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Cycles and Spectra

H. Blaine Lawson, Jr.

Abstract. Some recent work on spaces of algebraic cycles is surveyed. The main focus is on spaces of real and quaternionic cycles and their relation to equivariant Eilenberg-MacLane spaces.

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Roughly fifteen years ago a rather surprizing relationship was discovered between projective algebraic cycles and certain fundamental constructions in algebraic topology. The ideas involved were extensively developed in several directions. One led to a new homology/cohomology theory for algebraic varieties. Another led to the solution of an old conjecture of Graeme Segal. (See [31] for an account.) Recently these ideas have been revisited from the point of view of real and quaternionic algebraic geometry, and again the results were surprizing. One finds a rich structure which has no *a priori* reason to exist. This body of work, due to Pedro dos Santos, Paulo Lima-filho, Marie-Louise Michelsohn and myself, is the focus of this paper. I hope to introduce the fundamental ideas and survey the main results.

The principal theme here is that:

Algebraic cycles constitute natural models for classifying spaces in topology.

This in turn tells us much about spaces of cycles. The principle holds in the ordinary and also the G-equivariant categories where G is a finite group. It also holds when considering real structures. To illustrate the principle we will begin with an elementary but archetypal example.

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1 Points on the Projective Line

Recall that *d-fold symmetric product* of a topological space X is defined to be the quotient

$$SP^d(X) = X \times \cdots \times X / S_d$$

where the symmetric group S_d acts on the d-fold cartesian product by permutations of the factors. This construction is functorial and preserves the categories of analytic spaces and algebraic varieties. Recall also that complex projective n-space, the set of 1-dimensional subspaces of $\mathbb{C}^{n+1}$, can be expressed as the quotient

$$\mathbb{P}^n \equiv \mathbb{P}(\mathbb{C}^{n+1}) = \left(\mathbb{C}^{n+1} - \{0\}\right) / \mathbb{C}^\times$$

Proposition 1.1. *There is a homeomorphism, in fact an isomorphism of algebraic varieties:*

$$SP^d(\mathbb{P}^1) \cong \mathbb{P}^d$$

Proof. Let $p = \{p_1, \ldots, p_d\} \in SP^d(\mathbb{P}^1)$ be d unordered points in $\mathbb{P}^1$ with homogeneous coordinates $p_i = [-b_i : a_i]$. To p we associate the homogeneous polynomial of degree d

$$P(x, y) = \prod_{i=1}^{d}(a_i x + b_i y) = \sum_{k=0}^{d} c_k x^k y^{d-k}$$

where

$$c_k = \sum_{|I|=k} a_I b_{I'}$$

and the sum it taken over all multi-indices $I = \{0 \le i_1 < \cdots < i_k \le d\}$ of length $|I| = k$ and I' is the complementary multi-index with $|I'| = d - k$. The point $[c_0 : \cdots : c_d] \in \mathbb{P}^d$ is independent of the choices of homogeneous coordinates representing $p_1, ..., p_d$. The resulting map $SP^d(\mathbb{P}^1) \to \mathbb{P}^d$ has an inverse given by associating to any homogeneous polynomial $P(x, y)$ of degree d its roots (counted to multiplicity) as points in $\mathbb{P}^1$. $\square$

Alternativly, one could argue as follows. For any complex vector space V there is an embedding $SP^d(\mathbb{P}(V^*)) \hookrightarrow \mathbb{P}(\mathrm{Sym}^d(V^*))$ sending $\{[f_1], ..., [f_d]\}$ to $[f_1 \ldots f_d]$. By counting dimensions one sees that for $\dim(V) = 2$ this is an isomorphism.

There are two features of the general symmetric product that deserve notice.

Algebraic structure. One can write

$$SP^d(X) \;=\; \left\{ \sum_i n_i x_i \;:\; n_i \in \mathbb{Z}^+, x_i \in X \text{ and } \sum_i n_i = d \right\}$$

where the x_i are distict. Hence the disjoint union

$$SP^*(X) \;=\; \coprod_{d \geq 0} SP^d(X) \;=\; \left\{ \sum n_i x_i \;:\; n_i \in \mathbb{Z}^+ \right\}$$

has the structure of an *abelian topological monoid*. It is the free monoid generated by the points of X. It has a natural group completion

$$\mathbb{Z} \cdot X \;=\; \left\{ \sum n_i x_i \;:\; n_i \in \mathbb{Z} \right\} \;=\; SP^*(X) \times SP^*(X) / \sim$$

where $\sim$ is the obvious equivalence relation. This is an abelian topological group which, algebraically, is simply the free abelian group generated by the points of X.

This group can be considered as a "limit" of the $SP^d(X)$ as follows. Suppose X is compact and connected. Fix a base point $x_0 \in X$ and consider the family of translations $SP^*(X) \to SP^*(X)$ generated by $\sigma \mapsto \sigma + x_0$. This translation embeds $SP^d(X) \subset SP^{d+1}(X)$ and we define $SP^\infty(X) = \lim_d SP^d(X)$ with the compactly generated topology (cf. [53]). Then $\varinjlim SP^*(X) \cong \mathbb{Z} \times SP^\infty(X)$, and sending $(n, \sum n_i x_i) \mapsto (n - \sum n_i) x_0 + \sum n_i x_i$ yields a continuous map $\mathbb{Z} \times SP^\infty(X) \to \mathbb{Z} \cdot X$.

Theorem 1.2. (Dold-Thom 1954 [6]). *For any connected finite complex X, the mapping*

$$\varinjlim SP^*(X) = \mathbb{Z} \times SP^\infty(X) \xrightarrow{\;\cong\;} \mathbb{Z} \cdot X$$

is a homotopy equivalence.

In particular this shows that there is a homotopy equivalence

$$\mathbb{Z} \times \mathbb{P}^\infty \xrightarrow{\;\cong\;} \mathbb{Z} \cdot \mathbb{P}^1$$

The second feature it the following.

Real structures. A *real structure on a topological space* X is a continuous map $\psi : X \to X$ with $\psi^2 = \mathrm{Id}_X$. Any such map induces real structures $\psi_* : SP^*(X) \to SP^*(X)$ and $\psi_* : \mathbb{Z} \cdot X \to \mathbb{Z} \cdot X$ which are additive isomorphisms.

A *real structure on a complex algebraic variety* X is an algebraic variety $X_{\mathbb{R}}$ defined over $\mathbb{R}$ whose extension over $\mathbb{C}$ has a given isomorphism to X. In this case the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}_2$ acts on X by an antiholomorphic involution $\psi : X \to X$ which is a real struture in the topological sense. Now the variety $\mathbb{P}^1$ has two algebraic real structures reflecting the fact that the Brauer Group of $\mathbb{R}$ is $\mathbb{Z}_2$ (cf. [25]).

I. The standard real structure. This comes from the standard definition of projective space over a field. The involution is given by complex conjugation of homogenous coordinates. Its fixed-point set is the real projective line $\mathbb{P}^1_{\mathbb{R}} \subset \mathbb{P}^1$. One can see from the proof of Proposition 1.1 that the induced real structure on $SP^d(\mathbb{P}^1) = \mathbb{P}^d$ is the standard one.

II. The Brauer-Severi curve. Let $\mathbb{H}$ denote the quaternions and consider $\mathbb{P}^1 = \mathbb{P}_{\mathbb{C}}(\mathbb{H})$ to be the space of complex lines through 0 in $\mathbb{H}$. Then scalar multiplication $j : \mathbb{H} \to \mathbb{H}$ by the quaternion j is a $\mathbb{C}$-antilinear map with $j^2 = -1$. It induces an antiholomorphic involution $j : \mathbb{P}_{\mathbb{C}}(\mathbb{H}) \to \mathbb{P}_{\mathbb{C}}(\mathbb{H})$ without fixed points. Topologically this map is simply the antipodal mapping on S^2. To see that this comes from an algebraic real structure consider the Veronese embedding $\mathbb{P}^1 \subset \mathbb{P}^2$ given by $[z : w] \mapsto [z^2 + w^2 : i(z^2 - w^2) : 2izw]$ which realizes $\mathbb{P}^1$ as the quadric curve $Q = \{[X : Y : Z] \in \mathbb{P}^2 : X^2 + Y^2 + Z^2 = 0\}$. The involution is given by complex conjuation $(X, Y, Z) \mapsto (\overline{X}, \overline{Y}, \overline{Z})$.

