

Suppose a and $\bar{a}$ are equivalent sparks; i.e. $\bar{a} = a + db - S$, with $b \in \mathcal{D}'^{k-1}(X)$ and $S \in \mathcal{IF}^k(X)$. If $da = \phi - R$ then

$$d\bar{a} = \phi - (R + dS). \tag{1.7}$$

Let $\mathcal{Z}_0^{k+1}(X)$ denote the lattice of smooth d-closed, degree $k+1$ forms on X with integral periods. Also, given a cycle $R \in \mathcal{R}^{k+1}(X)$, let $[R]$ denote its class in $H^{k+1}(X, \mathbf{Z})$.

LEMMA 1.8. *The maps*

$$\delta_1 \colon \widehat{\mathbb{H}}^k(X) \to \mathcal{Z}_0^{k+1}(X) \quad \textit{and}$$
$$\delta_2 \colon \widehat{\mathbb{H}}^k(X) \to H^{k+1}(X, \mathbf{Z})$$

given by $\delta_1(\langle a \rangle) \stackrel{\text{def}}{=} \phi$ *and* $\delta_2(\langle a \rangle) \stackrel{\text{def}}{=} [R]$ *are well-defined surjections.*

Proof. Lemma 1.5 and (1.7) imply that δ_1 and δ_2 are well defined. Surjectivity is a direct consequence of the fundamental isomorphisms

$$H^*(\mathcal{D}'^*(X)) \cong H^*(X;\, \mathbf{R}) \qquad \text{and} \qquad H^*(\mathcal{IF}^*(X)) \cong H^*(X;\, \mathbf{Z}) \tag{1.9}$$

due to de Rham [deR] and Federer (cf. [HZ$_1$]) respectively. □

Definition 1.10. A differential character is said to be *smooth* if it can be represented by a smooth differential form. That is, the space $\widehat{\mathbb{H}}^k_\infty(X)$ of *smooth characters* of degree k on X is the image of the smooth forms $\mathcal{E}^k(X)$ in $\widehat{\mathbb{H}}^k(X)$. Note the natural isomorphism

$$\widehat{\mathbb{H}}^k_\infty(X) \cong \mathcal{E}^k(X)/\mathcal{Z}_0^k(X).$$

To see this suppose $a \in \mathcal{E}^k(X)$ represents the 0-character. Then $a = db + S$ where S is integrally flat. Thus b is a spark, and so $a \in \mathcal{Z}_0^k(X)$ by Lemma 1.5.

PROPOSITION 1.11. *There is a short exact sequence*

$$\text{(A)} \qquad 0 \longrightarrow \widehat{\mathbb{H}}^k_\infty(X) \longrightarrow \widehat{\mathbb{H}}^k(X) \xrightarrow{\delta_2} H^{k+1}(X, \mathbf{Z}) \longrightarrow 0.$$

Proof. If $\langle a \rangle \in \widehat{\mathbb{H}}^k(X)$ is represented by a smooth spark $a \in \mathcal{E}^k(X)$, then da is smooth and so $\delta_2(\langle a \rangle) = 0$. Conversely, suppose $da = \phi - R$ as in 1.5 and $[R] = 0$ in $H^k(X, \mathbf{Z})$. Then $R = dS$ with $S \in \mathcal{IF}^k(X)$ and hence $d(a + S) = \phi$. This proves that $[\phi] \in H^{k+1}(X, \mathbf{R})$ vanishes. Therefore, by de Rham there exist a smooth $\psi \in \mathcal{E}^k(X)$ with $d\psi = \phi$. Since $a + S - \psi$ is d-closed, the class of $a + S - \psi$ in $H^k(X, \mathbf{R})$ is represented by a smooth d-closed form $\eta \in \mathcal{E}^k(X)$. That is, $a + S - \psi = \eta + db$ for some $b \in \mathcal{D}'^{k-1}(X)$. The equivalent spark $\bar{a} = a + S - db = \psi + \eta$ representing $\langle a \rangle$ is smooth. This proves that the sequence is exact. □

PROPOSITION 1.12. *There is a short exact sequence*

$$\text{(B)} \qquad 0 \longrightarrow H^k(X, S^1) \longrightarrow \widehat{\mathbb{H}}^k(X) \xrightarrow{\delta_1} Z_0^{k+1}(X) \longrightarrow 0.$$

Proof. In keeping with our de Rham-Federer approach we give the following representation of $H^*(X; S^1)$ in terms of currents:

$$\text{(1.13)} \qquad H^k(X, S^1) \cong \frac{\{a \in \mathcal{D}'^k(X) : da \in \mathcal{IF}^{k+1}(X)\}}{d\mathcal{D}'^{k-1}(X) + \mathcal{IF}^k(X)}.$$

To prove this, consider the commutative diagram of sheaves on X:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbf{Z} & \longrightarrow & \mathcal{IF}^0 & \xrightarrow{d} & \mathcal{IF}^1 & \xrightarrow{d} & \mathcal{IF}^2 & \xrightarrow{d} & \cdots. \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{R} & \longrightarrow & \mathcal{D}'^0 & \xrightarrow{d} & \mathcal{D}'^1 & \xrightarrow{d} & \mathcal{D}'^2 & \xrightarrow{d} & \cdots.
\end{array}$$

The rows are acyclic resolutions of the constant sheaves $\mathbf{Z}$ and $\mathbf{R}$ respectively (cf. [HZ$_1$]). (This fact establishes the isomorphisms (1.9) above.) Taking the quotient we deduce from general arguments that

$$0 \longrightarrow \mathbf{R}/\mathbf{Z} \longrightarrow \mathcal{D}'^0/\mathcal{IF}^0 \xrightarrow{d} \cdots$$

is an acyclic resolution of the constant sheaf $S^1 = \mathbf{R}/\mathbf{Z}$, and therefore (1.13) holds.

To prove Proposition 1.12 it suffices by Lemma 1.8 to show that $\ker(\delta_1) \cong H^1(X, S^1)$. Since $\delta_1(\langle a\rangle) = \phi = 0$ if and only if $da = -R \in \mathcal{IF}^{k+1}(X)$, equation (1.13) shows that kernel $\delta_1 = H^k(X, S^1)$ as desired. □

Consider the direct sum of the maps δ_1 and δ_2 as a single map δ. Define

$$Q^k(X) = \{(\phi, u) \in Z_0^k(X) \times H^k(X, \mathbf{Z}) : [\phi] = u \otimes \mathbf{R} \text{ in } H^k(X, \mathbf{R})\}.$$

Following Cheeger-Simons this should be denoted $R^k(X)$. We have changed the notation to avoid confusion with the space of rectifiable currents.

PROPOSITION 1.14. *There is a short exact sequence*

$$0 \longrightarrow \frac{H^k(X;\, \mathbf{R})}{H^k_{\text{free}}(X;\, \mathbf{Z})} \longrightarrow \widehat{\mathbb{H}}^k(X) \xrightarrow{\delta} Q^{k+1}(X) \longrightarrow 0,$$

Proof. To see that δ is surjective consider a pair $(\phi, u) \in Q^{k+1}(X)$ and choose an integrally flat cycle R with $[R] = u$. Then $\phi - R$ is a d-closed current whose de Rham cohomology class is zero. By [deR] the cohomology of currents agrees with the real standard cohomology of X. Hence, there exists a current a with $da = \phi - R$, and surjectivity is proved. Suppose now that $a \in \mathcal{S}^k(X)$ and $\delta(\langle a\rangle) = 0$. Then $da = dS$ for some integrally flat current S. Let a denote the equivalent d-closed spark $\widetilde{a} = a - S$. Suppose a' is another d-closed spark in the spark class $\langle\widetilde{a}\rangle \in \widehat{\mathbb{H}}^k(X)$. Then $a' = \widetilde{a} + db + S_0$ with S_0 integrally flat. Since a and a' are d-closed, S_0 must be a cycle. Thus we see that $[\widetilde{a}] \in H^k(X, \mathbf{R})$ is determined by the class $\langle a\rangle \in \widehat{\mathbb{H}}^k(X)$ exactly up to translations by the subgroup $H^k_{\text{free}}(X;\, \mathbf{Z})$ generated by the integrally flat cycles. □

Remark 1.15. Via 1.14 the group of differential characters carries a *natural topology* coming from the C^∞-topology on forms in $\mathcal{Z}_0^{k+1}(X)$ and the standard topology on the torus $H^k(X;\, \mathbf{R})/H^k_{\text{free}}(X;\, \mathbf{Z})$. In this topology we see that $\widehat{\mathbb{H}}^k_\infty(X)$ is the *connected component of 0*, and the sequence (1.11) can be reinterpreted as the sequence obtained by dividing by this component.

The decompositions (A) and (B) given above can be further refined by restricting δ_1 to kernel $\delta_2 = \widehat{\mathbb{H}}^k_\infty(X)$ and δ_2 to kernel $\delta_1 \equiv H^k(X, S^1)$ and then also decomposing the images. This leads to a large diagram where (A) and (B) are the central column and row respectively.

Proposition 1.16. *The following diagram commutes, and each row and column is exact.*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^k(X,\mathbf{R})}{H^k_{\mathrm{free}}(X,\mathbf{Z})} & \longrightarrow & \widehat{\mathbb{H}}^k_\infty(X) & \xrightarrow{\delta_1} & d\mathcal{E}^k(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^k(X,S^1) & \longrightarrow & \widehat{\mathbb{H}}^k(X) & \xrightarrow{\delta_1} & \mathcal{Z}^{k+1}_0(X) & \longrightarrow & 0 \\
 & & {\scriptstyle\delta_2}\downarrow & & {\scriptstyle\delta_2}\downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{k+1}_{\mathrm{tor}}(X,\mathbf{Z}) & \longrightarrow & H^{k+1}(X,\mathbf{Z}) & \longrightarrow & H^{k+1}_{\mathrm{free}}(X,\mathbf{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Proof. In the top row and in the left column the kernel clearly coincides with the kernel of δ, given by the torus in Proposition 1.14. The surjectivity of δ_1 in the top row is immediate. The surjectivity of δ_2 in the left column follows directly from the long exact sequence based on $0 \to \mathbf{Z} \to \mathbf{R} \to S^1 \to 0$. However, we include a proof using the de Rham-Federer representation (1.13). Note first that if $\langle a \rangle \in H^k(X,S^1) = \mathrm{kernel}\,(\delta_1)$, then $da = -R \in \mathcal{IF}^{k+1}(X)$ and hence $[R] \in H^{k+1}_{\mathrm{tor}}(X,\mathbf{Z})$. Conversely, if $u \in H^{k+1}_{\mathrm{tor}}(X,\mathbf{Z})$ has representative $R \in \mathcal{IF}^{k+1}(X)$, then the equation $da = -R$ can be solved with $a \in \mathcal{D}'^k(X)$. The exactness of the bottom row and right column are obvious. The commutativity of the diagram is straightforward to verify. □

2. Isomorphisms of de Rham-Federer type. Among the deepest and most useful theorems of Federer and de Rham are those which assert that ordinary cohomology groups on a manifold can be computed in many ways as the cohomology of distinct complexes of currents. In this section we prove analogous results for differential characters, namely, we show that $\widehat{\mathbb{H}}^*(X)$ has several equivalent definitions whose interchangeablity is often useful in practice.

The definition of $\widehat{\mathbb{H}}^k(X)$ given in §1 used a "maximal" set of representatives. We begin by examining a nearly "minimal" set. Let $\widetilde{\mathcal{C}}^k(X) \subset \mathcal{D}'^k(X)$ denote the currents defined by locally finite sums of C^∞ singular $(n-k)$-chains on X (the *current chains* of de Rham [deR]). Note that

$$\widetilde{\mathcal{C}}^k(X) \subset \mathcal{R}^k(X) \subset \mathcal{IF}^k(X).$$

Definition 2.1.

$$\begin{aligned}\mathcal{S}^k_{\min}(X) &\equiv \{a \in \mathcal{E}^k_{L^1_{\rm loc}}(X) : da = \phi - R \text{ where } \phi \in \mathcal{E}^{k+1}(X) \text{ and } R \in \widetilde{\mathcal{C}}^{k+1}(X)\}\\ \widehat{\mathbb{H}}^k(X)_{\min} &\equiv \mathcal{S}^k_{\min}(X)/[d\mathcal{E}^{k-1}_{L^1_{\rm loc}}(X) + \widetilde{\mathcal{C}}^k(X)]_0\end{aligned}$$

where by definition $V/[W]_0 = V/(W \cap V)$ for subspaces V and W of $\mathcal{D}'^k(X)$.

Proposition 2.2. *The inclusion $\mathcal{S}^k_{\min}(X) \subset \mathcal{S}^k(X)$ induces an isomorphism of quotients*

$$\widehat{\mathbb{H}}^k(X)_{\min} \cong \widehat{\mathbb{H}}^k(X).$$

Proof. We first prove surjectivity. Suppose $\sigma \in \mathcal{D}'^k(X)$, with $d\sigma = \phi - R$ with $dR = d\phi = 0$ as in Lemma 1.5. Since $H^*(\widetilde{\mathcal{C}}^*(X)) \cong H^*(\mathcal{IF}(X))$, the class of R is represented by a chain $C \in \widetilde{\mathcal{C}}^{k+1}(X)$, i.e., $R = dS + C$, with $S \in \mathcal{IF}^k(X)$. Note that $\sigma + S \cong S$ in $\widehat{\mathbb{H}}^k(X)$ and $d(\sigma + S) = \phi - C$. Since $H^*(\mathcal{F}^*(X)) \cong H^*(\mathcal{D}'^*(X))$ and $\phi - C \in \mathcal{F}^{k+1}(X)$ represents the zero class in $H^*(\mathcal{D}'^{k+1}(X))$, we can solve the equation $df = \phi - C$ with $f \in \mathcal{F}^k(X)$. Now the current $\sigma + S - f$ is d-closed and therefore modulo $d\mathcal{D}'^{k-1}(X)$ it is represented by a flat current. That is, we have

$$\sigma + S - f = g + d\alpha$$

for some $g \in \mathcal{F}^k(X)$ and some $\alpha \in \mathcal{D}'^{k-1}(X)$. Hence, the current $\widetilde{\sigma} = \sigma + S - d\alpha$ is flat and represents the same class as σ in $\widehat{\mathbb{H}}^k(X)$.

Now by (1.1) the flat current $\widetilde{\sigma}$ can be written as $\widetilde{\sigma} = h + d\ell$ where h and ℓ are forms with $L^1_{\rm loc}$-coefficients. Thus $h \in \mathcal{E}^k_{L^1_{\rm loc}}(X)$ represents the same class as $\widetilde{\sigma}$ in $\widehat{\mathbb{H}}^k(X)$. Since $dh = \phi - C$ with $C \in \widetilde{\mathcal{C}}^{k+1}(X)$, we have $[h] \in \mathcal{S}^k_{\min}(X)$ and surjectivity is proved.

We now prove injectivity. Consider $h \in \mathcal{E}^k_{L^1_{\rm loc}}(X)$ with $dh = \phi - C$ where ϕ is smooth and $C \in \widetilde{\mathcal{C}}^{k+1}(X)$. Suppose that $h = d\alpha + R$ with $\alpha \in \mathcal{D}'^{k-1}(X)$ and $R \in \mathcal{IF}^k(X)$, i.e., suppose that h represents the zero class in $\widehat{\mathbb{H}}^k(X)$. Then $dR = dh = \phi - C$. By (1.4) this implies that $\phi = 0$ and $dR = -C$. Since

$$H^*(\widetilde{\mathcal{C}}^*(X)) \cong H^*(\mathcal{IF}^*(X)), \tag{2.3}$$

there exists a chain $K_1 \in \widetilde{\mathcal{C}}^k(X)$ satisfying $dK_1 = C$. Therefore $R + K_1$ is d-closed and integrally flat. Again using (2.3) we see that the class of $R + K_1$ is represented by a chain K_2, i.e., $R + K_1 = dS + K_2$ with $S \in \mathcal{IF}^{k-1}(X)$ and $K_2 \in \widetilde{\mathcal{C}}^k(X)$. Set $C' \equiv K_1 - K_2$. Then $h + C'$ represents the same class as $h \in \widehat{\mathbb{H}}^k(X)_{\min}$, and

$$\begin{aligned}h + C' &= d\alpha + R + K_1 - K_2\\ &= d(\alpha + S),\end{aligned}$$

with $\alpha + S \in \mathcal{D}'^{k-1}(X)$. Since $H^*(\mathcal{F}^*(X)) \cong H^*(\mathcal{D}'^*(X))$, we have $h + C' = dT$ for $T \in \mathcal{F}^{k-1}(X)$. Finally by (1.1) we have $T = f + dg$ where f and g are forms with L^1_{loc}-coefficients, and so $h + C' = df$ with $f \in \mathcal{E}^{k-1}_{L^1_{\text{loc}}}(X)$. This proves that $[h] = 0$ in $\widehat{\mathbb{H}}^k(X)_{\min}$. □

In practice the most useful spaces of representatives for differential characters are the following. (See [HL_{1-4}], [$\text{HZ}_{1,2}$] for example.)

Definition 2.4.

$$\widehat{\mathbb{H}}^k(X)_0 = \frac{\{a \in \mathcal{E}^k_{L^1_{\text{loc}}}(X) : da = \phi - R, \quad \phi \text{ is smooth and } R \text{ is rectifiable}\}}{[d\mathcal{E}^{k-1}_{L^1_{\text{loc}}}(X) + \mathcal{R}^k(X)]_0}$$

$$\widehat{\mathbb{H}}^k(X)_1 = \frac{\{a \in \mathcal{F}^k(X) : da = \phi - R, \quad \phi \text{ is smooth and } R \text{ is rectifiable}\}}{[d\mathcal{F}^{k-1}(X) + \mathcal{R}^k(X)]_0}$$

$$\widehat{\mathbb{H}}^k(X)_2 = \frac{\{a \in \mathcal{E}^k_{L^1_{\text{loc}}}(X) : da = \phi - I, \quad \phi \text{ is smooth and } I \text{ is integrally flat}\}}{[d\mathcal{E}^{k-1}_{L^1_{\text{loc}}}(X) + \mathcal{IF}^k(X)]_0}$$

$$\widehat{\mathbb{H}}^k(X)_3 = \frac{\{a \in \mathcal{F}^k(X) : da = \phi - I, \quad \phi \text{ is smooth and } I \text{ is integrally flat}\}}{d\mathcal{F}^{k-1}(X) + \mathcal{IF}^k(X)},$$

where by definition $V/[W]_0 = V/(W \cap V)$ for subspaces V and W of $\mathcal{F}^k(X)$. There is a natural commutative diagram of homomorphisms:

$$\begin{array}{ccccccc} & & & \widehat{\mathbb{H}}^k(X)_1 & & & \\ & & \nearrow & & \searrow & & \\ \widehat{\mathbb{H}}^k(X)_{\min} & \longrightarrow & \widehat{\mathbb{H}}^k(X)_0 & & \widehat{\mathbb{H}}^k(X)_3 & \longrightarrow & \widehat{\mathbb{H}}^k(X) \\ & & \searrow & & \nearrow & & \\ & & & \widehat{\mathbb{H}}^k(X)_2 & & & \end{array} \tag{2.5}$$

PROPOSITION 2.6. *All the mappings in diagram (2.5) are isomorphisms.*

Proof. Surjectivity is immediate from 2.2. Injectivity is straightforward to check. □

Note 2.7. An *integral current* is by definition a rectifiable current whose boundary is also rectifiable. Thus in Definition 2.4, “rectifiable” can be everywhere replaced by “integral” (in both the numerators and the denominators).

