RESEARCH INTERESTS AND PLANS

Christopher J. Bishop

My current research interests include various topics in geometric function theory, holomorphic dynamics, computational geometry, harmonic analysis and probability theory. In this statement I describe a few past highlights of my work, but I focus mainly on recent work and specific problems I am thinking about currently. More detailed descriptions can be found in my past and current NSF proposals, and the unifying role of harmonic measure in all these problems is described in my 2018 ICM article "Harmonic measure: algorithms and applications". Links to these and other documents are given at the end of this statement.

The statement is long, but broken into short sections that can be skipped or read independently, depending on the interests of the reader.

Teichmüller Space, Harmonic Analysis and Minimal Surfaces

Recently I have been working to give geometric characterizations of the class of Weil-Petersson (WP) curves in the plane. This class was defined in 2006 by Takhtajan and Teo to resolve certain problems arising in string theory by giving a Hilbert manifold structure to the space of smooth closed curves. However, the smooth curves are not complete in this metric and it was of interest to describe all curves in the closure. This question also arises in David Mumford's work on pattern recognition, and the WP class has natural connections to SLE (Schramm-Loewner Evolutions) by recent work of Rohde, Viklund and Wang.

• **Weil-Petersson curves and β-numbers:** Quasicircles are closed Jordan curves Γ in the plane that are quasiconformal images of circles. These can be characterized by the Ahlfors M-condition: if z is on the smaller diameter arc of Γ with endpoints x, y then $|z-x| \leq M|x-y|$ for some $M < \infty$. In general, such curves can be fractals, such as the von Koch snowflake. Quasicircles can be identified with points of universal Teichmüller space $T(1)$ (usually identified as quasi-symmetric homeomorphisms of the unit circle to itself), and it was a long standing problem to put a metric on this space that corresponds to the Weil-Petersson metric on finite dimensional Teichmüller spaces of Riemann surfaces. Such a metric was found by Takhtajan and Teo [45], but with this metric $T(1)$ is disconnected, and they asked for a geometric characterization of the connected component that contains all smooth curves. This component is the Weil-Petersson class, and [18] gives over twenty equivalent definitions of such curves. I will describe a few of these below.

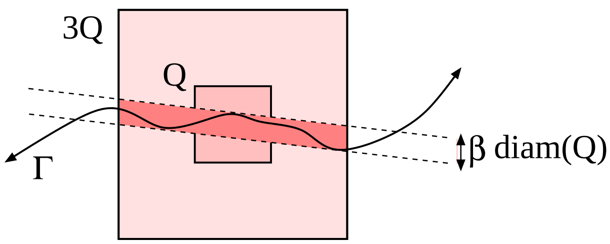
For a curve Γ, Peter Jones' β-numbers $\beta_\Gamma(Q)$ measures the deviation of $\Gamma \cap 3Q$ from a straight line when Q is a dyadic square in the plane. See the left-hand side of the figure below. Jones used these numbers to characterize rectifiable curves: his "traveling salesman theorem" implies that Γ has finite length iff

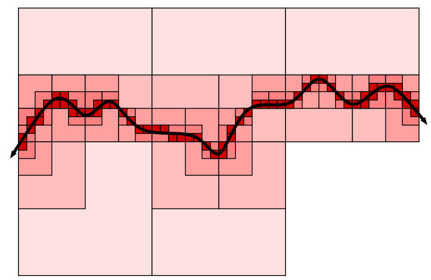
$$\sum_Q \beta_\Gamma(Q)^2 \mathrm{diam}(Q) < \infty,$$

where the sum is over all dyadic squares in the plane that hit Γ (see the right-hand figure below). I proved in [18] that Γ is Weil-Petersson iff

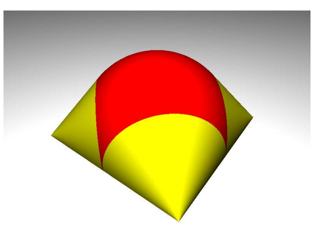
$$\sum_Q \beta_\Gamma(Q)^2 < \infty. \tag{1}$$

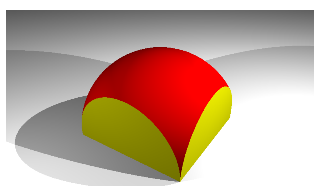
Thus WP curves have "finite total curvature" in an L^2 sense. The proof of (1) requires a strengthening of Jones's theorem for Jordan curves, which is of interest in its own right.