In this case the induced real structures on $SP^d(\mathbb{P}_{\mathbb{C}}(\mathbb{H}))$ depend on the degree. One can see from the proof of 1.1 that

$$SP^d(\mathbb{P}_{\mathbb{C}}(\mathbb{H})) = \begin{cases} \mathbb{P}^d \text{ (standard)} & \text{if } d \text{ is even} \\ \mathbb{P}^d_{\mathbb{C}}(\mathbb{H}^{\frac{1}{2}(d+1)}) & \text{if } d \text{ is odd.} \end{cases}$$

Therefore as real varieties the symmetric products have two distinct series giving two distinct stabilizations. Note that there is no j-fixed point to form the stabilization; there is only a fixed pair $\{x_0, jx_0\}$ which has degree 2. This dichotomy will reappear in our discussion of quaternionic cycles.

2 Algebraic Cycles

We have seen above that for a compact topological space X, we have

$$
\begin{aligned}
SP^*(X) &= \left\{ \sum_i n_i x_i \ : \ n_i \in \mathbb{Z}^+ \text{ and } x_i \text{ distinct points of } X \right\} \\
&\cap \\
\mathbb{Z} \cdot X &= \left\{ \sum_i n_i x_i \ : \ n_i \in \mathbb{Z} \text{ and } x_i \text{ distinct points of } X \right\}
\end{aligned}
$$

and by Dold-Thom there is a homotopy equivalence

$$\varinjlim SP^*(X) \cong \mathbb{Z} \cdot X.$$

Now when X is a projective algebraic variety Grothendeick's theory of schemes defines the "points" of X to be *all* the irreducible algebraic subvarieties of X – not just those of dimension 0. So in this context it is natural to consider the symmetric products of the p-dimensional points, that is, the set $C_p(X)$ of all finite formal sums $\sum n_i V_i$ where $n_i \in \mathbb{Z}^+$ and V_i are irreducible algebraic subvarieties of dimension p in X. A fundamental theorem of Chow and van der Waerden [3] asserts that if X is projective, $C_p(X)$ can be written as a countable disjoint union

$$C_p(X) = \coprod_\alpha C_{p,\alpha}(X)$$

where each $C_{p,\alpha}(X)$ has the structure of a projective algebraic variety. In particular, for varieties over $\mathbb{C}$, each $C_{p,\alpha}(X)$ is naturally a compact Hausdorff space, and $C_p(X)$ is an abelian topological monoid. It is natural to consider its group completion:

$$
\begin{aligned}
C_p(X) &= \Big\{ \sum_i n_i V_i \ : \ n_i \in \mathbb{Z}^+ \text{ and } V_i \\
&\qquad \text{is an irreducible } p\text{-dimensional subvariety of } X \Big\} \\
&\cap \\
\mathcal{Z}_p(X) &= \Big\{ \sum_i n_i x_i \ : \ n_i \in \mathbb{Z} \text{ and } V_i \\
&\qquad \text{irreducible } p\text{-dimensional subvariety of } X \Big\}
\end{aligned}
$$

This group $\mathcal{Z}_p(X)$, called the group of algebraic p-cycles on X carries a natural topology as the quotient of $C_p(X) \times C_p(X)$. There is an analogue of the Dold-Thom result.

Theorem 2.1. (P. Lima-Filho [41]). *There is a homotopy equivalence*

$$\varinjlim_{\alpha} C_p(X) \;\cong\; \mathcal{Z}_p(X)$$

This limit is taken over translations by the monoid of connected components of $C_p(X)$. The statement is equivalent to the assertion that $\Omega BC_p(X) \;\cong\; \mathcal{Z}_p(X)$ where BM denotes the classifying space of the monoid M. A proof of this result was also given in [18].

This theorem is important since it relates homotopy invariants of the Chow varieties to invariants of the limit. For example, one has that

$$\varinjlim_{\alpha} \pi_* C_{p,\alpha}(X) \;\cong\; \pi_* \mathcal{Z}_p(X)$$

3 Algebraic Suspension Theorems

A key to unlocking the structure of the groups $\mathcal{Z}_p(X)$ is the algebraic suspension theorem. It is based on the following construction. Let $X \subset \mathbb{P}^n$ be an algebraic variety. Choose an embedding $\mathbb{P}^n \subset \mathbb{P}^{n+1}$ and a disjoint base point $\mathbb{P}^0 \in \mathbb{P}^{n+1}$. Then the *algebraic suspension* $\Sigma\!\!\!/ X$ *of* X is defined to be the union of all lines in $\mathbb{P}^{N+1}$ joining X to $\mathbb{P}^0$. $\Sigma\!\!\!/ X$ is an algebraic subvariety of $\mathbb{P}^{n+1}$. To see this choose homogeneous coordinates $[z_0 : \cdots : z_{n+1}]$ so that $\mathbb{P}^n$ corresponds to points $[z_0 : \cdots : z_n : 0]$ and $\mathbb{P}_0$ corresponds to $[0 : \cdots : 0 : 1]$. Then $\Sigma\!\!\!/ X$ is defined by the same polynomials (in $z_0, \ldots, z_n$) that define X. This construction extends to a continuous homomorphism of cycle groups.

Theorem 3.1. (Lawson 1989). *The algebraic suspension homomorphism*

$$\Sigma\!\!\!/ : \mathcal{Z}_p(X) \longrightarrow \mathcal{Z}_{p+1}(\Sigma\!\!\!/ X)$$

is a homotopy equivalence.

Since $\Sigma\!\!\!/ \mathbb{P}^n = \mathbb{P}^{n+1}$ this immediately implies the following.

Corollary 3.2. *There are homotopy equivalences*

$$\mathcal{Z}_0(\mathbb{P}^n) \;\cong\; \mathcal{Z}_1(\mathbb{P}^{n+1}) \;\cong\; \mathcal{Z}_2(\mathbb{P}^{n+2}) \;\cong\; \ldots$$

Idea of the Proof. We consider the case $X = \mathbb{P}^n$; the general case follows the same lines of argument. The proof falls into two parts. We consider the subgroup $\mathcal{Z}_{p+1}(X)^{\pitchfork}$ of those $(p+1)$-cycles in $\mathbb{P}^{n+1}$ for which every component meets $\mathbb{P}^n$ in proper dimension. Consider the flow $\phi_t : \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$ which fixes $\mathbb{P}^n$ and $\mathbb{P}^0$ and is given in the above homogeneous coordinates by $\phi_t([z_0 : \cdots : z_n : z_{n+1}]) = [z_0 : \cdots : z_n : tz_{n+1}]$. Then ϕ_t acts on $\mathcal{Z}_{p+1}(X)$ preserving $\mathcal{Z}_{p+1}(X)^{\pitchfork}$ and fixing $\Sigma\!\!\!/\,\mathcal{Z}_p(X)$. As $t \to \infty$, each cycle $c \in \mathcal{Z}_{p+1}(X)^{\pitchfork}$ is pulled like “taffy” to a unique limit

$$\lim_{t\to\infty} \phi_t c \;=\; \Sigma\!\!\!/\,(c \bullet \mathbb{P}^n)$$

where $\bullet$ denotes the intersection product. This shows that $\Sigma\!\!\!/\,\mathcal{Z}_p(X)$is a deformation retract of $\mathcal{Z}_{p+1}(X)^{\pitchfork}$.

The second part of the proof consists in showing that the inclusion $\mathcal{Z}_{p+1}(X)^{\pitchfork} \subset \mathcal{Z}_{p+1}(X)$ is a homotopy equivalence. To do this we show that given any compact component $C_{p+1,\alpha}(X) \subset C_{p+1}(X)$ there exists an integer d and a continuous family of mappings $\Psi_t : C_{p+1,\alpha}(X) \to C_{p+1}(X)$, $0 \le t \le 1$, such that

$$\Psi_t\left(C_{p+1,\alpha}(X)\right) \;\subset\; C_{p+1}(X)^{\pitchfork} \qquad \text{for all } \; t > 0$$

and

$$\Psi_0 \;=\; d\,\cdot \qquad \text{(multiplication by } d).$$

It then follows from relatively standard arguments that the inclusion $\mathcal{Z}_{p+1}(X)^{\pitchfork} \subset \mathcal{Z}_{p+1}(X)$ induces an isomorphism on homotopy groups and is therefore a homotopy equivalence.

