Restriction and Removable Singularities for Fully Nonlinear Partial Differential Equations
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Reese Harvey
Our Outlook

To systematically take a potential theoretic approach to the study of nonlinear differential equations.
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In complex analysis and geometry, plurisubharmonic functions have been very effective.

Analogies can be developed in a surprisingly general context.

They apply, for example, to calibrated and symplectic geometry.
Consider $X^{\text{open}} \subset \mathbb{R}^n$. 

Classically, a fully nonlinear second-order equation is written

$$f(x, u(x), D_x u, D^2_x u) = 0,$$

where $f: X \times \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^n) \rightarrow \mathbb{R}$. 

Subsolutions:

$$f(x, u(x), D_x u, D^2_x u) \geq 0$$

Supersolutions:

$$f(x, u(x), D_x u, D^2_x u) \leq 0$$
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A Geometric Approach (cf. Krylov)

We fix a closed subset $F \subset J_2(\mathbb{X}) \equiv X \times \mathbb{R} \times \mathbb{R}^n \times \text{Sym}_2(\mathbb{R}^n)$ with certain mild properties:

- $F \times \{0,0\} \subset F$ for all $P \geq 0$
- $F \times \{r,0\} \subset F$ for all $r \leq 0$

$F$ is called a subequation.

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\[(P) \quad F_x + (0, 0, P) \subset F_x \quad \text{for all } P \geq 0\]
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\[(T)\quad F = \overline{\text{Int}F}, \quad F_x = \overline{\text{Int}_xF_x}, \quad \text{Int}_xF_x = \text{Int}F \cap F_x\]
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Definition. A function $u \in C^2(X)$ is $F$-subharmonic (a subsolution) if

$$J_x^2 u \equiv (x, u(x), D_x u, D_x^2 u) \in F \quad \forall x \in X.$$
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We want to extend the notion of $F$-subharmonicity to upper semi-continuous functions.
Viscosity Theory (Crandall, Ishii, Lions, Evans, et al.)

\[
\text{USC}(X) \equiv \{ u : X \to [-\infty, \infty) : u \text{ is upper semicontinuous} \}
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**Definition.** Fix \( u \in \text{USC}(X) \). A test function for \( u \) at a point \( x \in X \) is a function \( \varphi \), \( C^2 \) near \( x \), such that

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\begin{align*}
    u & \leq \varphi \quad \text{near } x \\
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J^2_x \varphi \in F.
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\( F(X) \equiv \) the set of these.
Remarkable Properties

\[ u, v \in \mathcal{F}(X) \implies \max\{u, v\} \in \mathcal{F}(X) \]

\[ \mathcal{F}(X) \text{ is closed under decreasing limits.} \]

\[ \mathcal{F}(X) \text{ is closed under uniform limits.} \]

If \( \mathcal{F} \subset \mathcal{F}(X) \) is locally uniformly bounded above, then \( u^* \in \mathcal{F}(X) \) where

\[ u(x) \equiv \sup_{v \in \mathcal{F}} v(x) \]

If \( u \in C^2(X) \), then

\[ u \in \mathcal{F}(X) \iff J^2_x u \in \mathcal{F} \forall x \in X. \]
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Subequations on Manifolds

The bundle of 2-jets on a manifold $X$ is the vector bundle $J^2(X) \rightarrow X$ whose fibre at $x \in X$ is $J^2_x(X) \equiv C^\infty_x / C^\infty_x$, where $C^\infty_x$ is the germs of smooth functions at $x$, and $C^\infty_x$,3 are those germs which vanish to order 3 at $x$.

There is a short exact sequence

$$0 \rightarrow \text{Sym}^2(T^*X) \rightarrow J^2(X) \rightarrow J^1(X) \rightarrow 0$$

and $J^1(X) = \mathbb{R} \oplus T^*X$. 

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The **bundle of 2-jets** on a manifold $X$ is the vector bundle

$$J^2(X) \longrightarrow X$$

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and $J^1(X) = \mathbb{R} \oplus T^*X$. 
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$$F \subset J^2(X)$$

which satisfies the three conditions (P), (N), and (T) above.
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The $F$-subharmonic functions $F(X)$ are defined as before. They have the same remarkable properties.
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The key to defining differential equations on $X$
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They have the same remarkable properties.