Note 2.8. (Concerning topologies) As pointed out in Remark 1.15, the group $\widehat{\mathbb{H}}^*(X)$ carries a natural topology for which the homomorphisms δ_1 and δ_2 are continuous. However this topology is far from being a quotient of any of the standard topologies on the representing spaces of currents discussed above. To see this consider the following basic examples of sparks. Let R be a rectifiable cycle of degree k on X, and let $H_\epsilon R$, $0 < \epsilon < 1$ be Federer's family of *smoothing homotopies* [F; 4.1.18]. Then $H_\epsilon R$ satisfies the spark equation

$$d(H_\epsilon R) \;=\; R_\epsilon - R$$

where R_ϵ is a smooth k-form and

$$\mathbf{M}(R_\epsilon) \leq \mathbf{M}(R) \qquad \text{and} \qquad \mathbf{M}(H_\epsilon R) \leq \epsilon \mathbf{M}(R)$$

where $\mathbf{M}(\bullet)$ denotes the mass norm. Clearly $H_\epsilon R \longrightarrow 0$ as $\epsilon \to 0$, and so $dH_\epsilon R \longrightarrow 0$ by continuity. However, $\delta_1(H_\epsilon R) = R_\epsilon$ and $\delta_2(H_\epsilon R) = R$ do not converge to 0. The topology on sparks which does descend to the topology on characters comes from the embedding $i \times d_1 \times d_2 : \mathcal{S}^*(X) \to \mathcal{D}'^*(X) \times \mathcal{E}^{*+1}(X) \times \mathcal{IF}^{*+1}(X)$.

We note that Federer defines his smoothing homotopies only in $\mathbf{R}^N$. However, they are easily transferred to any manifold X by embedding it in $\mathbf{R}^N$, smoothing the cycle in a tubular neighborhood and then projecting back to X.

Remark 2.9. (functoriality) The contravariant functoriality of differential characters is an immediate consequence of Theorem 4.1 below. However, this pull-back map can be constructed *directly* on the de Rham-Federer characters. In fact for a projection map $\pi\colon X \times Y \to Y$ there is a well-defined pull-back mapping on sparks which descends to a pull-back π^* on characters. For a general mapping $f\colon X \to Y$ between manifolds we would like to define the pull-back of a spark a on Y by the formula

$$f^*a = \pi'_*(\pi^*a \wedge [\Gamma_f]) \tag{2.10}$$

where $\pi'\colon X \times Y \to X$ is projection and $\Gamma_f \subset X \times Y$ is the graph of f. The intersection theory of Appendices A and B shows that given a this is well defined for almost all f. Furthermore, for fixed f, it is well defined for arbitrarily small deformations of π^*a on the product. This is enough to give the pull-back on characters. The details are given in the following paragraphs.

Consider first the projection map $\pi\colon X \times Y \to Y$, and note that push-forward (integration over the fibre) takes smooth forms with compact support on $X \times Y$ to smooth forms with compact support on Y. Hence there is a well-defined pull-back on currents. Now let $a \in \mathcal{S}^k(X)$ be an $L^1_{\rm loc}$-spark with $da = \phi - R$ as in 1.5. Then it is sraightforward to see that the pull-backs π^*a, $\pi^*\phi$, and π^*R are $L^1_{\rm loc}$, smooth and rectifiable respectively, and satisfy the equation $d\pi^*a = \pi^*\phi - \pi^*R$. Hence there is a well-defined mapping $\pi^*\colon\ \mathcal{S}^k(Y) \to \mathcal{S}^k(X \times Y)$ Furthermore, if $a' = a + db + S$

where S is rectifiable, then $\pi^*a' = \pi^*a + d\pi^*b + \pi^*S$, from which we see that our map on sparks determines a mapping of characters $\pi^*\colon\ \widehat{\mathbb{H}}^k(Y) \to \widehat{\mathbb{H}}^k(X \times Y)$

We now show that for any character $\alpha \in \widehat{\mathbb{H}}^k(X \times Y)$, there are sparks $a \in \alpha$ (in fact a dense set of them) with the property that $a \wedge [\Gamma_f]$ is well-defined flat current, and $R \wedge [\Gamma_f]$ is a well-defined rectifiable current, where $da = \phi - R$ as above. To start fix any $L^1_{\rm loc}$-spark $a_0 \in \alpha$. By Appendices A and B there is a diffeomorphism F of $X \times Y$, which can be taken arbitrarily close to the identity in the C^1-topology, such that $F_*a_0 \wedge \Gamma_f$ is a well-defined flat current, and $F_*R_0 \wedge \Gamma_f$ is a well-defined rectifiable current where $da_0 = \phi_0 - R_0$ as always. By de Rham-Federer theory we have $\phi_0 - F_*\phi_0 = d\psi$ where ψ is smooth and $R_0 - F_*R_0 = dS$ where S is rectifiable. Since $d(a_0 - F_*a_0 - \psi + S) = 0$, de Rham-Federer theory implies the existence of a smooth k-form η and a flat current b such that $a_0 - F_*a_0 - \psi + S = \eta + db$. The spark $a = F_*a_0 + \psi + \eta$, which is equivalent to a_0, does the trick.

Taking $\pi'_*(a \wedge \Gamma_f)$, where $\pi'\colon\ X \times Y \to X$ is projection, gives a spark on X. We claim that the associated character is independent of the choice of "good" spark a for α as above. Indeed if a' is another good spark, then by definition of the equivalence, $a - a' = dc + T$ where T is rectifiable. We again deform by a diffeomorphism F so that the deformed currents all meet Γ_f properly using Appendices A and B, and then we readjust F_*a and F_*a' as in the last paragraph. Intersecting with Γ_f and pushing down to X shows that the two sparks are equivalent. Thus we get a well-defined map $\Gamma_f^*\colon\ \widehat{\mathbb{H}}^k(X \times Y) \to \widehat{\mathbb{H}}^k(X)$. Composition with π^* gives a homomorphism $f^*\colon\ \widehat{\mathbb{H}}^k(Y) \to \widehat{\mathbb{H}}^k(X)$.

The proof that $(f \circ g)^* = g^* \circ f^*$ is straightforward.

3. Ring structure. In this section we introduce a densely defined product on sparks which descends to characters and makes $\widehat{\mathbb{H}}^*(\bullet)$ into a graded-commutative ring. We begin with the following Proposition which is proved in Appendix D.

PROPOSITION 3.1. *Given classes*

$$\alpha \in \widehat{\mathbb{H}}^k(X) \qquad \text{and} \qquad \beta \in \widehat{\mathbb{H}}^\ell(X).$$

there exist representatives $a \in \alpha$ and $b \in \beta$ with

$$da \;=\; \phi - R \qquad \text{and} \qquad db \;=\; \psi - S$$

so that $a \wedge S$, $R \wedge b$ and $R \wedge S$ are well-defined flat currents on X and $R \wedge S$ is rectifiable.

Definition 3.2. Given a and b as in Proposition 3.1, define products

$$a * b \;\stackrel{\rm def}{=}\; a \wedge \psi + (-1)^{k+1} R \wedge b$$

(3.3)
$$a\tilde{*}b \stackrel{\text{def}}{=} a \wedge S + (-1)^{k+1}\phi \wedge b$$

and note that by (B.3)

$$d(a * b) = d(a\tilde{*}b) = \phi \wedge \psi - R \wedge S. \tag{3.4}$$

THEOREM 3.5. *The classes* $\langle a * b\rangle$ *and* $\langle a\tilde{*}b\rangle$ *in* $\widehat{\mathbb{H}}^{k+\ell+1}(X)$ *agree and are independent of the choice of representatives* $a \in \alpha$ *and* $b \in \beta$. *Setting*

$$\alpha * \beta \stackrel{\text{def}}{=} \langle a * b\rangle = \langle a\tilde{*}b\rangle$$

gives $\widehat{\mathbb{H}}^*(X)$ *the structure of a graded commutative ring with unit such that* δ *is a ring homomorphism.*

Proof. To see that the two definitions agree first note that

$$\begin{aligned} a * b - a\tilde{*}b &= a \wedge \psi + (-1)^{k+1}R \wedge b - \{a \wedge S + (-1)^{k+1}\phi \wedge b\} \\ &= a \wedge (\psi - S) + (-1)^k(\phi - R) \wedge b \\ &= a \wedge db + (-1)^k da \wedge b. \end{aligned}$$

We now apply the following.

LEMMA 3.6. *Let* $a, b \in \mathcal{D}'^*(X)$ *be currents for which the products* $da \wedge b$ *and* $a \wedge db$ *are well defined. Then* $u = da \wedge b + (-1)^{\deg(a)}a \wedge db$ *is an exact current, i.e.,* $u = dv$ *for some current* v *on* X.

Proof. Choose smooth approximations $a_\epsilon, b_\epsilon \in \mathcal{E}^*(X)$ with $a_\epsilon \to a$ and $b_\epsilon \to b$ as $\epsilon \to 0$. Then $da_\epsilon \wedge b_\epsilon + (-1)^{\deg(a)}a_\epsilon \wedge b_\epsilon = d(a_\epsilon \wedge b_\epsilon)$, and the lemma now follows from the fact (cf. [deR]) that the range of d is closed. □

We now show that the product is independent of the choice of representatives a and b chosen. To begin suppose that $\tilde{a} = a + du$ for some current u. Then

$$\tilde{a} * b - a * b = du \wedge \psi = d(u \wedge \psi)$$

and so $\tilde{a} * b$ and $a * b$ define the same class in $\widehat{\mathbb{H}}^*(X)$.

Suppose now that $\tilde{a} = a + Q$ where Q is locally rectifiable, and write $da = \phi - R$ and $d\tilde{a} = \phi - \widetilde{R}$ as in Lemma 1.5. Note that by hypothesis $dQ = d\tilde{a} - da = R - \tilde{R}$ meets S properly. Using a deformation which keeps dQ fixed can find $\tilde{Q} = Q + d\sigma$ where $\tilde{Q}$ meets S properly. Using the paragraph above and the second definition of the product we have

$$\begin{aligned}\tilde{a} * b - a * b &= (\tilde{a} + d\sigma) * b - a * b \\ &= \tilde{Q} \wedge S \equiv 0 \qquad \text{in } \widehat{\mathbb{H}}^*(X).\end{aligned}$$

This shows that the product $[a*b]$ is independent of the choice of $a \in [a] \in \widehat{\mathbb{H}}^k(X)$. The argument for b is completely analogous.

A direct calculation using (3.3) now shows that

$$\alpha * \beta = (-1)^{(k+1)(\ell+1)} \beta \tilde{*} \alpha \equiv (-1)^{(k+1)(\ell+1)} \beta * \alpha \tag{3.7}$$

and the multiplication is graded-commutative as claimed. □

4. Cheeger-Simons characters. In their fundamental paper [CS] Cheeger and Simons defined the differential characters of degree k with coefficients in $\mathbf{R}/\mathbf{Z}$ on a manifold X to be the group

$$\widehat{H}^k(X;\mathbf{R}/\mathbf{Z}) = \{h \in \mathrm{Hom}(\mathcal{Z}_k(X), \mathbf{R}/\mathbf{Z}) : \delta h \equiv \phi (\mathrm{mod}\ \mathbf{Z}) \text{ for some } \phi \in \mathcal{E}^{k+1}(X)\}$$

where $\mathcal{Z}_k(X)$ denotes the smooth singular k-chains on X with $\mathbf{Z}$-coefficients and δ the standard codifferential. The first result of this section is the following.

THEOREM 4.1. *For any manifold X there is a natural isomorphism*

$$\Psi\colon \widehat{\mathbb{H}}^k(X) \xrightarrow{\cong} \widehat{H}^k(X;\mathbf{R}/\mathbf{Z})$$

induced by integration.

To define the homomorphism Ψ we will need the following.

PROPOSITION 4.2. *Given any class $\alpha \in \widehat{\mathbb{H}}^k(X)$ and any $Z \in \mathcal{Z}_k(X)$, there is a representative spark $a \in \alpha$ which is smooth on a neighborhood of supp(Z). Furthermore, given any two such representives $a, a' \in \alpha$, one has that*

$$\int_Z a \equiv \int_Z a' \ (\mathrm{mod}\ \mathbf{Z}).$$

Definition 4.3. Given $\alpha \in \widehat{\mathbb{H}}^k(X)$ and $Z \in \mathcal{Z}_k(X)$ we define

$$\Psi(\alpha)(Z) = \int_Z a \ (\mathrm{mod}\ \mathbf{Z})$$

where a is chosen as in Proposition 4.2. By 4.2 it is clear that $\Psi(\alpha)$ is well defined and that Ψ defines a homomorphism into $S^1 = \mathbf{R}/\mathbf{Z}$.

Proof of 4.2. We begin with the following assertion which is of independent interest.

LEMMA 4.4. *Given any locally rectifiable cycle* $R \in \delta_2(\alpha) \in H^{k+1}(X; \mathbf{Z})$*, there is an* L^1_{loc} *spark* $a \in \alpha$ *such that* $da = \delta_1(\alpha) - R$*.*

Proof. By Proposition 2.6 there exists an L^1_{loc}-form $\widetilde{a} \in \alpha$ with $d\widetilde{a} = \phi - R_0$ where $\phi = \delta_1(\alpha)$ is a smooth form and R_0 is a locally rectifiable cycle of dimension $n-k-1$. Since the integral cohomology class of R_0 is $\delta_2(\alpha)$, there exists a locally rectifiable current S with $dS = R_0 - R$. The current $\widetilde{a}+S$ is flat and can be written as $a + db$ where a and b are L^1_{loc}-forms (cf. (1.1)). Then $a \in \alpha$ has the asserted properties. □

Now the C^∞-singular chains over $\mathbf{Z}$ map to the rectifiable (in fact, integral) currents and their image computes integral cohomology. For dimensional reasons we may choose such a cocycle $R \in \delta_2(\alpha)$ with

$$\text{supp}(R) \cap \text{supp}(Z) = \emptyset. \tag{4.5}$$

We then choose $a \in \alpha$ as in Lemma 4.4. In $\Omega \equiv X - \text{supp}(R)$ we have $da = \phi$, a smooth form.

LEMMA 4.6. *Given* $a \in \mathcal{E}^k_{L^1_{\text{loc}}}(\Omega)$ *such that da is smooth, there exists* $b \in \mathcal{E}^{k-1}_{L^1_{\text{loc}}}(X)$ *such that* $a - db$ *is smooth.*

Proof. This follow directly form the fact that the cohomology of locally flat currents coincides with the de Rham cohomology of smooth forms [F]. □

Now choose b as in Lemma 4.6, and choose $f \in C^\infty_0(\Omega)$ with $f \equiv 1$ on a neighborhood of the compact subset $\text{supp}(Z) \subset \Omega$. Set $b_0 = fb$ and extend it by zero to all of X. Then $a_0 = a - db_0$ is equivalent to a and is smooth on a neighborhood of $\text{supp}(Z)$.

To prove the second assertion of Proposition 4.2 let $a' \in \alpha$ be another such choice. Then we have

$$da = \phi - R \qquad \text{and} \qquad da' = \phi - R'$$

where $\phi = \delta_2(\alpha)$, a and a' are both smooth on $\text{supp}(Z)$, and R and R' are C^∞-singular chains. Now $d(a - a') = R' - R = dS$ where S is a C^∞-singular chain which we may assume is transversal to Z. Thus $(a-a'-S, Z) \equiv (a-a', Z)$ mod $\mathbf{Z}$. However, since $[a] = [a'] = \alpha$ we have $a = a' + S_0 + db$ where S_0 is rectifable and b is flat. This shows that the cycle $a - a' - S$ is homologous to the rectifiable cycle $S_0 - S$, and so $(a - a' - S, Z) \equiv 0$ mod $\mathbf{Z}$ as desired. □

Note that the arguments above actually prove the following.

COROLLARY 4.7. *Fix $\alpha \in \widehat{\mathbb{H}}^k(X)$ and $Z \in \mathcal{Z}_k(X)$. Then given any rectifiable cycle $R \in \delta_2(\alpha)$, whose support is disjoint from* supp(Z)*, there is a representative $a \in \alpha$ with $\delta_2(a) = R$ which is smooth on a neighborhood of* supp(Z).

Proof of Theorem 4.1. We now recall that Theorem 1.1 of [CS] establishes a functorial short exact sequence:

$$0 \longrightarrow \frac{H^k(X;\,\mathbf{R})}{H^k_{\text{free}}(X;\,\mathbf{Z})} \longrightarrow \widehat{H}^k(X;\mathbf{R}/\mathbf{Z}) \xrightarrow{\delta} Q^{k+1}(X) \longrightarrow 0. \tag{4.8}$$

One checks directly that the mapping Ψ induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{H^k(X;\mathbf{R})}{H^k_{\text{free}}(X;\mathbf{Z})} & \longrightarrow & \widehat{\mathbb{H}}^k(X) & \xrightarrow{\delta} & Q^{k+1}(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \Psi & & \| & & \\ 0 & \longrightarrow & \frac{H^k(X;\mathbf{R})}{H^k_{\text{free}}(X;\mathbf{Z})} & \longrightarrow & \widehat{H}^k(X;\mathbf{R}/\mathbf{Z}) & \xrightarrow{\delta} & Q^{k+1}(X) & \longrightarrow & 0 \end{array} \tag{4.9}$$

where the top line is the sequence from 1.11, the bottom line is (4.8), and the left vertical arrow is an isomorphism. It follows immediately that Ψ is an isomorphism. This proves the theorem. □

From the definition it is obvious that the Cheeger-Simons characters are a contravariant functor on the category of smooth manifolds. What is not obvious is that they carry a graded ring structure for which δ_1 and δ_2 are ring homomorphisms. Such a product was established by Cheeger [C]. Our next result asserts that under the isomorphism Ψ, our product coincides with the Cheeger product. While our product is rather straightforward to define, the proof of its coincidence with the Cheeger product is not trivial. The difficulty stems from the fundamental difference between the intersection product of cochains, which is local in nature, and the cup product, which is not (cf. Note C.2).

THEOREM 4.10. *Under the isomorphism Ψ given in Theorem 4.1, the $*$-product on $\widehat{\mathbb{H}}^*(X)$ coincides with the Cheeger product defined on $\widehat{H}^*(X;\,\mathbf{R}/\mathbf{Z})$.*

Proof. Recall that the additive isomorphism $\Psi^{-1}\colon\ \widehat{H}^*(X;\,\mathbf{R}/\mathbf{Z}) \to \widehat{\mathbb{H}}^*(X)$ preserves δ_1 and δ_2 and therefore introduces two ring structures on $\widehat{\mathbb{H}}^*(X)$ for which these maps are ring homomorphisms. Thus the biadditive map

$$\rho\colon\ \widehat{\mathbb{H}}^*(X) \times \widehat{\mathbb{H}}^*(X) \longrightarrow \widehat{\mathbb{H}}^*(X) \qquad \text{given by} \qquad \rho(\alpha,\beta) \equiv \alpha * \beta - \alpha \hat{*} \beta$$

has the property that

$$\delta_1\rho(\alpha,\beta) = \delta_2\rho(\alpha,\beta) = 0$$

for all α, β. Thus the image of ρ is contained in the subgroup $H^*(X;\, \mathbf{R})/H^*(X;\, \mathbf{Z})$. Our next observation is the following.