The construction of Ψ_t entails a new “moving lemma” for cycles. For this one embeds $\mathbb{P}^{n+1} \subset \mathbb{P}^{n+2}$ and chooses two distinct base points x_0, x_1 in $\mathbb{P}^{n+2} - \mathbb{P}^{n+1}$. Let $\Sigma\!\!\!/_{x_k}$ denote the algebraic suspension of cycles to x_k and let $\pi_k : \mathbb{P}^{n+2} - \{x_k\} \to \mathbb{P}^{n+1}$ be the linear projection. Then for each positive divisor D on $\mathbb{P}^{n+2} - \{x_0, x_1\}$ we define a transformation of cycles

$$\Psi_D : \mathcal{Z}_{p+1}(\mathbb{P}^{n+1}) \;\longrightarrow\; \mathcal{Z}_{p+1}(\mathbb{P}^{n+1})$$

by

$$\Psi_D(c) \;\equiv\; (\pi_1)_* \left\{(\pi_0^* c) \bullet D\right\}.$$

Let tD, $0 \le t \le 1$ be the family of divisors obtained by applying the “scalar multiplication” flow ϕ_t for the x_0-suspension. Assume that x_0 and x_1 do not meet tD for any such t. Then we set $\Psi_t = \Psi_{tD}$. Careful estimates then show that for d sufficiently large, a generic choice of D has all the desired properties. □

The arguments involved in the second part of this proof play an important role in the proof of Chow's Moving Lemma for Families which was established by the author and Eric Friedlander [20]. This Lemma has many applications including a proof of Poincaré Duality in certain cycle-homology theories [21].

Subsequent developments of this subject have required enhanced versions of the Algebraic Suspension Theorem. For example there is an Equivariant Suspension Theorem [33] for varieties with a finite group of automorphisms. This result is far more delicate than the non-equivariant one. There are also versions of the Suspension Theorem for real and quaternionic cycles which are relevant to our discussion here.

4 Classifying Spaces

One of the fundamental and powerful ideas in algebraic topology is that of a classifying space. Suppose Φ is a contravariant functor from the category of compact topological spaces to the category of abelian groups. This assigns to each continuous map between topological spaces a homomorphism of groups

$$X \xrightarrow{f} Y \qquad\qquad \mapsto \qquad\qquad \Phi(Y) \xrightarrow{\Phi(f)} \Phi(X)$$

with the property that $\Phi(g \circ f) = \Phi(f) \circ \Phi(g)$ for $g : Y \to Z$. We shall assume that $\Phi(f)$ depends only on the homotopy class of f.

Definition 4.1. A topological space $\mathbb{Z}\Phi$ is a *classifying space* for Φ if there exists an equivalence of functors

$$\Phi(X) \;\cong\; [X, \mathbb{Z}_\Phi] \tag{4.1}$$

where $[X, Y]$ denotes the space of homotopy classes of continuous mapping from X to Y.

Notice in particular that $\Phi(\mathbb{Z}_\Phi) \;\cong\; [\mathbb{Z}_\Phi, \mathbb{Z}_\Phi]$ and so there is a distinguished element

$$\gamma \in \Phi(\mathbb{Z}_\Phi) \qquad \text{corresponding to} \qquad \mathrm{Id} \in [\mathbb{Z}_\Phi, \mathbb{Z}_\Phi]$$

called the *fundamental class*. Given $F : X \to \mathbb{Z}_\Phi$, one has $F^*(\mathrm{Id}_{\mathbb{Z}_\Phi}) = F$ and so under 4.1

$$F^*\gamma \in \Phi(X) \qquad \text{corresponds to} \qquad F \in [X, \mathbb{Z}_\Phi]$$

Example 4.2. If $\Phi(X) = H^1(X;\, \mathbb{Z})$, then $\mathbb{Z}_\Phi = S^1$. That is, there is an equivalence of functors

$$H^1(X;\, \mathbb{Z}) \cong [X, S^1]$$

which assigns to $F \in [X, S^1]$ the class $F^*\gamma \in H^1(X;\, \mathbb{Z})$ where γ is a chosen generator of $H^1(S^1;\, \mathbb{Z}) \cong \mathbb{Z}$. When X and F are smooth, this class can be represented by $F^*(\frac{1}{2\pi}d\theta)$ where $d\theta$ is the standard arc-length form on S^1.

Example 4.3. If $\Phi(X) = H^2(X;\, \mathbb{Z})$, then $\mathbb{Z}_\Phi = \mathbb{P}^\infty = \lim_{n\to\infty} \mathbb{P}^n$. This limit is taken over the family of linear inclusions $\mathbb{P}^1 \subset \mathbb{P}^2 \subset \mathbb{P}^3 \subset \ldots$ and given the *compactly generated topology* defined by declaring $C \subset \mathbb{P}^\infty$ to be closed iff $C \cap \mathbb{P}^n$ is closed for all n. In this topology a closed subset C is compact iff $C \subset \mathbb{P}^n$ for some n. Now there is an equivalence of functors

$$H^2(X;\, \mathbb{Z}) \cong [X, \mathbb{P}^\infty]$$

which assigns to $F \in [X, \mathbb{P}^\infty]$ the class $F^*\gamma \in H^2(X;\, \mathbb{Z})$ where γ is a chosen generator of $H^2(\mathbb{P}^\infty;\, \mathbb{Z}) \cong \mathbb{Z}$. When X and F are smooth, this class can be represented mod torsion by $F^*(\omega)$ where ω is the standard Kähler form (or "complex arc-length form") on $\mathbb{P}^n$.

Example 4.4. Let $\Phi(X) = \mathrm{Vect}^q_{\mathbb{C}}(X)$ be the set of isomorphism classes of q-dimensional complex vector bundles over X with $\Phi(f) \equiv f^*$ given by the induced-bundle construction. Then $\mathcal{Z}_\Phi = G^q(\mathbb{C}^\infty) = \lim_{n\to\infty} G^q(\mathbb{C}^n)$, where $G^q(\mathbb{C}^n)$ is the Grassmannian of codimension-q linear subspaces of $\mathbb{C}^n$. Over $G^q(\mathbb{C}^n)$ there is a q-dimensional vector bundle γ_q whose fibre at P is $\mathbb{C}^n/P$. This stabilizes to a *universal* bundle $\gamma_q \longrightarrow G^q(\mathbb{C}^\infty)$, and the equivalence of functors

$$\mathrm{Vect}^q_{\mathbb{C}}(X) \cong [X, G^q(\mathbb{C}^\infty)]$$

associates to $F : X \to G^q(\mathbb{C}^\infty)$ the bundle $F^*\gamma_q$. (See [**?**] or [28] for details.)

Example 4.5. Let $\Phi(X) = \widetilde{K}(X)$ be the *reduced K-theory* of X defined as follows. Let $K(X)$ denote the group completion of the additive monoid $\left(\coprod_{k\geq 0} \mathrm{Vect}^k_{\mathbb{C}}(X), \oplus\right)$ of vector bundles under Whitney sum $\oplus$. Then $\widetilde{K}(X)$ is the kernel of the dimension homomorphism $K(X) \to \mathbb{Z}$. One can show that $\mathcal{Z}_\Phi = G^\infty(\mathbb{C}^\infty) = \lim_{q\to\infty} G^q(\mathbb{C}^\infty)$, i.e., there is an equivalence of functors

$$\widetilde{K}(X) \cong [X, G^\infty(\mathbb{C}^\infty)]$$

(Again see [**?**] or [28] for details.)

Example 4.6. Let $\Phi(X)$ be the isomorphism classes of p-fold covering spaces of X where p is a prime. Then $\mathcal{Z}_\Phi \cong S^\infty/(\mathbb{Z}/p)$.

Group structure. Let Z be a topological space and $\Phi(X) \equiv [X, Z]$ the set-valued functor classified by Z. If Z is in fact a topological abelian group, then Φ is naturally a group-valued functor. Similarly, if Z is homotopy equivalent to the space of loops $Z \cong \Omega Z_1$ on a pointed topological space Z_1, then the loop product

$$Z \times Z \;\cong\; \Omega Z_1 \times \Omega Z_1 \;\longrightarrow\; \Omega Z_1 \;\cong\; Z$$

also makes Φ a group-valued functor.