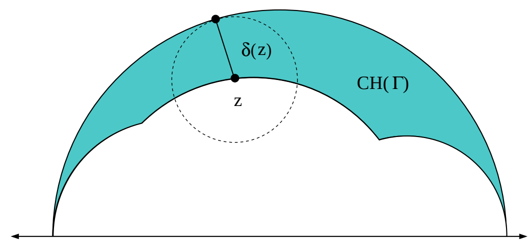




• **Hyperbolic convex hulls:** A surprising alternate characterizations of WP curves involves the hyperbolic convex hull of Γ in the upper half-space $\mathbb{R}^3_+ = \mathbb{R}^3_+$. This is the union of all hyperbolic geodesics (half-circles perpendicular to $\mathbb{R}^2$ with endpoints in Γ. If Γ is a circle, then its hyperbolic convex hull, $\mathrm{CH}(\Gamma)$ is a hemi-sphere, but otherwise it has non-empty interior and is bounded by two surfaces, each of which meets $\mathbb{R}^2$ along Γ, e.g., the left and center figures below show the lower and upper boundaries of the hyperbolic convex hull of a square. For $z \in \mathbb{R}^3_+$, we let $\delta(z)$ be the hyperbolic distance to farther boundary component; for z in $\mathrm{CH}(\Gamma)$, this measures the "width" of the convex hull near z (right figure below).







I prove in [18] that Γ is Weil-Petersson iff

$$\int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z) dA(z) < \infty, \tag{2}$$

where dA denotes hyperbolic surface area on the boundary of the convex hull. Condition (2) is the conformally invariant version of (1). This condition is very reminiscent of a result of Brock [24] that says the Weil-Petersson distance between two Riemann surfaces X, Y (the same topological surface Σ, but with different conformal structures) is approximately the volume of the convex core of an associated 3-manifold with X and Y as boundaries at infinity. In this case, the volume of the convex core is comparable to the L^2 norm of the function δ defined above, so it would be interesting to see if the two results can be unified.

• **Minimal surfaces and renormalized area:** It follows from the hyperbolic convex hull characterization of Weil-Petersson curves is that $\Gamma \subset \mathbb{R}^2$ is Weil-Petersson if and only if it is the boundary of a minimal surface $S \subset \mathbb{R}^3_+$ with finite total curvature, i.e., $\int_S |K|^2 dA$ where K is the norm of the second fundamental form of S. This also answers an old question of whether boundaries of surfaces of finite total curvature are differentiable: no, since WP curves can have infinite spirals. However, they are "almost" differentiable: I proved the arclength parameterization of a WP curve is in the Sobolev space $H^{3/2}$, and being in H^s for $s > 3/2$ would imply C^1. The Sobolev characterization also proves that a curve is Weil-Petersson iff it has finite Möbius energy, a concept that first arose in knot theory to find canonical representations of closed curves in $\mathbb{R}^3$ (it is the energy corresponding to a charge distributed like arclength on the curve, under an inverse cube repulsive force).

The curvature bound of a surface is related to the idea renormalized area. A surface $S \subset \mathbb{R}^3_+$ that has a non-degenerate curve $\Gamma \subset \mathbb{R}^2$ as its asymptotic boundary must have infinite hyperbolic area. However Robin Graham and Ed Witten [32] introduced the idea of a renormalized area that is finite if the boundary curve is sufficiently smooth (renormalized area is the limiting difference between the area and boundary length a certain sequence of compact subdomains that exhaust the surface).