PROPOSITION 4.11. *If either α or β is represented by a smooth differential form, then $\rho(\alpha, \beta) = 0$.*

Proof. Suppose β is represented by a smooth differential ℓ-form b. Via integration b defines a real cochain which when restricted to cycles gives the character β. Note that $\delta_1(\beta) = db \equiv \psi$ and $\delta_2(\beta) = 0$. Let $\delta_1(\alpha) \equiv \phi$. Then by the definition given in [C] we have

$$\alpha \hat{*} \beta \equiv (-1)^{k+1} \phi \cup b + E(\phi, \psi) \qquad \text{mod } \mathbf{Z}$$

where E is defined as follows. Let ϕ_1, ϕ_2 be smooth forms of degrees k_1, k_2 respectively. Let Δ^i denote the ith subdivision operator acting on C^∞ cubical chains. Let h be the standard chain homotopy on cubical chains satisfying $1 - \Delta = \partial \circ h + h \circ \partial$. Then $E(\phi_1, \phi_2)$ is a real cochain of degree $k_1 + k_2 + 1$ defined on a C^∞ cubical chain c by

$$E(\phi_1, \phi_2)(c) = -\sum_{i=0}^{\infty} (\phi_1 \cup \phi_2)(h\Delta^i c).$$

Cheeger proves that this series converges and when $d\phi_1 = d\phi_2 = 0$, one has $\delta E(\phi_1, \phi_2) = \phi_1 \wedge \phi_2 - \phi_1 \cup \phi_2$ where δ denotes the coboundary. Recalling that $d\phi = 0$ and $db = \psi$ we compute that

$$\begin{aligned}
\delta E(\phi, b)(c) &= -\sum_{i=0}^{\infty} (\phi \cup b)(h\Delta^i \partial c) = -\sum_{i=0}^{\infty} (\phi \cup b)(h\partial \Delta^i c) \\
&= -\sum_{i=0}^{\infty} (\phi \cup b)\{(1 - \Delta - \partial h)\Delta^i c\} \\
&= -\sum_{i=0}^{\infty} (\phi \cup b)\{(1 - \Delta)\Delta^i c\} + \sum_{i=0}^{\infty} \{d(\phi \cup b)\}(h\Delta^i c) \\
&= -\lim_{n\to\infty} \sum_{i=0}^{n} (\phi \cup b)\{(1 - \Delta)\Delta^i c\} + \sum_{i=0}^{\infty} \{d(\phi \cup b)\}(h\Delta^i c) \\
&= -\lim_{n\to\infty} \sum_{i=0}^{n} (\phi \cup b)(1 - \Delta^{n+1} c) + (-1)^{k+1} \sum_{i=0}^{\infty} (\phi \cup db)(h\Delta^i c) \\
&= (\phi \wedge b - \phi \cup b)(c) + (-1)^k E(\phi, \psi).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\alpha \hat{*} \beta - (-1)^k \delta E(\phi, b) &= (-1)^{k+1} \phi \cup b + E(\phi, \psi) \\
&\quad - (-1)^k \{\phi \wedge b - \phi \cup b + (-1)^k E(\phi, \psi)\}
\end{aligned}$$

$$= (-1)^{k+1}\phi \wedge b$$
$$= \alpha * \beta$$

Consequently, $\alpha \hat{*} \beta$ and $\alpha * \beta$ have the same values on cycles and therefore they define the same differential characters. □

From Propositions 1.11 and 1.14 we conclude that ρ descends to a biadditive mapping

$$\widetilde{\rho}\colon\ H^{k+1}(X;\,\mathbf{Z}) \times H^{\ell+1}(X;\,\mathbf{Z}) \longrightarrow H^{k+\ell+1}(X;\,\mathbf{R})/H^{k+\ell+1}_{\text{free}}(X;\,\mathbf{Z})$$

which we will show to be zero. To compute $\widetilde{\rho}$ fix classes $(u, v) \in H^{k+1}(X;\,\mathbf{Z}) \times H^{\ell+1}(X;\,\mathbf{Z})$ and choose good cycles $R \in u$ and $S \in v$, e.g., cycles from smooth triangulations such that all simplices in one triangulation are transversal to all simplices in the other. Let $H_\epsilon R$ and $H_\epsilon S$ be the smoothing homotopies from Note 2.8. We only need to evaluate $\widetilde{\rho}(R, S)$ on one cycle in each class of $H_{k+\ell+1}(X;\,\mathbf{Z})$, so we may assume these cycles are nice with respect to R and S (as above). For such a cycle Z it is an direct calculation that $(H_\epsilon R * H_\epsilon S - H_\epsilon R \hat{*} H_\epsilon S)(Z) \longrightarrow 0$ as $\epsilon \to 0$. □

5. Dual sequences. As noted in Remark 1.15 the group of characters carries a natural topology having $\widehat{\mathbb{H}}^*_\infty$ as the connected component of 0, and inducing the standard C^∞-topology on $\mathcal{Z}^*_0(X)$ in Proposition 1.12. It therefore makes sense to consider the *Pontrjagin dual* which we define to be the group

$$\widehat{\mathbb{H}}^k(X)^* \;\equiv\; \mathrm{Hom}(\widehat{\mathbb{H}}^k(X), S^1)$$

of *continuous* group homomorphisms from $\widehat{\mathbb{H}}^k(X)$ to the circle. The interesting fact is that this dual group sits in two fundamental exact sequences which resemble the fundamental sequences 1.11 and 1.12 for $\widehat{\mathbb{H}}^{n-k-1}(X)$ but which are interchanged under duality.

Here we examine these sequences and lay the groundwork for duality which will be established next. The symbol Hom shall always denote *continuous* homomorphisms. We assume throughout this section that X is compact, oriented and of dimension n.

LEMMA 5.1. *There are natural isomorphisms:*

$$\mathrm{Hom}(H^k(X;\,\mathbf{Z}),\,S^1) \;\cong\; H^{n-k}(X;\,S^1) \qquad \text{(Poincaré Duality)}$$
$$\mathrm{Hom}\left(\frac{H^k(X;\,\mathbf{R})}{H^k_{\text{free}}(X;\,\mathbf{Z})},\,S^1\right) \;\cong\; H^{n-k}_{\text{free}}(X;\,\mathbf{Z})$$
$$\mathrm{Hom}(d\mathcal{E}^k(X),\,S^1) \;\cong\; d\mathcal{D}'^{\,n-k-1}(X).$$

Proof. By Pontrjagin duality, classical Poincaré duality, $H^k(X;\mathbf{Z}) \cong H_{n-k}(X;\mathbf{Z})$, is equivalent to $\mathrm{Hom}(H^k(X;\mathbf{Z}), S^1) \cong \mathrm{Hom}(H_{n-k}(X;\mathbf{Z}), S^1)$. By the Universal Coefficient Theorem, $\mathrm{Hom}(H_{n-k}(X;\mathbf{Z}), S^1) \cong H^{n-k}(X;S^1)$. This proves the first assertion.

To prove the second assertion we recall that if $\Lambda \subset \mathbf{R}^m$ is a lattice of full rank, then

$$\mathrm{Hom}(\mathbf{R}^m/\Lambda, S^1) \cong \mathrm{Hom}(\Lambda, \mathbf{Z}) \quad \text{and} \quad \mathrm{Hom}(\Lambda, S^1) \cong \mathrm{Hom}(\Lambda, \mathbf{R})/\mathrm{Hom}(\Lambda, \mathbf{Z}). \tag{5.2}$$

Now by (5.2) and Poincaré duality we have $\mathrm{Hom}\left(\frac{H^k(X;\mathbf{R})}{H^k_{\text{free}}(X;\mathbf{Z})}, S^1\right) \cong \mathrm{Hom}(H^k_{\text{free}}(X;\mathbf{Z}), \mathbf{Z}) \cong \mathrm{Hom}(H_{n-k}(X;\mathbf{Z})_{\text{free}}, \mathbf{Z}) \cong H^{n-k}_{\text{free}}(X;\mathbf{Z})$.

For the third assertion first note that $\mathrm{Hom}(d\mathcal{E}^k, \mathbf{R}/\mathbf{Z}) = \mathrm{Hom}(d\mathcal{E}^k, \mathbf{R}) = (d\mathcal{E}^k)'$. To identify this vector space dual, consider the sequence $\mathcal{E}^k \xrightarrow{d} d\mathcal{E}^k \subset \mathcal{E}^{k+1}$. On any manifold the range of d is closed (cf. [deR]). Hence, by Hahn-Banach the right-hand mapping in the dual sequence $(\mathcal{E}^k)' \hookleftarrow (d\mathcal{E}^k)' \twoheadleftarrow (\mathcal{E}^{k+1})'$ is surjective. This composition is exactly the exterior derivative on currents, and so we conclude that $(d\mathcal{E}^k)' = d\{(\mathcal{E}^{k+1})'\} = d\{\mathcal{D}'^{n-k-1}\}$. □

Observation 5.3. There are (noncanonical) group homomorphisms:

$$\widehat{\mathbb{H}}^k(X) \cong d\mathcal{E}^k(X) \times \frac{H^k(X;\mathbf{R})}{H^k_{\text{free}}(X;\mathbf{Z})} \times H^{k+1}(X;\mathbf{Z})$$
$$\widehat{\mathbb{H}}^{n-k-1}(X) \cong d\mathcal{E}^{n-k-1}(X) \times H^{n-k}_{\text{free}}(X;\mathbf{Z}) \times H^{n-k-1}(X;S^1).$$

The first follows from Proposition 1.11 and the top row of Proposition 1.16, and the second from Proposition 1.12 and the right column of Proposition 1.16. Combined with Lemma 5.1 above this justaposition is quite suggestive. To pin down the implied duality we examine the Pontrjagin duals of the sequences in §1.

Taking the duals of the sequences from Proposition 1.11 and 1.16 (top row) gives us

$$0 \to \mathrm{Hom}(H^{k+1}(X;\mathbf{Z}), S^1) \to \mathrm{Hom}(\widehat{\mathbb{H}}^k(X), S^1) \to \mathrm{Hom}(\widehat{\mathbb{H}}^k_\infty(X), S^1) \to 0$$
$$0 \to \mathrm{Hom}(d\mathcal{E}^k(X), S^1) \to \mathrm{Hom}(\widehat{\mathbb{H}}^k_\infty(X), S^1) \to \mathrm{Hom}\left(\frac{H^k(X;\mathbf{R})}{H^k_{\text{free}}(X;\mathbf{Z})}, S^1\right) \to 0.$$

Applying Lemma 5.1 gives us:

PROPOSITION 5.4. *There are exact sequences*

$$0 \to H^{n-k-1}(X;S^1) \xrightarrow{\delta_2^*} \mathrm{Hom}(\widehat{\mathbb{H}}^k(X), S^1) \xrightarrow{\rho_2} \mathrm{Hom}(\widehat{\mathbb{H}}^k_\infty(X), S^1) \to 0 \tag{5.5}$$

$$0 \to d\mathcal{D}'^{n-k-1}(X) \xrightarrow{\delta_1^*} \operatorname{Hom}(\widehat{\mathbb{H}}^k_\infty(X), S^1) \xrightarrow{\rho_1} H^{n-k}_{\text{free}}(X; \mathbf{Z}) \to 0. \tag{5.6}$$

Similarly, taking the duals of Propositions 1.12 and 1.16 (right column) and then applying Lemma 5.1 and Pontrjagin duality gives:

PROPOSITION 5.7. *There are exact sequences*

$$0 \to \operatorname{Hom}(\mathcal{Z}_0^{k+1}(X), S^1) \xrightarrow{\delta_1^*} \operatorname{Hom}(\widehat{\mathbb{H}}^k(X), S^1) \xrightarrow{\rho_1} H^{n-k}(X; \mathbf{Z}) \to 0 \tag{5.8}$$

$$0 \to \frac{H^{n-k-1}(X; \mathbf{R})}{H^{n-k-1}_{\text{free}}(X; \mathbf{Z})} \to \operatorname{Hom}(\mathcal{Z}_0^{k+1}(X), S^1) \to d\mathcal{D}'^{n-k-1}(X) \to 0. \tag{5.9}$$

We now come to a central concept in this paper.

Definition 5.10. A homomorphism h: $\widehat{\mathbb{H}}^k_\infty(X) \to S^1$ is called *smooth* if there exists a smooth form $\omega \in \mathcal{Z}_0^{n-k}(X)$ such that

$$h(\alpha) \equiv \int_X a \wedge \omega \pmod{\mathbf{Z}} \tag{5.11}$$

for $a \in \alpha$. One checks easily that the right-hand side is independent of the choice of $a \in \alpha$. Let $\operatorname{Hom}_\infty(\widehat{\mathbb{H}}^k_\infty(X), S^1)$ denote the group of all such homomorphisms. We then define the *smooth Pontrjagin dual* of $\widehat{\mathbb{H}}^k(X)$ to be the group

$$\operatorname{Hom}_\infty(\widehat{\mathbb{H}}^k(X), S^1) \stackrel{\text{def}}{=} \rho_2^{-1}\operatorname{Hom}_\infty(\widehat{\mathbb{H}}^k_\infty(X), S^1)$$

where ρ_2 is the restriction homomorphism in (5.5).

The obvious map induces an isomorphism $\mathcal{Z}_0^{n-k}(X) \cong \operatorname{Hom}_\infty(\widehat{\mathbb{H}}^k_\infty(X), S^1)$ and the restriction of (5.6) to the smooth homomorphisms becomes

$$0 \to d\mathcal{E}^{n-k-1}(X) \to \operatorname{Hom}_\infty(\widehat{\mathbb{H}}^k_\infty(X), S^1) \to H^{n-k}_{\text{free}}(X; \mathbf{Z}) \to 0. \tag{5.12}$$

Let $\operatorname{Hom}_\infty(\mathcal{Z}_0^\ell(X), S^1) = \operatorname{Hom}(\mathcal{Z}_0^\ell(X), S^1) \cap \operatorname{Hom}_\infty(\widehat{\mathbb{H}}^{\ell-1}(X), S^1)$ in (5.8). Restricting (5.9) gives a short exact sequence

$$0 \to \frac{H^{n-k-1}(X; \mathbf{R})}{H^{n-k-1}_{\text{free}}(X; \mathbf{Z})} \to \operatorname{Hom}_\infty(\mathcal{Z}_0^{k+1}(X), S^1) \to d\mathcal{E}^{n-k-1}(X) \to 0. \tag{5.13}$$

PROPOSITION 5.14. *There are natural isomorphisms*

$$\mathcal{Z}_0^{n-k}(X) \cong \operatorname{Hom}_\infty(\widehat{\mathbb{H}}^k_\infty(X), S^1) \qquad \text{and} \qquad \widehat{\mathbb{H}}^{n-k-1}_\infty(X) \cong \operatorname{Hom}_\infty(\mathcal{Z}_0^{k+1}(X), S^1).$$

Proof. The first isomorphism is noted above. For the second we define a mapping $\widehat{\mathbb{H}}^{n-k-1}_\infty(X) \to \operatorname{Hom}_\infty(\mathcal{Z}_0^{k+1}(X), S^1)$ by formula (5.11) above. Note again that this integral mod $\mathbf{Z}$ does not depend on the choice of representative $a \in \alpha \in \widehat{\mathbb{H}}_\infty^{n-k-1}(X)$. We then observe that this mapping induces a mapping of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{H^{n-k-1}(X,\mathbf{R})}{H^{n-k-1}_{\text{free}}(X,\mathbf{Z})} & \longrightarrow & \widehat{\mathbb{H}}_\infty^{n-k-1}(X) & \xrightarrow{\delta_1} & d\mathcal{E}^{n-k-1}(X) & \longrightarrow & 0 \\ & & \cong\downarrow & & \downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & \frac{H^{n-k-1}(X;\mathbf{R})}{H^{n-k-1}_{\text{free}}(X;\mathbf{Z})} & \longrightarrow & \mathrm{Hom}_\infty(\mathcal{Z}_0^{k+1}(X),\, S^1) & \longrightarrow & d\mathcal{E}^{n-k-1}(X) & \longrightarrow & 0 \end{array}$$

from Proposition 1.16 (top row) to (5.13). The right and left vertical maps are isomorphisms and so therefore is the middle. □

This leads to the main result of this section.

THEOREM 5.15. *If X is compact and oriented, there are short exact sequences:*

$$0 \longrightarrow H^{n-k-1}(X;\, S^1) \longrightarrow \mathrm{Hom}_\infty(\widehat{\mathbb{H}}^k(X),\, S^1) \xrightarrow{\rho_2} \mathcal{Z}_0^{n-k}(X) \longrightarrow 0,$$
$$0 \longrightarrow \widehat{\mathbb{H}}_\infty^{n-k-1}(X) \longrightarrow \mathrm{Hom}_\infty(\widehat{\mathbb{H}}^k(X),\, S^1) \xrightarrow{\rho_1} H^{n-k}(X;\, \mathbf{Z}) \longrightarrow 0.$$

Proof. The first comes from (5.5) and Proposition 5.14; the second from (5.8) and Proposition 5.14. □

6. Duality for differential characters. Let X be a compact connected oriented manifold of dimension n. Then the top character group $\widehat{\mathbb{H}}^n(X) = \mathcal{D}'^n(X)/\{d\mathcal{D}'^{n-1}(X) + \mathcal{IF}^n(X)\}$ is easily seen to satisfy

$$\widehat{\mathbb{H}}^n(X) \cong \mathrm{Hom}(\mathcal{Z}_n(X), \mathbf{R}/\mathbf{Z}) \cong \mathrm{Hom}(\mathbf{Z}, \mathbf{R}/\mathbf{Z}) \cong \mathbf{R}/\mathbf{Z}.$$

As noted in Remark 1.15 each group of de Rham characters has a natural topology. In what follows Hom shall always denote *continuous* group homomorphisms.

Definition 6.1. For each integer k, $0 \le k < n$ we define the duality mapping

$$\mathcal{D}\colon\ \widehat{\mathbb{H}}^{n-k-1}(X) \longrightarrow \widehat{\mathbb{H}}^k(X)^* \equiv \mathrm{Hom}_\infty(\widehat{\mathbb{H}}^k(X), \mathbf{R}/\mathbf{Z})$$

by

$$\mathcal{D}(\alpha)(\beta) \overset{\text{def}}{=} \alpha * \beta \in \widehat{\mathbb{H}}^n(X) = \mathbf{R}/\mathbf{Z}.$$

THEOREM 6.2. (Poincaré-Pontrjagin Duality for Characters) *The duality mapping $\mathcal{D}$ is an isomorphism.*

Proof. Given $a \in \alpha \in \widehat{\mathbb{H}}^{n-k-1}(X)$ and $b \in \beta \in \widehat{\mathbb{H}}^{k}(X)$ we write $da = d_1 a - d_2 a$ and $db = d_1 b - d_2 b$ where $d_1 a, d_1 b$ are smooth and $d_2 a, d_2 b$ are rectifiable. Then (3.3) can be rewritten

$$a * b \;=\; a \wedge d_1 b + (-1)^{k+1} d_2 a \wedge b \tag{6.3}$$

$$a \tilde{*} b \;=\; a \wedge d_2 b + (-1)^{k+1} d_1 a \wedge b. \tag{6.4}$$

It is easily seen that a and b can be chosen so that the terms in these equations are well defined. Indeed, since $\dim d_2 a + \dim d_2 b = n-1$, we may choose a and b with $\mathrm{supp}(d_2 a) \cap \mathrm{supp}(d_2 b) = \emptyset$, and then by Corollary 4.7 we may assume that a is smooth on $\mathrm{supp}(d_2 b)$ and b is smooth on $\mathrm{supp}(d_2 a)$.

