

CHRISTOPHER J. BISHOP

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RESEARCH INTERESTS

Real and complex analysis, geometric function theory, conformal dynamics, probability theory, numerical analysis, analysis on fractals, quasiconformal geometry, computational geometry.

Some of my more particular interests have included: potential theory, fractal properties of harmonic measure, geometric properties of Brownian motion and other random processes, algebras generated by harmonic and holomorphic functions, geometry of hyperbolic manifolds and their covering groups, numerical computation of conformal mappings, multipole methods, optimal meshing algorithms, nonobtuse triangulation, quadrilateral meshing, iteration of entire functions and dimension distorting properties of quasiconformal maps. I program in C, Mathematica and Matlab as part of my investigations in these areas, and teach a class on experimental aspects of mathematics.

PROFESSIONAL EXPERIENCE

Sept. 1997 to present: Professor at Stony Brook University.
Sept. 1992 to Aug. 1997: Assoc. professor at Stony Brook.
Sept. 1991 to Aug. 1992: Asst. professor at Stony Brook.
Sept. 1988 to Aug. 1991: Hedrick Asst. professor at UCLA.
Sept. 1987 to Aug. 1988: NSF postdoc at MSRI, Berkeley.

EDUCATION

Cambridge University, Master of Advanced Study (MASt), 2011.
University of Chicago, Mathematics, Ph. D., 1987, Advisor Peter W. Jones.
Visiting graduate student and programmer, Dept. of Mathematics, Yale University, 1985-1987.
University of Chicago, Mathematics, Master of Science, 1984.
Cambridge University, Certificate of Advanced Study (Part III of Math. Tripos), 1983.
Michigan State University, Mathematics, Bachelor of Science, 1982.

PH.D. STUDENTS

Zsuzsanna Gonye, Ph. D. 2001, Geodesics in hyperbolic manifolds, Assoc. Professor, University of West Hungary

Karyn Lundberg, Ph. D. 2005, Boundary behavior of conformal mappings, researcher, Lincoln Labs

Hrant Hakobyan, Ph.D. 2007, Hausdorff dimension and quasimetric mappings, Assoc. Professor, Kansas State University

Chris Green, Ph.D. Dec. 2011, Ahlfors iteration for numerical conformal mapping, senior researcher at Twitter

Kirill Lazebnik, Ph.D. August 2017, Wandering domains for the Eremenko-Lyubich class Bate-man instructor at Cal Tech

Jack Burkart, Ph.D. expected May 2020, Packing dimension of transcendental Julia sets

POSTDOCS AND VISITORS

Maria Victoria Meliáan, 1994-1995

Torbjorn Lundh, 1996-1998

Jeremy Tyson, 1999-2002

Luke Rogers, Spring 2004

Mathew Badger, 2011-2014

Yaar Solomon, 2013-2016

Malik Younsi, 2014-2016

Simon Albrecht, Spring 2016

Dimitrios Ntalampekos, 2018-2021

Peter Lin, 2019-2022

Christina Karafyllia, 2019-2021

RECENT UNDERGRADUATE THESES AND RESEARCH

Ahmad Rafiqi, Summer 2011, Estimating conformal maps with accelerated random walks, Ph.D. program at Cornell.

Kevin Sackel, 2012-13, Geometric problems related to removability, Churchill Scholar at University of Cambridge, Ph.D. program MIT, postdoc at Stony Brook starting Fall 2019.

Abraham Rabinowitz, 2013-2014, Real Analysis, Ph.D. program at Northwestern.

Luke Green 2012-13, Real Analysis, data scientist for Openslate.

Jackson Rudd, Spring 2013, Functional Analysis, data intern for Disney/ESPN.

Shalin Parekh, 2014-15, Small dimension paths in the Brownian trace, one of ten students accepted special year in probability at University of Geneva run by Smirnov and Werner, currently in Ph.D. program at Columbia.

Christopher Dular, Fall 2015, Refining a triangulation to nonobtuse triangulation, masters program Georgia Tech.

William Vickery, Fall 2016, Generating random permutations using transpositions, Ph.D. program Northwestern.

Ray Zhang, Fall 2017, Functional analysis and machine learning, finance industry.

Joseph Suk, Fall 2017, Numerical approximation of shapes by dessins d'enfants, attending Carnegie-Mellon Ph.D. program.

Yugarshi Mondal, Spring 2018, Schramm-Loewner Evolutions.

Emi Brawley, Fall 2018, Keakeya sets, attending UC Davis PhD program in Fall 2019.

Hindy Drillick, Spring 2019, Removable sets and dimension, attending Columbia Ph.D. program in Fall 2019.

RECENT SERVICE

Department

Professional development seminar.
Search committee.
Graduate committee.
Hiring committee.
Simons lecture committee.
Math day lecturer/organizer.
Comprehensive exam writer/grader.

University

Organizer SCGP Spring School in Discrete and Computational Geometry, April 2017.
SBU Center for Finance committee.
Search committee for Frey chair in quantitative finance.
Mentor for senior honors theses.

Outside University

Reviewer for various journals.
NSF panelist and reviewer.
Math Reviews Reviewer.
Lecturer for Center for Talented Youth Program.
Organizer for Special Section at AMS sectional meeting, Spring 2016.
Organizer for International Research Symposium, KIAS Seoul South Korea, April 2017.
Outside hiring referee, Jyväskylä University.
Outside dissertation referee, Jyväskylä University.
Outside dissertation referee, Tel Aviv university.
Various promotion evaluations and letters.

AWARDS

Simons Fellow, 2019-20.

Fellow of the American Mathematical Society, 2019.

Invited speaker at 2018 ICM, Rio de Janeiro.

NSF standard grants (1991-present): DMS 91-00671, DMS 92-04092, DMS 95-00577, DMS 98-00924, DMS 01-03626, DMS 04-05578, DMS 07-05455, DMS 10-06309, DMS 13-05233, DMS 16-08577, DMS 19-06259. Recent proposal reviews available on personal website.

1992 Alfred P. Sloan Research Fellow

1987-91 NSF Postdoctoral Fellowship

1983-1986 McCormick Fellowship, NSF Fellowship, U. of Chicago

1982-1983 Churchill Fellowship, Cambridge England.

