Given With Deep Respect and Affection

for Marcel Berger

who was a father to so many

and one of the great geometers of his century
A MONGE AMPÈRE OPERATOR
in SYMPLECTIC GEOMETRY

with Reese Harvey
The Outline

1. LAGRANGIAN POTENTIAL THEORY

2. THE LAGRANGIAN MONGE-AMPÈRE OPERATOR

3. TRANSPLANTATION TO GROMOV MANIFOLDS

4. THE DIRICHLET PROBLEM

5. A FUNDAMENTAL SOLUTION IN $\mathbb{C}^n$

6. QUESTIONS
LAGRANGIAN POTENTIAL THEORY
“Geometrically Based” Potential Theories:

Consider a compact set

\[ G \subset G(p, \mathbb{R}^n) = \text{The Grassmannian of } p\text{-planes in } \mathbb{R}^n \]

Let

\[ \mathcal{P}(G) = \left\{ A \in \text{Sym}^2(\mathbb{R}^n) : \text{tr} (A|_W) \geq 0 \quad \forall W \in G \right\} \]

**Definition.** \( u \in C^2(\Omega^{\text{open}}) \) is **G-plurisubharmonic** if

\[ \text{tr} (D^2 u|_W) \geq 0 \quad \forall W \in G \]

i.e.,

\[ D^2 u \in \mathcal{P}(G) \]

on \( \Omega \).
Note

\[ u \in C^2(\Omega) \text{ is } \mathcal{G}-\text{psh} \quad \iff \quad u \big|_{\Omega \cap W} \text{ is subharmonic for all affine } \mathcal{G}\text{-planes } W \quad \iff \quad u \big|_M \text{ is subharmonic on every minimal } \mathcal{G}\text{-manifold } M \]
Definition. A function $u \in C^2(\Omega^{\text{open}})$ is **$G$-harmonic** if

$$D^2 u \in \partial \mathcal{P}(G)$$

on $\Omega$.

This means at every point $\exists \, W \in G$ with

$$\text{tr} \left( D^2 u \big|_W \right) = 0.$$
Example 1.

\[ G = [\mathbb{R}^n] = G(n, \mathbb{R}^n) \]

\[ \mathcal{P}(G) = \{ A : \text{tr}(A) \geq 0 \} \]

\[ u \text{ is } G\text{-psh} \iff \text{tr}(D^2 u) = \Delta u \geq 0. \]

\[ u \text{ is } G\text{-harmonic} \iff \text{tr}(D^2 u) = \Delta u = 0. \]

Classical Potential Theory
Example 2.

\[ \mathbf{G} = G(1, \mathbb{R}^n) \]

\[ \mathcal{P}(\mathbf{G}) = \{ A : \langle Av, v \rangle \geq 0 \ \forall \ v \in \mathbb{R}^n \} = \{ A \geq 0 \} \]

\[ u \text{ is } \mathbf{G}\text{-psh} \iff D^2 u \geq 0. \]

The Theory of Convex Functions

\[ u \text{ is } \mathbf{G}\text{-harmonic} \iff D^2 u \geq 0 \text{ and } \det(D^2 u) = 0 \]

The Real Monge-Ampère Equation
Example 3.

\[ \mathbf{G} = G_c(1, \mathbb{C}^n) \subset G(2, \mathbb{R}^{2n}) \]

Note the decomposition

\[ \text{Sym}^2(\mathbb{R}^{2n}) = \text{Herm}^{\text{sym}} \oplus \text{Herm}^{\text{skew}} \]

\[ A = \frac{1}{2}(A - JAJ) + \frac{1}{2}(A + JAJ) = A_c^{\text{sym}} + A_c^{\text{skew}} \]

\[ A_c^{\text{sym}} J = J A_c^{\text{sym}} \quad \text{and} \quad A_c^{\text{skew}} J = -J A_c^{\text{skew}} \]

\[ \mathcal{P}(\mathbf{G}) = \{ A_c^{\text{sym}} \geq 0 \} \]

\[ u \text{ is } \mathbf{G}\text{-psh} \iff (D^2 u)_c^{\text{sym}} \geq 0. \]

The Theory of Plurisubharmonic Functions

\[ u \text{ is } \mathbf{G}\text{-harmonic} \iff (D^2 u)_c^{\text{sym}} \geq 0 \text{ and } \det_c(D^2 u)_c^{\text{sym}} = 0 \]

The Complex Monge-Ampère Equation
Example 4. Calibrations.

\[ \phi \in \Lambda^p \mathbb{R}^n \quad \text{constant coefficient } p\text{-form} \]

is a **calibration** if

\[ \phi|_W \leq \text{dvol}_W \quad \text{for all oriented } p\text{-planes } W \]

