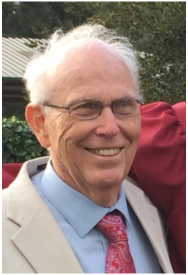
Given With Deep Respect and Affection

for Marcel Berger
who was a father to so many
and one of the great geometers of his century

A MONGE AMPÈRE OPERATOR in SYMPLECTIC GEOMETRY

with Reese Harvey



The Outline

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LAGRANGIAN POTENTIAL THEORY

“Geometrically Based” Potential Theories:

Consider a compact set

$$\mathbb{G} \subset G(p, \mathbf{R}^n) = \text{The Grassmannian of } p\text{-planes in } \mathbf{R}^n$$

Let

$$\mathcal{P}(\mathbb{G}) = \left\{ A \in \mathrm{Sym}^2(\mathbf{R}^n) : \mathrm{tr}\left(A\big|_W\right) \geq 0 \quad \forall\, W \in \mathbb{G} \right\}$$

Definition. $u \in C^2(\Omega^{\mathrm{open}})$ is **$\mathbb{G}$-plurisubharmonic** if

$$\mathrm{tr}\left(D^2 u\big|_W\right) \geq 0 \qquad \forall\, W \in \mathbb{G}$$

i.e.,

$$D^2 u \in \mathcal{P}(\mathbb{G})$$

on Ω.

Note

$$u \in C^2(\Omega) \text{ is } \mathbf{G}\text{-psh} \qquad \iff$$

$$u\big|_{\Omega \cap W} \text{ is subharmonic for all affine } \mathbf{G}\text{-planes } W \qquad \iff$$

$$u\big|_{M} \text{ is subharmonic on every minimal } \mathbf{G}\text{-manifold } M$$

$\mathbb{G}$-Harmonic

Definition. A function $u \in C^2(\Omega^{\text{open}})$ is **$\mathbb{G}$-harmonic** if

$$D^2 u \in \partial\mathcal{P}(\mathbb{G})$$

on Ω.

This means at every point $\exists\, W \in \mathbb{G}$ with

$$\operatorname{tr}\left(D^2 u\big|_W\right) = 0.$$

Example 1.

$$\mathbf{G} \;=\; [\mathbf{R}^n] \;=\; G(n, \mathbf{R}^n)$$

$$\mathcal{P}(\mathbf{G}) \;=\; \{A : \mathrm{tr}(A) \geq 0\}$$

u is $\mathbf{G}$-psh $\quad\Longleftrightarrow\quad \mathrm{tr}\,(D^2 u) \;=\; \Delta u \;\geq\; 0.$

u is $\mathbf{G}$-harmonic $\quad\Longleftrightarrow\quad \mathrm{tr}\,(D^2 u) \;=\; \Delta u \;=\; 0.$

Classical Potential Theory

Example 2.

$$\mathbf{G} = G(1, \mathbf{R}^n)$$

$$\mathcal{P}(\mathbf{G}) = \{A : \langle Av, v\rangle \geq 0 \ \ \forall\, v \in \mathbf{R}^n\} = \{A \geq 0\}$$

u is $\mathbf{G}$-psh $\iff$ $D^2 u \geq 0.$

The Theory of Convex Funtions

u is $\mathbf{G}$-harmonic $\iff$ $D^2 u \geq 0$ and $\det(D^2 u) = 0$

The Real Monge-Ampère Equation

Example 3.

$$\mathbf{G} = G_{\mathbf{C}}(1, \mathbf{C}^n) \subset G(2, \mathbf{R}^{2n})$$

Note the decomposition

$$\mathrm{Sym}^2(\mathbf{R}^{2n}) = \mathrm{Herm}^{\mathrm{sym}} \oplus \mathrm{Herm}^{\mathrm{skew}}$$

$$A = \tfrac{1}{2}(A - JAJ) + \tfrac{1}{2}(A + JAJ) = A_{\mathbf{C}}^{\mathrm{sym}} + A_{\mathbf{C}}^{\mathrm{skew}}$$

$$A_{\mathbf{C}}^{\mathrm{sym}} J = J A_{\mathbf{C}}^{\mathrm{sym}} \quad \text{and} \quad A_{\mathbf{C}}^{\mathrm{skew}} J = -J A_{\mathbf{C}}^{\mathrm{skew}}$$

$$\mathcal{P}(\mathbf{G}) = \{A_{\mathbf{C}}^{\mathrm{sym}} \geq 0\}$$

$$u \text{ is } \mathbf{G}\text{-psh} \iff (D^2 u)_{\mathbf{C}}^{\mathrm{sym}} \geq 0.$$