PUBLICATIONS

- [1] C. J. Bishop and J. K. Wetterer Planktivore prey selection: the relative field volume model vs. the apparent size model. *Ecology*, 66(2):457, 1985.

We compare two models for how fish choose what to eat. We show the predictions of the two models agree in deep water (fish select from a sphere), but differ in shallow water (fish select from a truncated sphere), and report on an experiment to distinguish the cases.

- [2] C. J. Bishop. A counterexample in conformal welding concerning Hausdorff dimension. *Michigan Math. J.*, 35(1):151–159, 1988.

I construct an example of a closed Jordan curve with the property that harmonic measure (i.e., first hitting distribution of Brownian motion) for the two sides are comparable for every subset, but the curve has Hausdorff dimension > 1 . This showed that even strong conditions on harmonic measure could not be used to characterize rectifiable curves.

- [3] C. J. Bishop. An element of the disk-algebra that is stationary on a set of positive length. *Algebra i Analiz*, 1(3):83–88, 1989.

This answered a question of Havin and Makarov, by constructing an example of a non-constant analytic function on the unit disk whose derivative extends to be zero on a positive length subset of the unit circle.

- [4] C. J. Bishop. Constructing continuous functions holomorphic off a curve. *J. Funct. Anal.*, 82(1):113–137, 1989.

This gives an explicit construction of certain space filling curves that arise naturally in function theory (they were previously known to exist by a result of Browder and Wermer using an indirect Hahn-Banach argument, but no one had “seen” one before).

- [5] C. J. Bishop. Approximating continuous functions by holomorphic and harmonic functions. *Trans. Amer. Math. Soc.*, 311(2):781–811, 1989.

Various results are proven about algebras generated by harmonic functions on plane domains which generalize well known results on the unit disk. For example, the closed algebra

generated by the bounded holomorphic functions and the complex conjugate of a single non-constant holomorphic function contains every uniformly continuous function on the domain.

- [6] C. J. Bishop, L. Carleson, J. B. Garnett, and P. W. Jones. Harmonic measures supported on curves. *Pacific J. Math.*, 138(2):233–236, 1989.

We geometrically characterize when the harmonic measure from two opposite sides of a closed curve are either mutually singular or mutually continuous. For example, if we start Brownian motions on opposite sides of fractal curve such as the von Koch snowflake, there are two disjoint sets which absorb the paths, i.e., the two hitting measures are mutually singular. I am fond of this paper because it gives me a co-authorship with Lennart Carleson.

- [7] C. J. Bishop. Bounded functions in the little Bloch space. *Pacific J. Math.*, 142(2):209–225, 1990.

Sarason gave a non-constructive proof that there exist Blaschke products in the little Bloch space (a class of holomorphic functions on the disk whose derivatives grow slowly near the boundary) but no explicit example was known. This paper gives such an example and characterizes all such examples in terms of their zero sets. This result has been extended to various spaces by others.

- [8] C. J. Bishop. Conformal welding of rectifiable curves. *Math. Scand.*, 67(1):61–72, 1990.

This answers a question of Walter Hayman. It constructs an example of a rectifiable curve that has a certain pathological property with respect to harmonic measure.

- [9] C. J. Bishop and P. W. Jones. Harmonic measure and arclength. *Ann. of Math. (2)*, 132(3):511–547, 1990.

This paper proves a conjecture of Øksendal that has various consequence in geometric function theory. We show that a set of zero length which lies on a rectifiable curve has zero harmonic measure in any simply connected domain.

- [10] C. J. Bishop and T. Steger. Three rigidity criteria for $\mathrm{PSL}(2, \mathbf{R})$. *Bull. Amer. Math. Soc. (N.S.)*, 24(1):117–123, 1991.

A summary of my Acta. Math. paper with Steger. See [14] below.

- [11] C. J. Bishop. A characterization of Poissonian domains. *Ark. Mat.*, 29(1):1–24, 1991.

A domain is called Poissonian if every bounded harmonic function on the domain is the Poisson extension of a bounded function on the boundary. This paper provides a characterization of such domains in terms of the geometry of the boundary. It also proves some bounds on the complexity of the Martin boundary of any Euclidean domain. Conjectures made in this paper led to work of Tsirelson on the impossibility of stochastic processes with certain symmetries. Some of the result of this paper have been generalized by Azzam, Mourougolou, and Tolsa.

- [12] C. J. Bishop. Brownian motion in Denjoy domains. *Ann. Probab.*, 20(2):631–651, 1992.

This considers a Cauchy process on the real line (which can be obtained from a Brownian motion in the plane restricted to the times when it is on the line). We show that for sets E of

positive length, such a process has a well defined “direction of approach” to its first hitting of E . This answers a question of Chris Burdzy.

- [13] C. J. Bishop. Some questions concerning harmonic measure. In *Partial differential equations with minimal smoothness and applications (Chicago, IL, 1990)*, volume 42 of *IMA Vol. Math. Appl.*, pages 89–97. Springer, New York, 1992.

A collection of open problems. One has been solved by Sunhi Choi, another by Azzam, Hofmann, Martell, Mayboroda, Mourougolou, Tolsa and Volberg.

- [14] C. J. Bishop and T. Steger. Representation-theoretic rigidity in $PSL(2, \mathbf{R})$. *Acta Math.*, 170(1):121–149, 1993.

Mostow rigidity implies that any two lattice inclusions of an abstract group into a connected simple Lie group are equivalent, if that group is not $PSL(2, R)$ (a better known consequence is that the geometry of a 3-dimensional hyperbolic manifold is determined by the fundamental group). This famous result fails for $PSL(2, R)$ because a surface can carry many non-equivalent Riemann surface structures. However, we show that a version of Mostow’s theorem is still true: two representations of $PSL(2, R)$ restricted to two lattices are equivalent iff they are the same representation and the two lattices are conjugate.

- [15] C. J. Bishop. An indestructible Blaschke product in the little Bloch space. *Publ. Mat.*, 37(1):95–109, 1993.

Another explicit construction of a holomorphic function with odd properties.

- [16] C. J. Bishop. How geodesics approach the boundary in a simply connected domain. *J. Anal. Math.*, 64:291–325, 1994.