We define

\[ \mathcal{G} = \mathcal{G}(\phi) = \{ W : \phi|_W = \text{dvol}_W \} \]
Lagrangian Planes

Here

\[ \mathcal{G} = \text{Lag} \subset G(n, \mathbb{R}^{2n}) \]

the set of Lagrangian \textit{n}-planes in \( \mathbb{C}^n = (\mathbb{R}^{2n}, J) \)

Recall

\( W \) is Lagrangian \iff \( \mathbb{C}^n = W \oplus J(W) \) (orthogonal direct sum)

As before

\[ \mathcal{P}(\text{Lag}) = \left\{ A \in \text{Sym}^2(\mathbb{R}^{2n}) : \text{tr} \left( A \big|_W \right) \geq 0 \ \forall \ W \in \text{Lag} \right\} \]

**Definition.** \( u \in C^2(\Omega) \) is \text{Lag-plurisubharmonic} if

\[ D^2u \in \mathcal{P}(\text{Lag}) \quad \text{on} \ \Omega. \]
$u \in C^2(\Omega)$ is Lag–psh

$\iff u \big|_{W \cap \Omega}$ is subharmonic for all affine Lagrangian planes $W$

$\iff u \big|_M$ is $M$-subharmonic for minimal Lagrangian submanifolds $M$

For the last

$$D^2 u \big|_M = \Delta_M u + H_M u$$
Semi-Continuous $G$-psh functions

We want to extend the notion of $G$-psh to non-differentiable functions. We use the notions from viscosity theory (Crandall, Ishii, Lions, Evans, ... ). For a domain $\Omega \subset \mathbb{R}^n$ we define

$$USC(\Omega) \equiv \{ u : \Omega \to [-\infty, \infty) : u \text{ is upper semi-continuous} \}$$

**Definition.** By a test function for $u \in USC(\Omega)$ at a point $x \in \Omega$ we mean a $C^2$-function $\varphi$ defined near $x$ with

$$u \leq \varphi \quad \text{and} \quad u(x) = \varphi(x).$$

**Definition.** A function $u \in USC(\Omega)$ is Lag-psh if for all $x \in \Omega$ and for each test function $\varphi$ for $u$ at $x$,

$$D^2 \varphi \in \mathcal{P}(\text{Lag})$$

$$\mathcal{P}_{\text{Lag}}(\Omega) = \text{the set of these}.$$
Note

Test functions **may not exist** for \( u \) at some point \( x \in \Omega \).

This is OK, and an important part of the definition.
Remarkable Properties

- \( u, v \in \mathcal{P}_{\text{Lag}}(\Omega) \Rightarrow \max\{u, v\} \in \mathcal{P}_{\text{Lag}}(\Omega) \)
- \( \mathcal{P}_{\text{Lag}}(\Omega) \) is closed under decreasing limits.
- \( \mathcal{P}_{\text{Lag}}(\Omega) \) is closed under uniform limits.
- If \( \mathcal{F} \subset \mathcal{P}_{\text{Lag}}(\Omega) \) is locally uniformly bounded above,

\[
\text{then} \quad U^* \in \mathcal{P}_{\text{Lag}}(\Omega) \quad \text{where}
\]

\[
U(x) \equiv \sup_{u \in \mathcal{F}} u(x) \quad \text{(Perron)}
\]
Also

- \( C^2 \) Lag-psh functions are in \( \mathcal{P}_{\text{Lag}}(\Omega) \).
- \( u \in \mathcal{P}_{\text{Lag}}(\Omega) \) is classically subharmonic on \( \Omega \).

For this note that \( C^n = W \oplus JW \) for \( W \in \text{Lag} \).
Viscosity Lag-Harmonics

**Dual Equation**

\[ \mathcal{P}(\text{Lag}) \equiv -\left( \sim \text{Int} \mathcal{P}(\text{Lag}) \right) = \sim \left( -\text{Int} \mathcal{P}(\text{Lag}) \right) \]

\[ \mathcal{P}(\text{Lag}) = \{ A : \text{tr} \left( A \big|_W \right) \geq 0 \text{ for some } w \in \text{Lag} \} \]

**Definition**  
*u* is **Lag-harmonic** on \( \Omega \) if

\[ u \in \mathcal{P}_{\text{Lag}}(\Omega) \quad \text{and} \quad -u \in \tilde{\mathcal{P}}_{\text{Lag}}(\Omega) \]

**Note that**

\[ \mathcal{P}(\text{LAG}) \cap \left( -\mathcal{P}(\text{Lag}) \right) = \partial \mathcal{P}(\text{Lag}) \]

**Otherwise said:** \( u \) is both a subsolution and a supersolution.
Lag-Convex Domains

**Definition**  
Consider $\Omega \subset \subset \mathbb{C}^n$ with smooth boundary $\partial \Omega$.

Then $\partial \Omega$ is **strictly Lag convex** if every point $x \in \partial \Omega$ has a smooth defining function which is strictly Lag-psh.