Theorem 1. *A curve $\Gamma \subset \mathbb{R}^2$ is Weil-Petersson iff it is the asymptotic boundary of a minimal surface in hyperbolic space that has finite renormalized area.*

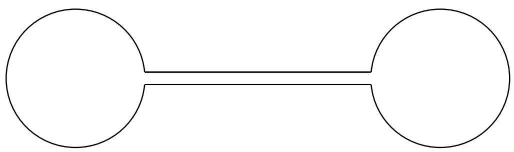
In fact, the proof shows that for a surface S with finite Euler characteristic $\chi(S)$ and asymptotic boundary Γ, the renormalized area is given by

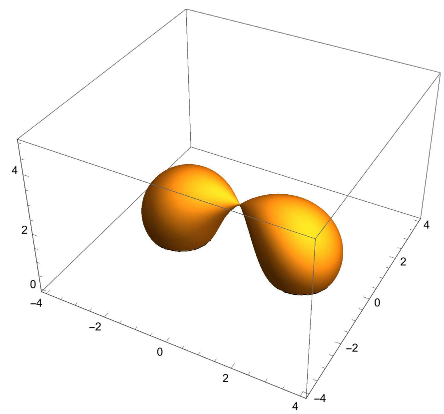
$$-2\pi\chi(S) - \int_S \kappa^2(z) d\mathrm{A}_\rho \tag{3}$$

This was proven by Alexakis and Mazzeo [2], [3] using the additional assumption that Γ is $C^{3,\alpha}$. My proof (based on isoperimetric inequalities for curved surfaces) works for any closed Jordan curve. Renormalized area is relevant conformal field theory and quantum entanglement, and I would like to understand the connections between the geometric characterizations of Weil-Petersson curves and these topics in physics. Formula (3) is also related to Willmore energy, another connection that awaits investigation.

• **Harmonic measure and exotic 4-manifolds:** Claude LeBrun and I recently constructed the first examples of anti-self-dual 4-manifolds N, so that the almost-Kähler metrics on N form an non-empty but proper subset of the moduli space. The simplest case is to start with a hyperbolic 3-manifold M that is homeomorphic to $\mathbb{R} \times \Sigma$ where Σ is a compact surface. Such a 3-manifold has a harmonic function u that tends to 0 in one end of M and tends to 1 in the other end. The function u is called the tunnel vision function and it measures the probability that a Brownian particle will tend to infinity down one end or the other (see the discussion of harmonic measure below). On the universal cover, u lifts to the harmonic function with boundary value 1 on one side of the limit set Γ of the Kleinian group associated to M (in this setting, this limit set is a Jordan curve, indeed, a quasicircle), and zero on the other side of Γ. To get a 4-manifold, we collapse the two ends of $M \times \mathbb{T}$ to two points; this gives a conformally flat 4-manifold N (but a hierarchy of topologically distinct non-flat examples also exists). One can show that this conformal metric is conformal to an almost-Kähler metric if and only if the tunnel vision function doesn't have any critical points.

Thus the construction is reduced to building quasi-Fuchsian Kleinian groups and associated harmonic functions and studying the critical points of these functions. Our example uses a group with a large number of generators and a limit set chosen to approximate a "dog-bone" contour (see left below). At a certain height, the level surfaces of the corresponding harmonic function switch from being disconnected to connected (see right below), proving the existence of a critical point.