This answers a question of Chris Burdzy about the geometric properties of a hyperbolic geodesics in a plane domain. In particular, if a geodesics passes within a small Euclidean distance of a boundary point, does it have to hit the boundary near this point? If not, how far away can the hitting point be on average? The most amusing aspect of this paper is that the answer involves the convergence or divergence of an integral of the form $\int_0^1 f(t)^{2/9} dt/t$.

- [17] C. J. Bishop and P. W. Jones. Harmonic measure, L^2 estimates and the Schwarzian derivative. *J. Anal. Math.*, 62:77–113, 1994.

This simplifies and extends results from [8] and is part of a broader program to use “ L^2 techniques” on sets rather than functions. Peter and I develop an analog of Littlewood-Paley theory based on the Schwarzian derivative (in place of the usual derivative) and use it to prove a number of conjectures from function theory, including a geometric characterization of “BMO domains” and an a.e. characterization of tangent points of a curve. Surprisingly, this paper is now cited in various papers dealing with large data sets because of a lemma that characterizes when a set in the plane can be approximated by a curve of finite length. This result was later generalized by others to estimate how well a set of points in a high dimensional space can be approximated by a lower dimensional (but not necessarily linear) submanifold.

- [18] C. J. Bishop. Some homeomorphisms of the sphere conformal off a curve. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 19(2):323–338, 1994.

This gives some examples of some unusual homeomorphisms of the 2-sphere to itself that are conformal except on a small set (dimension 1).

- [19] C. J. Bishop. A counterexample concerning smooth approximation. *Proc. Amer. Math. Soc.*, 124(10):3131–3134, 1996.

This answers a question of Stegenga and Smith about approximations in Sobolev spaces.

- [20] C. J. Bishop. A distance formula for algebras on the disk. *Pacific J. Math.*, 174(1):1–27, 1996.

This gives a characterization of certain algebras of functions on the unit disk which are generated by harmonic and holomorphic functions.

- [21] C. J. Bishop. Minkowski dimension and the Poincaré exponent. *Michigan Math. J.*, 43(2):231–246, 1996.

This gives a method of estimating the Hausdorff dimension of fractals known as Kleinian limit sets. The techniques here were adapted by Avila and Lyubich to study Julia sets.

- [22] C. J. Bishop. On a theorem of Beardon and Maskit. *Ann. Acad. Sci. Fenn. Math.*, 21(2):383–388, 1996.

This gives a characterization of geometrically finite Kleinian group. It is a cleaner version of an earlier result by Beardon and Maskit.

- [23] C. J. Bishop. Some characterizations of $C(\mathcal{M})$. *Proc. Amer. Math. Soc.*, 124(9):2695–2701, 1996.

This gives a simple characterization of which continuous functions on the unit disk are in the algebra generated by all bounded harmonic functions. This happens iff the level sets of the function can be approximated by curves on which arc length is a Carleson measure.

- [24] C. J. Bishop. Geometric exponents and Kleinian groups. *Invent. Math.*, 127(1):33–50, 1997.

Maskit showed the diameters of the components of the ordinary set sum the fourth power and conjectured the second power suffices; this is proven here. This shows that most components are not “long and skinny” in a precise way.

- [25] C. J. Bishop and Y. Peres. Packing dimension and Cartesian products. *Trans. Amer. Math. Soc.*, 348(11):4433–4445, 1996.

This gives optimal estimates for the dimension of the product of two sets. Previous results gave bounds which might or might not be sharp for a particular set, but this gives the sharp result for any set.

- [26] C. J. Bishop and Peter W. Jones. Hausdorff dimension and Kleinian groups. *Acta Math.*, 179(1):1–39, 1997.

This is my most cited paper. This proves three main results: the critical exponent of a Kleinian group equals the Hausdorff dimension of its radial limit set, the limit set of a finitely generated Kleinian group has Hausdorff dimension two iff the group is geometrically infinite,

and Hausdorff dimension of the limit set is upper semi-continuous. Each of these was a well known conjecture in the field. Our results have been extended to other settings such as symmetric spaces, variable curvature, Gromov hyperbolic groups and discrete quasiconformal groups.

- [27] C. J. Bishop and P. W. Jones. The law of the iterated logarithm for Kleinian groups. In *Lipa's legacy (New York, 1995)*, volume 211 of *Contemp. Math.*, pages 17–50. Amer. Math. Soc., Providence, RI, 1997.

We prove a conjecture of Dennis Sullivan regarding the exact Hausdorff dimension of certain Kleinian group limit sets by exploiting the connections between harmonic functions and martingales. We also answer a question of Curt McMullen regarding absolute continuity of conjugations between certain types of groups.

- [28] C. J. Bishop and P. W. Jones. Wiggly sets and limit sets. *Ark. Mat.*, 35(2):201–224, 1997.

We show that if a connected set deviates uniformly from a line segment at all points and all scales, then it must have dimension strictly larger than 1. The proof is a rather intricate argument involving the so called “traveling salesman theorem” of Peter Jones, but so far no one has yet found an easier proof of this simple sounding result.

- [29] C. J. Bishop, P. W. Jones, Robin Pemantle, and Yuval Peres. The dimension of the Brownian frontier is greater than 1. *J. Funct. Anal.*, 143(2):309–336, 1997.

We prove that the frontier of a Brownian motion (i.e., the boundary of its unbounded complementary component) has Hausdorff dimension strictly bigger than one, providing the first evidence for a conjecture of Mandelbrot that the dimension equals $4/3$. The full conjecture was later proven using Stochastic Loewner Evolutions (SLE) as developed by Lawler, Schramm and Wendelin (resulting in Fields medal for the latter). One fun aspect of our proof is a nested decomposition of the plane into fractal tiles, instead of the usual square grid (we needed to avoid straight lines in certain boundary estimates). I suspect this might be useful for other problems.

- [30] C. J. Bishop. Quasiconformal mappings which increase dimension. *Ann. Acad. Sci. Fenn. Math.*, 24(2):397–407, 1999.

This answers a question of Heinonen by showing that the dimension of any set $E \subset \mathbb{R}^n$ with $0 < \dim(E) < n$ can be increased by taking a quasiconformal image. This is interesting because the dimension cannot always be lowered (the infimum of dimension under quasiconformal maps is called the conformal dimension of the set and investigating it is currently a very active area).