Alternatively: if the second fundamental form of $\partial \Omega$ (w.r.t. inner normal) has strictly positive trace on every $W \in \text{Lag}$ which is tangent to $\partial \Omega$. 
The Dirichlet Problem

**Theorem** Let $\Omega \subset \subset \mathbb{C}^n$ have a smooth strictly Lag convex boundary $\partial \Omega$. Then for every $\varphi \in C(\partial \Omega)$ there exists a unique function $u \in C(\Omega)$, with

1. $u \big|_{\Omega}$ Lag-harmonic, and
2. $u \big|_{\partial \Omega} = \varphi$. 

THE LAGRANGIAN MONGE-AMPÈRE OPERATOR
Is There a Polynomial Differential Operator Whose Solutions are Lag-Harmonic?

\[ \text{Sym}^2(\mathbb{R}^{2n}) \text{ decomposes under } U(n): \]

\[ \text{Sym}^2(\mathbb{R}^{2n}) = \text{Herm}^{\text{sym}} \oplus \text{Herm}^{\text{skew}} \]

\[ = (\mathbb{R} \cdot \text{Id}) \oplus \text{Herm}_0^{\text{sym}} \oplus \text{Herm}^{\text{skew}} \]

\[ A = \frac{1}{2}(A - JAJ) + \frac{1}{2}(A + JAJ) = A_c^{\text{sym}} + A_c^{\text{skew}} \]

\[ = \left( \frac{\text{tr}A}{2n} \right) \text{Id} + (A_c^{\text{sym}})_0 + A_c^{\text{skew}} \]

**Basic Fact**

The Lag-Analysis is independent of \( \text{Herm}_0^{\text{sym}} \).
Basic Fact

The Lag-Analysis is independent of \( \text{Herm}_0^{\text{sym}} \).

Proof. Let

\[ E \in \text{Herm}_0^{\text{sym}} \quad \text{and} \quad W \in \text{Lag.} \]

Choose orthonormal basis \( \{e_k\}_k \) for \( W \). Then

\[
\text{tr} \left( E \mid_W \right) = \sum_k \langle Ee_k, e_k \rangle = \frac{1}{2} \left\{ \sum_k \langle Ee_k, e_k \rangle + \sum_k \langle JEe_k, Je_k \rangle \right\} = \frac{1}{2} \left\{ \sum_k \langle Ee_k, e_k \rangle + \sum_k \langle EJe_k, Je_k \rangle \right\} = \frac{1}{2} \text{tr}(E) = 0 \quad \blacksquare
\]
So we see:

\[ A^\text{sym}_C \text{ plays a central role in } \mathbf{C}\text{-psh functions} \]
and the complex Monge-Ampère equation,
but \( A^\text{skew}_C \) is invisible (trace = 0 on complex lines)

\[ A_{\text{Lag}} \equiv \left( \frac{\text{tr}A}{2n} \right) \text{Id} + A^\text{skew}_C \text{ plays a central role in Lag-psh functions} \]
and the Lagrangian Monge-Ampère equation,
but \( (A^\text{sym}_C)_0 \) is invisible (trace = 0 on Lagrangians).
The Lagrangian Monge-Ampère Operator

Suppose

\[ B \in \text{Herm}^{\text{skew}} \quad BJ = -JB \]

\[ B(e) = \lambda e \quad \Rightarrow \quad B(Je) = -\lambda Je \]

\[
\begin{pmatrix}
\lambda_1 \\
-\lambda_1 \\
\vdots \\
\ldots \\
\lambda_n \\
-\lambda_n
\end{pmatrix}
\]

Assume \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_n \)

If \( W \) is Lagrangian, \( \text{tr} \left( B \big|_W \right) \geq -(\lambda_1 + \cdots + \lambda_n) \)
The Lagrangian Monge-Ampère Operator