This second construction generalizes the first, since for any topological group Z there exists a *classifying space* $Z_1 = BZ$ and a weak homotopy equivalence $Z \sim \Omega BZ$ (See [44], [46] for example).

5 Spectra

Suppose $\{Z_n\}_{n=0}^\infty$ is a sequence of pointed spaces provided with homotopy equivalences

$$Z_n \;\cong\; \Omega Z_{n+1}$$

for all n, so we have equivalences

$$Z \;\equiv\; Z_0 \;\cong\; \Omega Z_1 \;\cong\; \Omega^2 Z_2 \;\cong\; \Omega^3 Z_3 \;\cong\; \dots .$$

Then $\{Z_n\}_{n=0}^\infty$ is called an Ω-*spectrum* and Z is called an *infinite loop space*. Under these conditions the graded group-valued functor

$$\Phi^n(X) \;\equiv\; [X,\; Z_n]$$

satisfies all the axioms for a cohomology theory except the dimension axiom (cf. [17], and so $\Phi^*(\bullet)$ is a *generalized cohomology theory*.

6 Eilenberg-Maclane Spaces

The defining "universal" property of a classifying space $\mathcal{Z}_\Phi$ usually implies directly that it is unique up to homotopy equivalence. Moreover, there often exists a nice homotopy characterization of $\mathcal{Z}_\Phi$. A basic example is the following.

Example 6.1. Let $\Phi(X) = H^n(X;\ \Lambda)$ where Λ is a finitely generated abelian group. The corresponding classifying space $\mathcal{Z}_\Phi = K(\Lambda, n)$, called the *Eilenberg-Maclane space* of type (Λ, n), is uniquely characterized up to homotopy equivalence, in the category of countable CW complexes, by the property that

$$\pi_k\, K(\Lambda, n) = \begin{cases} \Lambda & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases} \tag{6.1}$$

Thus for example we have homotopy equivalences:

$$K(\mathbb{Z}, 1) \cong S^1, \qquad K(\mathbb{Z}, 2) \cong \mathbb{P}^\infty, \qquad K(\mathbb{Z}/p, 1) \cong S^\infty/(\mathbb{Z}/p)$$

Of course the classifying property means that there is an equivalence of functors

$$H^n(X;\ \Lambda) \cong [X, K(\Lambda, n)] \tag{6.2}$$

Since $\pi_k\Omega X = \pi_{k+1}X$ for any pointed space X, the characterization (6.1) shows that

$$K(\Lambda, n) \cong \Omega K(\Lambda, n+1)$$

for all n. Hence, $\{K(\Lambda, n)\}_{n=0}^\infty$ forms an Ω-spectrum called the Eilenberg-MacLane spectrum, classifying the cohomology theory $H^*(X;\ \Lambda)$.

7 The Importance of Classifying Spaces and Spectra, I

Having an explicit classifying space $\mathcal{Z}_\Phi$ for a functor Φ can be quite useful. Many of basic properties of Φ, as well as natural transformations $\Phi \to \Psi$ to other functors, can be determined geometrically at the universal level from the structure of $\mathcal{Z}_\Phi$.

Example 7.1. (Characteristic classes). A *characteristic class* for vector bundles of rank n is a natural transformation which assigns to each vector bundle $E \to X$, a cohomology class $u(E) \in H^k(X;\ \Lambda)$ for some fixed k and Λ. By definition of a natural tranformation, $u(f^*E) = f^*u(E)$ for any continuous map $f : Y \to X$. In light of 4.4 we see that u is therefore completely determined by the cohomology class $u(\gamma_q) \in H^k(G^q(\mathbb{C}^\infty);\ \Lambda)$. Thus:

$$\begin{aligned} &\text{Characteristic classes of } \mathbb{C} \text{ vector bundles} \\ &\cong \text{ cohomology of } G^q(\mathbb{C}^\infty) \\ &\cong [G^q(\mathbb{C}^\infty),\ K(k,\ \Lambda)] \end{aligned}$$

for various choices of k and Λ.

Example 7.2. (Cohomology operations). A *cohomology operation* is a simply a natural transformation of functors $H^k(\bullet; \Lambda) \to H^{k'}(\bullet; \Lambda')$. Just as above we find that:

$$\begin{aligned} &\text{Cohomology operations} \\ &\quad\cong \text{ cohomology of } K(k, \Lambda) \\ &\quad\cong [K(k, \Lambda), K(k', \Lambda')] \end{aligned}$$

for various choices of k, k' and Λ, Λ'.

8 The Importance of Classifying Spaces and Spectra, II

Although classifying spaces are often characterized by simple homotopy conditions, it can be quite useful to find **good models** for them. One obtains a two-way flow of information:

$$\textbf{MODELS} \quad \longleftrightarrow \quad \textbf{THEORY}$$

Explicit constructions of models can lead to nice representations of such things as characteristic classes and cohomology operations. For example, the natural harmonic forms on Grassmann manifolds give rise to Chern-Weil Theory which represents characteristic classes of smooth vector bundles as explicit polynomials in the curvature of a given connection.

In the other direction, if one determines that a particular space $\mathcal{Z}$ is a classifying space for some functor Φ, then our knowledge of $\mathcal{Z}_\Phi$ tells us much about the topological structure of Z.

The flow of information in both directions will play a role in our subsequent discussion.

9 Cycles and Eilenberg-MacLane Spaces

In 1954 A. Dold and R. Thom gave the following beautiful models for the Eilenberg-MacLane spaces.

Theorem 9.1. [6] *For all $n > 0$ there is a homotopy equivalence*

$$SP^\infty(S^n) \cong K(\mathbb{Z}, n).$$

More generally, for any finite complex Y there are homotopy equivalences

$$\mathbb{Z} \cdot Y \cong \mathbb{Z} \times SP^\infty(Y) \cong \prod_{n \geq 0} K((H_n(Y; \mathbb{Z}), n).$$

In particular this shows that for any finite complex Y one has an isomorphism of graded groups

$$\pi_*(\mathbb{Z} \cdot Y) \cong H_*(Y;\, \mathbb{Z}).$$

The functor $Y \mapsto \mathbb{Z}\cdot Y$ has the effect of converting homology groups to homotopy groups.

Note that if Y is an algebraic variety, then $\mathbb{Z} \cdot Y$ is just the group of 0-cycles on Y. In light of the discussion in §2 one might ask whether analogues of Theorem 9.1 hold for algebraic cycles of higher dimension. Indeed this is the case. We adopt the notation

$$\mathcal{Z}^q(\mathbb{P}^n) \equiv \mathcal{Z}_{n-q}(\mathbb{P}^n)$$

for the group of algebraic cycles of codimension-q on $\mathbb{P}^n$. Then The Algebraic Suspension Theorem 3.1 together with 9.1 above leads to the following.

Theorem 9.2. [30]. *For each integer q, $0 \le q \le n$, there is a canonical homotopy equivalence*

$$\boxed{\mathcal{Z}^q(\mathbb{P}^n) \cong \prod_{k=0}^{q} K(\mathbb{Z}, 2k)} \tag{9.1}$$

In fact one has that for each $n \ge q$ there is a homotopy equivalence

$$\mathcal{Z}^q(\mathbb{C}^n) \equiv \mathcal{Z}^q(\mathbb{P}^n)/\mathcal{Z}^{q-1}(\mathbb{P}^{n-1}) \cong K(\mathbb{Z}, 2q). \tag{9.2}$$

where the quotient $\mathcal{Z}^q(\mathbb{C}^n)$ can be identified with the group of algebraic cycles on $\mathbb{C}^n$. Thus the affine algebraic cycles, suitably topologized, give models for the Eilenberg-MacLane spaces and thus represent integral cohomology in even degrees.

Stabilizing the equivalence (9.1) to the limit

$$\mathcal{Z}^\infty \equiv \mathcal{Z}^\infty(\mathbb{P}^\infty) \equiv \lim_{n,q\to\infty} \mathcal{Z}^q(\mathbb{P}^n) \tag{9.3}$$

classifies the functor $H^{2*}(X;\, \mathbb{Z})$. We shall see that this space $\mathcal{Z}^\infty$ carries additional beautiful properties related to the cup product in the ring $H^{2*}(X;\, \mathbb{Z})$.