The key to defining differential equations on $X$
is to use subequations and duality.
Duality and $F$-Harmonics

Define the dual of $F \subset J^2(X)$ by

\[ \tilde{F} \equiv \sim(\text{Int} F) = -\sim(\text{Int} F) \]

• $F$ is a subequation $\iff \tilde{F}$ is a subequation.

• In this case $\tilde{\tilde{F}} = F$.

• In our analysis, the roles of $F$ and $\tilde{F}$ are often interchangeable.

Note that $F \cap -\tilde{F} = \partial F$. 

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Let $F \subset J^2(X)$ be a subequation.

Definition. A function $u$ on $X$ is $F$-harmonic if $u \in F(X)$ and $-u \in \tilde{F}(X)$.

These are our solutions.

$u \in C^2(X)$ is $F$-harmonic $\iff J^2 x u \in \partial F \forall x \in X$. 

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These are our solutions.

$$u \in C^2(X) \text{ is } F\text{-harmonic} \quad \iff \quad J^2_x u \in \partial F \quad \forall \ x \in X.$$
Examples $P$ and $\tilde{P}$

Constant coefficient, pure second-order in $\mathbb{R}^n$:

\[
\begin{align*}
\text{Define } P & \subset \text{Sym}_2(\mathbb{R}^n) \text{ by } P \equiv \{ A : A \geq 0 \} \\
\tilde{P} & = \{ A : A \text{ has at least one eigenvalue } \geq 0 \}.
\end{align*}
\]

Proposition.

For an open set $X \subset \mathbb{R}^n$:

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\begin{align*}
P(X) & = \text{the convex functions on } X \\
\tilde{P}(X) & = \text{the subaffine functions on } X.
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The homogeneous real Monge-Ampère Equation

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D^2 u \geq 0 \text{ and } \det(D^2 u) = 0.
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Examples: Other Branches of the MA Equation
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For $A \in \text{Sym}^2(\mathbb{R}^n)$ let

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be the ordered eigenvalues of $A$. 
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P_{k} \equiv \{ \lambda_{k}(A) \geq 0 \}
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$$P_k \equiv \{ \lambda_k(A) \geq 0 \}$$

$$\tilde{P}_k = P_{n-k+1}$$
Examples: Other Elementary Symmetric Functions

\[ S_k \equiv \{ A : \sigma_1(A) \geq 0, \ldots, \sigma_k(A) \geq 0 \} \]

\[ \sigma_k(A) \equiv \sigma_k(\lambda_1(A), \ldots, \lambda_n(A)) \]
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This is the **principal branch** of the equation

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The equation has \((k - 1)\) other branches.
Examples: p-Convexity

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For each real number \( p \in [1, n] \), define

\[
\mathcal{P}_p \equiv \left\{ A : \lambda_1(A) + \cdots + \lambda_p(A) + (p - [p])\lambda_{p+1}(A) \geq 0 \right\}
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where \( \lambda_1(A) \leq \cdots \leq \lambda_n(A) \) are the ordered eigenvalues of \( A \).
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The $\mathcal{P}_p$-subharmonics are $p$-convex functions.
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**Theorem.** For $p$ an integer:
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The \( \mathcal{P}_p \)-subharmonics are \( p \)-convex functions.

**Theorem.** For \( p \) an integer:

The restriction of a \( \mathcal{P}_p \)-subharmonic to any minimal \( p \)-dimensional submanifold \( Y \) is subharmonic in the induced metric on \( Y \).
Examples: p-Convexity

For each real number $p \in [1, n]$, define

$$
\mathcal{P}_p \equiv \left\{ A : \lambda_1(A) + \cdots + \lambda_p(A) + (p - [p])\lambda_{[p]+1}(A) \geq 0 \right\}
$$

where $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ are the ordered eigenvalues of $A$.

The $\mathcal{P}_p$-subharmonics are $p$-convex functions.

**Theorem.** For $p$ an integer:

The restriction of a $\mathcal{P}_p$-subharmonic to any minimal $p$-dimensional submanifold $Y$ is subharmonic in the induced metric on $Y$.