Consider now the composition $\rho_1 \circ \mathcal{D}\colon\ \widehat{\mathbb{H}}^{n-k-1}(X) \longrightarrow \mathcal{Z}_0^{n-k}(X)$. Note that ρ_1 is defined by restriction of homomorphisms to the subgoup $\ker \delta_2 \subset \widehat{\mathbb{H}}^{k}(X)$. Now if $\delta_2 \beta = 0$, we may choose $b \in \beta$ to be a smooth form. In particular, $d_2 b = 0$ and by (6.4)

$$\mathcal{D}(a)(b) = (-1)^{k+1} d_1 a \wedge b$$

which implies that

$$\rho_1 \circ \mathcal{D} = (-1)^{k+1} d_1. \tag{6.5}$$

Similarly we have the composition $\rho_2 \circ \mathcal{D}\colon\ \widehat{\mathbb{H}}^{n-k-1}(X) \longrightarrow H^{n-k}(X;\,\mathbf{Z})$ where ρ_2 is defined by restriction to $\ker \delta_1$. If $\delta_1 \beta = 0$, we may choose $b \in \beta$ with $d_1 b = 0$, and by (6.3) we have

$$\mathcal{D}(a)(b) \;=\; (-1)^{k+1} d_2 a \wedge b.$$

This shows that

$$\rho_2 \circ \mathcal{D} = (-1)^{k+1} d_2. \tag{6.6}$$

Combining (6.5) and (6.6) and using Theorem 5.15 we get a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{H^{n-k-1}(X;\mathbf{R})}{H^{n-k-1}_{\mathrm{free}}(X;\mathbf{Z})} & \longrightarrow & \widehat{\mathbb{H}}^{n-k-1}(X) & \xrightarrow{\delta=(\delta_1,\delta_2)} & R^{n-k}(X) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \epsilon} & & \Big\downarrow & & \Big\downarrow{\scriptstyle (-1)^{k+1}} & & \\
0 & \longrightarrow & \dfrac{H^{n-k-1}(X;\mathbf{R})}{H^{n-k-1}_{\mathrm{free}}(X;\mathbf{Z})} & \longrightarrow & \mathrm{Hom}_\infty(\widehat{\mathbb{H}}^{k}(X),\,\mathbf{R}/\mathbf{Z}) & \xrightarrow{\rho=(\rho_1,\rho_2)} & R^{n-k}(X) & \longrightarrow & 0.
\end{array}$$

It remains to show that the mapping ϵ is an isomorphism. However, this map asociates to a smooth closed $(n-k-1)$-form ψ the element

$$[\psi] \in H^{n-k-1}(X;\mathbf{R})/H^{n-k-1}_{\text{free}}(X;\mathbf{Z}) \cong \text{Hom}(H_{n-k-1}(X;\mathbf{Z})_{\text{free}}, \mathbf{R}/\mathbf{Z})$$

given by

$$[\psi](z) \equiv \int_z \psi \ (\text{mod } \mathbf{Z}).$$

This map is clearly an isomorphism. □

A second proof of duality. We present here a second proof of Theorem 6.2 which does not use the dual sequences. This argument will be used in a subsequent article to establish a general "Alexander-Pontrjagin" duality for characters.

We first prove that the pairing is nondegenerate. Fix $\alpha \in \widehat{\mathbb{H}}^{n-k-1}(X)$ and assume that

$$(\alpha * \beta)\,[X] = 0$$

for all $\beta \in \widehat{\mathbb{H}}^k(X)$. We shall prove that $\alpha = 0$.

To begin consider a class β which is represented by a smooth form b. Then

$$(\alpha * \beta)[X] = (-1)^{n-k} \int_X \delta_1\alpha \wedge b \ \equiv \ 0 \ (\text{mod } \mathbf{Z}).$$

Since this holds for all smooth k-forms b, we conclude that $\delta_1\alpha = 0$. Hence, there is a representative $a \in \alpha$ with $da = R$ a rectifiable cycle whose integral homology class is torsion.

We shall show that this homology class is zero. Choose a class $u \in H^{k+1}(X;\mathbf{Z})_{\text{tor}}$ of order m, and let T be a rectifiable cycle with $[T] = u$. Since $m[T] = 0$ there exists a rectifiable current S of degree k with $mT = dS$. Set $\beta = [\frac{1}{m}S] \in \widehat{\mathbb{H}}^k(X)$ and note that $\delta_1\beta = 0$. Now

$$\begin{aligned}(\alpha * \beta)[X] &= (-1)^{n-k}(d_2 a \wedge b)[X] = (-1)^{n-k}(R \wedge \tfrac{1}{m}S)[X] \\ &= (-1)^{n-k}\tfrac{1}{m}\{\text{intersection number of } R \text{ with } S\} \\ &= (-1)^{n-k} Lk([R], u) \\ &\equiv 0 \ (\text{mod } \mathbf{Z}),\end{aligned}$$

where Lk denotes the Seifert-deRham linking number. Since this holds for all u, the nondegeneracy of this linking pairing on torsion cycles implies that $[R] = 0$ in $H^{n-k}(X;\mathbf{Z})$. Hence $\delta_2\alpha = 0$ and so after adding an exact current we may assume that a is a smooth d-closed differential form of degree $n-k-1$.

Now choose any class $v \in H^{k+1}(X; \mathbf{Z}) \cong H_{n-k-1}(X; \mathbf{Z})$ and let S be a rectifiable cycle in v. Let ϕ be a smooth d-closed form representing $v \otimes \mathbf{R}$ and choose a current b with $db = \phi - S$. Let $\beta = [b] \in \widehat{\mathbb{H}}^k(X)$. Then

$$(\alpha * [b])[X] = (a \wedge d_2 b)[X] \equiv \int_S a \equiv 0 \pmod{\mathbf{Z}}.$$

Hence, $[a] = 0$ in $H^{n-k-1}(X; \mathbf{R})/H^{n-k-1}_{\text{free}}(X; \mathbf{Z}) = \ker \delta$, and so $\alpha = 0$ as claimed.

By the commutativity of the $*$-product we may interchange α and β above and conclude that the pairing is nondegenerate as asserted.

Finally we recall that $\widehat{\mathbb{H}}^k_\infty(X) \subset \widehat{\mathbb{H}}^k(X)$ is a closed subgroup with discrete quotient, and $\widehat{\mathbb{H}}^k_\infty(X) \cong \mathcal{E}^k(X)/\mathcal{Z}^k_0(X) \cong d\mathcal{E}^k(X) \oplus \{H^k(X; \mathbf{R})/H^k_{\text{free}}(X; \mathbf{Z})\}$. Recall, by Lemma 5.1 that

$$\text{Hom}(d\mathcal{E}^k(X), S^1) = d\mathcal{D}'^{n-k-1}(X).$$

It follows that the image of the homomorphism $\mathcal{D}$ consists exactly of the smooth homomorphisms. □

Remark 6.7. Evaluation homomorphisms lie at the edge of the smooth dual. Note that for any rectifiable cycle Z of dimension k there is a natural *evaluation homomorphism*

$$h_Z \colon \widehat{\mathbb{H}}^k(X) \longrightarrow S^1 \qquad \text{given by} \quad h_Z(\beta) \equiv \Psi(\beta)(Z)$$

where Ψ is the map to Cheeger-Simons characters defined in §4. *These homomorphisms are not in the smooth dual of* $\widehat{\mathbb{H}}^k(X)$*, however they lie at the boundary of the smooth dual. In fact elements of the smooth dual can all be considered as generalized smoothing homotopies associated to these homomorphisms.* To see this let $a \in \mathcal{S}^{n-k-1}(X)$ be an L^1_{loc}-spark which satisfies the equation

$$da = \psi - Z$$

for a smooth $(n-k)$-form ψ. Fix $\beta \in \widehat{\mathbb{H}}^k(X)$ and choose an L^1_{loc}-spark $b \in \beta$ which is smooth on a neighborhood of supp(Z) with spark equation $db = \phi - R$. Then by Definition 3.2

$$(6.8) \qquad \begin{aligned} (\alpha * \beta)[X] &\equiv \int_X a \wedge \psi + h_Z(\beta) \pmod{\mathbf{Z}} \\ &\equiv \int_R a + \int_X \phi \wedge b \pmod{\mathbf{Z}} \end{aligned}$$

where $\alpha = \langle a \rangle$ and where we assume a to be smooth on supp(R).

Now suppose we take a family of Federer smoothing homotopies $a_\epsilon = H_\epsilon(Z)$ as in Note 2.8. Then a_ϵ is an L^1_{loc}-form of degree $n-k-1$ with support in an ϵ-neighborhood of Z. It satisfies the equation

$$da_\epsilon = Z_\epsilon - Z$$

where Z_ϵ is a smooth $(n-k)$-form which is a local smoothing of the cycle Z. This family has the property that

$$\lim_{\epsilon \to 0} a_\epsilon = 0 \qquad \text{in the flat topology.}$$

and in particular

$$\lim_{\epsilon \to 0} Z_\epsilon = Z.$$

Now $\text{supp}(Z) \cap \text{supp}(R) = \emptyset$ since b is smooth near Z. Hence $\text{supp}(a_\epsilon) \cap \text{supp}(R) = \emptyset$ for all ϵ sufficiently small. With $\alpha_\epsilon = \langle a_\epsilon \rangle$, equation (6.8) becomes

$$(6.9) \qquad \begin{aligned} (\alpha_\epsilon * \beta)\,[X] &\equiv \int_X a_e \wedge \psi + h_Z(\beta) \pmod{\mathbf{Z}} \\ &\equiv \int_X Z_\epsilon \wedge b \pmod{\mathbf{Z}} \end{aligned}$$

and we see (in two ways) that as $\epsilon \to 0$ the smooth homomorphisms $(a_\epsilon * \bullet)[X]$ converge to h_Z. Thus the evaluation homomorphisms lie at the boundary of the smooth ones.

7. Examples. We examine the groups of differential characters on some standard spaces to illustrate the duality established above. Given a closed subspace V of differential forms, let $V'_\infty = \text{Hom}_\infty(V, S^1) = \text{Hom}_\infty(V, \mathbf{R})$ denote the smooth Pontrjagin dual of V. We begin with the following basic observation.

LEMMA 7.1. *Let X be a compact oriented manifold of dimension n. Then for each k,*

$$\{d\mathcal{E}^k(X)\}'_\infty = d\mathcal{E}^{n-k-1}(X).$$

Proof. By Lemma 5.1 we know that $(d\mathcal{E}^k)' = d\mathcal{D}'^{n-k-1} \subset \mathcal{D}'^{n-k}$. Hence, $(d\mathcal{E}^k)'_\infty = (\mathcal{E}^k)'_\infty \cap d\mathcal{D}'^{n-k-1} = \mathcal{E}^{n-k} \cap d\mathcal{D}'^{n-k-1} = d\mathcal{E}^{n-k-1}$ since by the basic results of de Rham any smooth form which is weakly exact is smoothly exact. ◻

Example 7.2. (Surfaces) Let Σ be a compact oriented surface of genus g. The duality result asserts that $\widehat{\mathbb{H}}^1(\Sigma)$ is the smooth Pontrjagin dual of $\widehat{\mathbb{H}}^0(\Sigma)$. We see this explicitly as follows. From Proposition 1.14 we have the (split) short exact sequences

$$0 \to \frac{H^1(\Sigma;\, \mathbf{R})}{H^1(\Sigma;\, \mathbf{Z})} \to \widehat{\mathbb{H}}^1(\Sigma) \to d\mathcal{E}^1(\Sigma) \times \mathbf{Z} \to 0$$
$$0 \to \frac{H^0(\Sigma;\, \mathbf{R})}{H^0(\Sigma;\, \mathbf{Z})} \to \widehat{\mathbb{H}}^0(\Sigma) \to d\mathcal{E}^0(\Sigma) \times \mathbf{Z}^{2g} \to 0$$

from which we deduce that

$$\widehat{\mathbb{H}}^1(\Sigma) \cong (S^1)^{2g} \times \mathbf{Z} \times d\mathcal{E}^1(\Sigma) \qquad \text{and} \qquad \widehat{\mathbb{H}}^0(\Sigma) \cong \mathbf{Z}^{2g} \times S^1 \times d\mathcal{E}^0(\Sigma).$$

Example 7.3. (3-manifolds) Let M be a compact oriented 3-manifold. Then by Observation 5.3, $\widehat{\mathbb{H}}^1(M) \cong \{H^1(M;\, \mathbf{R})/H^1(M;\, \mathbf{Z})_{\text{free}}\} \times d\mathcal{E}^1 \times H^2(M;\, \mathbf{Z})$ which using Poincaré duality can be rewritten

$$\widehat{\mathbb{H}}^1(M) \cong \text{Hom}(H_1(M;\, \mathbf{Z})_{\text{free}}, S^1) \times H_1(M;\, \mathbf{Z})_{\text{free}} \times d\mathcal{E}^1(M) \times H_1(M;\, \mathbf{Z})_{\text{torsion}}$$

where (via Lemma 7.1) the self-duality is manifest.

Example 7.4. (Complex Projective Space $\mathbb{P}^n_{\mathbb{C}}$) If $X = \mathbb{P}^n_{\mathbb{C}}$, we see that

$$\frac{H^k(X;\, \mathbf{R})}{H^k(X;\, \mathbf{Z})} = \begin{cases} S^1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

and

$$\mathcal{Z}_0^{k+1}(X) = \begin{cases} td\mathcal{E}^k(X) & \text{if } k \text{ is even} \\ \mathbf{Z} \times d\mathcal{E}^k(X) & \text{if } k \text{ is odd.} \end{cases}$$

From Proposition 1.14 we then compute the following:

k	$\widehat{\mathbb{H}}^k(X)$
-1	$\mathbf{Z}$
0	$S^1 \times d\mathcal{E}^0$
1	$\mathbf{Z} \times d\mathcal{E}^1$
2	$S^1 \times d\mathcal{E}^2$
3	$\mathbf{Z} \times d\mathcal{E}^3$
.	.
.	.
.	.
$2n-3$	$\mathbf{Z} \times d\mathcal{E}^{2n-3}$
$2n-2$	$S^1 \times d\mathcal{E}^{2n-2}$
$2n-1$	$\mathbf{Z} \times d\mathcal{E}^{2n-1}$
$2n$	S^1

Example 7.5. (Real Projective Space $\mathbb{P}^{2n+1}_{\mathbf{R}}$) Proceeding as above we calculate that when $X = \mathbb{P}^{2n+1}_{\mathbf{R}}$, one has

k	$\widehat{\mathbb{H}}^k(X)$
-1	$\mathbf{Z}$
0	$S^1 \times d\mathcal{E}^0$
1	$\mathbf{Z}_2 \times d\mathcal{E}^1$
2	$d\mathcal{E}^2$
3	$\mathbf{Z}_2 \times d\mathcal{E}^3$
4	$d\mathcal{E}^4$
5	$\mathbf{Z}_2 \times d\mathcal{E}^5$
6	$d\mathcal{E}^6$
.	.
.	.
.	.
$2n-4$	$d\mathcal{E}^{2n-4}$
$2n-3$	$\mathbf{Z}_2 \times d\mathcal{E}^{2n-3}$
$2n-2$	$d\mathcal{E}^{2n-2}$
$2n-1$	$\mathbf{Z}_2 \times d\mathcal{E}^{2n-1}$
$2n-0$	$\mathbf{Z} \times d\mathcal{E}^{2n}$
$2n+1$	S^1

Recall that $\mathrm{Hom}(\mathbf{Z}_2, S^1) = \mathbf{Z}_2$.

Example 7.6. (Products of projective spaces) When $X = \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$, we find that:

k	$\widehat{\mathbb{H}}^k(X)$
-1	$\mathbf{Z}$
0	$S^1 \times d\mathcal{E}^0$
1	$\mathbf{Z} \times \mathbf{Z} \times d\mathcal{E}^1$
2	$S^1 \times S^1 \times d\mathcal{E}^2$
3	$\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times d\mathcal{E}^3$
4	$S^1 \times S^1 \times S^1 \times d\mathcal{E}^4$
5	$\mathbf{Z} \times \mathbf{Z} \times d\mathcal{E}^5$
6	$S^1 \times S^1 \times d\mathcal{E}^6$
7	$\mathbf{Z} \times d\mathcal{E}^7$
8	S^1

8. Characters with compact support. The theory developed in the previous sections can be carried through for characters with compact support. This represents the "homology version" of the theory. Let X be a smooth n-manifold as before and let $\mathcal{D}'^k_{\mathrm{cpt}}(X) \subset \mathcal{D}'^k(X)$ denote the *currents with compact support* on X. For each of the subspaces $W \subset \mathcal{D}'^k(X)$ introduced in §1, we denote

$$W_{\mathrm{cpt}} \stackrel{\mathrm{def}}{=} W \cap \mathcal{D}'^k_{\mathrm{cpt}}(X).$$

The analogues of statements (1.1) hold for currents with compact support, and one has:

Definition 8.1. (Sparks with compact support)

$$\mathcal{S}^k_{\mathrm{cpt}}(X) \stackrel{\mathrm{def}}{=} \{a \in \mathcal{D}'^k_{\mathrm{cpt}}(X) : da = \phi - R \text{ where } \phi \in \mathcal{E}^{k+1}_{\mathrm{cpt}}(X) \text{ and } R \in \mathcal{IF}^{k+1}_{\mathrm{cpt}}(X)\}$$

Definition 8.2. The space

$$\widehat{\mathbb{H}}^k_{\mathrm{cpt}}(X) \stackrel{\mathrm{def}}{=} \mathcal{S}^k_{\mathrm{cpt}}(X) / \{d\mathcal{D}'^{k-1}_{\mathrm{cpt}}(X) + \mathcal{IF}^k_{\mathrm{cpt}}(X)\}$$

will be called the group of *de Rham-Federer characters with compact support* on X.

The analogues of Propositions 1.11, 1.12, 1.14 hold for $\widehat{\mathbb{H}}^k_{\text{cpt}}(X)$. In particular we have:

PROPOSITION 8.3. *There is a functorial short exact sequence*

$$0 \longrightarrow \frac{H_{n-k}(X;\, \mathbf{R})}{H_{n-k}(X;\, \mathbf{Z})_{\text{free}}} \longrightarrow \widehat{\mathbb{H}}^k_{\text{cpt}}(X) \xrightarrow{\delta} Q^{k+1}_{\text{cpt}}(X) \longrightarrow 0$$

where

$$Q^{\ell}_{\text{cpt}}(X) \stackrel{\text{def}}{=} \{(\phi, u) \in \mathcal{Z}^{\ell}_{0,\text{cpt}}(X) \times H_{n-\ell}(X;\, \mathbf{Z}) : [\phi] = u \otimes \mathbf{R}\}.$$

The analogues of the results in §2 and §3 carry over to these spaces.