The Theory of Plurisubharmonic Functions

$$u \text{ is } \mathbf{G}\text{-harmonic} \iff (D^2 u)_{\mathbf{C}}^{\mathrm{sym}} \geq 0 \text{ and } \det{}_{\mathbf{C}}(D^2 u)_{\mathbf{C}}^{\mathrm{sym}} = 0$$

The Complex Monge-Ampère Equation

Example 4. Calibrations.

$$\phi \in \Lambda^p \mathbf{R}^n \qquad \text{constant coefficient } p\text{-form}$$

is a **calibration** if

$$\phi\big|_W \leq \mathrm{dvol}_W \qquad \text{for all oriented } p\text{-planes } W$$

We define

$$\mathbb{G} = \mathbb{G}(\phi) = \left\{ W : \phi\big|_W = \mathrm{dvol}_W \right\}$$

Lagrangian Planes

Here

$$\mathbf{G} = \mathrm{Lag} \subset G(n, \mathbf{R}^{2n})$$

the set of **Lagrangian n-planes** in $\mathbf{C}^n = (\mathbf{R}^{2n}, J)$

Recall

$$W \text{ is Lagrangian} \iff \mathbf{C}^n = W \oplus J(W) \text{ (orthogonal direct sum)}$$

As before

$$\mathcal{P}(\mathrm{Lag}) = \left\{ A \in \mathrm{Sym}^2(\mathbf{R}^{2n}) : \mathrm{tr}\left(A\big|_W\right) \geq 0 \ \forall\, W \in \mathrm{Lag} \right\}$$

Definition. $u \in C^2(\Omega)$ is **Lag-plurisubharmonic** if

$$D^2 u \in \mathcal{P}(\mathrm{Lag}) \qquad \text{on } \Omega.$$

$$
\begin{aligned}
& u \in C^2(\Omega) \quad \textit{is } \textbf{Lag} - \textbf{psh} \\
\iff \quad & u\big|_{W\cap\Omega} \text{ is subharmonic for all affine Lagrangian planes } W \\
\iff \quad & u\big|_{M} \text{ is } M\text{-subharmonic for minimal Lagrangian submanifolds } M
\end{aligned}
$$

For the last

$$
D^2 u\big|_M = \Delta_M u + H_M u
$$

Semi-Continuous G-psh functions

We want to extend the notion of G-psh to non-differentiable functions.

We use the notions from viscosity theory (Crandall, Ishii, Lions, Evans, ...).

For a domain $\Omega \subset \mathbf{R}^n$ we define

$$\mathrm{USC}(\Omega) \;\equiv\; \{u : \Omega \to [-\infty, \infty) : \; u \text{ is upper semi-continuous}\}$$

Definition. By a **test function** for $u \in \mathrm{USC}(\Omega)$ at a point $x \in \Omega$ we mean a C^2-function φ defined near x with

$$u \;\leq\; \varphi \qquad \text{and} \qquad u(x) \;=\; \varphi(x).$$

Definition. A function $u \in \mathrm{USC}(\Omega)$ is **Lag-psh** if for all $x \in \Omega$ and for each test function φ for u at x,

$$D^2\varphi \in \mathcal{P}(\mathrm{Lag})$$

$\mathcal{P}_{\mathrm{Lag}}(\Omega)$ = the set of these.

Note

Test functions **may not exist** for u at some point $x \in \Omega$

This is OK, and an important part of the definition.