Suppose

\[ A \in \text{Sym}^2(\mathbb{R}^{2n}) \quad \text{and} \quad W \in \text{Lag} \]

\[ \text{tr} (A|_W) = \text{tr} \left\{ \left( \frac{\text{tr}A}{2n} \text{Id} + A_{\text{skew}}^c \right)|_W \right\} \]

\[ = \frac{\text{tr}A}{2} + \text{tr} (A_{\text{skew}}^c|_W) \]

\[ \geq \mu - (\lambda_1 + \cdots + \lambda_n) \]

where \( \mu \equiv \frac{\text{tr}A}{2} \) and

\[ 0 \leq \lambda_1 \leq \cdots \leq \lambda_n \quad \text{are the non-negative e-values of} \quad A_{\text{skew}}^c. \]
The Lagrangian Monge-Ampère Operator

\[ \text{tr} \left( A \mid_W \right) \geq \mu - (\lambda_1 + \cdots + \lambda_n) \]

\[ \mu \equiv \frac{\text{tr} A}{2} \quad \text{and} \quad 0 \leq \lambda_1 \leq \cdots \leq \lambda_n \quad \text{are e-values of } A_{c_{\text{skew}}}^\text{skew}. \text{ So} \]

\[ \text{tr} \left( A \mid_W \right) \geq 0 \quad \forall \ W \in \text{Lag} \quad \iff \quad \mu - (\lambda_1 + \cdots + \lambda_n) \geq 0 \]

Consider the operator

\[ \text{MA}_{\text{Lag}}(A) \equiv \prod_{\pm \pm \cdots \pm} (\mu \pm \lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_n) \]

This is a polynomial in \( \mu \)

coefficients are symmetric functions of \( \lambda_1^2, \ldots, \lambda_n^2 \)
The Lagrangian Monge-Ampère Operator

\[ \text{MA}_{\text{Lag}}(A) \equiv \prod_{\pm} (\mu \pm \lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_n) \]

- \( \mathcal{P}(\text{Lag}) = \text{Closure}\{\text{MA}_{\text{Lag}}(A) > 0\}\text{Id} \)

- A Lag-Harmonic \( u \) is a \textit{viscosity solution} of

\[ (D^2u)_{\text{Lag}} \geq 0 \quad \text{and} \quad \text{MA}_{\text{Lag}}(D^2u) = 0. \]
An Invariant Definition, I

\[ A \in \text{Sym}^2(\mathbb{R}^{2n}) \quad \text{and} \quad A_{\text{Lag}} = \frac{\text{tr}A}{2n} \text{Id} + A_{\text{C}}^{\text{skew}} \]

\[ D_{A_{\text{Lag}}} \equiv \text{the derivation on} \quad \Lambda^n \mathbb{R}^{2n} \]

\[ M_{A_{\text{Lag}}}(A) = \text{a factor of} \quad \det(D_{A_{\text{Lag}}}) \]
An Invariant Definition, II (Spinors)

\[ A \in \text{Sym}^2(\mathbb{R}^{2n}) \quad \text{and} \quad B = A^{\text{skew}}_c \]

\[ B \cong \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \]

Set \( |B| = \sqrt{B^2} \). Then

\[ |B|J = \begin{pmatrix} \lambda & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} \]

and hence defines an element in \( \Lambda^2 \mathbb{R}^{2n} \):

\[ \mathbf{B} = \sum \lambda_k e_k \wedge Je_k \in \Lambda^2 \mathbb{R}^{2n} \subset \mathbb{C}\ell(\mathbb{R}^{2n}) \]
An Invariant Definition, II (Spinors)

Let $S$ be an irreducible complex representation of $\mathcal{C}l(\mathbb{R}^{2n})$

$$(S = \Lambda^{0,*})$$

Then $B$ acts by Clifford multiplication on $S$ and

$$MA_{\text{Lag}}(A) = \det(\mu \text{Id} + iB)$$
TRANSPLANTATION TO GROMOV MANIFOLDS
Gromov Manifolds

**Definition**

A **Gromov manifold** is a triple $(X, \omega, J)$ where

$(X, \omega)$ is a symplectic manifold and

$J$ is an almost complex structure on $X$ with:

$$\omega(v, w) = \omega(Jv, Jw) \quad \text{and} \quad \omega(Jv, v) > 0$$

The riemannian metric

$$\langle v, w \rangle \equiv \omega(Jv, w) \quad \text{has} \quad \langle Jv, Jw \rangle = \langle v, w \rangle.$$  

By Gromov any compact symplectic manifold admits such a structure.