Arguments parallel to those of §4 establish the following. We shall say that a Cheeger-Simons differential character $a\colon\ \mathcal{Z}_k(X) \to \mathbf{R}/\mathbf{Z}$ has *compact support* if there is a compact subset $K \subset X$ such that $a(Z) = 0$ for all cycles Z with $\text{supp}(Z) \cap K = \emptyset$.

THEOREM 8.4. *Let $\widehat{H}^k_{\text{cpt}}(X; \mathbf{R}/\mathbf{Z}) \subset \widehat{H}^k(X; \mathbf{R}/\mathbf{Z})$ denote the subgroup of differential characters of degree k with compact support on X. Then there is a natural isomorphism*

$$\Psi\colon\ \widehat{\mathbb{H}}^k_{\text{cpt}}(X) \xrightarrow{\cong} \widehat{H}^k_{\text{cpt}}(X; \mathbf{R}/\mathbf{Z})$$

induced by integration.

The definition of the $*$-product given in §3 extends straightforwardly to give a pairing

$$*\colon\ \widehat{\mathbb{H}}^k(X) \times \widehat{\mathbb{H}}^{\ell}_{\text{cpt}}(X) \longrightarrow \widehat{\mathbb{H}}^{k+\ell+1}_{\text{cpt}}(X) \tag{8.5}$$

which in particular makes $\widehat{\mathbb{H}}^*_{\text{cpt}}(X)$ a graded commutative ring.

Suppose now that X is connected, oriented, and of dimension n.

Definition 8.6. For each integer k, $0 \leq k < n$ we define duality mappings

$$\mathcal{D}\colon\ \widehat{\mathbb{H}}^{n-k-1}_{\text{cpt}}(X) \longrightarrow \widehat{\mathbb{H}}^k(X)^* \equiv \text{Hom}_\infty(\widehat{\mathbb{H}}^k(X), \mathbf{R}/\mathbf{Z})$$

and

$$\mathcal{D}\colon\ \widehat{\mathbb{H}}^{n-k-1}(X) \longrightarrow \widehat{\mathbb{H}}^k_{\text{cpt}}(X)^* \equiv \text{Hom}_\infty(\widehat{\mathbb{H}}^k_{\text{cpt}}(X), \mathbf{R}/\mathbf{Z})$$

by

$$\mathcal{D}(\alpha)(\beta) \stackrel{\text{def}}{=} \alpha * \beta \in \widehat{\mathbb{H}}^n_{\text{cpt}}(X) = \mathbf{R}/\mathbf{Z}.$$

where $\text{Hom}_\infty(\bullet, \mathbf{R}/\mathbf{Z})$ denotes the smooth homomorphisms, defined as in 5.10.

THEOREM 8.7. (Poincaré-Pontrjagin Duality on Noncompact Manifolds) *For any connected, oriented manifold X the duality mappings above are isomorphisms.*

Proof. The argument follows directly the second proof given in §6. Full details appear in [HL_5]. □

9. The Thom homomorphism. An interesting feature of the de Rham theory is that one easily constructs Thom isomorphisms for characters.

Let X be a smooth n-manifold and

$$\pi\colon\ E \longrightarrow X$$

a smooth oriented Riemannian vector bundle of rank d. Once and for all we fix a *compact approximation mode* in the sense of [HL_1; §I.4]. This amounts to choosing a C^∞-function $\chi\colon\ [0,\infty] \to [0,1]$ with $\chi(0) = 0$, $\chi' \geq 0$ and $\chi(t) \equiv 1$ for $t \geq 1$. Then from [HL_1; §§IV1-2] we have the following:

THEOREM 9.1. (Harvey-Lawson [HL_1; §§IV1-2]). *To each orthogonal connection D_E on E there is a canonically associated* Thom form $\tau \in \mathcal{E}^d(E)$ *such that*

(1) $d\tau = 0$,

(2) $\text{supp}(\tau) \subset D(E) = \{e \in E : \|e\| \leq 1\}$,

(3) $\pi_*(\tau) = 1$,

(4) $\tau|_X = \chi(D_E)$,

where $\chi(D_E)$ is the Chern-Euler form of D_E when d is even and is 0 when d is odd.

There is furthermore a canonically associated Thom spark $\gamma \in \mathcal{E}^{d-1}_{L^1_{\text{loc}}}$ *which is smooth outside the zero section $X \subset E$, has support in the unit disk bundle $D(E)$ and satisfies the spark equation*

$$d\gamma = \tau - [X] \qquad \text{on} \quad E.$$

Remark 9.2. For d even, τ is the Chern-Euler characteristic form of the time-1 pushforward connection on π^*E associated to the tautological cross-section.

Remark 9.3. Since $\pi_*\colon\ \mathcal{E}^{k+d}_{\text{cpt}}(E) \longrightarrow \mathcal{E}^{k}_{\text{cpt}}(X)$ (integration over the fibre), there is a pull-back map on general currents

$$\pi^*\colon\ \mathcal{D}'^k(X) \longrightarrow \mathcal{D}'^{k+d}(E).$$

This map preserves the properties of local integrability, local rectifiability, local flatness and local integral flatness.

As a consequence we have an induced homomorphism

$$\pi^*\colon\ \widehat{\mathbb{H}}^*(X) \longrightarrow \widehat{\mathbb{H}}^{*+d}(E).$$

We would like the corresponding mapping for the functor $\widehat{\mathbb{H}}^*_{\text{cpt}}$.

Definition 9.4. We define a *Thom mapping*

$$\mathcal{T}: \ \mathcal{S}^k_{\text{cpt}}(X) \longrightarrow \mathcal{S}^{k+d}_{\text{cpt}}(E)$$

as follows. For $a \in \mathcal{S}^k_{\text{cpt}}(X)$ write $da = \phi - R$ as in Lemma 1.5. Then

$$\mathcal{T}(a) \stackrel{\text{def}}{=} \pi^* a \wedge \tau + (-1)^{k+d+1} \pi^*(R) \wedge \gamma$$

where τ and γ are the Thom form and the Thom spark associated to the connection in Theorem 9.1. Under a local orthogonal trivialization $\pi^{-1}U \cong U \times \mathbf{R}^d$ of E we have $\pi^* R \cong R \times [\mathbf{R}^d]$ and $\gamma = \text{pr}^* \gamma_0$ where γ_0 is the universal Thom spark on $\mathbf{R}^d$. Thus $\pi^*(R) \wedge \gamma$ is well defined. Note that

$$d\mathcal{T}(a) = \pi^* \phi \wedge \tau - \pi^* R \wedge [X]. \tag{9.5}$$

THEOREM 9.6. *Let $E \longrightarrow X$ be as above. Then for all $k \geq 0$ the Thom mapping $\mathcal{T}$ induces an injective homomorphism*

$$\widehat{\mathbb{T}}: \ \widehat{\mathbb{H}}^k_{\text{cpt}}(X) \longrightarrow \widehat{\mathbb{H}}^{k+d}_{\text{cpt}}(E)$$

with the following properties:

(i) $\widehat{\mathbb{T}}(1) = [\gamma]$

(ii) $\widehat{\mathbb{T}}(\alpha) = (\pi^*\alpha) * \widehat{\mathbb{T}}(1) = (\pi^*\alpha) * [\gamma]$

(iii) $\delta_1 \widehat{\mathbb{T}}(\alpha) = \mathbb{T}_{\text{deR}}(\delta_1 \alpha) \stackrel{\text{def}}{=} \pi^*(\delta_1 \alpha) \wedge \omega$ and $\delta_2 \widehat{\mathbb{T}}(\alpha) = \mathbb{T}(\delta_2 \alpha)$

where $\mathbb{T}$ denotes the Thom isomorphism on integral cohomology.

Note 9.7. The unit $1 \in \widehat{\mathbb{H}}^{-1}(X)$ is the deRham character γ_0 with

$$d\gamma_0 = 1 - [X] \qquad \text{on } X.$$

Of course, $1 = [X]$ and $\gamma_0 = 0$ as currents. Think of γ_0 as the identity spark. Part (i) says that the image of γ_0 is represented by the Thom spark γ which satisfies

$$d\gamma = \tau - [X] \qquad \text{on } E.$$

The class $[\gamma] \in \widehat{\mathbb{H}}^{d-1}_{\text{cpt}}(E)$ will be called the *Thom character*.

Note 9.8. Part (ii) states that the Thom homomorphism for characters is a $\widehat{\mathbb{H}}^*(X)$-module mapping. The image is the free rank-1 module generated by the Thom character.

Note 9.9. Part (iii) states that the differential δ carries the Thom homomorphism on characters over to the classical Thom isomorphisms.

Proof of Theorem 9.6. We have directly from Definitions 3.2 and 9.4 that

$$\mathcal{T}(1) = \gamma \qquad \text{and} \qquad \mathcal{T}(a) = a * \gamma,$$

which proves (i) and (ii). Part (iii) follows immediately from (9.5). It remains to prove that $\widehat{\mathbb{T}}$ is injective. For this we observe that $\widehat{\mathbb{T}}$ induces a map of the short exact sequences from Propositions 8.3 and 1.14:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \frac{H_{n-k}(X;\mathbf{R})}{H_{n-k}(X;\mathbf{Z})_{\text{free}}} & \longrightarrow & \widehat{\mathbb{H}}^k(X) & \xrightarrow{\delta} & Q^{k+1}(X) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow\widehat{\mathbb{T}} & & \downarrow \mathbb{T}_{\text{deR}}\times\mathbb{T} & & \\
0 & \longrightarrow & \frac{H_{n-k}(E;\mathbf{R})}{H_{n-k}(E;\mathbf{Z})_{\text{free}}} & \longrightarrow & \widehat{\mathbb{H}}^{k+d}_{\text{cpt}}(E) & \xrightarrow{\delta} & Q^{k+d+1}_{\text{cpt}}(E) & \longrightarrow & 0.
\end{array}$$

The map on the left is a quotient of Thom isomorphisms under Poincaré duality. The map on the right is a product of $\mathbb{T}_{\text{deR}}(\phi) = \pi^*(\phi) \wedge \tau$, which is clearly injective, and the Thom isomorphism $\mathbb{T}$ on integral cohomology. It follows that $\widehat{\mathbb{T}}$ is injective. □

Remark 9.10. (Atomic sections) A smooth cross-section $s\colon X \to E$ is called *atomic* if, whenever it is written in a local frame $e_1, \ldots, e_d$ for E as $s = u_1 e_1 + \cdots + u_d e_d$, it has the property that $du^I/|u|^{|I|} \in L^1_{\text{loc}}$ for all $|I| < d$. Associated to an atomic section s is a codimension-d integrally flat current $\text{Div}(s)$ defined by the vanishing of s [HS]. Any section s with nondegenerate zeros is atomic, and $\text{Div}(s)$ is the manifold defined by $s = 0$.

If $s\colon X \to E$ is atomic and d is even, then $s^*\tau = \chi(D_s)$ is the Euler form of the time-1 push-forward connection D_s associated to s (cf. Remark 9.2). Furthermore, $s^*\gamma$ is the associated Euler spark as defined in [HZ$_2$]. This is a canonical L^1_{loc}-form on X which satisfies

$$ds^*\gamma = \chi(D_s) - \text{Div}(s).$$

Note that in general

$$s^*\mathcal{T}(a) = a * (s^*\gamma). \tag{9.11}$$

We now examine how $\widehat{\mathbb{T}}$ changes under a change of connection on E. Suppose D_0 and D_1 are two orthogonal connections on E with Thom sparks γ_0 and γ_1 satisfying

$$d\gamma_0 = \tau_0 - [X] \qquad \text{and} \qquad d\gamma_1 = \tau_1 - [X].$$

Consider the convex family of connections $D_t = (1-t)D_0 + tD_1$, and define a connection D on the bundle $E \times \mathbf{R} \longrightarrow X \times \mathbf{R}$ by $D = D_t + dt \otimes (\partial/\partial t)$. Let γ and τ be the Thom spark and Thom form associated to D. They satisfy the equation

$$d\gamma = \tau - [X \times \mathbf{R}].$$

Consider the flat current β and the C^∞-form γ_{01} defined by

$$\beta \overset{\text{def}}{=} \mathrm{pr}_*(\psi\gamma) \qquad \text{and} \qquad \gamma_{10} \overset{\text{def}}{=} -\mathrm{pr}_*(\psi\tau),$$

where ψ is the characteristic function of $E \times [0,1] \subset E \times \mathbf{R}$ and pr: $E \times \mathbf{R} \to E$ is the projection. Since $\mathrm{pr}_*(\psi[X \times \mathbf{R}]) = 0$, we see that $d\beta = \mathrm{pr}_*(d(\psi\gamma)) = \gamma_1 - \gamma_0 + \mathrm{pr}_*(\psi\tau)$, and so

$$\gamma_1 - \gamma_0 = \gamma_{10} + d\beta. \tag{9.12}$$

Taking d gives the equation

$$\tau_1 - \tau_0 = d\gamma_{10}. \tag{9.13}$$

PROPOSITION 9.14. *Let $\widehat{\mathbb{T}}_k$ be the Thom homomorphism on differential characters associated to the connection D_k on E for $k = 0, 1$. Then*

$$\widehat{\mathbb{T}}_1(\alpha) - \widehat{\mathbb{T}}_0(\alpha) = \pi^*(\alpha) * [\gamma_{10}]$$

where $[\gamma_{10}]$ denotes the character defined by the smooth transgression form γ_{10}.

Proof. From (9.12) we have

$$\begin{aligned}\widehat{\mathbb{T}}_1(\alpha) - \widehat{\mathbb{T}}_0(\alpha) &= \pi^*(\alpha) * [\gamma_1] - \pi^*(\alpha) * [\gamma_0] \\ &= \pi^*(\alpha) * ([\gamma_{10} + d\beta]) = \pi^*(\alpha) * ([\gamma_{10}]).\end{aligned}$$

□

10. Gysin maps. One advantage of the de Rham point of view is that it is easy to define Gysin maps. For example if $f\colon X \to Y$ is a smooth fibre bundle with compact oriented fibres of dimension d, then push-forward of currents defines "Gysin" homomorphisms

$$f_!\colon \widehat{\mathbb{H}}^{k+d}(X) \longrightarrow \widehat{\mathbb{H}}^k(Y) \qquad \text{and} \qquad f_!\colon \widehat{\mathbb{H}}^{k+d}_{\mathrm{cpt}}(X) \longrightarrow \widehat{\mathbb{H}}^k_{\mathrm{cpt}}(Y). \tag{10.1}$$

This raises the question of whether there exist Gysin maps of differential characters for more general mappings, and in particular for smooth embeddings. Such maps do exist—the Thom homomorphism of §9 is a basic example. It depends on a choice of connection. In general the Gysin homomorphism depends on a choice of "normal geometry" to the embedding.

Definition 10.2. Suppose X and Y are manifolds and $f\colon X \hookrightarrow Y$ is a smooth embedding with oriented normal bundle N. Given an orthogonal connection on N and a smooth embedding $F\colon N \hookrightarrow Y$ extending f, there is a Gysin homomorphism

$$f_!\colon \widehat{\mathbb{H}}^*_{\rm cpt}(X) \longrightarrow \widehat{\mathbb{H}}^{*+d}_{\rm cpt}(Y),$$

where $d = \dim(Y) - \dim(X)$, defined by

$$f_! = F_* \circ \widehat{\mathbb{T}}$$

where $\widehat{\mathbb{T}}$ is the Thom homomorphism for N defined in §9.

Example 10.3. Suppose X and Y are Riemannian and $f\colon X \hookrightarrow Y$ is an isometric embedding. Let ρ_f be the injectivity radius of f, i.e., the sup of the numbers ρ such that the exponential mapping $\exp\colon N \to Y$ is injective on $D_\rho(N) = \{v \in N : \|v\| \le \rho\}$. Then for any $\rho < \rho_f$ the mapping $F\colon N \to Y$ given by $F = \exp \circ \mu_\rho$ is an embedding on a neighborhood of $D_1(N)$ and since the Thom spark of N has support in $D_1(N)$ the composition $f_! = F_* \circ \widehat{\mathbb{T}}$ is well defined.

Definition 10.4. Let $f\colon X \to Y$ be *any* smooth mapping between compact oriented manifolds. Assume Y is Riemannian with injectivity radius i_Y. Consider the smooth embedding $\gamma_f\colon X \hookrightarrow X \times Y$ defined by graphing $\gamma_f(x) = (x, f(x))$ for $x \in X$. Now the normal bundle N to γ_f can be identified with the restriction of TY to $\gamma_f(X)$, and we have a *canonical* identification of $\{v \in N : \|v\| < i_Y\}$ with a tubular neighborhood of $\gamma_f(x) \subset X \times Y$ which associates to $v \in T_{f(x)}Y$ the point $\exp_{f(x)}(v)$. Thus as above we have a canonically defined Gysin mapping $(\gamma_f)_!$. Following Grothendieck we combine this with the projection $\mathrm{pr}\colon X \times Y \to Y$ to get a Gysin mapping

$$f_! \stackrel{\rm def}{=} \mathrm{pr}_! \circ (\gamma_f)_!.$$

Remark 10.5. If X and Y are compact and oriented, then Gysin maps for ordinary cohomology can be defined using Poincaré duality $D\colon H^k(X; \mathbf{Z}) \to H_{n-k}(X; \mathbf{Z})$ by setting $f_! = D^{-1} \circ f_* \circ D$. This does not quite work for differential characters because the inverse $\mathcal{D}^{-1}$ to the duality map is only densely defined. However if one takes the family of Gysin maps $f_!^t$, $0 < t \le 1$ obtained via the family of Thom forms $\tau_t = \rho_t^* \tau$ (where ρ_t is scalar multiplication by t in the normal bundle), then these maps converge to the "nonsmooth character" $\mathcal{D}^{-1} \circ f_* \circ \mathcal{D}$, i.e.,

$$\lim_{t \to 0} \mathcal{D} \circ f_!^t = f_* \circ \mathcal{D}$$

on $\widehat{\mathbb{H}}^*(X)$.

11. Morse sparks. Sparks represent differential characters just as forms (or currents) represent cohomology, and they arise naturally in many situations. Interestingly a rich source of sparks is provided by Morse Theory. In [HL_4] it is shown that for each Morse function f on X there is a subcomplex $\mathcal{E}^*_{0,f}(X) \subset \mathcal{E}^*(X)$ of finite codimension with a surjection P: $\mathcal{E}^*_{0,f}(X) \to \mathbb{S}^{\mathbf{Z}}_f$ onto the *integral* Morse complex, which has affine fibres and induces an isomorphism in cohomology. Furthermore, if $\mathcal{Z}^*_{0,f}(X) \subset \mathcal{Z}^*_0(X)$ denotes the closed forms in $\mathcal{E}^*_{0,f}(X)$, then there is a continuous linear mapping T: $\mathcal{Z}^*_{0,f}(X) \to S^{*-1}(X)$ into the space of sparks with $d(T\phi) = \phi - P(\phi)$. This inverts the differential δ_1 and canonically splits the characters on this set. We present below a summary of this construction.