Remarkable Properties

- $u, v \in \mathcal{P}_{\text{Lag}}(\Omega) \quad \Rightarrow \quad \max\{u, v\} \in \mathcal{P}_{\text{Lag}}(\Omega)$
- $\mathcal{P}_{\text{Lag}}(\Omega)$ is closed under decreasing limits.
- $\mathcal{P}_{\text{Lag}}(\Omega)$ is closed under uniform limits.
- If $\mathcal{F} \subset \mathcal{P}_{\text{Lag}}(\Omega)$ is locally uniformly bounded above,

$$\text{then} \quad U^* \in \mathcal{P}_{\text{Lag}}(\Omega) \quad \text{where}$$

$$U(x) \equiv \sup_{u \in \mathcal{F}} u(x)$$

(Perron)

Also

- C^2 Lag-psh functions are in $\mathcal{P}_{\rm Lag}(\Omega)$.
- $u \in \mathcal{P}_{\rm Lag}(\Omega)$ is classically subharmonic on Ω

For this note that $\mathbf{C}^n = W \oplus JW$ for $W \in \rm Lag$

Viscosity Lag-Harmonics

Dual Equation

$$\widetilde{\mathcal{P}(\mathrm{Lag})} \equiv -(\sim \mathrm{Int}\mathcal{P}(\mathrm{Lag})) = \sim (-\mathrm{Int}\mathcal{P}(\mathrm{Lag}))$$

$$\widetilde{\mathcal{P}(\mathrm{Lag})} = \left\{A : \mathrm{tr}\left(A\big|_W\right) \geq 0 \text{ for some } W \in \mathrm{Lag}\right\}$$

Definition u is **Lag-harmonic** on Ω if

$$u \in \mathcal{P}_{\mathrm{Lag}}(\Omega) \qquad \text{and} \qquad -u \in \widetilde{\mathcal{P}}_{\mathrm{Lag}}(\Omega)$$

Note that $\mathcal{P}(\mathrm{LAG}) \cap (-\widetilde{\mathcal{P}(\mathrm{Lag})}) = \partial\mathcal{P}(\mathrm{Lag})$

Otherwise said: u is both a subsolution and a supersolution.

Lag-Convex Domains

Definition Consider $\Omega \subset\subset \mathbf{C}^n$ with smooth boundary $\partial\Omega$.

Then $\partial\Omega$ is **strictly Lag convex** if every point $x \in \partial\Omega$

has a smooth defining function which is strictly Lag-psh.

Alternatively: if the second fundamental form of $\partial\Omega$ (w.r.t. inner normal)

has strictly positive trace on every $W \in \mathrm{Lag}$ which is tangent to $\partial\Omega$.

The Dirichlet Problem

Theorem Let $\Omega \subset\subset \mathbf{C}^n$ have a smooth strictly Lag convex boundary $\partial\Omega$.

Then for every $\varphi \in C(\partial\Omega)$

there exists a unique function $u \in C(\overline{\Omega})$, with

(1) $u\big|_{\Omega}$ Lag-harmonic, and

(2) $u\big|_{\partial\Omega} = \varphi$.

THE LAGRANGIAN MONGE-AMPÈRE OPERATOR

Is There a Polynomial Differential Operator Whose Solutions are Lag-Harmonic?

$\mathrm{Sym}^2(\mathbf{R}^{2n})$ decomposes under $\mathrm{U}(n)$:

$$\begin{aligned}\mathrm{Sym}^2(\mathbf{R}^{2n}) &= \mathrm{Herm}^{\mathrm{sym}} \oplus \mathrm{Herm}^{\mathrm{skew}} \\ &= (\mathbf{R}\cdot \mathrm{Id}) \oplus \mathrm{Herm}_0^{\mathrm{sym}} \oplus \mathrm{Herm}^{\mathrm{skew}}\end{aligned}$$

$$\begin{aligned}A &= \tfrac{1}{2}(A - JAJ) + \tfrac{1}{2}(A + JAJ) = A_{\mathbf{C}}^{\mathrm{sym}} + A_{\mathbf{C}}^{\mathrm{skew}} \\ &= \left(\frac{\mathrm{tr}A}{2n}\right)\mathrm{Id} + (A_{\mathbf{C}}^{\mathrm{sym}})_0 + A_{\mathbf{C}}^{\mathrm{skew}}\end{aligned}$$

Basic Fact

The Lag-Analysis is independent of $\mathrm{Herm}_0^{\mathrm{sym}}$.

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The Lag-Analysis is independent of $\mathrm{Herm}_0^{\mathrm{sym}}$.