- [31] C. J. Bishop. A quasisymmetric surface with no rectifiable curves. *Proc. Amer. Math. Soc.*, 127(7):2035–2040, 1999.

Builds a pathological surface in 3-space that is nice in certain respects, but which contains no connected set of finite length. This answered a question of Rohde. This result was extended by David and Toro.

- [32] C. J. Bishop, A. Böttcher, Yu. I. Karlovich, and I. Spitkovsky. Local spectra and index of singular integral operators with piecewise continuous coefficients on composed curves. *Math. Nachr.*, 206:5–83, 1999.

This discusses the properties of a certain class of operators. My contribution consisted of the proof of certain estimates regarding harmonic measure.

- [33] C. J. Bishop and J. T. Tyson. Conformal dimension of the antenna set. *Proc. Amer. Math. Soc.*, 129(12):3631–3636, 2001.

We compute the minimal possible dimension over all possible quasiconformal images of a certain fractal. In trying to understand how the dimension of a set can be lowered by QC mappings, there are very few non-trivial cases where we actually know what the minimum is. This provides one such example. This has been extended to metric spaces by Azzam, who also answers a question from this paper.

- [34] C. J. Bishop and J. T. Tyson. Locally minimal sets for conformal dimension. *Ann. Acad. Sci. Fenn. Math.*, 26(2):361–373, 2001.

This gives more examples relevant to dimension lowering by quasiconformal maps.

- [35] C. J. Bishop. Bi-Lipschitz homogeneous curves in \mathbb{R}^2 are quasicircles. *Trans. Amer. Math. Soc.*, 353(7):2655–2663 (electronic), 2001.

This proves that if a Jordan curve is biLipschitz homogeneous then it is “nice” (i.e., satisfies Ahlfors’ 3-point condition). Homogeneous means that any point in the set can be mapped to any other point by a map which is biLipschitz on the set. This had been an open problem in the field. It is open whether an exotic continua can be biLipschitz homogeneous.

- [36] C. J. Bishop. Divergence groups have the Bowen property. *Ann. of Math. (2)*, 154(1):205–217, 2001.

This completes a series of papers by Rufus Bowen, Dennis Sullivan, Kari Astala and Michel Zinsmeister by showing that a Fuchsian group is divergence type iff it has Bowen’s property (this says that any deformation of the group has a limit set which is either a circle or has Hausdorff dimension > 1). A novelty of the proof is the use of Dennis Sullivan’s “Convex Hull Theorem” in hyperbolic 3-space and I give a new proof of Sullivan’s result.

- [37] C. J. Bishop. BiLipschitz approximations of quasiconformal maps. *Ann. Acad. Sci. Fenn. Math.*, 27(1):97–108, 2002.

This that shows quasiconformal maps in two dimensions can be approximated by bi-Lipschitz maps in a certain precise sense.

- [38] C. J. Bishop. Quasiconformal mappings of Y -pieces. *Rev. Mat. Iberoamericana*, 18(3):627–652, 2002.

This describes the optimal way to deform a Riemann surface by shrinking closed loops. It is used in [39]. It has also been quoted widely in the literature on Teichmüller spaces of infinite area Riemann surfaces.

- [39] C. J. Bishop. Non-rectifiable limit sets of dimension one. *Rev. Mat. Iberoamericana*, 18(3):653–684, 2002.

Answers a question of Astala and Zinsmeister by constructing certain deformations of Fuchsian groups whose limit sets are non-rectifiable curves of dimension 1.

- [40] C. J. Bishop and P. W. Jones. Compact deformations of Fuchsian groups. *J. Anal. Math.*, 87:5–36, 2002. Dedicated to the memory of Thomas H. Wolff.

We show that a deformation of a Fuchsian group which only changes the conformal structure on a compact set gives a limit set where the escaping limit set had dimension 1.

- [41] C. J. Bishop. Quasiconformal Lipschitz maps, Sullivan’s convex hull theorem and Brennan’s conjecture. *Ark. Mat.*, 40(1):1–26, 2002.

This gives a new proof of a result of Dennis Sullivan about convex sets in hyperbolic space and gives applications to conformal maps in the plane. One such is the fact that any simply connected plane domain can be mapped to a disk by a Lipschitz homeomorphism.

- [42] C. J. Bishop, V. Ya. Gutlyanskiĭ, O. Martio, and M. Vuorinen. On conformal dilatation in space. *Int. J. Math. Math. Sci.*, (22):1397–1420, 2003.

This is my second most cited paper. This shows that if the dilatation of a quasiconformal map in space satisfies certain integrability conditions then the map is pointwise differentiable.

- [43] C. J. Bishop. Big deformations near infinity. *Illinois J. Math.*, 47(4):977–996, 2003.

This proves the existence of certain deformations of a Fuchsian group. These are used in [44] to study the analytic dependence of a limit set as a function of a deformation parameter.

- [44] C. J. Bishop. δ -stable Fuchsian groups. *Ann. Acad. Sci. Fenn. Math.*, 28(1):153–167, 2003.

This introduces the idea of a δ -stable group and gives examples of such things.

- [45] C. J. Bishop. An explicit constant for Sullivan’s convex hull theorem. In *In the tradition of Ahlfors and Bers, III*, volume 355 of *Contemp. Math.*, pages 41–69. Amer. Math. Soc., Providence, RI, 2004.

Improves the constant ($K = 7.82$) in Dennis Sullivan’s convex hull theorem. The size of this constant has implications for planar conformal mappings. The constant has since been improved by Bridgeman, Canary and Yarmola.

- [46] C. J. Bishop. The linear escape limit set. *Proc. Amer. Math. Soc.*, 132(5):1385–1388 (electronic), 2004.

Proves some results concerning the Hausdorff dimension of special limit sets. In particular, the dimension of the limit set either equals the dimension of the bounded geodesics, or the dimension of the geodesics that tend to infinity as fast as possible (linear speed).

- [47] C. J. Bishop. Orthogonal functions in H^∞ . *Pacific J. Math.*, 220(1):1–31, 2005.