Let f: $X \to \mathbf{R}$ be a Morse function on a compact manifold and choose a gradient-like flow φ_t: $X \to X$ satisfying the Morse-Stokes Axioms in [HL_4] (such flows always exist). Let Cr_f denote the set of critical points of f and for each $p \in \mathrm{Cr}_f$ let S_p, U_p denote the stable and unstable manifolds of p respectively. These are manifolds of finite volume in X and so define integral currents $[S_p]$ and $[U_p]$. Consider the finite-dimensional vector space and the integral lattice inside it given by

$$\mathbb{S}_f = \mathrm{span}_{\mathbf{R}}\{[S_p]\}_{p\in \mathrm{Cr}_f} \qquad \text{and} \qquad \mathbb{S}^{\mathbf{Z}}_f = \mathrm{span}_{\mathbf{Z}}\{[S_p]\}_{p\in \mathrm{Cr}_f}.$$

Each of these spaces of currents is d invariant and there are natural isomorphisms

$$H^*(\mathbb{S}_f) \cong H^*(X;\, \mathbf{R}) \qquad \text{and} \qquad H^*(\mathbb{S}^{\mathbf{Z}}_f) \cong H^*(X;\, \mathbf{Z}).$$

In fact the continuous linear projection

$$P\colon\ \mathcal{E}^*(X) \longrightarrow \mathbb{S}_f$$

given by

$$P(\phi) = \sum_{p\in \mathrm{Cr}_f} \left\{ \int_{U_p} \phi \right\} [S_p] \tag{11.1}$$

is chain homotopic to the identity. That is, there exists a continuous linear mapping T: $\mathcal{E}^*(X) \to \mathcal{D}'^*(X)$ of degree -1, such that

$$d \circ T + T \circ d = I - P, \tag{11.2}$$

where I: $\mathcal{E}^*(X) \to \mathcal{D}'^*(X)$ is the obvious inclusion.

Let us denote by $Z\mathbb{S}_f$ and $Z\mathbb{S}^{\mathbf{Z}}_f$ the cycles in $\mathbb{S}_f$ and $\mathbb{S}^{\mathbf{Z}}_f$ respectively.

Definition 11.3. Fix an integral cycle $R = \sum n_p[S_p] \in Z\mathbb{S}^{\mathbf{Z}}_f$. A smooth form $\phi \in \mathcal{Z}^k_0(X)$ is called a *Thom form for R* if $P(\phi) = R$, i.e., if $\int_{U_p} \phi = n_p$ for all p.

Let $\mathcal{Z}^*_{0,f}(X) = \mathcal{Z}^*_0(X) \cap P^{-1}(Z\mathbb{S}_f^{\mathbf{Z}})$ denote the set of all such forms. We now show that Thom forms exists for every cycle in the Morse complex.

LEMMA 11.4. *The restricted mapping* P: $\mathcal{Z}^*_{0,f}(X) \longrightarrow Z\mathbb{S}_f^{\mathbf{Z}}$ *is surjective. Its fibres are affine subspaces of finite codimension in* $\mathcal{Z}^*_0(X)$.

Proof. For a Morse-Stokes flow we have that S_p intersects U_p transversely in one point and that $\overline{S_p} \cap \overline{U_{p'}} = \emptyset$ for all distinct pairs $p, p' \in \mathrm{Cr}_f$ of the same index. Fix a cycle $R \in Z\mathbb{S}_f^{\mathbf{Z}}$ of degree k, and denote by $H_\epsilon(R)$ the Federer homotopy smoothing supported in an ϵ-neighborhood of R (see Note 2.8). Since $\mathrm{supp}(R) \cap \partial U_p = \emptyset$ for all p of index $n-k$, for ϵ sufficiently small we also have $\mathrm{supp}(H_\epsilon(R)) \cap \partial U_p = \emptyset$ for all such p. Now $dH_\epsilon(R) = R_\epsilon - R$ where R_ϵ is a smooth d-closed k-form. Now

$$\begin{aligned}\int_{U_p} R_\epsilon &= (R \wedge [U_p])[X] + (dH_\epsilon(R) \wedge [U_p])[X] \\ &= \sum_{p'} n_{p'}([S_{p'}] \wedge [U_p])[X] + (H_\epsilon(R) \wedge \partial[U_p])[X] = n_p + 0\end{aligned}$$

as desired. Note that $P^{-1}(R)$ is defined by the affine equations $\int_{U_p} \phi = n_p$ for all $p \in \mathrm{Cr}_f$ whereas $\mathcal{Z}^k_0(X)$ is defined by the weaker conditions

$$\sum_{p \in \mathrm{Cr}_f} m_p \int_{U_p} \phi \in \mathbf{Z} \qquad \text{whenever } d\left(\sum m_p U_p\right) = 0 \text{ and all } m_p \in \mathbf{Z}.$$

This proves the second assertion of the lemma. □

Observe now that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{Z}^*_{0,f}(X) & \xrightarrow{\subset} & \mathcal{Z}^*_0(X) \\ {\scriptstyle P}\big\downarrow & & \big\downarrow{\scriptstyle P_0} \\ Z\mathbb{S}_f^{\mathbf{Z}} & \xrightarrow{\mathrm{pr}_0} & H^*_{\mathrm{free}}(X; \mathbf{Z}) \end{array} \tag{11.5}$$

where P_0: $\mathcal{Z}^*_0(X) \to \mathrm{Hom}(H_*(X; \mathbf{Z}), \mathbf{Z}) \cong H^*_{\mathrm{free}}(X; \mathbf{Z})$ is the *period mapping* and pr_0 is the obvious projection. Both P and P_0 are surjective with affine fibres. They represent a "fattening" by smooth forms of free abelian groups $Z\mathbb{S}_f^{\mathbf{Z}}$ and $H^*_{\mathrm{free}}(X; \mathbf{Z})$. Note that $\mathcal{Z}^*_{0,f}(X) = \mathcal{Z}^*_0(X)$ if and only if $\mathbb{S}_f = Z\mathbb{S}_f$, i.e., iff f is a perfect Morse function and $H^*(X; \mathbf{Z})$ is torsion-free.

The map T from (11.2) associates a Morse spark to each Thom form on X. This gives the following:

THEOREM 11.6. [HL$_4$] *Let* $f\colon X \to \mathbf{R}$ *be a Morse function as above. Then there is a continuous linear operator of degree* -1

$$T\colon\ \mathcal{Z}^*_{0,f}(X) \longrightarrow \mathcal{S}^{*-1}(X)$$

with the property that for each $\phi \in \mathcal{Z}^*_{0,f}(X)$

$$d\{T(\phi)\} = \phi - \sum_{p\in \mathrm{Cr}(f)} n_p\,[S_p] \qquad \textit{where} \qquad n_p = \int_{U_p} \phi.$$

Remark 11.7. Note that T is a right inverse of the mapping $d_1\colon\ \mathcal{S}^{*-1}(X) \to \mathcal{Z}^*_0(X)$ on the subset $\mathcal{Z}^*_{0,f}(X)$, and so it splits the fundamental sequence from Proposition 1.12 on this set. To be more explicit we define the subgroup of *f-characters*:

$$\widehat{\mathbb{H}}^*_f(X) \equiv \delta_1^{-1}\{\mathcal{Z}^*_{0,f}(X)\} \subset \widehat{\mathbb{H}}^*(X).$$

This is a union of subspaces of constant finite codimension in $\widehat{\mathbb{H}}^*(X)$. The map T above gives a canonical splitting

$$\widehat{\mathbb{H}}^*_f(X) = H^*(X;\,S^1) \times \mathcal{Z}^*_{0,f}(X)$$

under which δ_1 becomes projection to the second factor and $\delta_2 = \mathrm{pr}\circ P\circ\delta_1$ where pr: $Z\mathbb{S}^{\mathbf{Z}}_f \to H^*(X;\,\mathbf{Z})$ is the projection.

12. Hodge sparks and Riemannian Abel-Jacobi mappings. Another important source of sparks is provided by Hodge theory. Suppose X is a compact Riemannian manifold. Recall (cf. [HP]) that any current R on X, not just the L^2-forms, has a *Hodge decomposition*

$$R = H(R) + dd^*G(R) + d^*dG(R) \tag{12.1}$$

where H is harmonic projection and G is the Greens operator. Also recall that d commutes with G, so that if R is a cycle, then $dG(R) = 0$.

Definition 12.2. Given an integrally flat cycle $R \in \mathcal{IF}^{k+1}(X)$, we define its *Hodge spark* $\sigma(R) \in \mathcal{S}^k(X)$ by

$$\sigma(R) \equiv -d^*G(R)$$

and observe that it satisfies the spark equation

$$d\sigma(R) = H(R) - R. \tag{12.3}$$

Note that $d_1 \circ \sigma = H$ and $d_2 \circ \sigma = \mathrm{Id}$ on the space of integrally flat cycles, where d_1 and d_2 are the differentials introduced in §1. In particular σ is a right inverse to d_2.

Let $\widehat{\sigma}(R) \in \widehat{\mathbb{H}}^k(X)$ denote the differential character corresponding to the Hodge spark $\sigma(R)$. Define the kth *character tJacobian* of X to be the torus

$$\mathrm{Jac}^k(X) \equiv \ker \delta \cong H^k(X;\, \mathbf{R})/H^k_{\mathrm{free}}(X;\, \mathbf{Z})$$

contained in $\widehat{\mathbb{H}}^k(X)$. If we restrict $\widehat{\sigma}$ to the space $\mathcal{B}^{k+1}(X) = d\mathcal{I}\mathcal{F}^k(X) = d\mathcal{R}^k(X)$ of integrally flat boundaries then

$$\widehat{\sigma}\colon\ \mathcal{B}^{k+1}(X) \to \mathrm{Jac}^k(X). \tag{12.4}$$

To see that $R = d\Gamma$ implies that $\delta_1\widehat{\sigma}(R) = 0$ note that $H(R) = 0$ and apply (12.3). Obviously $\delta_2(\widehat{\sigma}(R)) = 0$. The map $\widehat{\sigma}$ will be called the *character Abel-Jacobi map*.

Definition 12.5. An integrally flat boundary $R = d\Gamma \in \mathcal{B}^{k+1}(X)$ is *linearly equivalent to zero* or a *principal boundary* if $\widehat{\sigma}(R) = 0$. Let $\mathcal{B}^{k+1}_P(X)$ denote the space of principal boundaries, and define the k^{th} *Picard space of* X to be $\mathrm{Pic}^k(X) = \mathcal{B}^{k+1}(X)/\mathcal{B}^{k+1}_P(X)$.

In summary, $\widehat{\sigma}$ induces an injection (also denoted $\widehat{\sigma}$) on the quotient $\mathrm{Pic}^k(X)$. That is

$$\mathrm{Pic}^k(X) \overset{\hat{\sigma}}{\hookrightarrow} \mathrm{Jac}^k(X). \tag{12.6}$$

The character Jacobian is independent of the Riemannian metric on X. However, fixing a metric, and letting $\mathrm{Har}^k(X)$ and $\mathrm{Har}^k_0(X)$ denote the spaces of harmonic k forms and harmonic k forms with integral periods respectively, we have isomorphisms

$$\mathrm{Jac}^k(X) \cong \mathrm{Har}^k(X)/\mathrm{Har}^k_0(X) \cong \mathrm{Har}^{n-k}(X)^*/H_{n-k}(X, \mathbb{Z}) \cong \mathrm{Hom}(\mathrm{Har}^{n-k}_0, \mathbb{R}/\mathbb{Z}).$$

Definition 12.8. The map

$$\widehat{\sigma}_r\colon\ \mathcal{B}^{k+1}(X) \longrightarrow \mathrm{Hom}(\mathrm{Har}^{n-k}_0, \mathbb{R}/\mathbb{Z})$$

induced by $\widehat{\sigma}$ will be called the kth *Riemannian Abel-Jacobi map*.

In order to compute $\widehat{\sigma}_r$ more explicitly we must find a harmonic spark which is equivalent to the spark $\sigma(R)$.

LEMMA 12.9. *Suppose $R = d\Gamma$ where Γ is integrally flat. Then the Hodge decomposition of Γ has the form*

$$\Gamma = H(\Gamma) + dB - \sigma(R), \tag{12.10}$$

where $B = d^*G(\Gamma)$.

Proof. Note that $d^*dG(\Gamma) = d^*G(d\Gamma) = -\sigma(R)$. □

COROLLARY 12.11. *The sparks $\sigma(R)$ and $H(\Gamma)$ determine the same differential character, i.e.,*

$$\widehat{\sigma}(R) = \widehat{H}(\Gamma) \in \widehat{\mathbb{H}}^k(X).$$

In particular, $\widehat{H}(\Gamma)$ is independent of the choice of integrally flat Γ with $d\Gamma = R$.

Consequently, the mapping from $\mathcal{B}^{k+1}(X)$ to $\mathrm{Har}^k(X)/\mathrm{Har}_0^k(X)$ induced by $\widehat{\sigma}$ is given by sending $R = d\Gamma \in \mathcal{B}^{k+1}(X)$ to $H(\Gamma) \in \mathrm{Har}^k(X)$. Utilizing the duality theorem between $\mathrm{Har}^k(X)$ and $\mathrm{Har}^{n-k}(X)$ we see that:

$$\widehat{\sigma}_r(R)(\theta) \equiv \int H(\Gamma) \wedge \theta \mod \mathbb{Z} \tag{12.12}$$

for all $\theta \in \mathrm{Har}_0^{n-k}(X)$.

Since θ is d and d^* closed, equation (12.10) implies that

$$\widehat{\sigma}_r(R)(\theta) \equiv \int_\Gamma \theta \mod \mathbb{Z}, \quad \text{for all } \theta \in \mathrm{Har}_0^{n-k}(X).$$

Of course, $\widehat{\sigma}$ and $\widehat{\sigma}_r$ have the same kernel, yielding a generalization of a result of Chatterjee [Cha].

PROPOSITION 12.14. *Let $R = d\Gamma$ where Γ is integrally flat. Then R is linearly equivalent to zero if and only if $\int_\Gamma \theta \in \mathbf{Z}$ for all $\theta \in \mathrm{Har}_0^{n-k}(X)$.*

Example 12.15. (The case of dimension 0) For each point $p \in X$, considered as a cycle of degree n, equation (12.3) takes the form $d\sigma(p) = \omega - p$ where ω is the Riemannian volume element nomalized to have total volume one. For any other point q we have $\sigma(q - p) = \sigma(q) - \sigma(p)$ and the Abel-Jacobi map in this case restricts to a map

$$J\colon X \longrightarrow \mathrm{Jac}^{n-1}(X)$$

given by

$$J(q) = \widehat{\sigma}(q - p) = \widehat{H}(\Gamma_{pq})$$

where Γ_{pq} is any smooth curve in X joining p to q. Note that q is linearly equivalent to p iff $\int_{\Gamma_{pq}} \theta \equiv 0 \mod \mathbf{Z}$ for all harmonic 1-forms θ with integral periods. The map J extends linearly to all 0-cycles on X and in particular to symmetric powers of X. This strictly generalizes the classical map defined in Riemann surface theory. For 3-manifolds it appeared in Hitchin's discussion of gerbes and the mirror symmetry conjecture [Hi].

Example 12.16. Let X be a compact Kähler n-manifold and denote by $Z_{m,0}$ the holomorphic m-chains on X which are homologous to zero. (A *holomorphic m-chain* is a rectifiable cycle of the form $z = \sum_i n_i[V_i]$ where $n_i \in \mathbf{Z}$ and the V_i are irreducible complex analytic subvarieties of dimension m.) The restriction of $\widehat{\sigma}$ to $Z_{m,0} \subset d\mathcal{R}_{2m}$ can be seen to coincide with the classical mapping into the Griffiths mth intermediate Jacobian

$$\mathrm{Har}^{2(n-m)-1}/\mathrm{Har}_0^{2(n-m)-1} \cong (H^{2m+1,0} \oplus H^{2m,1} \oplus \cdots \oplus H^{m+1,m})^*/H_{2m+1}(X;\mathbf{Z}).$$

To see this isomorphism note first that the map $\phi \mapsto h_\phi$ defined by $h_\phi(\psi) = \int_X \phi \wedge \psi$ gives an isomorphism $\mathrm{Har}^{2(n-m)-1}/\mathrm{Har}_0^{2(n-m)-1} \cong (\mathrm{Har}^{2m+1})^*/H_{2m+1}(X;\mathbf{Z})$ and then use the canoncial isomorphism $\mathrm{Har}^{2m+1} = H^{2m+1,0} \oplus H^{2m,1} \oplus \cdots \oplus H^{m+1,m}$.

13. Characteristic sparks and degeneracy sparks. Sparks occur naturally in the theory of singular connections developed by the authors in [HL_1]. They appear as canonical coboundary, or "transgression" terms relating the classical Chern-Weil forms with certain characteristic currents defined by the singularites of a bundle map. We present some basic examples here.

Suppose that E and F are vector bundles with connection over a manifold X, and let G denote the Grassmann compactification of $\mathrm{Hom}(E, F)$. There is a flow φ_t on G engendered by the flow $\widetilde{\varphi}_t(e,f) = (te,f)$ on $E \oplus F$. Constructing operators like those in §11 from the flow φ_t and applying them to characteristic forms of the tautological bundle $U \to G$ leads to a generalized Chern-Weil theory relating characteristic forms and singularities of bundle maps $\alpha\colon\ E \to F$ (See [HL_{1-4}]). In general, for each characteristic form $\Phi(\Omega)$ for E or F one obtains a formula of the type

$$dT = \Phi(\Omega) - \sum_k \mathrm{Res}_{k,\Phi}\,[\Sigma_k(\alpha)] \tag{13.1}$$

where $T \in \mathcal{E}^k_{L^1_{\mathrm{loc}}}(X)$, $[\Sigma_k(\alpha)]$ is a current defined by the dropping by k of the rank of α, and $\mathrm{Res}_{k,\Phi}$ is a smooth residue form defined along $\Sigma_k(\alpha)$.

In many important cases the residues are constants. Whenever this is true, T defines a spark, and so in particular a differential character. Here are some examples.

Example 13.2. (Euler sparks) Let E be an oriented real vector bundle of even rank with an orthogonal connection, and let s be a cross-section with nondegenerate zeros, or more generally an *atomic* cross-section. Then as pointed out in Remark 9.10 there is a canonically defined *Euler spark* $\sigma(s) \in \mathcal{E}^{n-1}_{L^1_{\mathrm{loc}}}(X)$ satisfying the spark equation

$$d\sigma(s) = \chi(\Omega) - \mathrm{Div}(s)$$

where $\chi(\Omega)$ is the Euler form, which depends on the connection but not on the section s, and $\mathrm{Div}(s)$ depends on s but is independent of the connection. The comparison formula for Euler sparks (see [HZ$_2$, Thm. 7.6]) says that the Euler differential character $\widehat{\chi}(E, D) = \widehat{\sigma}(s)$ determined by $\sigma(s)$ is independent of the section s.