Proof. Let

$$E \in \mathrm{Herm}_0^{\mathrm{sym}} \quad \text{and} \quad W \in \mathrm{Lag}.$$

Choose orthonormal basis $\{e_k\}_k$ for W. Then

$$\mathrm{tr}\left(E\big|_W\right) = \sum_k \langle Ee_k, e_k\rangle = \frac{1}{2}\left\{\sum_k \langle Ee_k, e_k\rangle + \sum_k \langle JEe_k, Je_k\rangle\right\}$$
$$= \frac{1}{2}\left\{\sum_k \langle Ee_k, e_k\rangle + \sum_k \langle EJe_k, Je_k\rangle\right\}$$
$$= \frac{1}{2}\mathrm{tr}(E) = 0 \qquad \blacksquare$$

So we see:

$A^{\text{sym}}_{\mathbf{C}}$ plays a central role in **C**-psh functions
and the complex Monge-Ampère equation,
but $A^{\text{skew}}_{\mathbf{C}}$ is invisible (trace = 0 on complex lines)

$A_{\text{Lag}} \equiv \left(\frac{\text{tr}A}{2n}\right) \text{Id} + A^{\text{skew}}_{\mathbf{C}}$ plays a central role in Lag-psh functions
and the Lagrangian Monge-Ampère equation,
but $\left(A^{\text{sym}}_{\mathbf{C}}\right)_0$ is invisible (trace = 0 on Lagrangians).

The Lagrangian Monge-Ampère Operator

Suppose

$$B \in \mathrm{Herm}^{\mathrm{skew}} \qquad BJ = -JB$$

$$B(e) = \lambda e \quad \Rightarrow \quad B(Je) = -\lambda Je$$

$$B \cong \begin{pmatrix} \lambda_1 & & & & & & \\ & -\lambda_1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \lambda_n & \\ & & & & & & -\lambda_n \end{pmatrix}$$

Assume $0 \le \lambda_1 \le \cdots \le \lambda_n$

If W is Lagrangian, $\quad \mathrm{tr}\left(B\big|_W\right) \ge -(\lambda_1 + \cdots + \lambda_n)$

The Lagrangian Monge-Ampère Operator

Suppose

$$A \in \mathrm{Sym}^2(\mathbf{R}^{2n}) \qquad \text{and} \qquad W \in \mathrm{Lag}$$

$$\begin{aligned}
\mathrm{tr}\left(A\big|_W\right) &= \mathrm{tr}\left\{\left(\frac{\mathrm{tr}A}{2n}\,\mathrm{Id} \;+\; A_{\mathbf{C}}^{\mathrm{skew}}\right)\Big|_W\right\} \\
&= \frac{\mathrm{tr}A}{2} \;+\; \mathrm{tr}\left(A_{\mathbf{C}}^{\mathrm{skew}}\big|_W\right) \\
&\geq \mu - (\lambda_1 + \cdots + \lambda_n)
\end{aligned}$$

where $\qquad \mu \equiv \frac{\mathrm{tr}A}{2} \qquad$ and

$0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ are the non-negative e-values of $A_{\mathbf{C}}^{\mathrm{skew}}$.

The Lagrangian Monge-Ampère Operator

$$\mathrm{tr}\left(A\big|_W\right) \;\geq\; \mu - (\lambda_1 + \cdots + \lambda_n)$$

$\mu \equiv \frac{\mathrm{tr}A}{2}$ and $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ are e-values of $A_{\mathbf{C}}^{\mathrm{skew}}$. So

$$\mathrm{tr}\left(\mathbf{A}\big|_{\mathbf{W}}\right) \;\geq\; \mathbf{0} \quad \forall\, \mathbf{W} \in \mathrm{Lag} \qquad \Longleftrightarrow \qquad \mu - (\lambda_1 + \cdots + \lambda_n) \;\geq\; \mathbf{0}$$

Consider the operator

$$\mathrm{MA}_{\mathrm{Lag}}(\mathbf{A}) \;\equiv\; \prod_{\pm\pm\cdots\pm} (\mu \pm \lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_n)$$