This answers a well known question of Walter Rudin: if f is holomorphic on the unit disk and the sequence of powers f, f^2, f^3, \dots are orthogonal, must f be an inner function (i.e., $|f| = 1$ a.e. on the circle)? In fact, I provide large families of counter-examples, essentially

one for every finite, radial measure in any annulus. An independent solution was obtained by Carl Sundberg.

- [48] C. J. Bishop. Boundary interpolation sets for conformal maps. *Bull. London Math. Soc.*, 38(4):607–616, 2006.

This characterizes interpolating sets for conformal maps of the disk, i.e., sets E on the circle so that given any homeomorphism there is a conformal map on the disk whose restriction to E agrees with the given map.

- [49] C. J. Bishop. A criterion for the failure of Ruelle’s property. *Ergodic Theory Dynam. Systems*, 26(6):1733–1748, 2006.

We give examples to show that even for divergence type groups, the dimension of the limit set need not be an analytic function of the deformation.

- [50] C.J. Bishop. Harmonic measure by Garnett and Marshall. *Book review in Bull. Amer. Math. Soc.* 44(2):267-276, 2007.

This is an expository survey of recent results in geometric function theory.

- [51] C.J. Bishop. An A_1 weight not comparable to any quasiconformal Jacobian. *In the tradition of Ahlfors-Bers, IV*, volume 432 of *Contemp. Math.*, pages 7–18. Amer. Math. Soc., Providence, RI. 2007

Gives the example described by the title. This solved a problem of Stephen Semmes. Characterizing the Jacobians of quasiconformal maps has been one of the driving problems in the field, and this example shows that no sufficient condition only in terms of the distribution function is possible. I also construct a surface in Euclidean 3-space which is quasisymmetrically equivalent to the plane, but which is not bi-Lipschitz equivalent to the plane. No previous example was even embeddable in any Banach space.

- [52] C.J. Bishop and H. Hakobyan. A central set of dimension 2. *Proc. Amer. Math. Soc.*, pages 2453-2461, 136(2008), no. 7.

The medial axis of a domain is the set of centers of disks in the domain which hit the boundary in two or more points. The central set is the set of centers of maximal disks in the domain. They are distinct sets in general, but are sometimes identified by mistake in the literature. Here we emphasize the difference by constructing a Lipschitz domain where the medial axis has dimension 1 and the central set has dimension 2.

- [53] C.J. Bishop. Conformal welding and Koebe’s theorem. *Ann. of Math.* 166(2): 613–656, 2007.

A conformal welding is a homeomorphism of the unit circle to itself of the form $h = g^{-1} \circ f$ where f and g are conformal maps of the two sides of the circle to two sides of a closed Jordan curve. It is known that not every homeomorphism has this form, but I prove that every homeomorphism is “almost” a conformal welding in a precise sense. This paper also contains a new and simple proof of the famous result that every quasisymmetric circle mapping is a conformal welding. It also contains a characterization of the weldings of “flexible curves”;

the images of these curves under homeomorphisms conformal off the curve are dense in all closed curves.

- [54] C.J. Bishop. Decreasing dilatations can increase dimension. *Proc. Amer. Math. Soc.*, 136: 2453–2461, 2008.

Answers a question of Zinsmeister about quasiconformal maps and dimension.

- [55] C.J. Bishop. A set containing rectifiable arcs locally but not globally. *Pure and Applied Math. Quarterly*, 7(1): 121-138, 2011. Special issue in honor of Fred Gehring, part 1 of 2.

I construct a compact set E of zero area and a $K > 1$ so that any K -quasiconformal image of E contains a rectifiable arc, but so that some quasiconformal image of E does not. This is related to the quasiconformal Jacobian problem, in that any function that blows up as it approaches E cannot be the Jacobian of any K -quasiconformal mapping.

- [56] C.J. Bishop. Conformal mapping in linear time. *Discrete and Computational Geometry*, 44(2) 330-428, 2010.

Gives a linear time algorithm for computing conformal maps. The motivation for this method comes from hyperbolic manifolds and computational geometry. Moreover, the method is near optimal. It will find an ϵ approximation to the conformal map onto an n -gon in time $O(n|\log \epsilon \log \log \epsilon|)$. The techniques applied come from hyperbolic 3-manifolds, computational geometry, numerical analysis and geometric function theory. I would like to think this is one of my best results.

- [57] C.J. Bishop. Bounds for the CRDT algorithm. *Computational Methods in Function Theory*, 10(1): 325-366, 2010.

In 1998 Driscoll and Vavasis introduced the CRDT algorithm for computing conformal maps. Although it works well in practice, there is no proof that it converges to the correct answer. We partially address this by showing the CRDT algorithm will always give an answer that is close to the correct answer in a certain precise sense (with bounds independent of the domain)

- [58] C.J. Bishop. Optimal angle bounds for quadrilateral meshes. *Discrete and Computational Geometry*, 44(2): 308-329, 2010.

This shows that any polygon has a linear size quadrilateral mesh so that every new angle used is between 60 and 120 degrees. The angle bounds are optimal. This answers a question of Bern and Eppstein. Previously, it was not even known if there was a linear size mesh with all new angles bounded from below.

- [59] C.J. Bishop. Tree-like decompositions and conformal maps. *Annals Acad. Sci. Fenn.*, 35(2): pages 389-404, 2010.

A companion paper showed that every planar simply connected domain has a collection of disjoint crosscuts (Jordan arcs with endpoints on the boundary) that divide the domain into uniformly chord-arc pieces. This paper uses such a decomposition to construct an approximation to the conformal map onto a disk that is within a uniformly bounded quasiconformal

distance from the true map. Some examples of how this can be used to construct an iterative algorithm for conformal mapping are discussed.

- [60] C.J. Bishop. A random walk in analysis. In the collection *All That Math: portraits of mathematicians as young readers*, 2011, a special volume of *Revisita Matematica Iberoamericana*, celebrating the Centennial of the Real Sociedad Matematica Espanola, Edited by Antonio Cordoba, Jose L. Fernandez and Pablo Fernandez

This is an essay written for a special issue of *Revista Mat Iberoamericana* describing how certain papers have influenced the course of my own work. Of particular importance were the papers of Makarov on harmonic measure, Jones on rectifiable sets and Sullivan on hyperbolic convex hulls.