$\mathcal{P}_p$-harmonics are solutions of the polynomial equation

$$
MA_p(A) = \prod_{i_1 < \cdots < i_p} (\lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A)) = 0.
$$
Examples: p-Convexity

The Riesz kernel

\[ K_p(x) \equiv \frac{-1}{(p-2)|x|^{p-2}} \quad \text{if } p \neq 2 \]
Examples: p-Convexity

The Riesz kernel

\[ K_p(x) \equiv \frac{-1}{(p-2)|x|^{p-2}} \quad \text{if } p \neq 2 \quad \text{and} \quad K_2(x) \equiv \log|x| \]
Examples: p-Convexity

The Riesz kernel

\[ K_p(x) \equiv -\frac{1}{(p-2)|x|^{p-2}} \quad \text{if } p \neq 2 \quad \text{and} \quad K_2(x) \equiv \log|x| \]

is $\mathcal{P}_p$-harmonic in $\mathbb{R}^n - \{0\}$

and $\mathcal{P}_p$-subharmonic across 0.
Examples: Complex Analogues

\[ \mathbb{C}^n = (\mathbb{R}^{2n}, J). \]
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\[ C^n = (\mathbb{R}^{2n}, J). \]

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Examples: Complex Analogues

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Examples: Complex Analogues

\[ C^n = (R^{2n}, J). \]

\[ \text{Sym}_C^2(C^n) \subset \text{Sym}^2(R^{2n}) \]

\[ A_C \equiv \frac{1}{2}(A - JAJ) \]

\[ A_C \] has complex eigenspaces and ordered eigenvalues

\[ \lambda_1^C(A) \leq \cdots \leq \lambda_n^C(A) \]
Examples: Complex Analogues

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\[ \text{Sym}_c^2(C^n) \subset \text{Sym}^2(\mathbb{R}^{2n}) \]

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All the O(n)-invariant subequations given in terms of the \( \lambda_k(A) \)
Examples: Complex Analogues

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\( A_c \) has complex eigenspaces and ordered eigenvalues

\[ \lambda^c_1(A) \leq \cdots \leq \lambda^c_n(A) \]

All the \( O(n) \)-invariant subequations given in terms of the \( \lambda_k(A) \)
have \( U(n) \)-invariant analogues given by the same conditions on the \( \lambda^c_k(A) \)
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Example: The homogeneous complex Monge-Ampère equation

\[ P^C = \{ A : A_C \geq 0 \}, \]

and its branches

\[ P^C_k = \{ A : \lambda^C_k(A) \geq 0 \}, \]
Examples: Complex Analogues

\[ \mathbb{C}^n = (\mathbb{R}^{2n}, J). \]

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\[ P_k^C = \{ A : \lambda_k^C(A) \geq 0 \}, \]

(Note: \( P_C(X) \) = the plurisubharmonic functions on \( X \))
Examples: Quaternionic Analogues

\[ H^n = (\mathbb{R}^{4n}, I, J, K). \]

\[ A_H \equiv \frac{1}{4}(A - |A| - JAJ - KAK) \]

\( A_C \) has quaternionic eigenspaces and ordered eigenvalues

\[ \lambda_1^H(A) \leq \cdots \leq \lambda_n^H(A) \]

All the \( O(n) \)-invariant subequations given in terms of the \( \lambda_k(A) \) have \( \text{Sp}(n) \)-invariant analogues given by same conditions on the \( \lambda_k^H(A) \)

Example: The quaternionic Monge-Ampère equation (Alesker, Verbitsky)

\[ P^H = \{ A : A_H \geq 0 \}, \]

and its branches

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Fix a compact set

$$G \subset G(\rho, \mathbb{R}^n) \quad \text{(Grassmannian)}$$
Examples: Geometric Cases – Calibrations

Fix a compact set

\[ \mathbf{G} \subset G(\rho, \mathbb{R}^n) \quad \text{(Grassmannian)} \]

and define

\[ F_{\mathbf{G}} \equiv \{ A : \text{tr} \left( A \big|_{\mathcal{W}} \right) \geq 0 \text{ for all } \mathcal{W} \in \mathbf{G} \} \]
Examples: Geometric Cases – Calibrations

Fix a compact set

$$G \subset G(p, \mathbb{R}^n)$$ (Grassmannian)

and define

$$F_G \equiv \{ A : \text{tr} (A|_W) \geq 0 \text{ for all } W \in G \}$$

Examples:

$$G = G(1, \mathbb{R}^n) \Rightarrow F_G = \mathcal{P}.$$
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- $$G = G(1, \mathbb{R}^n) \Rightarrow F_G = P.$$
- $$G = G(p, \mathbb{R}^n) \Rightarrow F_G = \mathcal{P}_p.$$
Examples: Geometric Cases – Calibrations