This is a polynomial in μ

coefficients are symmetric functions of $\lambda_1^2, ..., \lambda_n^2$

The Lagrangian Monge-Ampère Operator

$$\mathrm{MA_{Lag}}(\mathbf{A}) \equiv \prod_{\pm\pm\cdots\pm} (\mu \pm \lambda_{\mathbf{1}} \pm \lambda_{\mathbf{2}} \pm \cdots \pm \lambda_{\mathbf{n}})$$

- $\mathcal{P}(\mathrm{Lag}) = \mathrm{Closure}\{\mathrm{MA_{Lag}}(A) > 0\}_{\mathrm{Id}}$

- A Lag-Harmonic u is a **viscosity solution** of

$$(D^2u)_{\mathrm{Lag}} \geq 0 \qquad \text{and} \qquad \mathrm{MA_{Lag}}(D^2u) = 0.$$

An Invariant Definition, I

$$A \in \mathrm{Sym}^2(\mathbf{R}^{2n}) \qquad \text{and} \qquad A_{\mathrm{Lag}} = \frac{\mathrm{tr}A}{2n}\,\mathrm{Id} + A_{\mathbf{C}}^{\mathrm{skew}}$$

$$D_{A_{\mathrm{Lag}}} \equiv \text{ the derivation on } \Lambda^n \mathbf{R}^{2n}$$

$$\mathrm{MA}_{\mathrm{Lag}}(A) = \textbf{ a factor of } \det\left(D_{A_{\mathrm{Lag}}}\right)$$

An Invariant Definition, II (Spinors)

$$A \in \mathrm{Sym}^2(\mathbf{R}^{2n}) \qquad \text{and} \qquad B = A^{\text{skew}}_{\mathbf{C}}$$

$$B \cong \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

Set $|B| = \sqrt{B^2}$. Then

$$|B|J = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}$$

and hence defines an element in $\Lambda^2\mathbf{R}^{2n}$:

$$\mathbf{B} = \sum \lambda_k e_k \wedge Je_k \in \Lambda^2\mathbf{R}^{2n} \subset C\ell(\mathbf{R}^{2n})$$

An Invariant Definition, II (Spinors)

Let S be an irreducible complex representation of $C\ell(\mathbf{R}^{2n})$

$(S = \Lambda^{0,*})$

Then $\mathbf{B}$ acts by Clifford multiplication on S and

$$\mathrm{MA}_{\mathrm{Lag}}(A) = \det(\mu \,\mathrm{Id} + i\mathbf{B})$$

TRANSPLANTATION TO GROMOV MANIFOLDS

Gromov Manifolds

Definition

A **Gromov manifold** is a triple (X, ω, J) where

(X, ω) is a symplectic manifold and

J is an almost complex structure on X with:

$$\omega(v, w) \;=\; \omega(Jv, Jw) \qquad \text{and} \qquad \omega(Jv, v) \;>\; 0$$

The riemannian metric

$$\langle v, w\rangle \;\equiv\; \omega(Jv, w) \qquad \text{has} \quad \langle Jv, Jw\rangle \;=\; \langle v, w\rangle.$$

By Gromov any compact symplectic manifold admits such a structure.

This structure pushes forward under symplectomorphisms.

Riemannian Hessian

Definition

For $u \in C^\infty(X)$, the **hessian** of u is a section of $\mathrm{Sym}^2(T^*X)$ defined on vector fields v, w by

$$(\mathrm{Hess} f)(v, w) \ \equiv \ vwf - (\nabla_v w)f$$

This Hessian gives a canonical splitting of the 2-jet bundle of X:

$$J^2(X) \ = \ \mathbf{R} \oplus T^*X \oplus \mathrm{Sym}^2(T^*X),$$

Via the metric and J

$$\mathrm{Sym}^2(T^*X) \ = \ \mathbf{R} \oplus \mathrm{Herm}_0^{\mathrm{sym}}(TX) \oplus \mathrm{Herm}^{\mathrm{skew}}(TX)$$

Everything above carries over –

including the Lag Monge-Ampère operator.

Lagrangian Pseudoconvexity

$\exists$ Lag analogues of pseudoconvexity and total reality from complex analysis.

For example.