More generally one can replace projective space with an arbitrary algebraic variety X, and consider the topological group $\mathcal{Z}_p(X)$ of algebraic p-cycles on X. Taking homotopy groups yields a bigraded homology theory $L_\bullet H_{2\bullet+*}(X) = \pi_*\mathcal{Z}_\bullet(X)$ which Theorem 3.1 shows to have particularly nice properties. A general survey of this theory can be found in [31] and [43].

10 Cycles and Chern Classes

The simplest of all algebraic subvarieties in $\mathbb{P}^n$ are the linear subspaces, and this observation gives a natural embedding $G^q(\mathbb{P}^n) \longrightarrow \mathcal{Z}^q(\mathbb{P}^n)$ of the Grassmannian of codimension-q linear subspaces into cycles of degree one. Now with respect to the canonical homotopy equivalence (9.1) this map represents a cohomology class in $H^{2*}(G^q(\mathbb{P}^n);\ \mathbb{Z})$ (cf. (6.2)).

Theorem 10.1. [36]. *With respect to (9.1) the map*

$$G^q(\mathbb{P}^n) \longrightarrow \mathcal{Z}^q(\mathbb{P}^n)_{\deg 1} \tag{10.1}$$

classifies the total Chern class of the "universal" q-plane bundle $\gamma_q \to G^q(\mathbb{C}^n)$.

Taking the limit as $n \to \infty$ in (10.1) yields a map of classifying spaces

$$G^q(\mathbb{P}^\infty) \longrightarrow \mathcal{Z}^q(\mathbb{P}^\infty)_{\deg 1}. \tag{10.2}$$

which represents a natural transformation of the corresponding functors. As seen in 4.4 the first space classifies vector bundles, and via (9.1) the second space classifies integral cohomology. The import of Theorem 10.1 is that this map represents the total Chern class. In other words, for every finite complex X, (10.2) induces a mapping

$$\begin{array}{ccc} [X, G^q(\mathbb{P}^n)] & \longrightarrow & [X, \mathcal{Z}^q(\mathbb{P}^n)]_{\deg 1} \\ \| & & \| \\ \mathrm{Vect}^q(X) & \longrightarrow & \{1\} \times H^2(X;\ \mathbb{Z}) \times \cdots \times H^{2q}(X;\ \mathbb{Z}) \end{array}$$

which sends

$$E \ \mapsto \ c(E) = 1 + c_1(E) + \cdots + c_q(E)$$

Taking the limit as $q \to \infty$ gives a map of classifying spaces

$$G^\infty(\mathbb{P}^\infty) \longrightarrow \mathcal{Z}^\infty(\mathbb{P}^\infty)_{\deg 1}. \tag{10.3}$$

which represents the natural transformation of functors

$$K(X) \longrightarrow H^{2*}(X;\ \mathbb{Z})$$

corresponding to the total Chern class.

11 Cycles and the Cup Product

On cycles in projective space there is an elementary biadditive pairing

$$\# : \mathcal{Z}^q(\mathbb{P}^n) \times \mathcal{Z}^{q'}(\mathbb{P}^{n'}) \longrightarrow \mathcal{Z}^{q+q'}(\mathbb{P}^{n+n'+1})$$

called the *algebraic join* which is constructed as follows. Embed $\mathbb{P}^n$ and $\mathbb{P}^{n'}$ into $\mathbb{P}^{n+n'}$ as disjoint linear subspaces. Then for irreducible subvarieties $V \subset \mathbb{P}^n$ and $V' \subset \mathbb{P}^{n'}$, the subvariety $V\#V' \subset \mathbb{P}^{n+n'}$ is defined to be the union of all lines joining V to V'.

Theorem 11.1. [36]. *With respect to the canonical homotopy equivalences (9.1), the join pairing* $\# : \mathcal{Z}^q \times \mathcal{Z}^{q'} \longrightarrow \mathcal{Z}^{q+q'}$ *classifies the cup product.*

One checks directly that if V and V' are linear subspaces, so is $V\#V'$, and under the embeddings (10.1) the join restricts to a mapping

$$\oplus : G^q \times G^{q'} \longrightarrow G^{q+q'}$$

which represents the Whitney sum of vector bundles. From the discussion of §10 we obtain the classical result:

Corollary 11.2. *For vector bundles E and F over a finite complex,* one has

$$c(E \oplus F) = c(E)c(F).$$

12 Cycles and Spectra

The join pairing extends to the stabilized spaces $\mathcal{Z}^\infty$, defined in (9.3), to give a map $\# : \mathcal{Z}^\infty \times \mathcal{Z}^\infty \longrightarrow \mathcal{Z}^\infty$. Now the space $\mathcal{Z}^\infty$ breaks into connected components

$$\mathcal{Z}^\infty = \coprod_{d=-\infty}^{\infty} \mathcal{Z}^\infty(d)$$

where $\mathcal{Z}^\infty(d)$ corresponds to the cycles of degree d, and one finds that

$$\mathcal{Z}^\infty(d)\#\mathcal{Z}^\infty(d') \subset \mathcal{Z}^\infty(dd').$$

Adopting the standard notation $BU = G^\infty(\mathbb{P}^\infty)$ we have the following.

Theorem 12.1. [2]. *There is an infinite loop structure on $\mathcal{Z}^\infty(1)$ extending the cup product mapping*

$$\mathcal{Z}^\infty(1) \times \mathcal{Z}^\infty(1) \longrightarrow \mathcal{Z}^\infty(1)$$

and making the total Chern class

$$c : BU \longrightarrow \mathcal{Z}^\infty(1)$$

an infinite loop map.

Consequently there is a transformation of generalized cohomology theories

$$K^* \longrightarrow h^*$$

(where h^* is classified by $\mathcal{Z}^\infty(1)$ with its infinite loop structure) which at level 0 is just the transformation

$$K(X) \longrightarrow H^{\text{even}}(X, \mathbb{Z})$$

given by the total Chern class. This fact has useful consequences. For example it implies that this classical map commutes with the transfer homomorphisms in the respective theories.

13 Real Algebraic Cycles

We now turn to the topic of real and quaternionic cycles which we alluded to in §1. Let $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ as before. Then complex conjugation of the homogeneous coordinates $\mathbb{C}^{n+1}$ gives an antiholomorphic map

$$c : \mathbb{P}^n \longrightarrow \mathbb{P}^n \qquad \text{with} \quad c^2 = \text{Id}$$

and with fixed-point set $\mathbb{P}^n_{\mathbb{R}} = \mathbb{P}(\mathbb{R}^{n+1})$. This involution c extends to a $\mathbb{Z}_2$-action on $\mathbb{Z}^q(\mathbb{P}^n)$ whose fixed-point set

$$\mathbb{Z}^q_{\mathbb{R}}(\mathbb{P}^n) \subset \mathbb{Z}^q(\mathbb{P}^n)$$

consists of the *real algebraic cycles* of codimension-q. These are simply the complex algebraic cycles on $\mathbb{P}^n$ which can be defined over $\mathbb{R}$. Note that a real subvariety may have no real points. Consider for example the hyperquadric $\{[Z] \in \mathbb{P}^n : \sum Z_k^2 = 0\}$.

Now the spaces $\mathbb{Z}^q_{\mathbb{R}}(\mathbb{P}^n)$ represent families of basic objects, and in light of the results above, it is natural to ask about their topological structure. This structure has now been completely determined and the results are rather interesting.

The first step in the analysis was to establish an Equivariant Algebraic Suspension Theorem which extended Theorem 3.1 to the case where there was a finite group acting on the space X. This result, which appeared in [33], is much more delicate than its non-equivariant cousin. In general the homotopy equivalences appear only in a certain stable range. However, for involutions coming from a real structure on a variety X, the theorem holds exactly as in the non-equivariant case. In particular we have the following.

Theorem 13.1. [33]. *The algebraic suspension map* $\not\Sigma : \mathcal{Z}^q(\mathbb{P}^n) \to \mathcal{Z}^q(\mathbb{P}^{n+1})$ *is a* $\mathbb{Z}_2$*-equivariant homotopy equivalence.*

This shows that the homotopy type of $\mathbb{Z}^q_{\mathbb{R}}(\mathbb{P}^n)$ is independent of $n \geq q$, and it reduces its computation to the case of 0-cycles.