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Examples:

\[ G = G(1, \mathbb{R}^n) \Rightarrow F_G = \mathcal{P}. \]
\[ G = G(p, \mathbb{R}^n) \Rightarrow F_G = \mathcal{P}_p. \]
\[ G = G^c(1, \mathbb{C}^n) \Rightarrow F_G = \mathcal{P}_c. \]
Examples: Geometric Cases – Calibrations

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$$G = G^c(1, C^n) \Rightarrow F_G = \mathcal{P}^c.$$  

$$G = \text{LAG} = \text{the lagrangian } n \text{ planes in } C^n.$$
Examples: Geometric Cases – Calibrations

Fix a compact set

$$\mathbf{G} \subset G(p, \mathbb{R}^n) \text{ (Grassmannian)}$$

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Examples:

- $$\mathbf{G} = G(1, \mathbb{R}^n) \Rightarrow F_\mathbf{G} = \mathcal{P}.$$
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- $$\mathbf{G} = G^c(1, \mathbb{C}^n) \Rightarrow F_\mathbf{G} = \mathcal{P}^c.$$
- $$\mathbf{G} = \text{LAG} = \text{the lagrangian } n \text{ planes in } \mathbb{C}^n.$$
- $$\mathbf{G} = \mathbf{G}(\phi) = \text{the } \phi\text{-planes associated to a calibration } \phi$$
Riemannian manifolds and the decomposition of $J^2(X)$

When $X$ is Riemannian, there is a canonical bundle splitting $J^2(X) = \mathbb{R} \oplus T^*X \oplus \text{Sym}^2(T^*X)$ given by

$$J^2_xu = (u(x), (du)_x, \text{Hess}_xu)$$

where $\text{Hess}_u \in \Gamma(\text{Sym}^2(T^*X))$ is the Riemannian hessian defined by

$$(\text{Hess}_u)(V, W) = VWu - (\nabla V W)u$$

for vector fields $V, W$. 
When $X$ is **Riemannian**, there is a canonical bundle splitting

$$J^2(X) = \mathbb{R} \oplus T^*X \oplus \text{Sym}^2(T^*X)$$
Riemannian manifolds and the decomposition of $J^2(X)$

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Riemannian manifolds and the decomposition of $J^2(X)$

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$$(\text{Hess} \ u)(V, W) = VWu - (\nabla_V W)u$$

for vector fields $V, W$. 
Suppose $F \subset J \equiv \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^n)$ is closed, $O(n)$-invariant, and satisfies (P), (N) and (T). Then $F$ canonically determines a subequation $F_X \subset J^2(X)$ on any Riemannian manifold $X$. 

Example. $F \equiv \mathbb{R} \times \mathbb{R}^n \times \{\text{tr} A \geq 0\}$ gives $\text{tr}(\text{Hess} u) = \Delta u \geq 0$. 
Universal Riemannian subequations

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Then \( F \) **canonically determines a subequation**

\[ F_X \subset \mathcal{J}^2(X) \]

on any riemannian manifold \( X \).
Universal Riemannian subequations

Suppose

\[ F \subset J \equiv R \times R^n \times \text{Sym}^2(R^n) \]

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- \(O(n)\)-invariant
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Then \( F \) canonically determines a subequation

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Example. \( F \equiv R \times R^n \times \{\text{tr}A \geq 0\} \) gives

\[ \text{tr}(\text{Hess } u) = \Delta u \geq 0. \]
Universal Hermitian subequations

Let $C^n = (\mathbb{R}^{2n}, J)$. If

$$ F \subset J \equiv \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^{2n}) $$

- is closed
- $U(n)$-invariant
- and satisfies (P), (N) and (T)

Then $F$ canonically determines a subequation

$$ F_X \subset J^2(X) $$

on any almost complex, hermitian manifold $X$.

Example. $F \equiv \mathbb{R} \times \mathbb{R}^n \times \{A_c \geq 0\}$ gives the homogeneous complex Monge-Ampère subequation.
Manifolds with Topological Structure Group G

Let

\[ G \subset O(n) \]

be a closed subgroup. If

\[ F \subset J \equiv \mathbb{R} \times \mathbb{R}^n \times \text{Sym}^2(\mathbb{R}^{2n}) \]

- is closed
- \( G \)-invariant
- and satisfies (P), (N) and (T)

Then \( F \) canonically determines a subequation \( F_\chi \) on any riemannian manifold with topological structure group \( G \).
Automorphisms