Definition The **Lagrangian hull** of a compact subset $K \subset X$ is

$$\widehat{K} \equiv \{x \in X : u(x) \leq \sup_K u \ \ \forall \text{ Lag-psh } u \text{ on } X\}$$

Theorem The following are equivalent.

1) If $K \subset\subset X$, then $\widehat{K} \subset\subset X$.

2) There exists a Lag-psh proper exhaustion function f on X.

This defines Lag-pseudoconvixity.

Freeness

A manifold $M \subset X$ is **Lag-free** if it has no tangent Lagrangian planes (always true if $\dim(M) < n$)

If M is free, then

$$M_\epsilon = \{x : \mathrm{dist}(x, M) < \epsilon\} \quad \text{is Lag-convex.}$$

In fact, M has a fundamental neighborhood system of Lag-convex neighborhoods

THE DIRICHLET PROBLEM

The Inhomogeneous Dirichlet Problem

THEOREM. Let $\Omega \subset X$ be a Lag-convex domain. Then for every

$$\psi \in C(\overline{\Omega}), \ \ \psi \geq 0 \qquad \text{and} \qquad \varphi \in C(\partial\Omega)$$

there exists a unique

$$H \in C(\overline{\Omega}) \cap \mathcal{P}_{\text{Lag}}(\Omega)$$

with

$$\text{MA}_{\text{Lag}}(H) \ = \ \psi, \qquad \text{and}$$
$$H\big|_{\partial\Omega} \ = \ \varphi.$$

The (Homog) Dirichlet Problem for Other Branches

Given $A \in \mathrm{Sym}^2(\mathbf{R}^{2n})$, let

$$\Lambda_1 \le \Lambda_2 \le \cdots$$

be the ordered eigenvalues of $\mathrm{MA}_{\mathrm{Lag}}(A)$, and set

$$\mathcal{P}^k_{\mathrm{Lag}} \;=\; \{A : \Lambda_k(A) \ge 0\}$$

THEOREM. Let $\Omega \subset X$ be a Lag-convex domain. Then for every

$$\varphi \in C(\partial\Omega)$$

there exists a unique

$$H \in C(\overline{\Omega}) \cap \mathcal{P}^k_{\mathrm{Lag}}(\Omega)$$

with

$$\mathrm{MA}_{\mathrm{Lag}}(H) \;=\; 0, \qquad \text{and} \qquad H\big|_{\partial\Omega} \;=\; \varphi.$$

A FUNDAMENTAL SOLUTION IN EUCLIDEAN SPACE

Riesz Kernels

The **Riesz charateristic** of the subequation $\mathcal{P}_{\rm Lag}$ in $\mathbf{C}^n$ is n.

The **Riesz kernel**

$$K_n(x) \equiv \begin{cases} -\frac{1}{|x|^{n-2}} & \text{for } n \geq 3 \\ \log|x| & \text{for } n = 2 \end{cases}$$

is **Lag-harmonic** in $\mathbf{C}^n - \{0\}$ and **Lag-psh** across 0.

THEOREM.

$$\mathrm{MA}_{\rm Lag}(\mathbf{K_n})^\alpha = \mathbf{c}\,\delta_{\mathbf{0}} \qquad (\mathbf{c} > \mathbf{0})$$

where $\alpha = \frac{1}{2^{n-1}}$

.

This Means:

$$K_{n,\epsilon}(x) \;=\; -\frac{1}{(|x|^2+\epsilon^2)^{\frac{n}{2}}} \qquad \text{is Lag-psh and} \downarrow K_n(X)$$

and

$$\mathrm{MA}_{\mathrm{Lag}}(K_{n,\epsilon})^{\alpha} \;=\; \frac{1}{\epsilon^{2n}}\varphi\left(\frac{|x|}{\epsilon}\right) \;\rightarrow\; c\,\delta_0$$

where $\varphi \geq 0$ is integrable on $\mathbf{C}^n$.

QUESTIONS

Questions

1. **Regularity** of the solution to (DP) when $\psi > 0$?

   We note that $\mathrm{MA}_{\mathrm{Lag}}$ is not uniformly elliptic.

   However its linearization at a solution is elliptic.

2. Is there a **foliation** (generically) attached to a Lag-harmonic?

3. Can one establish useful **capacities** using Lag-harmonics?