Before discussing the full computation, let's examine an interesting "reduced" problem. Consider the quotient topological group

$$\widetilde{\mathcal{Z}}^q_{\mathbb{R}} \;\equiv\; \mathcal{Z}^q_{\mathbb{R}}/\mathcal{Z}^q_{\text{ave}}$$

where $\mathcal{Z}^q_{\text{ave}} \equiv \{z + c(z) : \; z \in \mathcal{Z}^q\}$ is the subgroup of *averaged cycles*.

Theorem 13.2. [29]. *For each q there is a canonical homotopy equivalence*

$$\widetilde{\mathcal{Z}}^q_{\mathbb{R}} \;\cong\; \prod_{k=0}^{q} K(\mathbb{Z}_2,\ k)$$

Analogues of all the results discussed above carry over to this case:

1. The degree-1 inclusion of the real Grassmannian $G^q_{\mathbb{R}}(\mathbb{P}^n) \to \widetilde{\mathcal{Z}}^q_{\mathbb{R}}$ classifies the total Stiefel-Whitney class of the universal real q-plane bundle over $G^q_{\mathbb{R}}(\mathbb{P}^n)$.
2. The algebraic join pairing $\# : \widetilde{\mathcal{Z}}^q_{\mathbb{R}} \times \widetilde{\mathcal{Z}}^{q'}_{\mathbb{R}} \to \widetilde{\mathcal{Z}}^{q+q'}_{\mathbb{R}}$ classifies the cup product in $\mathbb{Z}_2$-cohomology.
3. The space $\widetilde{\mathcal{Z}}^{\infty}_{\mathbb{R}}(1)$ carries an infinite loop space structure making the the total Stiefel-Whitney map $BO \to \widetilde{\mathcal{Z}}^{\infty}_{\mathbb{R}}(1)$ (arising in part 1) an infinite loop map.

The first two results are due to Lam [29] and the last appears in [2].

It turns out that the full homotopy type of $\mathcal{Z}^q_{\mathbb{R}}$ is substantially more complicated.

Theorem 13.3. [34]. *There is a canonical homotopy equivalence*

$$\mathcal{Z}^q_{\mathbb{R}} \cong \prod_{n=0}^{q} \prod_{k=0}^{n} K(I_{n,k}, n+k)$$

where

$$I_{n,k} = \begin{cases} 0 & , \text{ if } k \text{ is odd or } k > n; \\ \mathbb{Z} & , \text{ if } k = n \text{ and } k \text{ is even;} \\ \mathbb{Z}_2 & , \text{ if } k < n \text{ and } k \text{ is even.} \end{cases}$$

Here are the groups $I_{n,k}$ located on the (n, k)-coordinate grid.

$$\begin{array}{lcccccccccccl}
k \uparrow & & & & & & & & & & & \dots & \\
 & & & & & & & & & & \mathbb{Z} & \dots & \\
 & & & & & & & & & 0 & 0 & \dots & \\
 & & & & & & & & \mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2 & \dots & \\
 & & & & & & & 0 & 0 & 0 & 0 & \dots & \\
 & & & & & & \mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \dots & \\
 & & & & & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \\
 & & & & \mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \dots & \\
 & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \\
 & & \mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \dots & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \\
\mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \dots & \xrightarrow[n]{}
\end{array}$$

Just as above the $\mathcal{Z}^q_{\mathbb{R}}$ yield natural classifying spaces which have multiplicative and infinite loop structures and give interesting new characteristic classes for real vector bundles. (See [34].)

This homotopy picture of the space of real cycles is, in a certain sense, complete. However, recently we have reached a deeper understanding of the situation. These new insights are due to Paulo Lima-Filho and Pedro dos Santos who considered $\mathcal{Z}^q$ as a $\mathbb{Z}_2$-space and determined its complete equivariant homotopy type. Their results are beautiful and unexpected. Stating them properly will require an excursion through equivariant homotopy theory.

14 Equivariant Homotopy Theory

We now fix a finite group G and leave ordinary topology for its more exotic G-equivariant analogue. We plunge into the world of G-spaces, G-maps, G-homotopies, G-homotopy types, etc. and search for theorems which reduce (when $G = \{1\}$) to our cherished classical results. Many such theorems have been proved and the general theory has been carried to a high degree of sophistication (see [39], [48] for example).

An interesting facet of this theory is that the analogues of classical invariants indexed by the integers are now indexed by real representations of G.

An instructive example is provided by homotopy groups. For ordinary spaces we have the groups

$$\pi_n(X) \ = \ [S^n, X]$$

defined for non-negative integers n. When X is a G-space we can define more general groups

$$\pi_V(X) \ = \ [S^V, X]_G$$

where V is a finite-dimensional real representation space for G, $S^V = V \cup \{\infty\}$ is the one-point compactification of V, and $[Y, X]_G$ denotes G-homotopy classes of G-equivariant maps from Y to X. One retrieves the first set of groups on a trivial G-space X by taking $V = \mathbb{R}^n$ to be the trivial real representation of dimension n.

The homology and cohomology functors in this theory are similarly indexed by such representations V (in fact by all virtual representations in $RO(G)$). In general these are complicated objects. One reason is that the coefficients in the theory are themselves quite complicated. We can see motivation for this by recalling that a natural approach to homology starts by taking a cell decomposition of the space and defining chain groups. In an *equivariant* cell decomposition the cells are acted upon by G and are thereby organized into orbits of the form

$$G \cdot e^n \ \cong \ \coprod_{\alpha \in G/H} e^n_\alpha$$

where $H = \{g \in G : g(e^n) = e^n\}$. The boundary (or "attaching") maps in this complex are G-maps. Hence the natural coefficients to consider for the theory are functors which map the category of finite G-sets into abelian groups and have certain additional desirable properties. The good objects of this type are called *Mackey functors* whose full definition we will not give. (It can be found in [**?**].) However, for every Mackey functor $\underline{M}$ and every real representation V there are well-defined ordinary homology and cohomology groups $H_V(X; \ \underline{M})$

and $H^V(X;\ \underline{M})$ which enjoy the properties of their non-equivariant analogues and form basic invariants in the theory [38].

One of the deeper results in equivariant homotopy theory is the existence and homotopy characterization of **Eilenberg-MacLane spaces** $K(\underline{M}, V)$ classifying the corresponding cohomology groups. That is, one has an equivalence of functors

$$H_V(X;\ \underline{M}) \ \cong\ [X, K(\underline{M}, V)]_G.$$

One of the simplest Mackey functors is the one which assigns the group $\mathbb{Z}$ to every finite G-set and behaves in a simple way on G-maps consistent with requirements. It is called the *Mackey functor constant at* $\mathbb{Z}$ and is denoted $\underline{\mathbb{Z}}$.

One of the beautiful results in this theory is the following **Equivariant Dold-Thom Theorem** pioneered by Paulo Lima-Filho.

Theorem 14.1. [42] and [8]. *Let V be a finite-dimensional real representation of G and denote by $\mathbb{Z}\cdot S^V$ the free abelian group on the V-sphere. Let $(\mathbb{Z}\cdot S^V)_0$ denote the connected component of 0. Then there is an equivariant homotopy equivalence*

$$(\mathbb{Z}\cdot S^V)_0 \ \cong\ K(\underline{\mathbb{Z}},\ V)$$

15 Real Cycles from the Equivariant Point of View

With this understood, Pedro dos Santos gave the following beautiful result.

Theorem 15.1. [7], [12]. *Let $\mathcal{Z}^q = \mathcal{Z}^q(\mathbb{P}^n)$ denote the group of algebraic cycles of codimension q under the involution induced by complex conjugation on $\mathbb{P}^n$. Then there is a $\mathbb{Z}_2$-homotopy equivalence*

$$\boxed{\mathcal{Z}^q \ \cong\ \prod_{k=0}^{q} K(\underline{\mathbb{Z}},\ \mathbb{R}^{k,k})} \tag{15.1}$$

where $\mathbb{R}^{k,k} = \mathbb{R}^k \oplus i\mathbb{R}^k = \mathbb{C}^k$ is the representation of $\mathbb{Z}_2$ given by complex conjugation.

Note the analogy with (9.1).