Example 13.3. (Chern sparks) Let E be a complex vector bundle of rank n with unitary connection. Let $s = (s_0, \ldots, s_{n-k})$ be a set of $n - k + 1$ sections with a well-defined linear dependency current $\mathbb{D}(s)$. This is an integrally flat cycle essentially defined as the locus where $s_0, \ldots, s_{n-k}$ are linearly dependent (See [HL$_1$, p. 260]). Then there is a canonically defined kth *Chern spark* $C_k \in \mathcal{E}^{2k-1}_{L^1_{\mathrm{loc}}}(X)$ with

$$dC_k = c_k(\Omega) - \mathbb{D}(s)$$

where $c_k(\Omega)$ is the kth Chern form of the connection. Here $[\mathbb{D}(s)] = c_k(E) \in H^{2k}(X; \mathbf{Z})$.

Example 13.4. (Pontrjagin-Ronga sparks) Let E be a real vector bundle of rank n with orthogonal connection. Let $s = (s_1, \ldots, s_{n-2k+2})$ be a set of sections of E with a well-defined 2-dependency current $\mathbb{D}_2(s)$. This is an integrally flat cycle essentially defined as the locus where $\dim(\mathrm{span}\{s_1, \ldots, s_{n-2k+2}\}) \leq n - 2k$ (see [HL$_2$]). Then there is a canonically defined kth *Pontrjagin-Ronga spark* $P_k \in \mathcal{E}^{4k-1}_{L^1_{\mathrm{loc}}}(X)$ with

$$dP_k = p_k(\Omega) - \mathbb{D}_2(s)$$

where $p_k(\Omega)$ is the kth Pontrjagin form of the connection. It was shown by Ronga [R] that

$$[\mathbb{D}_2(s)] = p_k(E) + W_{2k-1}(E)\, W_{2k+1}(E) \in H^{4k}(X, \mathbf{Z})$$

where p_k denotes the kth integral Pontrjagin class and W_j is the jth integer Stiefel-Whitney class defined by $W_j := \delta_*(w_{j-1})$ where w_{j-1} is the mod 2 Stiefel-Whitney class and δ_* is the Bockstein coboundary from the exact sequence $0 \to \mathbf{Z} \xrightarrow{2} \mathbf{Z} \to \mathbf{Z}_2 \to 0$. The characters determined by C_k and P_k are independent of the choice of sections (see [HZ$_2$]).

Example 13.5. (Thom-Porteous sparks) Let $\alpha\colon\ E \to F$ be a bundle map between complex vector bundles with unitary connections where $\mathrm{rank}(E) = m$ and $\mathrm{rank}(F) = n$. Suppose α is k-atomic (see [HL$_1$, p. 260]) so the kth degeneracy current $\Sigma_k(\alpha)$ is well-defined. This is an integrally flat cycle which measures the dropping of α in rank by k. Then there is a canonical kth *Thom-Porteous spark*

$TP_k \in \mathcal{E}^{2(m-k)(n-k)-1}_{L^1_{\mathrm{loc}}}(X)$ with

$$dTP_k = \Delta^{(m-k)}_{n-k}\{c(\Omega^F)c(\Omega^E)^{-1}\} - \Sigma_k(\alpha)$$

where $\Delta^{(m-k)}_{n-k}$ denotes the Shur polynomial in the total Chern form of $E - F$ (see [HL_2]).

There is an analogue of Example 13.5 for real bundles which involves the Pontrjagin forms. These degeneracy sparks appear in many geometric situations. For example the associated degeneracy currents can be measuring:

(i) the higher complex tangencies of an immersion $f\colon X \to Z$ of a real manifold into a complex manifold,

(ii) the higher order tangencies of a pair of foliations on X,

(iii) the higher order singularities of mappings $f\colon X \to X'$ between smooth manifolds,

(iv) the higher degeneracies of k-frame fields,

etc. See [HL_2] for a full discussion.

14. Characters in degree zero. In low dimensions differential characters have particularly nice interpretations, and so do the corresponding spaces of sparks. These cases nicely illustrate the relationship between sparks and characters, and also the duality we have established here.

We begin with degree zero. Recall that $\widehat{\mathbb{H}}^0(X) = \mathcal{S}^0(X)/\mathcal{R}^0(X)$ where $\mathcal{S}^0(X)$ denotes the space of L^1_{loc}-sparks and $\mathcal{R}^0(X)$ is the group of rectifiable currents of degree 0 on X. Note that $\mathcal{R}^0(X) = L^1_{\mathrm{loc}}(X, \mathbf{Z})$, the space of integer-valued functions in $L^1_{\mathrm{loc}}(X)$.

For a manifold X let $G(X) = C^\infty(X, S^1)$ denote the smooth maps to the circle $S^1 = \mathbf{R}/\mathbf{Z}$ (the gauge group for complex line bundles) and define

$$\widetilde{G}(X) = \{f \in L^1_{\mathrm{loc}}(X) : \pi \circ f \text{ is smooth}\}$$

where $\pi\colon \mathbf{R} \to \mathbf{R}/\mathbf{Z}$ is the projection.

Proposition 14.1. *The spaces $\widetilde{G}(X)$ and $\mathcal{S}^0(X)$ agree as subspaces of $L^1_{\mathrm{loc}}(X)$. In particular, there is an isomorphism of short exact sequences:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^1_{\mathrm{loc}}(X;\, \mathbf{Z}) & \longrightarrow & \widetilde{G}(X) & \xrightarrow{\pi_*} & G(X) & \longrightarrow & 0 \\
 & & \downarrow = & & \downarrow = & & \downarrow \cong & & \\
0 & \longrightarrow & L^1_{\mathrm{loc}}(X;\, \mathbf{Z}) & \longrightarrow & \mathcal{S}^0(X) & \xrightarrow{q} & \widehat{\mathbb{H}}^0(X) & \longrightarrow & 0
\end{array}$$

where $\pi_*(f) \equiv \pi \circ f$, and q is the canonical quotient.

Proof. For each point $p \in S^1$ choose $r \in \mathbf{R}$ with $\pi(r) = p$, and let θ_p: $S^1 \to [r, r+1)$ be the unique function with $\pi \circ \theta_p$ =Id. This L^1_{loc}-function is smooth outside p and satisfies the spark equation

$$d\theta_p = \Theta - [p] \tag{14.2}$$

where Θ is the unit volume (arc-length) form on S^1. Given $g \in G(X)$, pick $p \in S^1$ a regular value of g. Define $\overline{f} = g^*\theta_p$, $\overline{\phi} = g^*\Theta$, and $\overline{R} = [g^{-1}(p)]$. Then the equation

$$d\overline{f} = \overline{\phi} - \overline{R} \tag{14.3}$$

is the pull-back of (14.2) by g. Note that $\pi \circ \overline{f} = g$. In particular, π_* is surjective. Now given any $f \in \widetilde{G}(X)$ such that $\pi \circ f = g$, we see that $f = \overline{f} + f_0$ with $f_0 \in L^1_{\text{loc}}(X, \mathbf{Z})$. Hence, by (14.3), $df = \overline{\phi} - \overline{R} + df_0$ and since $\overline{R} + df_0$ is integrally flat, we conclude that $\widetilde{G}(X) \subset \mathcal{S}^0(X)$.

Now for any given $f \in \mathcal{S}^0(X)$, $d_1\langle f \rangle \equiv \phi$ belongs to $\mathcal{Z}^1_0(X)$ and hence ϕ determines a smooth map g: $X \to S^1$, unique up to rotation, with $\phi = \overline{\phi} = g^*\Theta$. Therefore $d(f - \overline{f}) = R - \overline{R}$. This implies that $f - \overline{f} + c \in L^1_{\text{loc}}(X, \mathbf{Z})$ for some constant c since $R - \overline{R}$ is zero in integral cohomology and the kernel of d on $\mathcal{D}'^0$ is the constants. Therefore, $\mathcal{S}^0(X) \subset \widetilde{G}(X)$. □

Remark 14.4. For every pair $(g, p) \in G(x) \times S^1$ where p is a regular value of g, the construction above gives a $\mathbf{Z}$-family of particularly nice sparks $\overline{f}$ with $\pi_*(\overline{f}) = g$.

Remark 14.5. Note that in this degree regularity implies that every generalized spark $a \in \mathcal{D}'^0(X)$ is actually in L^1_{loc} since $da = \phi - R$ is a measure.

We now examine duality in this degree. Note that there is a basic family of continuous homomorphisms

$$h_x\colon\ C^\infty(X, S^1) \longrightarrow S^1 \qquad \text{for } x \in X \tag{14.6}$$

given by $h_x(g) = g(x)$. These homomorphisms do *not* lie in the smooth dual of $\widehat{\mathbb{H}}^0(X)$. However, as noted in Remark 6.7 *any Federer smoothing homotopy of the current* $[x]$ *does give a smooth homomorphism.* By Note 2.8, or by standard constructions, there are families of L^1_{loc} $(n-1)$-forms a_ϵ with $\text{supp}(a_\epsilon) \subset B_\epsilon(x)$, an ϵ-ball about x in local coordinates, such that

$$da_\epsilon = \Omega_\epsilon - [x] \qquad \text{and} \qquad \lim_{\epsilon \to 0} a_\epsilon = 0$$

where the limit is taken in the *flat* topology. In particular $\Omega_\epsilon \to [x]$ as $\epsilon \to 0$.

Now for each $f \in C^\infty(X, S^1)$ we have a well-defined integral

$$h_{\epsilon,x}(f) \equiv \int_X \bar{f}\Omega_\epsilon \qquad (\text{mod } \mathbf{Z})$$

where $\bar{f}$: $X \to \mathbf{R}$ is any lift of f which is continuous on $B_\epsilon(x) \supset \text{supp}(\Omega_\epsilon)$. By (6.9) we see that $h_{\epsilon,x}(f) \equiv (a_\epsilon * f)$. Now each of the homomorphisms

$$h_{\epsilon,x}\colon\ C^\infty(X, S^1) \longrightarrow S^1$$

is smooth, and we have that $h_{\epsilon,x} \to h_x$ as $\epsilon \to 0$. Thus the homomorphisms (14.6) lie at the boundary of the smooth ones.

One could consider any spark $a \in \mathcal{S}^{n-1}(X)$ with $da = \Omega - [x]$ to be a generalized smoothing homotopy of the point mass $[x]$. Note that $\mathcal{S}^{n-1}(X)$ is exactly the group generated by such things. The smooth dual of $C^\infty(X, S^1)$ is exactly the group generated by these smoothing homotopies. Notice that for $f \in \widetilde{G}(X) = \mathcal{S}^0(X)$ with $df = \phi - R$ as above, we have by Definition 3.2 that

$$\begin{aligned}\langle a\rangle * \langle f\rangle\,[X] &\equiv \int_X a \wedge \phi + (-1)^n f(x) \qquad (\text{mod } \mathbf{Z})\\ &\equiv \int_R a + (-1)^n \int_X f\Omega \qquad (\text{mod } \mathbf{Z})\end{aligned}$$

where we assume a to be smooth on the support of R (cf. Corollary 4.7). This shows explicitly how the smooth homomorphism corresponding to a character $\alpha \in \widehat{\mathbb{H}}^{n-1}(X)$ differs from point evaluation on $\delta_2(\alpha)$ or integration against the form $\delta_1(\alpha)$.

When $H^1(X; \mathbf{Z}) = 0$, we have $C^\infty(X,\ S^1) = C^\infty(X)/\mathbf{Z}$ whose smooth dual is just $\mathcal{Z}_0^n(X)$ where the pairing is given by integration. In general there is a short exact sequence

$$0 \longrightarrow C^\infty(X)/\mathbf{Z} \longrightarrow C^\infty(X,\ S^1) \longrightarrow H^1(X;\ \mathbf{Z}) \longrightarrow 0$$

whose dual sequence

$$0 \longleftarrow \mathcal{Z}_0^n(X) \longleftarrow \text{Hom}_\infty(C^\infty(X,\ S^1),\ S^1) \longleftarrow \text{Hom}(H_{n-1}(X,\ \mathbf{Z}),\ S^1) \longleftarrow 0$$

corresponds to Proposition 1.12.

15. Characters in degree 1. We now consider degree one. Let $E \to X$ be a smooth complex line bundle with unitary connection D. Then to each atomic cross-section σ (for example any section which vanishes nondegenerately), there is a canonically associated spark $T(\sigma)$ with the property that

$$dT(\sigma) = c_1(\Omega) - \mathrm{Div}(\sigma) \tag{15.1}$$

where Ω is the curvature form of D ([HL_1, II.5]). It is given locally by the formula

$$T(\sigma) = \mathrm{Re}\left\{\frac{1}{2\pi i}\left(\omega - \frac{du}{u}\right)\right\} \tag{15.2}$$

where $\sigma = ue$, $De = -\omega \otimes e$, and e is a local section of E with $\|e\| \equiv 1$. The smooth section σ is defined to be *atomic* if $du/u \in L^1_{\mathrm{loc}}$, and by definition

$$\mathrm{Div}(\sigma) = d\,\mathrm{Re}\left\{\frac{1}{2\pi i}\frac{du}{u}\right\}. \tag{15.3}$$

Denote by $L(X)$ the set of (isomorphism classes of) complex line bundles with unitary connection up to gauge equivalence on X, and set

$$\widetilde{L}(X) = \{(E, D, \sigma)\colon\ E,\ D \text{ are as above and } \sigma \text{ is an atomic cross-section of } E\}.$$

Proposition 15.4. *There is a commutative diagram*

$$\begin{array}{ccc} \widetilde{L}(X) & \xrightarrow{\ p\ } & L(X) \\ {\scriptstyle T}\big\downarrow & & \cong\big\downarrow{\scriptstyle T_0} \\ \mathcal{S}^1(X) & \xrightarrow{\ q\ } & \widehat{\mathbb{H}}^1(X) \end{array}$$

where $p(E, D, \sigma) \equiv (E, [D])$, q is the canonical quotient, and T_0 is an isomorphism.

Proof. The comparison formula for Euler sparks proven in [HZ_2, Thm. 7.6] says that the map $q \circ T$ is independent of the choice of cross-section σ and therefore descends to the mapping T_0. To see that T_0 is surjective, fix $\alpha \in \widehat{\mathbb{H}}^1(X)$ with $\delta\alpha = \Phi - e$. The class $e \in H^2(X; \mathbf{Z})$ corresponds to a complex line bundle E on X and the de Rham class of the smooth form Φ is the real Chern class $c_1(E)$. Hence there exists a unitary connection D on E whose associated Chern form $c_1(\Omega) = \Phi$. To see this note that for any unitary connection D' one has $c_1(\Omega') - \Phi = d\phi$ for a smooth real-valued 1-form ϕ, and then set $D = D' - \phi$. For any atomic section σ of E we have $\delta\{\alpha - q \circ T(\sigma, E, D)\} = 0$. Thus by 1.14, $\alpha - q \circ T(\sigma, E, D) = [\psi] \in H^1(X; \mathbf{R})/H^1_{\mathrm{free}}(X; \mathbf{Z})$ for a smooth closed 1-form ψ. By (15.2), $T(\sigma, E, D + \psi) = T(\sigma, E, D) + \psi$, and so $q \circ T(\sigma, E, D + \psi) = \alpha$ as desired.

We now prove injectivity. Suppose $T_0(E, D) = 0$. Then since $\delta T_0(E, D) = 0$, E is topologically trivial and D is flat. Choose a section e of E with $\|e\| \equiv 1$, i.e., a trivialization of E. Then $T(e, E, D)$ is a smooth, d-closed 1-form whose class in $H^1(X;\,\mathbf{R})/H^1_{\text{free}}(X;\,\mathbf{Z})$ represents the holonomy of D. Since this is 0, the connection is gauge equivalent to the trivial one. □

Remark 15.5. Given any $(E, D) \in L(X)$ one can choose a section σ of E with regular zeros, so that $\text{Div}(\sigma)$ is a smooth submanifold of codimension-2 in X. These give particularly nice sparks in $\widetilde{L}(X)$ representing the character associated to (E, D). They are analogous to the nice sparks constructed in the last section (cf. (13.4)).

Remark 15.6. Essentially all sparks in $\mathcal{S}^1(X)$ can be represented by triples (E, D, σ) where the σ's are certain measurable sections of the unit circle bundle of E. In particular all sparks a such that $d_2 a$ has support of measure zero can be so constructed.

We now consider the dual to $\widehat{\mathbb{H}}^1(X) \cong L(X)$. As in degree 0 there is a natural family of continuous homomorphisms which are not smooth; namely for each piecewise-C^1-loop γ in X we have the *holonomy mapping*

$$h_\gamma\colon\; L(X) \longrightarrow S^1.$$

The characters $\widehat{\mathbb{H}}^{n-2}(X)$ represent generalized smoothing homotopies of these holonomy homomorphisms. They are represented by L^1_{loc}-forms a of degree $n-2$ with

$$da = \Omega - \gamma$$

where γ is such a loop and Ω is a smooth $(n-1)$-form.

Notice that for $e \equiv (E, D, \sigma) \in \widetilde{L}(X)$ whose associated spark $T(\sigma) \in \mathcal{S}^1(X)$ satisfies (15.1), we have by Definition 3.2 that

$$\begin{aligned} \langle a\rangle * \langle e\rangle\,[X] \;&\equiv\; \int_X a \wedge c_1(\Omega) + h_\gamma(e) \qquad (\text{mod } \mathbf{Z}) \\ &\equiv\; \int_{\text{Div}(\sigma)} a + \int_X \Omega \wedge T(\sigma) \qquad (\text{mod } \mathbf{Z}). \end{aligned}$$

If we take a local Federer family of smoothing homotopies a_ϵ with $da_\epsilon = \gamma_\epsilon - \gamma$ where a_ϵ is supported in an ϵ-neighborhood of γ and $\lim_{\epsilon\to 0} a_\epsilon = 0$ in the flat topology, the first equation above becomes

$$\langle a_\epsilon\rangle * \langle e\rangle\,[X] \equiv \int_X a_\epsilon \wedge c_1(\Omega) + h_\gamma(e) \qquad (\text{mod } \mathbf{Z})$$

and we see that as $\epsilon \to 0$ the smooth homomorphisms given by a_ϵ converge to h_γ. Thus the holonomy homomorphisms lie at the boundary of the smooth ones.

16. Characters in degree 2 (gerbes). Differential characters in degree 2 are intimately connected to *gerbes with connection*. These objects generalize the previous two cases, and have been much discussed in the recent literature. We shall briefly present them and their relationship to characters. For a rounded discussion with applications see the excellent article of Hitchin [Hi] and the book of Brylinski [Br].