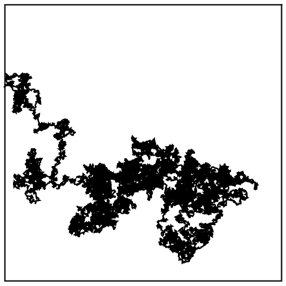
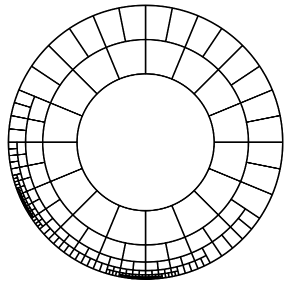
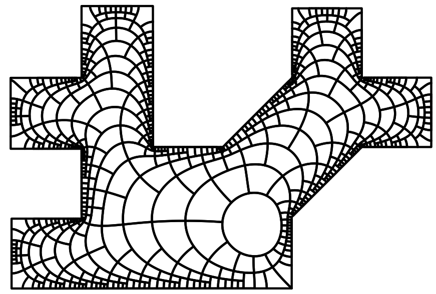


Many questions remain open. For which planar domains Ω does this function have a critical point? Does it matter whether we consider Euclidean or hyperbolic harmonic functions? How few generators are needed to create an example? Can we locate "small" examples using numerical experiments? Are critical points common for groups near the boundary of Teichmüller space for any large G, e.g., near degenerate limit sets in the boundary? Are such examples "common" or "'rare" in Teichmüller space.

Harmonic Measure, Trees and Triangulations

• **Harmonic measure:** The most intuitive definition of harmonic measure is as the boundary hitting distribution of Brownian motion. More precisely, suppose $\Omega \subset \mathbb{R}^n$ is a domain (open and connected) and $z \in \Omega$. We start a random particle at z and let it run until the first time it hits $\partial\Omega$. See the left figure below. We will assume this happens almost surely; this is true for all bounded domains in $\mathbb{R}^n$ and many, but not all, unbounded domains. Then the first hit defines a probability measure on $\partial\Omega$. The measure of $E \subset \partial\Omega$ is usually denoted $\omega(z, E, \Omega)$ or $\omega_z(E)$. For E fixed, $\omega(z, E, \Omega)$ is a harmonic function of z on Ω, hence the name "harmonic measure". The "tunnel vision" function of the last section is an example of harmonic measure on a manifold: there the boundary consisted of the two infinite ends of the manifold.

For simply connected planar domains, harmonic measure for z is the image of normalized length measure on the unit circle under a conformal map from the disk to Ω taking 0 to z (because Brownian motion is conformally invariant). This allows allows much of complex function theory to be applied. The two right figures below illustrate a conformal map from the disk to a polygon; harmonic measures of various edges can be estimated by the number of grid boxes covering them. It also illustrates the idea conformally transferring meshes from the disk to a polygon, an idea we shall return to later.

• **Singularity of harmonic measures:** Harnack's inequality implies that $\omega(z, E, \Omega)$ is either the constant 0 or 1 or is strictly between 0 and 1 on all of Ω. Thus the null sets for harmonic measure are independent of z. If $\Omega \subset \mathbb{R}^2$ is a Jordan domain, this implies two base points on the same side of $\Gamma = \partial\Omega$ give mutually continuous harmonic measures. However, this need not happen for points on opposite sides of Γ.

Theorem 2. *Harmonic measures for two sides of a closed curve γ are mutually singular iff the set of tangent points of γ has zero linear measure.*

This was the main part of my PhD thesis, answering a question raised by Lennart Carleson (and resulting in a paper [10] with him, John Garnett and Peter Jones). Thus for a fractal curve, Brownian motions started on opposite sides of the curve almost surely hit disjoint sets. At the other extreme, if Γ is rectifiable, the 1916 F. and M, Riesz theorem says harmonic measures for both sides are mutually continuous to arclength, hence to each other. Peter Jones and I generalized this, verifying a conjecture of Øksendal [19], [20]:

Theorem 3. *(Local F. and M. Riesz Theorem) If $\Omega \subset \mathbb{C}$ is simply connected and $E \subset \partial\Omega$ is a zero length subset of a rectifiable curve, then E has zero harmonic measure in Ω.*

Recall that a set Γ is Alhfors regular if $\Gamma \cap D(x, r)$ has length $O(r)$ for every disk. The methods developed to prove the theorem above also gave an extension of a famous result of Walter Hayman and Jang-Mei Wu.

