Research Highlights
Christopher J. Bishop

This is a brief summary of some of my favorite results. Abstracts of all my papers can be found on my webpage at http://www.math.sunysb.edu/~bishop/papers/papers.html as well as downloadable versions of many of the more recent ones (roughly 1996 and later). Citations refer to the list of references attached to my vita.

• **Thesis, Chicago 1987 [1],[2],[3]**: In my thesis I characterized when the harmonic measures corresponding to the two sides of a closed curve are mutually singular: this happens iff the set of tangent points of the curve has zero linear measure. I used this and $L^\infty$ estimates for the $\overline{\partial}$-problem to give a constructive proof of a theorem of A. Brower and J. Wermer on function algebras. I also gave an example of a curve with large Hausdorff dimension for which harmonic measures on opposite sides are $L^\infty$ related.

• **Geometric function theory [8],[16]**: Peter Jones and I proved

**Theorem 1.** Suppose $\Gamma$ is connected. There is a constant $C_\Gamma < \infty$ such that

$$\ell(\Phi^{-1}(\Gamma \cap \Omega)) \leq C_\Gamma,$$

for every simply connected domain $\Omega$ and Riemann mapping $\Phi : \mathbb{D} \to \Omega$ iff $\Gamma$ is Ahlfors regular, i.e., there is an $M > 0$ such that $\ell(\Gamma \cap D(x,r)) \leq Mr$ for every disk $D(x,r)$.

This completed a line research on the Hayman-Wu problem of determining the most general sets $\Gamma$ with this property. It is closely related to a problem of Øksendal which we solved by proving the following local version of the famous F. and M. Riesz theorem.

**Theorem 2.** Suppose that $\Omega$ is a simply connected plane domain and that $\Gamma$ is a rectifiable curve in the plane. If $E \subset \partial \Omega \cap \Gamma$ has positive harmonic measure in $\Omega$ then it has positive length.

The new ideas here are to use geometric square functions to obtain $L^2$ estimates for the Schwarzian derivative of a conformal map $f$ and then use a non-linear version Littlewood-Paley theory to control the boundary behavior $f$. The main result of this non-linear theory is that

**Theorem 3.** If $\Phi$ is univalent and

$$A = A(\Phi) = |\Phi'(0)| + \iint_D |\Phi'(z)||S(\Phi)(z)|^2(1 - |z|^2)^3dxdy < \infty,$$

then $\Phi' \in H^{1-\eta}_{\frac{1}{2}}$ for every $\eta > 0$ and $\|\Phi'\|_{1-\eta} \leq C(\eta)A$.

This shows, for example, that an $L^2$ estimate on the Schwarzian implies that the map will have radial boundary values a.e.. Various local versions are also obtained.
• **Representation theoretic rigidity** [9],[14]: Mostow rigidity fails for discrete subgroups of $PSL(2,\mathbb{R})$, but Tim Steger proved that a different kind of rigidity does hold.

**Theorem 4.** Suppose that $\pi_1$ and $\pi_2$ are irreducible unitary representations of $PSL$, not in the discrete series. Then $\pi_1 \circ \iota_1$ and $\pi_2 \circ \iota_2$ are equivalent representations of $\Gamma$ if and only if $\iota_1$ and $\iota_2$ are equivalent inclusions and $\pi_1$ and $\pi_2$ are equivalent representations of $PSL$.

• **Kleinian groups** [27],[28],[29]: In a series of papers, Peter Jones and I resolved several open problems about the geometry of Kleinian limit sets. Among our results are:

**Theorem 5.** If $G$ is finitely generated and $\Lambda \neq S^2$ then $\dim(\Lambda) = 2$ iff $G$ is geometrically infinite.

**Theorem 6.** For any non-elementary Kleinian group $G$, the critical exponent $\delta$ equals $\dim(\Lambda_c)$, the dimension of the conical limit set.

**Theorem 7.** For finitely generated Kleinian groups, both $\dim(\Lambda)$ and $\delta$ are lower semi-continuous with respect to algebraic convergence.

**Theorem 8.** If $G$ is finitely generated then $\Lambda$ is either a circle, a Cantor set or has dimension $> 1$.

**Theorem 9.** If $G$ is a topologically tame group and there is a positive lower bound on the injectivity radius of the quotient manifold then the limit set $\Lambda$ has positive, finite Hausdorff measure for the function $\varphi(t) = t^2 \sqrt{\log t \log \log \log t}$.

The new idea in several of these is to study the heat kernel on the quotient manifold.

• **Maskit’s conjecture**: I proved that

**Theorem 10.** Suppose $\Lambda$ is the limit set of an analytically finite Kleinian group and that $\{\Omega_j\}$ is an enumeration of the components of $\Omega = S^2 \setminus \Lambda$. Then $\sum_j \diam(\Omega_j)^2 < \infty$.

This was Maskit’s conjecture.

• **Bowen’s dichotomy** [36],[38],[39]: In 1979 Rufus Bowen proved that if $G$ is a cocompact Fuchsian group then the limit set of any quasiconformal deformation $G'$ is either a circle or has Hausdorff dimension strictly greater than one. Later Dennis Sullivan extended this to cofinite groups and I showed that

**Theorem 11.** Bowen’s property holds it iff $G$ is divergence type.

Astala and Zinsmeister proved Bowen’s property fails for all convergence groups; there is always a deformation to a group which has a rectifiable, but non-circular, limit set. In I show that Bowen’s property can fail in a different way: if $G$ is a convergence group with bounded injectivity radius then there is a deformation whose limit set has dimension one, but is nowhere rectifiable.
This requires several new results of interest on their own involving deformations of surfaces and estimates of Green's function.