An automorphism of the 2-jet bundle is an intrinsically defined bundle isomorphism \( \Phi : J^2(X) \rightarrow J^2(X) \) with the property that for any splitting \( J^2(X) = R \oplus T^*X \oplus \text{Sym}^2(T^*X) \) of the short exact sequence, \( \Phi \) has the form

\[
\Phi(r, p, A) = (r, gp, hAh + L(p))
\]

where \( g, h : T^*X \rightarrow T^*X \) are bundle isomorphisms and \( L : T^*X \rightarrow \text{Sym}^2(T^*X) \) is a smooth bundle map.
Automorphisms

An **automorphism** of the 2-jet bundle is an intrinsically defined bundle isomorphism

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is a smooth bundle map.
Affine Automorphisms

An affine automorphism of the 2-jet bundle is a bundle isomorphism $\Psi : J^2(X) \rightarrow J^2(X)$ of the form $\Psi = \Phi + J$ where $\Phi$ is an automorphism and $J$ is a section of $J^2(X)$.
An **affine automorphism** of the 2-jet bundle is a bundle isomorphism

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$$

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Affine Automorphisms

An **affine automorphism** of the 2-jet bundle is a bundle isomorphism

\[ \psi : J^2(X) \longrightarrow J^2(X) \]

of the form

\[ \psi = \Phi + J \]

where \( \Phi \) is an automorphism and \( J \) is a section of \( J^2(X) \).
Jet Equivalence of Subequations

Definition. Two subequations $F_1, F_2 \subset J_2(X)$ are affinely jet equivalent if there exists an affine automorphism $\Psi : J_2(X) \rightarrow J_2(X)$ such that $\Psi(F_1) = F_2$. 
Definition. Two subequations

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**Definition.** Two subequations

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Jet Equivalence of Subequations

• Jet equivalence does \textbf{not} send $F_1(X)$ to $F_2(X)$. 

• The universal equations above are all locally jet equivalent to constant coefficient equations in local coordinates.

• Affine jet equivalence converts homogeneous equations to inhomogeneous equations, e.g., $\det C = e^u$ with $\text{Hess} C u \geq 0$. 

$\lambda^k \left( \text{Hess} u \right) = f(x)$
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Jet Equivalence of Subequations

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- Affine jet equivalence converts **homogeneous** equations to **inhomogeneous** equations, e.g.,

$$\det_{\mathbb{C}}(\text{Hess}_{\mathbb{C}} u) = e^u \quad \text{with} \quad \text{Hess}_{\mathbb{C}} u \geq 0.$$ 

$$\lambda_k(\text{Hess}u) = f(x)$$
Monotonicity Cones – An Important Concept

Definition. A subset $M \subset J^2(X)$ is a monotonicity cone for $F$ if

(i) $M_x$ is a convex cone with vertex at 0, and

(ii) $F_x + M_x \subset F_x$

Example 1. In $\mathbb{R}^n$, $P \equiv \{A \geq 0\}$ is a monotonicity cone for every pure second-order subequation.

Example 2. In $\mathbb{C}^n$, $P_C \equiv \{A_C \geq 0\}$ is a monotonicity cone for every pure second-order hermitian subequation.
Monotonicity Cones – An Important Concept

\[ F \subset J^2(X) \] a subequation on a manifold \( X \).
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Example 1. In \( \mathbb{R}^n \), \( P = \{ A \geq 0 \} \) is a monotonicity cone for every pure second-order subequation.

Example 2. In \( \mathbb{C}^n \), \( P_{\mathbb{C}} = \{ A_{\mathbb{C}} \geq 0 \} \) is a monotonicity cone for every pure second-order hermitian subequation.
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Example 1. In \( \mathbb{R}^n \), \( \mathcal{P} \equiv \{ A \geq 0 \} \) is a monotonicity cone for every pure second-order subequation.

Example 2. In \( \mathbb{C}^n \), \( \mathcal{P}^\mathbb{C} \equiv \{ A_{\mathbb{C}} \geq 0 \} \) is a monotonicity cone for every pure second-order hermitian subequation.
The Dirichlet Problem

Let $\mathcal{F} \subset \mathcal{J}^2(X)$ be a subequation with monotonicity cone $\mathcal{M}$.