Theorem 4. *(Generalized Hayman-Wu Theorem) A connected set Γ is Ahlfors regular iff $f^{-1}(\Gamma)$ has uniformly bounded length for every conformal map f on $\mathbb{D}$.*

• **Conformal welding:** The question of "how singular" the harmonic measures for different sides of a curve can be, may be formulated as a question about how singular conformal weldings can be. Given the curve Γ we can define conformal maps f, g from the inside and outside of $\mathbb{T}$ to the inside and outside of Γ respectively. The composition $h = g^{-1} \circ f$ defines a homeomorphism of $\mathbb{T}$ to itself, called a conformal welding homeomorphism. Not every circle homeomorphism is a conformal welding, but large classes are known to be (e.g., every smooth circle homeomorphism or every quasisymmetric one). The best known result in this direction is:

Theorem 5. [11] *For every circle homeomorphism h and $\epsilon > 0$ there is a conformal welding g so that $\{h \neq g\} \subset \mathbb{T}$ has length $< \epsilon$.*

The map from curves to weldings is known not to be injective. Indeed, I have constructed weldings whose preimage curves are dense in the set of all closed curves for the Hausdorff metric, hence the map is non-injective in a strong way. The following conjecture attempts to explain why understanding conformal weldings has proven so difficult.

Conjecture 1. *Conformal weldings form a non-Borel subset of all circle homeomorphism.*

As a first step, I have shown that conformally removable sets are a non-Borel subset of all compact sets (with respect to the Hausdorff metric). A set K is conformally removable if every homeomorphism of the sphere that is conformal off K is conformal everywhere, hence Möbius. The two problems are connected because if the map taking a closed curve to its conformal welding was 1-1 in an appropriate sense, then weldings would be a Borel subset of all homeomorphisms. The injectivity fails for conformally non-removable curves, so the set theoretic complexity of these classes should be closely related. By comparison, it's easy to see that removable sets for bounded holomorphic functions are merely a G_δ subset of all compact sets, but Tolsa's characterization of this class is still a monumental result.

• **Fast conformal mapping via hyperbolic geometry:** There are many practical methods for numerically computing conformal maps from the unit disk to a given polygon; this has been an active area of research with important applications for over a 100 years. Because of the Schwarz-Christoffel formula, it suffices to find the preimages on $\mathbb{T}$ of the polygon's vertices (known as the SC-parameters). However, very few of the underlying algorithms come with a guarantee of convergence, or an estimate of the work needed to reach a given accuracy. The following is the fast mapping theorem from [14].

Theorem 6. *Given $\epsilon > 0$ and an n-gon P, there is $\mathbf{w} = \{w_1, \dots, w_n\} \subset \mathbb{T}$ so that*

1. *$\mathbf{w}$ can be computed in at most Cn steps, where $C = O(1 + \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$,*
2. *$d_{QC}(\mathbf{w}, \mathbf{z}) < \epsilon$ where $\mathbf{z}$ are the true SC-parameters.*

Here a step means an infinite precision arithmetic operation or function evaluation. The error in Theorem 6 is measured using a distance between n-tuples defined by

$$d_{QC}(\mathbf{w}, \mathbf{z}) = \inf\{\log K : \exists\ K\text{-quasiconformal } h : \overline{\mathbb{D}} \to \overline{\mathbb{D}} \text{ such that } h(\mathbf{z}) = \mathbf{w}\}.$$

A K-quasiconformal map is a homeomorphism that is differentiable almost everywhere and that sends infinitesimal circles to ellipses with eccentricity at most K. These generalize conformal maps, which infinitesimally send circles to circles. It turns out that this is a natural and useful distance between n-tuples, and it dominates the Hausdorff distance between the n-tuples in a conformally invariant way.