Just as a manifold is presented by an atlas of coordinate charts, a gerbe X is presented by Čech 2-cocycle with values in S^1. Thus the data of a gerbe is an open covering $\{U_\alpha\}_{\alpha\in A}$ of a manifold X (which for simplicity we assume to be acyclic) and continuous functions

$$g_{\alpha\beta\gamma}\colon\ U_\alpha \cap U_\beta \cap U_\gamma \longrightarrow S^1$$

with $g_{\alpha\beta\gamma} = g^{-1}_{\beta\alpha\gamma} = g^{-1}_{\gamma\beta\alpha} = g^{-1}_{\alpha\gamma\beta}$ and satisfying the cocycle condition:

$$g_{\beta\gamma\delta} \cdot g^{-1}_{\alpha\gamma\delta} \cdot g_{\alpha\beta\delta} \cdot g^{-1}_{\alpha\beta\gamma} \equiv 1 \qquad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta.$$

Gerbes form an abelian group under "tensor product," and they can be pulled back by a continuous map $f\colon\ Y \to X$. Gerbes arise naturally in many contexts including Spin^c-geometry, the study of Calabi-Yau manifolds, geometric quantization, and the topological Brauer group (see [Hi], [Br]). They are most interesting when equipped with a connection. A *connection* on the gerbe $g_{\alpha\beta\gamma}$ defined on a manifold X consists of the following:

(a) $\phi \in \mathcal{E}^3(X)$ (called the *curvature* of the gerbe),

(b) $A_\alpha \in \mathcal{E}^2(U_\alpha)$ for each α

(c) $A_{\alpha\beta} \in \mathcal{E}^1(U_\alpha \cap U_\beta)$ for each α, β.

with the property that:

(i) $\phi = dA_\alpha$ on U_α

(ii) $A_\beta - A_\alpha = dA_{\alpha\beta}$ on $U_\alpha \cap U_\beta$

(iii) $A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = -id \log g_{\alpha\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

Note 16.1. This is "one step up" from a connection on a line bundle. A line bundle L is presented by a Čech 1-cocycle $\{g_{\alpha\beta}\}$ with values in S^1. A connection on L consists of:

(a′) $\phi \in \mathcal{E}^2(X)$ (called the *curvature* of the line bundle),

(b′) $A_\alpha \in \mathcal{E}^1(U_\alpha)$ for each α

with the property that:

$$
\begin{array}{lll}
\text{(i}'\text{)} & \phi = dA_\alpha & \text{on } U_\alpha \text{, and} \\
\text{(ii}'\text{)} & A_\beta - A_\alpha = -id \log g_{\alpha\beta} & \text{on } U_\alpha \cap U_\beta .
\end{array}
$$

A slight modifiction. To make stronger analogy with our de Rham-Federer theory, we modify the above notion of a gerbe with connection to one which is essentially equivalent. We now consider a gerbe with connection to be a triple

$$
A = (\{A_\alpha\}, \{A_{\alpha\beta}\}, \{A_{\alpha\beta\gamma}\}) \tag{16.2}
$$

where $\{A_\alpha\}, \{A_{\alpha\beta}\}$ are as above and $A_{\alpha\beta\gamma} \in \mathcal{E}^0(U_\alpha \cap U_\beta \cap U_\gamma)$ is some choice

$$
A_{\alpha\beta\gamma} = -i \log g_{\alpha\beta\gamma}
$$

Gauge equivalence. Two gerbes with connection A and A' as in (16.2) are said to be *gauge equivalent* (written $A \sim A'$) if there exist

$$
\begin{array}{lll}
\text{(a)} & B_\alpha \in \mathcal{E}^1(U_\alpha) & \text{for each } \alpha \\
\text{(b)} & B_{\alpha\beta} \in \mathcal{E}^0(U_\alpha \cap U_\beta) & \text{for each } \alpha, \beta .
\end{array}
$$

with the property that:

$$
\begin{array}{lll}
\text{(i)} & dB_\alpha = A_\alpha - A'_\alpha & \text{on } U_\alpha \\
\text{(ii)} & B_\beta - B_\alpha - dB_{\alpha\beta} = A_{\alpha\beta} - A'_{\alpha\beta} & \text{on } U_\alpha \cap U_\beta \\
\text{(iii)} & B_{\alpha\beta} + B_{\beta\gamma} + B_{\gamma\alpha} + 2\pi n_{\alpha\beta\gamma} = A_{\alpha\beta\gamma} - A'_{\alpha\beta\gamma} & \text{on } U_\alpha \cap U_\beta \cap U_\gamma
\end{array}
$$

where $n_{\alpha\beta\gamma}$ is a locally constant integer-valued function. This directly generalizes the notion of gauge equivalence for line bundles with connection.

The Čech-de Rham form bicomplex. The ideas above can be nicely packaged by considering the double complex $(\mathcal{C}^*(\mathcal{U}, \mathcal{E}^*), \mathbb{D})$ of the open cover $\mathcal{U} = \{U_\alpha\}$ where $\mathbb{D} = d + (-1)^k \delta$ and δ denotes the Čech differential. A gerbe with connection is then simply an element

$$
A \in \bigoplus_{j+k=2} \mathcal{C}^j(\mathcal{U}, \mathcal{E}^k)
$$

which satisfies the equation

$$
\mathbb{D}A = \phi - R \tag{16.3}
$$

where

$$\phi \in \mathcal{Z}^0(\mathcal{U}, \mathcal{Z}^3) \subset \mathcal{C}^0(\mathcal{U}, \mathcal{E}^3) \qquad \text{and} \qquad R \in \mathcal{Z}^3(\mathcal{U}, 2\pi\mathbf{Z}) \subset \mathcal{C}^3(\mathcal{U}, \mathcal{E}^0).$$

Thus ϕ is a globally defined closed 3-form and $R = \delta\{A_{\alpha\beta\gamma}\}$ is an integral Čech 3-cocycle.

Two such creatures A and A' are gauge equivalent if there exist

$$B \in \bigoplus_{j+k=1} \mathcal{C}^j(\mathcal{U}, \mathcal{E}^k) \qquad \text{and} \qquad S \in \mathcal{C}^2(\mathcal{U}, 2\pi\mathbf{Z})$$

such that

$$A - A' = \mathbb{D}B + S. \tag{16.4}$$

Note 16.5. The reader will certainly have noticed the analogy of equation (16.3) with the spark equation (0.1), and the analogy of (16.4) with the equivalence relation on sparks introduced in Definitions 1.6 and 2.4. Indeed, as one might guess at this point, gauge equivalence classes of gerbes with connection coincide with differential characters of degree-2. The key to this observation is the following lemma whose proof is left to the reader.

LEMMA 16.6. *Let A be a gerbe with connection with $\mathbb{D}A = \phi - R$ as above.*
(i) *If $R = 0$, then $A \sim (F, 0, 0)$ where $F \in \mathcal{E}^2(X)$ is a global smooth 2-form.*
(ii) *If $\phi = 0$, then $A \sim (0, 0, T)$ where $T \in \mathcal{C}^2(\mathcal{U}, \mathbf{R})$ is a cochain with constant coefficients.*

The holonomy map. Just as connections on line bundles have holonomy on each oriented closed curve in X, gerbes with connection have holomony on each oriented closed surface in X. It is defined as follows. Let $\Sigma \subset X$ be such a surface and A a gerbe with connection as above. The restriction of A to Σ satisfies the hypothesis of Lemma 16.6(ii) since $\phi|_\Sigma = 0$. Hence $A|_\Sigma \sim (0, 0, T)$ where $\delta T \equiv 0 \pmod{2\pi\mathbf{Z}}$. Now T determines Čech 2-cocycle on Σ with values in the constant sheaf $2\pi\mathbf{R}/2\pi\mathbf{Z} = S^1$, which is unique up to Čech coboundaries. Evaluating on the fundamental cycle $[\Sigma]$ gives the *holonomy*

$$h_A(\Sigma) \in H^2(\Sigma; S^1) = S^1.$$

One verifies that the holonomy h_A *depends only on the gauge equivalence class of A*.

PROPOSITION 16.7. *There is a natural equivalence of functors:*

$$\widehat{\mathbb{H}}^2(X) \cong \{\text{gauge equivalence classes of gerbes on } X\}.$$

In the Cheeger-Simons picture this equivalence associates to a gerbe with connection A the homomorphism from integral 2-cycles to S^1 given by the holonomy h_A.

Remark 16.8. The *Čech-de Rham form* presentation of differential characters sketched here for degree-2 extends easily to all degrees. A theory of this type was first mentioned in [FW, §6]. Full details will appear in a sequel [HL_6] to this paper.

Obvious question. *Is there a direct relationship between the Čech-de Rham form theory and the de Rham-Federer theory of differential characters?* The answer is yes—there exists a large *Čech-de Rham current* double complex which presents differential characters and from which one can distill these two theories as special cases. Details of this will appear in [HL_6].

17. Some historical remarks on the complex analogue. Jeff Cheeger realized in 1972 that characters could be represented by forms with singularities, and explicit reference is made to this in [CS]. However, the first appearance of the spark equation (0.1) occured in the work of Gillet-Soulé on intersection theory for arithmetic varieties [GS_2]. They began with a smooth complex projective variety X defined over $\mathbf{Z}$, and introduced currents G of bidegree (k,k), called *Green currents*, on X with the property that

$$dd^C G = \phi - C \tag{17.1}$$

where ϕ is a smooth $2k$-form and C is an algebraic cycle of codimension-k on X. They then divided the group of Green currents by the image of ∂ and $\overline{\partial}$ and a certain subgroup which yields linear equivalence on the cycles C. The resulting quotient was called the kth *arithmetic Chow group of* X. These groups formed a contravariant functor with a nontrivial ring structure which allowed Gillet and Soulé to extend previous results of Arakelov, Deligne, Beilinson and others to a full arithmetic intersection theory. Gillet and Soulé went on to show that algebraic bundles with hermitian metrics have refined characteristic classes in these arithmetic Chow groups [GS_3]. Eventually much deep work of Bismut, Gillet, Soulé and Lebeau led to a refined "arithmetic" version of the Riemann-Roch-Grothendieck Theorem [BGS_*], [BL], [GS_4] and to arithmetic analogues of the standard conjectures [GS_5].

The close similarity of arithmetic Chow groups with differential characters and the parallel nature of the exact sequences in the two theories led Gillet-Soulé and B. Harris to explore the relationship. They realized that the natural analogue of equation (17.1) in real geometry is the spark equation, and they took this approach to differential characters. They all realized that for algebraic cycles of degree 0, differential characters should be related to the Abel-Jacobi mapping into intermediate Jacobians. In [GS_1] the relationship between the two theories is sketched and, among other things, a refined version of Riemann-Roch-Grothendieck is proved for Kähler fibrations. In [H] it is shown that differential characters gave a simplified and more conceptual approach to certain formulas in algebraic geometry originally proved by Harris via Chen's iterated integrals.

The authors came independently to the spark equation while developing the theory of singular connections and characterstic currents ([HL_{1-4}] and [HZ]) where they arise quite naturally.

Appendix A. Slicing currents by sections of a bundle. Let $E \to X$ be a smooth real vector bundle of rank n and let R be p-dimensional current on X which is either flat or rectifiable. Let $\upsilon_1, \dots, \upsilon_k \in \Gamma(E)$ be smooth sections such that

$$\text{span}\{\upsilon_1(x), \dots, \upsilon_k(x)\} = E_x \qquad \text{for all } x \in X. \tag{A.1}$$

Define a map

$$\psi\colon\ X \times \mathbf{R}^k \to E$$

by $\psi(x, \xi) = \sum_i \xi_i \upsilon_i(x)$. This gives a short exact sequence of vector bundles over X:

$$0 \to F \to X \times \mathbf{R}^k \xrightarrow{\psi} E \to 0.$$

Consider the current $\widetilde{R} \equiv p^*R$ where $p\colon\ F \to X$ is the bundle projection. Then $\widetilde{R}$ is a flat or rectifiable current (depending on R) of dimension $p+k-n$ in $X \times \mathbf{R}^k$. Applying Federer slicing theory [F, 4.3] to the projection $\pi\colon\ X \times \mathbf{R}^k \to \mathbf{R}^k$ gives the following:

THEOREM A.2. *The slice*

$$R_\xi = \langle \widetilde{R}, \pi, \xi \rangle$$

exists in $\mathcal{F}_{p-n}(X)$ for almost all $\xi \in \mathbf{R}^k$ and is rectifiable if R is rectifiable. Furthermore, for almost all $\xi \in \mathbf{R}^k$ we have

$$dR_\xi = \langle d\widetilde{R}, \pi, \xi \rangle. \tag{A.3}$$

In particular, by (1.1) if R is integrally flat so is R_ξ for almost all ξ.

Note that

$$\text{supp}(R_\xi) \subseteq \left\{(x, \xi) \in \text{supp}(R) \times \mathbf{R}^k : \sum \xi_i \upsilon_i(x) = 0\right\}.$$

Note also that the maps ψ and $t\psi$ have the same kernel for all $t \in \mathbf{R} - \{0\}$. Thus we may reduce our parameter space from $\mathbf{R}^k$ to $\mathbb{P}^{k-1}_{\mathbf{R}}$.

Appendix B. Intersecting currents. Suppose now that R and S are currents of dimensions p and q respectively on a compact n-manifold X. Consider $R\times S\subset X\times X$. Let $\Delta\subset X\times X$ be the diagonal, and choose an identification of a tubular neighborhood Δ_ϵ of Δ in $X\times X$ with its normal bundle $N\equiv N(\Delta)\cong T(X)$. Let $p\colon N\to\Delta\cong X$ be the bundle projection. Then $\Delta\subset N$ is the divisor $\mathrm{Div}(\upsilon_0)$ of the tautological cross-section of $p^*N\to N$.

We now apply Appendix A to the current $(R\times S)\cap\Delta_\epsilon\cong(R\times S)\cap N$ and the bundle $p^*N\to N$. We choose sections $\upsilon_1,\dots,\upsilon_k\in\Gamma(N)$ which satisfy (A.1) and proceed as above using the family $(\upsilon_0,\upsilon_1,\dots,\upsilon_k)$. We find that for almost all $\xi=(\xi_1,\dots,\xi_k)\in\mathbf{R}^k$ sufficiently small the slice $(R\times S)_\xi$ of $R\times S$ by

$$\upsilon_\xi=\upsilon_0+\sum_{i=1}^k\xi_i\upsilon_i$$

exists in $\mathcal{F}_{p+q-n}(N)$ and is rectifiable if R and S are rectifiable. Furthermore, for almost all ξ we have

$$d\{(R\times S)_\xi\}=\{d(R\times S)\}_\xi=\{(dR)\times S+(-1)^{n-p}R\times dS\}_\xi. \tag{B.1}$$

Now since υ_0 is a transversal cross section of p^*N we have that for all $\|\xi\|$ sufficiently small, $\mathrm{Div}(\upsilon_\xi)$ is a small perturbation of the diagonal, and thus

$$\mathrm{Div}(\upsilon_\xi)=\mathrm{graph}(f_\xi)$$

where $f_\xi\colon X\to X$ is a diffeomorphism close to the identity.

Definition B.2. We define the *intersection*

$$R\wedge(f_\xi)_*S$$

of the currents R and $(f_\xi)_*S$ in X to be the ξ-slice of the current $R\times S$ in $X\times X$. This exists for almost all $\xi\in\mathbf{R}^k$ with $\|\xi\|$ sufficiently small. Furthermore for almost all such ξ we have by (B.1) that

$$d\{R\wedge(f_\xi)_*S\}=(dR)\wedge(f_\xi)_*S+(-1)^{n-p}R\wedge(f_\xi)_*(dS). \tag{B.3}$$

Appendix C. The de Rham-Federer approach to integral cohomology. On any manifold X the complex of integrally flat currents computes the integral cohomology $H^*(X;\mathbf{Z})$. This follows from the fact that the sheaves of such currents give an acyclic resolution of the constant sheaf $\mathbf{Z}$ (See [HZ$_1$]). These comments apply also to the subcomplex of integral currents.

THEOREM C.1. (The intersection product on de Rham-Federer $H^*(X;\mathbf{Z})$) *The intersection of currents introduced in B.2 gives a densely defined pairing on in-*

tegral and on integrally flat currents which descends to an associative, graded-commutative multiplication on $H^*(X; \mathbf{Z})$.

Proof. This is a straightforward consequence of Theorem A.2. □

Note C.2. The intersection product on $H^*(X; \mathbf{Z})$ coincides with the cup product. Classical facts show that for certain nice subcomplexes of the de Rham-Federer complex, where intersection is well defined, the two products agree on cohomology. Hence they must agree for general de Rham-Federer currents. However, proving the coincidence of these products in any setting is not trivial. The basic reason is that the intersection of currents, when defined, is local in nature, generalizing the wedge product on smooth currents, whereas the cup product of cochains is not. This important difference makes Cheeger's definition [C] of the $*$-product on differential characters nontrivial—it involves an infinite series like the one in §4. This difference also accounts for the difficulty in §4 of proving the coincidence of his product with the one introduced here.

Note C.3. Federer proves in [F, 4.4] that the complex of integral (or integrally flat) currents with compact support computes the *homology* of X. Thus Theorem A.1 also yields an intersection ring structure on $H_*(X; \mathbf{Z})$. When X is compact and oriented, the implied isomorphism $H_{n-k}(X; \mathbf{Z}) \cong H^k(X; \mathbf{Z})$ is Poincaré duality.

Note C.4. The contravariant functoriality of de Rham-Federer cohomology is an immediate consequence of its sheaf-theoretic definition. However, this pull-back can be defined more directly, as we did for characters in Remark 2.9.

Appendix D. The spark product.

THEOREM D.1. *For given sparks $\alpha \in \widehat{\mathbb{H}}^k(X)$ and $\beta \in \widehat{\mathbb{H}}^\ell(X)$ with $k + \ell \le n$, there exist representatives $a \in \mathcal{F}^k(X)$ and $b \in \mathcal{F}^\ell(X)$ for α and β respectively, with*

$$da = \phi - R \qquad \text{and} \qquad db = \psi - S, \tag{D.2}$$

with ϕ, ψ smooth and R, S rectifible, such that the products $a \wedge b$, $a \wedge S$, $R \wedge b$ and $R \wedge S$ are well-defined and flat and $R \wedge S$ is rectifiable.

Proof. Let $a \in \alpha$ and $b \in \beta$ be any flat representatives with exterior derivatives given as in (D.2). Choose ξ as above so that $a \wedge f_{\xi *} b$, $a \wedge f_{\xi *} S$, $R \wedge f_{\xi *} b$ and $R \wedge f_{\xi *} S$ are well-defined, flat and the last is rectifiable. Note that

$$d(f_{\xi *} b) = f_{\xi *} \psi - f_{\xi *} S.$$

Now $\psi - f_{\xi *}\psi = d\chi$ where χ is a smooth $(\ell - 1)$-form. Therefore, $\widetilde{b} = f_{\xi *} b + \chi$ satisfies the equation $d\widetilde{b} = \psi - f_{\xi *} S$. Then, since S and $f_{\xi *} S$ are homologous integral cycles, we have $\delta\langle \widetilde{b} \rangle = \delta\langle b \rangle = \delta\beta$. Therefore, by the exact sequence from Proposition 1.14, $\beta - \langle \widetilde{b} \rangle = d\langle \eta \rangle$ where η is a smooth d-closed ℓ-form. Hence,

$$\beta = \langle f_{\xi *} b + \chi + \eta \rangle$$

and the flat form $b' \equiv f_{\xi *} b + \chi + \eta$ has the property that the products $a \wedge b'$, $a \wedge d_2 b'$, $d_2 a \wedge b'$ and $d_2 a \wedge d_2 b'$ are well-defined flat currents and the last is rectifiable. □

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