**Theorem.** Suppose $\mathcal{F}$ is locally affinely jet-equivalent to a constant coefficient subequation. Suppose also that $X$ supports a strictly $\mathcal{M}$-harmonic function. Then for any domain $\Omega \subset X$ whose boundary is both $\mathcal{F}$ and $\tilde{\mathcal{F}}$ strictly convex, the Dirichlet Problem for $\mathcal{F}$-harmonic functions is uniquely solvable for all continuous boundary functions $\phi \in C(\partial \Omega)$.
Let $F \subset J^2(X)$ be a subequation with monotonicity cone $M$. 
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The Dirichlet Problem

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**THEOREM.** Suppose $F$ is locally affinely jet-equivalent to a constant coefficient subequation. Suppose also that $X$ supports a strictly $M$-harmonic function.

The Dirichlet Problem

Let $F \subset J^2(X)$ be a subequation with monotonicity cone $M$.

**THEOREM.** Suppose $F$ is locally affinely jet-equivalent to a constant coefficient subequation. Suppose also that $X$ supports a strictly $M$-harmonic function.

Then for any domain $\Omega \subset X$ whose boundary is both $F$ and $\tilde{F}$ strictly convex,
Let $F \subset J^2(X)$ be a subequation with monotonicity cone $M$.

**THEOREM.** Suppose $F$ is locally affinely jet-equivalent to a constant coefficient subequation. Suppose also that $X$ supports a strictly $M$-harmonic function.

Then for any domain $\Omega \subset\subset X$ whose boundary is both $F$ and $\tilde{F}$ strictly convex, the Dirichlet Problem for $F$-harmonic functions is **uniquely solvable** for all continuous boundary functions $\varphi \in C(\partial \Omega)$. 
Removable Singularities – General Results

Definition. A subset $E \subset X$ is $M$-polar if $E = \{ \psi = -\infty \}$ for an $M$-subharmonic function which is smooth on $X - E$.

THEOREM A. Suppose $E \subset X$ is a closed subset which is locally $M$-polar. Then $E$ is removable for $F$-subharmonic functions which are locally bounded above across $E$. That is, if $u \in F(X - E)$ and locally bounded above across $E$, then its canonical upper semi-continuous extension is $F$-subharmonic on $X$.

THEOREM B. Suppose $E$ is a closed set with no interior, which is locally $M$-polar. Then for $u \in C(X)$, $u$ is $F$-harmonic on $X - E \Rightarrow u$ is $F$-harmonic on $X$. 
Removable Singularities – General Results

Let $F \subset J^2(X)$ be a subequation with monotonicity cone $M$. 

**Definition.** A subset $E \subset X$ is $M$-polar if $E = \{ \psi = -\infty \}$ for an $M$-subharmonic function which is smooth on $X - E$.

**Theorem A.** Suppose $E \subset X$ is a closed subset which is locally $M$-polar. Then $E$ is removable for $F$-subharmonic functions which are locally bounded above across $E$. That is, if $u \in F(X - E)$ and locally bounded above across $E$, then its canonical upper semi-continuous extension is $F$-subharmonic on $X$.

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Restriction and Removable Singularities
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**THEOREM.** Pluripolar sets (i.e. \(P^C\)-polar sets) in \(\mathbf{C}^n\) are removable for all branches of the homogeneous Monge-Ampère equation.
Fix $2 \leq n$ and recall

$$P_p \equiv \{ A: \lambda_1(A) + \cdots + \lambda_{\lfloor p \rfloor}(A) + (p - \lfloor p \rfloor) + 1(A) \geq 0 \}.$$ 

**Theorem C.** A closed set $E$ with locally finite Hausdorff $(p - 2)$-measure is locally $P_p$-polar.

Thus, if $F$ is a subequation with $F + P_p \subset F$, then $E$ is removable for $F$-subharmonics and $F$-harmonics as before.

The proof uses Riesz potentials $\mu^* K_p$ where $K_p(x) = -\frac{1}{|x|^{p-2}}.$
Fix $2 < p \leq n$ and recall

$$\mathcal{P}_p \equiv \left\{ A : \lambda_1(A) + \cdots + \lambda_{[p]}(A) + (p - [p])\lambda_{[p]+1}(A) \geq 0 \right\}$$
Removable Singularities – Riesz Potentials

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The proof uses Riesz potentials

$$\mu \ast K_p \quad \text{where} \quad K_p(x) = \frac{-1}{|x|^{p-2}}.$$
Riesz Characteristics

Lemma. \( p \in \mathbb{P} \iff I - p \pi e \in \mathcal{M} \) for all unit vectors \( e \in \mathbb{R}^n \).