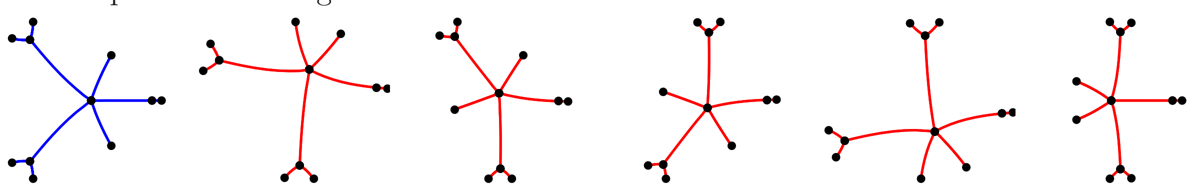
The proof of the fast mapping theorem is an iteration involving approximate solutions of a Beltrami equation using fast multipole expansions. The key to getting uniform time bounds, however, is finding a QC mapping to use as a initial point for the iteration that is fast to compute and uniformly close to being conformal. This is given by 3-dimensional hyperbolic geometry. The "dome" of a planar domain Ω is the surface $S = S(\Omega) \subset \mathbb{R}^3_+ = \mathbb{R}^3_+ = \{(x, y, t) : t > 0\}$ that is the boundary of the union of all hemispheres whose base disk is contained in Ω. This object was mentioned earlier in connection to Weil-Petersson curves. The dome of Ω is also the boundary of the hyperbolic convex hull in $\mathbb{R}^3_+$ of $\Omega^c = \mathbb{C} \setminus \Omega$. We define the "nearest point retraction" $R : \Omega \to S(\Omega)$ by expanding a horo-sphere in $\mathbb{R}^3_+$ tangent to $\mathbb{R}^2$ at $z \in \Omega$ until it first hits S at a point $R(z)$. Dennis Sullivan's convex hull theorem states that R is a quasi-isometry from the hyperbolic metric on Ω to the hyperbolic path metric on the dome. Thurston observed the dome is isometric to the hyperbolic disk. The composition of the retraction map with the isometry gives a provably good starting point for our iteration. Moreover, this "approximate conformal map" can be computed in linear time using the medial axis of Ω (a concept from computation geometry).

• **True trees:** It is not hard to prove that if the harmonic measures for two points on opposite sides of a closed Jordan curve are exactly the same measure, then the curve is a circle, and the points are reflections of each other. A more interesting version is to ask if any finite planar tree T can be drawn so that harmonic measure is equal on "both sides" of each edge? More precisely, with base point equal to infinity, can we draw T so that

(1) every edge has equal harmonic measure,

(2) any subset of any edge has equal harmonic measure from both sides?

Perhaps surprisingly, the answer is yes, every finite planar tree T has such drawing, unique up to similarities, called the "true form of the tree" (or a "true tree" for short). Here are a few examples with 10 edges.



The fact that every tree can be drawn in this way is well known, and closely related to the uniformization theorem: given a planar tree T, we can connect each vertex to infinity, so as to obtain a topological triangulation of the sphere where each side of the tree occurs as one side of a triangle. If we identify each topological triangle with an equilateral triangle, this gives the sphere a conformal structure, which must agree with the usual one by the uniformization theorem, and the true form of T is its image under identification with the usual Riemann sphere. Alex Eremenko asked what restrictions exist on the possible shapes of a true tree, and I showed there are none.

Theorem 7. *(True trees are dense) For every continuum $K \subset \mathbb{R}^2$ and every $\epsilon > 0$, there is a true tree that approximates K to within ϵ in the Hausdorff metric.*

True trees are a special case of Grothendieck's theory of *dessins d'enfants*, in which a finite graph on a topological surface imparts a conformal structure to the surface (in our case, a tree on the sphere). Associated to each true tree T is a polynomial p that has critical values exactly ± 1 and so that $T = p^{-1}([-1, 1])$. This polynomial has algebraic coefficients and this gives rise to an action of the absolute Galois group on planar trees, i.e., an action that permutes finite trees, although a general description of the orbits is unknown e.g., two trees in the same orbit must have the same vertex degree sequences, but this is not sufficient. The blue tree (leftmost) above is a fixed point of the action, and the red ones (the rest) form a single orbit. Because of the relation to balanced harmonic measure, conformal maps between the exteriors of true trees with the same number of vertices have analytic extensions across the interiors of every edge; this allows us to "mate" the exteriors in a natural way to form Riemann surfaces covering a punctured plane. Perhaps the orbit structure of the trees be related to these surfaces and relations among them?