This structure pushes forward under symplectomorphisms.
Riemannian Hessian

Definition

For $u \in C^\infty(X)$, the **hessian** of $u$ is a section of $\text{Sym}^2(T^*X)$ defined on vector fields $v, w$ by

$$(\text{Hess} f)(v, w) \equiv vw f - (\nabla_v w) f$$

This Hessian gives a canonical splitting of the 2-jet bundle of $X$:

$$J^2(X) = \mathbb{R} \oplus T^*X \oplus \text{Sym}^2(T^*X),$$

Via the metric and $J$

$$\text{Sym}^2(T^*X) = \mathbb{R} \oplus \text{Herm}^\text{sym}_0(TX) \oplus \text{Herm}^\text{skew}(TX)$$

Everything above carries over – including the Lag Monge-Ampère operator.
Lagrangian Pseudoconvexity

∃ Lag analogues of pseudoconvexity and total reality from complex analysis.

For example.

**Definition** The **Lagrangian hull** of a compact subset $K \subset X$ is

$$
\hat{K} \equiv \{ x \in X : u(x) \leq \sup_{K} u \ \forall \text{Lag-psh } u \text{ on } X \}
$$

**Theorem** The following are equivalent.

1) If $K \subset X$, then $\hat{K} \subset X$.

2) There exists a Lag-psh proper exhaustion function $f$ on $X$.

**This defines Lag-pseudoconvexity.**
Freeness

A manifold $M \subset X$ is **Lag-free** if it has no tangent Lagrangian planes
(always true if $\dim(M) < n$)

If $M$ is free, then

$$M_\varepsilon = \{ x : \text{dist}(x, M) < \varepsilon \}$$

is Lag-convex.

In fact, $M$ has a fundamental neighborhood system
of Lag-convex neighborhoods
THE DIRICHLET PROBLEM
THEOREM. Let $\Omega \subset X$ be a Lag-convex domain. Then for every

$$\psi \in C(\overline{\Omega}), \quad \psi \geq 0 \quad \text{and} \quad \varphi \in C(\partial \Omega)$$

there exists a unique

$$H \in C(\overline{\Omega}) \cap \mathcal{P}_{\text{Lag}}(\Omega)$$

with

$$\text{MA}_{\text{Lag}}(H) = \psi, \quad \text{and} \quad H\big|_{\partial \Omega} = \varphi.$$
The (Homog) Dirichlet Problem for Other Branches

Given $A \in \text{Sym}^2(\mathbb{R}^{2n})$, let

$$\Lambda_1 \leq \Lambda_2 \leq \cdots$$

be the ordered eigenvalues of $\mathcal{M}_{\text{Lag}}(A)$, and set

$$\mathcal{P}^k_{\text{Lag}} = \{ A : \Lambda_k(A) \geq 0 \}$$

**THEOREM.** Let $\Omega \subset X$ be a Lag-convex domain. Then for every

$$\varphi \in C(\partial \Omega)$$

there exists a unique

$$H \in C(\overline{\Omega}) \cap \mathcal{P}^k_{\text{Lag}}(\Omega)$$

with

$$\mathcal{M}_{\text{Lag}}(H) = 0, \quad \text{and} \quad H\big|_{\partial \Omega} = \varphi.$$
A FUNDAMENTAL SOLUTION
IN EUCLIDEAN SPACE
Riesz Kernels

The **Riesz characteristic** of the subequation $\mathcal{P}_{\text{Lag}}$ in $\mathbb{C}^n$ is $n$.

The **Riesz kernel**

$$K_n(x) \equiv \begin{cases} -\frac{1}{|x|^{n-2}} & \text{for } n \geq 3 \\ \log|x| & \text{for } n = 2 \end{cases}$$

is **Lag-harmonic** in $\mathbb{C}^n - \{0\}$ and **Lag-psh** across 0.

**THEOREM.**

$$\text{MA}_{\text{Lag}}(K_n)^\alpha = c \delta_0 \quad (c > 0)$$

where $\alpha = \frac{1}{2n-1}$.
This Means:

\[ K_{n,\epsilon}(x) = -\frac{1}{(|x|^2 + \epsilon^2)^{n/2}} \]

is Lag-psh and \( \downarrow K_n(X) \)

and

\[ \text{MA}_{\text{Lag}}(K_{n,\epsilon})^\alpha = \frac{1}{\epsilon^{2n}} \varphi \left( \frac{|x|}{\epsilon} \right) \to c \delta_0 \]

where \( \varphi \geq 0 \) is integrable on \( \mathbb{C}^n \).
QUESTIONS
Questions

1. **Regularity** of the solution to (DP) when $\psi > 0$?
   
   We note that $\text{MA}_{\text{Lag}}$ is not uniformly elliptic.
   
   However its linearization at a solution is elliptic.

2. Is there a **foliation** (generically) attached to a Lag-harmonic?

3. Can one establish useful **capacities** using Lag-harmonics?