The results of section 13 can be deduced from this theorem by determining the homtopy-type of the fixed-point sets $K(\underline{\mathbb{Z}},\ \mathbb{R}^{k,k})^{\mathbb{Z}_2}$, or equivalently by calculating the homotopy groups $\pi_n K(\underline{\mathbb{Z}},\ \mathbb{R}^{k,k}) = \pi_n\{K(\underline{\mathbb{Z}},\ \mathbb{R}^{k,k})^{\mathbb{Z}_2}\}$ for the trivial representations.

This sets the stage for investigating analogues of the results in §§10 and 11. Dos Santos also carried this out in his thesis [7, 12], and we shall present part of that here. To begin, note that any real representation of $\mathbb{Z}_2$ is of the form $\mathbb{R}^{k,\ell} \equiv \mathbb{R}^k \oplus i\mathbb{R}^\ell = k$ times the trivial representation plus ℓ times the non-trivial one. This means, in light of our discussion above, that $\mathbb{Z}_2$-equivariant cohomology is indexed by pairs of integers (k, ℓ). The "coefficients" in $\mathbb{Z}_2$-equivariant cohomology theory are the bigraded ring

$$R \equiv H^{*,*}(\mathrm{pt}, \underline{\mathbb{Z}}).$$

Both dos Santos and Dan Dugger showed the following.

Proposition 15.2. [12], [14]. *The $\mathbb{Z}_2$-equivariant cohomology of the Grassmannian is a polynomial ring*

$$H^{*,*}(G^q(\mathbb{P}^\infty); \underline{\mathbb{Z}}) = R[\overline{c}_1, \ldots, \overline{c}_q]$$

for canonical classes $\overline{c}_k \in H^{k,k}(G^q(\mathbb{P}^\infty); \underline{\mathbb{Z}})$.

Now the space $G^q(\mathbb{P}^\infty) \equiv BU_q$ classifies Real rank q vector bundles in the sense of Atiyah [**?**], [34]. These are complex q-plane bundles $E \to X$ over a $\mathbb{Z}_2$-space X with a complex *antilinear* involution covering the one given on X. The class is defined to be the k^{th} equivariant Chern class for such bundles.

Theorem 15.3. [12]. *With respect to (15.1), the equivariant inclusion*

$$G^q(\mathbb{P}^n) \longrightarrow \mathcal{Z}^q(\mathbb{P}^n)$$

given by considering $G^q(\mathbb{P}^n)$ to consist of cycles of degree one, represents the total equivariant Chern class $c = 1 + \overline{c}_1 + \cdots + \overline{c}_q$ of the universal q-plane bundle over $G^q(\mathbb{P}^n)$. In particular, its stabilization

$$BU_q = G^q(\mathbb{P}^\infty) \longrightarrow \mathcal{Z}^q(\mathbb{P}^\infty)$$

as $n \to \infty$ represents the total equivariant Chern class of Real rank q bundles.

Therefore as $q \to \infty$, the cycle inclusion becomes an equivariant map

$$BU = G^\infty(\mathbb{P}^\infty) \longrightarrow \mathcal{Z}^\infty(\mathbb{P}^\infty) \cong \prod_{k=0}^{\infty} K(\underline{\mathbb{Z}}, \mathbb{R}^{k,k})$$

classifying the total equivariant Chern class map in Atiyah's KR-theory:

$$\widetilde{KR}(X) \to \prod_{k \geq 0} H^{k,k}(X; \underline{\mathbb{Z}}).$$

16 Quaternionic Cycles from the Equivariant Point of View

Recall that the other real structure on projective space is the antiholomorphic involution

$$\mathbf{j} : \mathbb{P}_{\mathbb{C}}(\mathbb{H}^n) \longrightarrow \mathbb{P}_{\mathbb{C}}(\mathbb{H}^n)$$

given by quaternion scalar multiplication by the quaternion j in homogeneous coordinates $\mathbb{H}^n \cong \mathbb{C}^n \oplus j \cdot \mathbb{C}^n$. Note that $\mathbf{j}$ is fixed-point free. In fact there is a smooth fibration $\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n) \longrightarrow \mathbb{P}_{\mathbb{H}}(\mathbb{H}^n)$ whose fibres are complex projective lines, and $\mathbf{j}$ is equivalent to the antipodal map on these fibres, considered as 2-spheres.

Now $\mathbf{j}$ induces a $\mathbb{Z}_2$-action on $\mathcal{Z}^q(\mathbb{P}^{2n-1})$ where $\mathbb{P}^{2n-1} \equiv \mathbb{P}_{\mathbb{C}}(\mathbb{H}^n)$ whose fixed-point set

$$\mathcal{Z}^q_{\mathbb{H}}(\mathbb{P}^{2n-1}) \subset \mathcal{Z}^q(\mathbb{P}^{2n-1})$$

is the group of *quaternionic algebraic cycles* of codimension-q.

The obvious natural questions now are:

(1) What is the homotopy type of $\mathcal{Z}^q_{\mathbb{H}}(\mathbb{P}^{2n-1})$?

(2) What is the $\mathbb{Z}_2$-equivariant homotopy type of $\mathcal{Z}^q(\mathbb{P}^{2n-1})$ under the involution induced by $\mathbf{j}$?

(3) Are there relations of these spaces to some form of K-theory?

There is a complete answer to the first question.

Theorem 16.1. [33]. *Quaternionic algebraic suspension*

$$\Sigma\!\!\!/_{\mathbb{H}} : \mathcal{Z}^q(\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n)) \longrightarrow \mathcal{Z}^q(\mathbb{P}_{\mathbb{C}}(\mathbb{H}^{n+1})),$$

given in homogeneous coordinates by product with a quaternion line, is a $\mathbb{Z}_2$-homotopy equivalence.

Note that quaternionic suspension increases the complex dimension of the underlying complex projective space by 2. Thus Theorem 16.1 allows one to reduce the determination of $\mathcal{Z}^q(\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n))$ to the case of 0-cycles when q is odd, but only to 1-cycles when q is even. When q is odd, the determination of the fixed-point set, that is, the group of quaternionic algebraic cycles $\mathcal{Z}^q_{\mathbb{H}}(\mathbb{P}^{2n-1})$ is straightforwardly computed in [33]. The corresponding determination of q even is substantially harder and is given in [35].

In [35] the concept of quaternionic subvarieties and cycles, relations among the real and quaterionic cycle spaces, and their relations with various K-theories are discussed in detail.

Recently P. Lima-Filho and P. dos Santos have determined the complete equivariant homotopy type of cycles under the quaternion involution. I shall state the results in completely stabilized form. However, as above there are theorems for cycles of finite codimension which are simply truncations of the "stable" case.

Recall from above that we must be careful to distinguish cycles of even and odd codimension under quaternionic suspension. Set

$$\mathcal{Z}^{\text{odd}} \equiv \lim_{q\to\infty} \mathcal{Z}^{2q+1}(\mathbb{P}_{\mathbb{C}}(\mathbb{H}^\infty)) \qquad \text{and} \qquad \mathcal{Z}^{\text{even}} \equiv \lim_{q\to\infty} \mathcal{Z}^{2q}(\mathbb{P}_{\mathbb{C}}(\mathbb{H}^\infty)).$$

Theorem 16.2. [10]. *There are $\mathbb{Z}_2$-equivariant homotopy equivalences:*

$$\mathcal{Z}^{\text{odd}} \cong \prod_{k\geq 1} \text{Map}(\mathbb{P}_{\mathbb{C}}(\mathbb{H})_+,\ K(\underline{\mathbb{Z}}, \mathbb{C}^{2k-1}))$$
$$\mathcal{Z}^{\text{even}} \cong \prod_{k\geq 1} \text{Map}(\mathbb{P}_{\mathbb{C}}(\mathbb{H})_+,\ K(\underline{\mathbb{Z}}, \mathbb{C}^{2k}))$$

Note. By X_+ we mean the space X with a disjoint base point added.