• Rudin's conjecture [48]: A bounded, holomorphic function \( f \) on the unit disk is called orthogonal if the sequence of powers \( \{ f^n \} \) is orthogonal in the Hardy space \( H^2 \). Inner functions (i.e., \( |f| = 1 \) a.e. on the boundary) with \( f(0) = 0 \) have this property. Walter Rudin conjectured that these were the only orthogonal functions, but I constructed non-inner examples. Such an example was obtained independently by Carl Sundberg. In addition, I characterize which measures occur as the push forward of Lebesgue measure under an orthogonal function. In particular, normalized area measure occurs. This has the surprising consequence that the Bergman space embeds isometrically as a closed subspace of the Hardy space via a composition operator by \( f \).

• Conformal welding [47]: A circle homeomorphism \( h \) is called a generalized conformal welding on \( E \subset \mathbb{T} \) (denoted \( h \in \text{GCW}(E) \)) if \( h = g^{-1} \circ f \), where \( f \) and \( g \) are univalent maps from \( \mathbb{D}, \mathbb{D}^* \) to disjoint domains \( \Omega, \Omega^* \), and the composition exists for radial limits of \( f \) on \( E \) and \( g \) on \( h(E) \) (this was invented by David Hamilton). Moreover, \( h \) is a (standard) conformal welding (denoted \( h \in \text{CW} \)) if \( E = \mathbb{T} \) and \( \Omega, \Omega^* \) are the two sides of a closed Jordan curve \( \Gamma \). Determining which maps are conformal weldings is a fundamental problem with many applications. Not every \( h \) is a conformal welding, but I proved

**Theorem 12.** Every \( h \) agrees with some \( H \in \text{CW} \) off a set \( E \) of arbitrarily small measure.

**Theorem 13.** Every \( h \) is in \( \text{GCW}(\mathbb{T} \setminus E_1 \cup E_2) \) for some sets with \( \text{cap}(E_1) = \text{cap}(h(E_2)) = 0 \).

**Theorem 14.** If \( \mathbb{T} = E_1 \cup E_2 \) with \( \text{cap}(E_1) = \text{cap}(h(E_2)) = 0 \) then \( h \in \text{CW} \)

The last condition is called “log-singular” and is quite different from the usual sufficient condition of \( h \) being quasisymmetric. I also give a new, short proof that quasisymmetric homeomorphisms are conformal weldings using Koebe's circle domain theorem (avoiding the use of the measurable Riemann mapping theorem).

• Factorization and Sullivan's theorem [36],[40],[44]: I show

**Theorem 15.** Any conformal map \( f : \mathbb{D} \to \Omega \) can be written as \( f = g \circ h \) where \( h \) is a \( K \)-quasiconformal self-map of the disk and \( |g'| \) is bounded away from zero uniformly. We can always take \( K < 8 \), independent of \( f \).

This factorization is closely related to 3-dimensional hyperbolic geometry and actually originates in a theorem of Dennis Sullivan's about boundaries of hyperbolic and among the geometric consequences: any bounded quasirectangle maps to a circle by a Lipschitz homeomorphism of the plane.

• Fast approximation of conformal maps [52]: Suppose \( \Omega \) is a plane domain bounded by a simple \( n \)-gon \( P \). To apply the Schwarz-Christoffel formula one needs to know the conformal
preimages of the vertices, but there is no simple formula for these. In practice they are often found by an iterative method from some initial guess (e.g., \( n \) evenly distributed points on the circle). I show

**Theorem 16.** In time \( O(n) \) we can compute \( n \) points on the unit circle which are close to the true prevertices in the sense that there is a \( K \)-quasiconformal map of the disk sending our points to the true prevertices and that \( K \) is independent of the geometry of \( P \).

So far as I know, there was no previous way of making a good initial guess for the conformal prevertices.

- **Non-removable sets** [53]: A set \( E \subset \mathbb{R}^n \) is called non-removable for quasiconformal mappings if there is a homeomorphism of \( \mathbb{R}^n \) which is quasiconformal off \( E \) but not on the whole space. Thus such sets are closely related to the natural domains of definition of quasiconformal and various Sobolev type functions.

**Theorem 17.** There is a totally disconnected set in \( \mathbb{R}^3 \) which is non-removable for quasiconformal mappings.

There does not seem to be any other non-trivial examples known in dimensions \( \geq 2 \).

- **Algebras of analytic functions** [4],[24]: Some of my earlier papers deal with the closed (in the supremum norm) algebras generated by \( H^\infty(\Omega) \) (the bounded holomorphic functions on a domain \( \Omega \)) and a collection of harmonic functions. Two sample theorems of this type are

**Theorem 18.** If \( \Omega \) is any open set in the plane and \( f \in H^\infty(\Omega) \) is non-constant on every component of \( \Omega \) then \( C(\overline{\Omega}) \subset H^\infty(\Omega)\{f\} \).

**Theorem 19.** A function \( f \) on the unit disk is in the closed algebra generated by all bounded harmonic functions iff for every \( \epsilon > 0 \) there is a union of curves \( \Gamma \subset \mathbb{D} \) so that (1) arclength on \( \Gamma \) is a Carleson measure and (2) \( f \) is with \( \epsilon \) of a constant on each component of \( \mathbb{D} \setminus \Gamma \).