- [61] C.J. Bishop. True trees are dense. *Invent. Mat.* 197(2): pages 433-452, 2014.

We show that any compact, connected set in the plane can be approximated by the critical points of a polynomial with only two critical values. This is related to the ‘true form’ of a finite tree in the plane, i.e., I show that such true forms are dense in all compact connected sets.

- [62] C.J. Bishop with E. Feinberg and J. Zhang. Examples concerning Abel and Cesaro limits. *J. Math. Analysis and App.*, 420(2): pages 1654-1661, 2014.

This note describes examples of all possible equality and strict inequality relations between upper and lower Abel and Cesaro limits of sequences bounded above or below. It also provides applications to Markov Decision Processes. This paper has been cited in the Economics literature on optimal utility.

- [63] C.J. Bishop. The order conjecture fails in S. *Journal d’Analyse*, 127(1): pages 283–302, 2015.

We show there is an entire function f with only four critical values and no finite asymptotic values whose order can change under a quasiconformal equivalence. This disproves the so called ‘order conjecture’ in the Speiser class. A counterexample in the Eremenko-Lyubich class had been independently found by Epstein and Rempe.

- [64] C.J. Bishop. Constructing entire functions by quasiconformal folding. *Acta. Math.*, 214(1): pages 1–60, 2015.

We give a method for constructing transcendental entire functions with good control of both the singular values of f and the geometry of the tracts of f . The method consists of first building a quasiregular map by “gluing together” copies of the right half-plane that have each been quasiconformally “folded” into themselves. The measurable Riemann mapping theorem is then invoked to produce an entire function with similar geometry. Applications include constructing transcendental wandering domains and counterexamples (with only finite many singular values) to Wiman’s conjecture, the area conjecture, the strong Eremenko conjecture and the order conjecture.

- [65] C.J. Bishop and K. Pilgrim. Dynamical dessins are dense. *Rev. Mat. Iberoamericana*, 31(3): pages 1033-1040, 2015.

We show that every compact connected set in the plane can be approximated in the Hausdorff metric by the Julia set of a post-critically finite polynomial (and thus the Julia set is a dendrite). This uses the result from “True trees are dense” [61] that true trees approximate any continuum, combined with work of Pilgrim that every true tree is approximated by a Julia set of the type above.

- [66] C.J. Bishop. Models for the Eremenko-Lyubich class *J. London Math. Soc.*, 92(1): 202-221, 2015.

This paper deals with approximation in a quasiconformal sense of models: a model domain is a union of disjoint simply connected domains and a model function is a conformal map of each connected component to the right half-plane followed by exponentiation. I show that any model can be approximated by models arising from Eremenko-Lyubich functions (entire functions with bounded singular sets). This reduces the construction of Eremenko-Lyubich entire functions with certain properties to the construction of models with these properties and this is often much easier.

- [67] C.J. Bishop. Nonobtuse triangulations of PSLGs *Discrete and Computational Geometry*, 56(1): pages 43–92, 2016.

We show that any planar PSLG with n vertices has a conforming triangulation by $O(n^{2.5})$ nonobtuse triangles; they may be chosen to be all acute or all right. This is the first polynomial bound for nonobtuse triangulation of PSLGs. This result also improves a previous $O(n^3)$ bound of Eldesbrunner and Tan for conforming Delaunay triangulations. In the special case that the PSLG is the triangulation of a simple polygon, I show that only $O(n^2)$ elements are needed, improving an $O(n^4)$ bound of Bern and Eppstein. We also show that for any $\epsilon > 0$, every PSLG has a conforming triangulation with $O(n^2/\epsilon^2)$ elements and with all angles bounded above by $90^\circ + \epsilon$. This improves a result of S. Mitchell when $\epsilon = \frac{3}{8}\pi = 67.5^\circ$ and Tan when $\epsilon = \frac{7}{30}\pi = 42^\circ$.

- [68] C.J. Bishop. Quadrilateral meshes for PSLGs *Discrete and Computational Geometry*, 56(1): pages 1–42, 2016.

We prove that any PSLG has a conforming quadrilateral mesh with $O(n^2)$ elements and all new angles between 60° and 120° (the complexity and angle bounds are both sharp). Moreover, all but $O(n)$ of the angles may be taken in a smaller interval, say $[89^\circ, 91^\circ]$. This paper uses a result from [67], “Nonobtuse triangulation of PSLGs”.

- [69] C.J. Bishop, H. Hakobyan and M. Williams. Frequency of dimension distortion under quasimetric mappings, *Geometric and Functional Analysis (GAFA)*, 26(2): pages 379–421, 2016.

It is well known that a planar quasiconformal map f can send the segment $[0, 1]$ to a fractal curve of dimension d greater than 1, but it has been an open problem whether this can happen simultaneously for an uncountable set of parallel segments, i.e., can we increase the dimension of every component of $E \times [0, 1]$ for some uncountable E ? It was even unknown whether

one could make the image purely unrectifiable (no rectifiable subarcs). This paper proves that this is indeed possible and we give estimates for how much dimension can be increased. For each d between 0 and 1 we construct a set E of dimension d so that every component of $E \times [0, 1]$ is mapped to a curve of dimension as close to $2/(d + 1)$ as we wish, and we prove this upper bound is sharp. We also give a number of other results concerning purely unrectifiable images, sets whose dimension can't be lowered by QC maps, and dimension bounds in higher dimensions and for maps into metric spaces. The paper also proves that if E is Ahlfors regular in R^n and f is quasiconformal then $\dim(f(E + x)) = \dim(E)$ for Lebesgue almost every x . Indeed, this equality holds not just for Euclidean space, but on any Carnot group, and the inequality \leq holds even more generally.

- [70] C.J. Bishop. Models for the Speiser class. *Proc. London Math. Soc.*, 114(3), 765–797, 2017.

This is the Speiser class version of [66] dealing with the Eremenko-Lyubich class. Both papers deal with approximation in a QC sense of models: a model domain is a union of disjoint simply connected domains and a model function is a conformal map of each connected component to the right half-plane followed by exponentiation. In this paper I show that any model function can be approximated by models arising from the Speiser class, but the Speiser model domain many have more tracts than the target being approximated; this is unavoidable in some cases. However, I prove the sharp result that at most twice as many tracts are needed for the approximation. I also give some results that place geometric restrictions on Speiser models that distinguish them from Eremenko-Lyubich models.