Note 16.3. Considering theorem 16.2 as giving an equivalence of classifying spaces, we obtain an equivalence of functors

$$[X, \mathcal{Z}^{\text{odd}}] \cong \prod_{k\geq 1} H^{2k-1,\,2k-1}(\mathbb{P}_{\mathbb{C}}(\mathbb{H}) \times X;\ \underline{\mathbb{Z}})$$
$$[X, \mathcal{Z}^{\text{even}}] \cong \prod_{k\geq 1} H^{2k,\,2k}(\mathbb{P}_{\mathbb{C}}(\mathbb{H}) \times X;\ \underline{\mathbb{Z}}).$$

Let us now consider the (effective) cycles of degree one in $\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n)$ and stabilize. This yields a pair of stabilized Grassmannians $G^{\text{even}}_{\mathbb{C}}(\mathbb{H}^\infty)$, $G^{\text{even}}_{\mathbb{C}}(\mathbb{H}^\infty)$ with involution. As a $\mathbb{Z}_2$-space their disjoint union classifies the Dupont KH-groups, which are constructed as follows (cf. [13]). Let X be a $\mathbb{Z}_2$-space with action given by an involution $j : X \to X$. Then by a *quaternionic bundle* on X we mean a complex vector bundle $E \to X$ with a $\mathbb{C}$-antilinear bundle mapping $\widetilde{j} : E \to E$ such that

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{j}} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{j} & X \end{array}$$

commutes and

$$\widetilde{j}^2 = -\mathrm{Id}.$$

Isomorphism classes of these form an abelian monoid whose associated Grothendieck group is denoted $KH(X)$. This is a natural functor on the category of $\mathbb{Z}_2$-spaces which for spaces with trivial involution becomes simply the Grothendeick group of quaternionic vector bundles in the usual sense.

Note however that the (complex) tensor product of two quaternionic bundles is not quaternionic, but is instead a Real bundle in the sense of Atiyah (i.e., the involution $\widetilde{j}$ now satisfies $\widetilde{j}^2 = \mathrm{Id}$). One finds in fact that the direct sum $KR(X) \oplus KH(X)$ is a $\mathbb{Z}_2$-graded ring. The interesting fact (see [15]) is that there is a natural isomorphism

$$KR(\mathbb{P}_{\mathbb{C}}(\mathbb{H}) \times X) \cong KR(X) \oplus KH(X).$$

(A striking analogy of this formula also appears in Quillen's computation [51] of the algebraic K-theory of the Brauer-Severi curve over a scheme X). Comparing with 16.3 we see that once again the spaces of algebraic cycles, quite unreasonably, represent some of the most basic objects in algebraic topology.

17 Chern Classes in Quaternionic K-Theory

Now the functor $KH(X)$ is classified by equivariant maps of X into certain infinite Grassmannians. However one must be careful; parity of dimension plays a role. Let $G^q_{\mathbb{C}}(\mathbb{H}^n)$ denote the Grassmann manifold of complex linear subspaces of $\mathbb{H}^n$ under the quaternionic involution. There are equivariant embeddings $G^q_{\mathbb{C}}(\mathbb{H}^n) \to G^{q+2}_{\mathbb{C}}(\mathbb{H}^{n+1})$ and $G^q_{\mathbb{C}}(\mathbb{H}^n) \to G^q_{\mathbb{C}}(\mathbb{H}^{n+1})$ given by inclusion and suspension respectively. Taking the limit as $q, n \to \infty$, we obtain two spaces:

$$G^{\mathrm{even}}_{\mathbb{C}}(\mathbb{H}^\infty) \qquad \text{and} \qquad G^{\mathrm{odd}}_{\mathbb{C}}(\mathbb{H}^\infty).$$

There is an equivalence of functors

$$KH(X) \cong [X, G^\bullet_{\mathbb{C}}(\mathbb{H}^\infty)]_{\mathbb{Z}_2} \tag{17.1}$$

where $G^\bullet_{\mathbb{C}}(\mathbb{H}^\infty) = \coprod_{k=-\infty}^{\infty} G^k_{\mathbb{C}}(\mathbb{H}^\infty)$ where by definition

$$G^k_{\mathbb{C}}(\mathbb{H}^\infty) = \begin{cases} G^{\mathrm{even}}_{\mathbb{C}}(\mathbb{H}^\infty) & \text{if } k \text{ is even} \\ G^{\mathrm{odd}}_{\mathbb{C}}(\mathbb{H}^\infty) & \text{if } k \text{ is odd.} \end{cases}$$

Considering linear subspaces as cycles gives $\mathbb{Z}_2$-equivariant embeddings

$$G^{\mathrm{even}}_{\mathbb{C}}(\mathbb{H}^\infty) \longrightarrow \mathcal{Z}^{\mathrm{even}} \qquad \text{and} \qquad G^{\mathrm{odd}}_{\mathbb{C}}(\mathbb{H}^\infty) \longrightarrow \mathcal{Z}^{\mathrm{odd}}$$

and therefore an embedding

$$G^{\bullet}_{\mathbb{C}}(\mathbb{H}^{\infty}) \longrightarrow \mathcal{Z}^{\bullet} \tag{17.2}$$

where $\mathcal{Z}^{\bullet}$ is defined in analogy with $G^{\bullet}$. By 16.3 this map classifies certain equivariant cohomology classes of $\mathbb{P}_{\mathbb{C}}(\mathbb{H}) \times G^{\bullet}$. The first problem is to compute the equivariant cohomology of these spaces. Set

$$R_{\mathbb{H}} \equiv H^{*,*}(\mathbb{P}_{\mathbb{C}}(\mathbb{H});\ \underline{\mathbb{Z}}).$$

Theorem 17.1. (dos Santos and Lima-Filho [10]). *There is a $\mathbb{Z}_2$-homotopy equivalence $\mathbb{P}_{\mathbb{C}}(\mathbb{H}) \times G^{even}_{\mathbb{C}}(\mathbb{H}^{\infty}) \cong \mathbb{P}_{\mathbb{C}}(\mathbb{H}) \times G^{odd}_{\mathbb{C}}(\mathbb{H}^{\infty})$, and the $\mathbb{Z}_2$-equivariant cohomology of this space is a polynomial ring*

$$H^{*,*}(\mathbb{P}_{\mathbb{C}}(\mathbb{H}) \times G^{even}_{\mathbb{C}}(\mathbb{H}^{\infty});\ \underline{\mathbb{Z}}) \cong R_{\mathbb{H}}[\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \dots]$$

for canonical classes $\mathbf{d}_k \in H^{k,k}$.

Theorem 17.2. (dos Santos and Lima-Filho [10]). *The map (17.2) given by cycle inclusion represents the class*

$$1 + \mathbf{d}_2 + \mathbf{d}_4 + \mathbf{d}_6 + \dots \qquad \text{on } G^{\text{even}}_{\mathbb{C}}(\mathbb{H}^{\infty}) \qquad \text{and}$$

$$\mathbf{d}_1 + \mathbf{d}_3 + \mathbf{d}_5 + \dots \qquad \text{on } G^{\text{odd}}_{\mathbb{C}}(\mathbb{H}^{\infty})$$

In light of (17.1) this gives a definition of equivariant **characteristic classes for quaternionic bundles** and settles a problem posed by Dupont.

In the same paper dos Santos and Lima-Filho show that the algebraic join pairing on $\mathcal{Z}^{\bullet}$ classifies a certain enhanced cup product. Hence, with repect to this product the characteristic classes $\mathbf{d}_*$ satisfy the usual formulas for the direct sum of bundles.

In [10] it is furthermore shown that there is are equivariant infinite loop space structures on $G^{\bullet}$ and $\mathcal{Z}^{\bullet}$ making the map (17.2) an infinite loop map. Hence **(17.2) enhances to a map of equivariant spectra**.

18 Algebraic or "Motivic" Directions

It has probably occured to the reader that much of the discussion here might usefully be carried over from projective space to more general varieties. Indeed this is true, and quite interesting results have been established. In the classification process one can replace spaces of continuous maps with spaces of algebraic maps.

This leads to interesting cohomology theories for varieties which are Poincaré dual to the homology theories defined by cycles [19], [21]. These theories have been extended to real varieties [12] and are related to Grothendieck-Galois cohomology [26], [27]. Analogous constructions lead to interesting forms of K-theory on varieties [19], [4, 5], [22, 23, 24], [11]. In the quaternionic case this is related to Quillen's computation [51] of the algebraic K-theory of the Brauer-Severi curve over a scheme X [11].

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H. Blaine Lawson, Jr.
address: ???
E-mail: ???