Definition. The Riesz characteristic of \( \mathcal{M} \) is \( p_{\mathcal{M}} \equiv \sup \{ p : I - p \pi e \in \mathcal{M} \text{ for all } |e| = 1 \} \).

Theorem D. Suppose \( \mathcal{F} \subset \text{Sym}_2(\mathbb{R}^n) \) is a subequation with monotonicity cone \( \mathcal{M} \). Then any closed set \( E \) of locally finite Hausdorff \( (p_{\mathcal{M}} - 2) \) measure is removable for \( \mathcal{F} \)-subharmonics and \( \mathcal{F} \)-harmonics.
Riesz Characteristics

Let $M \subset \text{Sym}^2(\mathbb{R}^n)$ be a convex cone subequation and $1 \leq p \leq n$. 

Lemma.

$P^p \subset M \Leftrightarrow I - p \pi e \in M$ for all unit vectors $e \in \mathbb{R}^n$.

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The Riesz characteristic of $M$ is $p_M \equiv \sup \{ p : I - p \pi e \in M \text{ for all } |e| = 1 \}$.

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\[\mathcal{P}_p \subset M \iff l - p \pi e \in M \text{ for all unit vectors } e \in \mathbb{R}^n.\]

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Examples

Example 1. (The \(\delta\)-Uniformly Elliptic Cone). For \(\delta > 0\) define

\[
P(\delta) \equiv \{ A \in \text{Sym}^2(\mathbb{R}^n) : A + \delta (\text{tr} A) \cdot I \geq 0 \}
\]

\(P(\delta)\) has Riesz characteristic \(1 + \delta n^{1+\delta}\).

Example 2. (The Pucci Cone). For \(0 < \lambda < \Lambda\) define

\[
P_{\lambda, \Lambda} \equiv \{ A \in \text{Sym}^2(\mathbb{R}^n) : \lambda \text{tr} A + \Lambda \text{tr} A - \geq 0 \},
\]

\(P_{\lambda, \Lambda}\) has Riesz characteristic \(\lambda \Lambda (n-1)^{1} +^{1}\).

Example 3. (The \(k\)th Elementary Symmetric Cone). For \(k = 1, \ldots, n\) define

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F_k \equiv \{ A \in \text{Sym}^2(\mathbb{R}^n) : \sigma_1(A) \geq 0, \ldots, \sigma_k(A) \geq 0 \},
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Given a subequation $F \subset J^2(X)$ and a submanifold $i: Y \hookrightarrow X$, there is an induced subequation $i^*F \subset J^2(Y)$.

**Problem:**
Given an (u.s.c.) $u \in F(X)$, when is $u \big|_{Y} \in (i^*F)(Y)$?

There is a general answer based on a technical restriction hypothesis. This leads to a number of more specific interesting results. For constant coefficient subequations restriction always holds. For linear equations there is a simple and useful linear restriction hypothesis.
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For linear equations there is a simple and useful linear restriction hypothesis.
Consider a closed subset $G_l \subset (k, X)$ and let $F_{G_l} \equiv \{ J_2 u: \text{tr} (\text{Hess} u | W) \geq 0 \}$.

A $G_l$-submanifold of $X$ is defined to be a $k$-dimensional submanifold $Y \subset X$ such that $T_y Y \in G_l$ for all $y \in Y$.

Theorem. Let $Y \subset X$ be a $G_l$-submanifold which is minimal. Then restriction to $Y$ holds for $F_{G_l}$.

In other words, the restriction of any $F_{G_l}$-plurisubharmonic function to $Y$ is subharmonic in the induced Riemannian metric on $Y$. 

Blaine Lawson

Restriction and Removable Singularities

October 27, 2013
Restriction Theorem 1

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In other words, the restriction of any \( F_{\mathcal{G}} \)-plurisubharmonic function to \( Y \) is subharmonic in the induced riemannian metric on \( Y \).
There is a second restriction theorem which assumes local jet equivalence mod $\mathcal{C}$ to a constant coefficient subequation. One consequence is:

**Theorem.** Let $X$ be a riemannian manifold of dimension $N$ and $F \subset \mathcal{C}^2(X)$ a subequation canonically determined by an $O(N)$-invariant universal subequation $F \subset \mathcal{C}^2_N$. Then restriction holds for $F$ on any totally geodesic submanifold $Y \subset X$. 