- [71] C.J. Bishop. A transcendental Julia set of dimension 1. *Inventiones Math.*, 212(2), 407–460, 2018.

In 1975 Baker proved that if f is a transcendental entire function, then Julia set of f contains a continuum and hence has Hausdorff dimension at least 1. Misiurewicz and McMullen had each given examples showing dimension 2 is possible and Stallard showed every value in $(1,2]$ is possible, but the question of whether dimension 1 can be attained has remained open. I show the answer is yes in the strongest possible way by showing there is a transcendental entire function whose Julia set has finite length on the sphere. The packing dimension of this example is also 1, the first known example where the dimension is less than 2. This example does lie on a rectifiable curve on the sphere; whether this is possible remains open.

- [72] C.J. Bishop and C. LeBrun. Anti-Self-Dual 4-manifolds, Quasi-Fuchsian groups and almost-Kähler geometry to appear in *Communications in Analysis and Geometry*.

This paper gives an application of harmonic measure in three dimensions to the geometry of 4-manifolds. The analytic part of the paper is the construction of a closed Jordan curve in the plane that is the limit set of finitely generated, co-compact Fuchsian group and so that the harmonic measure of one side of the curve defines a hyperbolically harmonic function in the upper half-space that has a critical point. From the existence of this harmonic function, we deduce the existence of a compact Riemannian 4-manifold that is anti-self-dual but not

almost-Kähler. The example is interesting because the almost-Kähler metrics form a non-empty, open subset of the anti-self-dual metrics on such a manifold, and this example shows, for this first time, that this can be a proper open subset. A more careful argument (proving that the critical point is non-degenerate, hence persists under small perturbations) shows that the complement (the non-almost-Kähler metrics) also has interior, so it would be interesting in the future to investigate the properties of the boundary between these regions.

- [73] C.J. Bishop and K. Lazebnik. Prescribing the Postsingular Dynamics of Meromorphic Functions, to appear in *Math. Annalen*, 375(3), 1761-1782, 2019.

We show that any dynamics on any discrete planar sequence can be realized by the postsingular dynamics of some transcendental meromorphic function, provided we allow for small perturbations of the set. This work is motivated by an analogous result of DeMarco, Koch and McMullen in for rational functions. The proof contains a variation of quasiconformal folding for constructing meromorphic functions and an argument of the second author involving an infinite dimensional fixed point theorem.

PREPRINTS

- [74] C.J. Bishop. Non-removable sets for quasiconformal and quasi-isometric mappings in \mathbb{R}^3 .

We construct a Cantor set in 3-space which is not removable for QC mappings. This is the only known non-trivial example of a non-removable set in dimension higher than 2.

- [75] C.J. Bishop. Interpolating sequences for the Dirichlet space and its multipliers

We give a description of the universal interpolating sequences for the Dirichlet space and give partial results for the usual interpolating sequences for the Dirichlet space and for the space of its multipliers. This preprint was never published because it has significant overlap with independent work of Marshall and Sundberg, but has been cited numerous times.

- [76] C.J. Bishop. Distortion of disks by conformal maps.

Answers a question of Astala about how a collection of disks can be distorted by a QC map which is conformal outside the disks.

- [77] C.J. Bishop. A fast quasiconformal mapping theorem for polygons.

Gives a linear time approximation to the Riemann map. This is an introduction to the longer paper “Conformal mapping in linear time”, [56].

- [78] C.J. Bishop. Estimates for harmonic conjugation.

Gives a geometric characterization of the planar domains for which harmonic conjugation defines a L^2 bounded operator on the boundary. The answer is in terms on a decomposition of the domain into pieces and roughly says that conjugation is bounded iff the Poincaré inequality holds on a weighted tree related to the decomposition.

- [79] C.J. Bishop. Another Besicovitch-Keakeya set We give a 1-page proof that there is a planar set of zero area that contains a unit segment in every direction. Moreover, I show this set has near optimal Minkowski dimension.

- [80] C.J. Bishop. Quasiconformal maps with dilatations of small support

We give an estimate that quantifies the fact that a normalized quasiconformal map whose dilatation is non-zero only on a set of small area approximates the identity on the whole plane.

- [81] C.J. Bishop. Conformal images of Carleson curves

We show that arclength on curve inside the unit disk is a Carleson measure iff the image of the curve has finite length under every conformal map onto a domain with rectifiable boundary. This answers a question posed by Percy Deift.

- [82] C.J. Bishop. A curve with no simple crossings by segments

We construct a closed Jordan curve in plane that has an uncountable intersection with any closed line segment whose endpoints are in different complementary components of the curve. This answers a question posed to me by Percy Deift. Some additional questions are listed at the end of the note.

- [83] C.J. Bishop. A Speiser class Julia set with dimension near one, with S. Albrecht

In 1975 Baker proved that the Julia set of a non-polynomial (transcendental) entire function contains a non-trivial continuum, and hence has Hausdorff dimension at least 1. Stallard gave examples with all dimensions strictly between 1 and 2, and I constructed a transcendental entire function whose Julia set has dimension equal to 1. Stallard's examples are in the Eremenko-Lyubich class, that is, their singular sets are bounded, and she proved that the Julia set of any EL function has dimension strictly larger than 1. In this paper Albrecht and I construct functions in the Speiser class (finite singular sets) whose Julia sets have dimension as close to 1 as desired, strengthening Stallard's examples in the EL class. The construction is an application of my quasiconformal folding method, perhaps the most intricate application to date.

- [84] C.J. Bishop. Curves of finite total curvature and the Weil-Petersson class.

The Weil-Petersson class is collection of rectifiable planar quasicircles that arise naturally in Takhtajan and Teo's work on Teichmüller space, as well as in operator theory, probability theory, and SLE. However all previously known characterizations of this class of curves are in terms of function theory, e.g., in terms of the conformal maps associated to the curve, and it has been an open problem to give an intrinsic geometric characterization. This paper gives several such characterizations in terms of Peter Jones' beta-numbers, curvature estimates, and hyperbolic geometry. One of the tools needed is a refinement of Jones' traveling salesman theorem in the plane, and one of the consequences is a connection between the Weil-Petersson class, minimal surfaces in hyperbolic 3-space and renormalized area, a concept arising from conformal field theory.

- [85] C.J. Bishop, H. Drillick and D. Ntalampekos. Falconers' distance set conjecture can fail for strictly convex sets in \mathbb{R}^d .

For any norm on \mathbb{R}^d with countably many extreme points, there is a compact set E whose distance set with respect to this norm has zero linear measure. This was previously known only for norms associated to certain finite polygons in the plane. Similar examples exist for norms that are very well approximated by polyhedral norms, including some examples where the unit ball is strictly convex and has C^1 boundary. This paper corrects a gap in a paper of Falconer noticed by the second author while writing her undergraduate honors thesis at Stony Brook.

BOOKS

- [86] C.J. Bishop and Yuval Peres. Fractals in Analysis and Probability, Cambridge University Press, 2017

This is an introduction to Hausdorff, Minkowski and packing dimension, illustrated by examples drawn from analysis, dynamics and probability theory. It covers topics including self-similar sets, self-affine sets, the Weierstrass nowhere differentiable function, Brownian motion, Markov chains, Kakeya sets and Peter Jones' traveling salesman theorem.

- [87] C.J. Bishop and Yuval Peres. Conformal Fractals (in preparation)

In the same spirit as the book above, but dealing with fractals that arise from conformal dynamics or involve conformal invariance. Currently planned chapters include: introduction to conformal invariants, diffusion limited aggregation, Makarov's theorems on planar, simply connected harmonic measure, the Jones-Wolff theorem on multiply connected domains, Wolff snowflakes, Julia sets, Kleinian limit sets, the Gaussian free field.

- [88] C.J. Bishop. The Riemann Mapping Theorem (in preparation)

This gives an introduction to conformal mapping with an emphasis on various methods for numerically computing such maps.

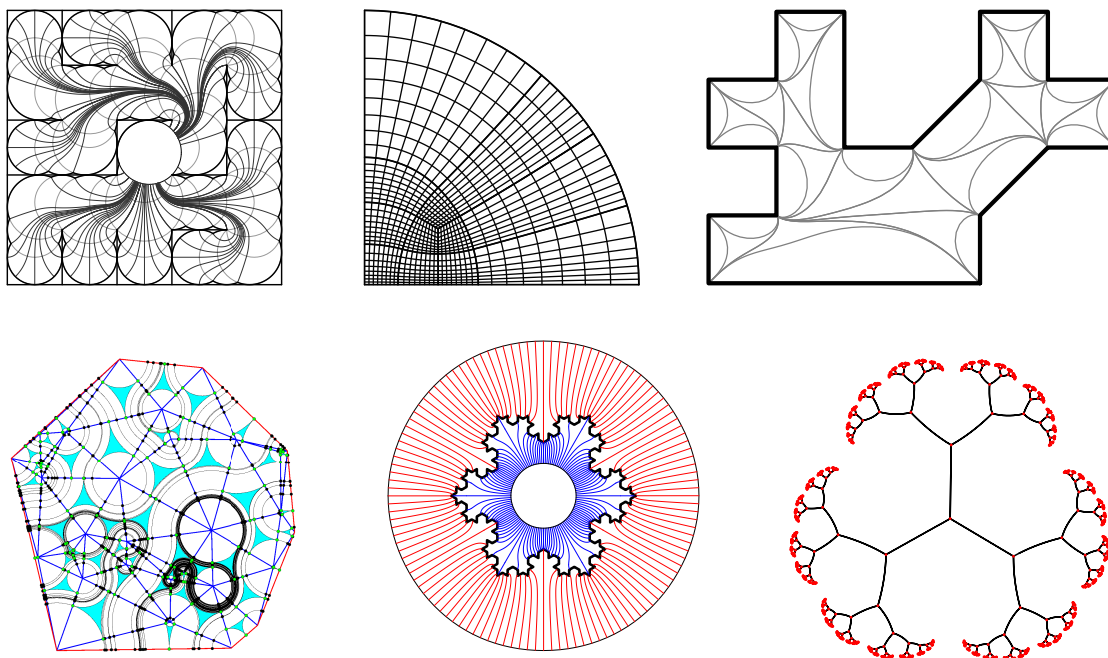
- [89] C.J. Bishop. Introduction to Transcendental Dynamics (in preparation)

This is an introduction to the iteration theory of entire functions and also includes introduction to the basic techniques of geometric function theory needed to understand them, such as hyperbolic geometry, normal families, potential theory, extremal length and quasiconformal maps in the plane.

- [90] C.J. Bishop. Quasiconformal Mappings (in preparation)

A self-contained introduction to planar quasiconformal mappings up through the proof of the measurable Riemann mapping theorem and some of its applications to rational and transcendental dynamics.

POSTSCRIPT



Some illustrations from my recent work. In the top row, the left figure illustrate a flow from the boundary of a domain to an interior circle. This flow has a simple geometric definition (inspired by a result of Dennis Sullivan on hyperbolic 3-manifolds), but gives a uniform approximation to the Riemann mapping, with estimates independent of the domain and forms the first step of an algorithm that converges quadratically to the conformal map. In the center is a quadrilateral meshing of a circular triangle which forms a “worst case” in my algorithm which finds a quadrilateral mesh of any polygon with all new angles between 60° and 120° . These angle bounds are best possible and so is the required time, $O(n)$ for an n -gon. On the right is a “hyperbolic triangulation” of a polygon computed using conformal mapping software I wrote.

On the bottom row, the left figure shows a natural flow associated to any triangulation; my NOT-theorem depends on estimating the lengths of the flow lines and perturbing them to make them terminate more quickly. The center shows conformal images of radial lines corresponding to the two sides of the von Koch snowflake; the landing points illustrate the fact that the harmonic measures for the two sides are mutually singular. The rightmost is the true form of a finite binary tree, e.g., every edge has equal harmonic measure from infinity. This is related to Grothendieck’s theory of *dessins d’enfants*.

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