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Restriction Theorem 2

There is a second restriction theorem which assumes

\textbf{local jet equivalence mod } Y

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**Theorem.**  *Let $X$ be a riemannian manifold of dimension $N$ and $F \subset J^2(X)$ a subequation canonically determined by an $O_N$-invariant universal subequation $F \subset J^2_N$. Then restriction holds for $F$ on any totally geodesic submanifold $\mathcal{Y} \subset X$.***
Almost Complex Manifolds and the Pali Conjecture

There are three definitions of plurisubharmonic functions.

1. There is an intrinsic subequation defined by
   \[ i \frac{\partial}{\partial u} \geq 0, \]
   or equivalently
   \[ H(u)(V, V) = (VV + (JV)(JV) + J[V, JV]) \cdot u \geq 0. \]

   One now applies the viscosity definition as before.
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On any *almost complex* manifold \((X, J)\)

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Almost Complex Manifolds and the Pali Conjecture

2. A classical result of Nijenhuis and Woolf states that

Given $x \in X$ and a complex tangent line $\ell$ at $x$
there exists a (pseudo-)holomorphic curve $\Sigma \subset X$ through $x$ with tangent $\ell$. 

We now define $u \in USC(X)$ to be plurisubharmonic if its
restriction to each $\Sigma \subset X$ through $x$ with tangent $\ell$.

3. A distribution $u \in D'(X)$ is plurisubharmonic if
$i \partial \partial u = \mu \geq 0$ (a positive $(1,1)$−current).

THEOREM. These three definitions are equivalent.
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Given \( x \in X \) and a complex tangent line \( \ell \) at \( x \) there exists a (pseudo-)holomorphic curve \( \Sigma \subset X \) through \( x \) with tangent \( \ell \).

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**THEOREM.** These three definitions are equivalent.
Almost Complex Manifolds and the Pali Conjecture

That (1) ⇔ (2) uses the Restriction Theorem.
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**THEOREM.**

(a) Suppose $u$ is $\mathcal{P}^c(J)$-plurisubharmonic.

That (1) $\Rightarrow$ (3) was proven years ago by Nefton Pali.

Pali conjectured (3) $\Rightarrow$ (1) and proved it under certain assumptions on $u$. Blaine Lawson

Restriction and Removable Singularities

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(a) Suppose \( u \) is \( \mathcal{P}^c(J) \)-plurisubharmonic. Then \( u \in L^1_{\text{loc}}(X) \subset \mathcal{D}'(X) \), and \( u \) is distributionally \( J \)-plurisubharmonic.
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Fix a smooth volume form $\lambda$ and a continuous function $f \geq 0$ on $X$.

Fix a domain $\Omega \subset X$ with smooth boundary $\partial \Omega$.

For $\phi \in C(\partial \Omega)$, consider the Dirichlet Problem:

Find $u \in PC(\Omega)$ with $(i \partial / \partial n) u = f \lambda$ (viscosity sense) on $\Omega$ and $u \mid_{\partial \Omega} = \phi$.

THEOREM. (a) Uniqueness holds for the Dirichlet Problem if $(X, J)$ supports a $C^2$-strictly plurisubharmonic function.

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Fix a smooth volume form $\lambda$ and a continuous function $f \geq 0$ on $X$. 

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Consider a conical subequation $F \subset \text{Sym}_2(\mathbb{R}^n)$ which has Riesz characteristic $p$, $2 \leq p \leq n$, and is $P_p$-monotone.

Let $\Omega \subset \mathbb{R}^n$ be a domain with a smooth boundary which is both $F$ and $\tilde{F}$ strictly convex.

**Theorem.** Suppose $0 \in \Omega$ and let $h \in C(B_{\epsilon} - \{0\})$ be an $F$-harmonic function with $\lim_{x \to 0} h(x) = -\infty$. Fix $\phi \in C(\partial \Omega)$.

**Existence.** There exists $H \in C(\Omega - \{0\})$ such that:

1. $H$ is $F$-harmonic on $\Omega - \{0\}$,
2. $H|_{\partial \Omega} = \phi$,
3. $h(x) + c \leq H(x) \leq h(x) + C$ on a neighborhood of 0 for some constants $c$, $C$.

**Uniqueness.** There is at most one function $h \in C(\Omega - \{0\})$ satisfying (1), (2), and (3).
The Dirichlet Problem with Prescribed Asymptotics

Consider a conical subequation $F \subset \text{Sym}^2(\mathbb{R}^n)$ which has Riesz characteristic $p, 2 \leq p \leq n$, and is $\mathcal{P}_p$-monotone.
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