• **Equilateral triangulations of surfaces:** The construction proving the density of true trees is based on quasiconformal mappings and can be extended to Riemann surfaces. A holomorphic function $f : X \to \mathbb{S}^2$ on a Riemann surface X is called a Belyi function if f is branched only over 0, 1 and ∞, and f has no removable singularities at punctures of X. The latter condition implies that f cannot be holomorphically extended to a Riemann surface properly containing X. For example, the polynomials associated to true trees are Belyi functions for the Riemann sphere. If the surface X has a Belyi function f, then the preimages of the upper and lower half-planes are topological triangles in G which are conformally equivalent to equilateral triangles glued (according to arclength) along their edges. Lasse Rempe and I have proved the following.

Theorem 8. *Every open Riemann surface has an equilateral triangulation.*

In particular, every non-compact surface has a Belyi function. This is not true for compact surfaces: by a famous theorem of Belyi [6], a compact surface X has a Belyi function iff it is algebraic. Another consequence of our result is the following

Corollary 9. *Every Riemann surface is a branched cover of the 2-sphere with only finitely many branch points.*

For compact surfaces, this follows from the classical Riemann-Roch theorem, but seems new in general. Pommerenke asked if every hyperbolic Riemann surface has a meromorphic Lipschitz map to the Riemann sphere. Proving this should be a matter of controlling the sizes of the triangles used in the theorem above. Our methods should also have applications to equidistribution properties of Belyi surfaces within Teichmüller space and to the embedding of Riemann surfaces into $\mathbb{C}^2$. Rempe and I have also constructed examples finite type holomorphic dynamical systems on various hyperbolic Riemann surfaces, generalizing the classical examples of iterating rational functions on the Riemann sphere, or finite type entire functions on the complex plane. This is a new, completely unexplored, sub-field of holomorphic dynamics.

• **Revisiting some classical theorems:** The quasiconformal folding method mentioned above has a number of other consequences. My former PhD student Kirill Lazebnik and I have recently proven new, more precise versions of the classical approximation theorems of Weierstrass, Runge and Mergelyan. Each of these says that (under certain conditions) a continuous function on a compact planar set K can be uniformly approximated by a polynomial or rational function. The usual versions of these results do not say much about what the approximating functions look like away from the set K, but Kirill and I have been able to give a precise geometric description of the approximating functions everywhere in the plane. In particular, we prove that the critical points of the approximating polynomial or rational function can all be taken close to K and the critical values all close to $f(K)$. These are conditions that should be very useful in holomorphic dynamics, where approximation theorems are often used to construct exotic examples, and where understanding the location and behavior of critical points and critical values is essential to understanding the dynamical properties of a function.

We also use quasiconformal folding to extend a classical result of Hilbert on polynomial lemniscates (sets of the form $\{z : |p(z)| = 1\}$ for a polynomial p) to rational lemniscates. Hilbert's theorem says that any closed Jordan curve can be approximated by a polynomial lemniscate, with a parameterization that is uniformly close. We prove that any disjoint, finite collection of closed Jordan curves can be similarly approximated by a rational lemniscate.

Conformal Dynamics

• **Kleinian groups:** A Kleinian group is a discrete group of isometries acting on hyperbolic 3-space. The quotient of hyperbolic space by such a group is a hyperbolic manifold. A Fuchsian group is similar, but acting on hyperbolic 2-space, and the quotient is a Riemann surface. A finitely generated group is called geometrically finite if there is a finite sided fundamental region for the group action. For Fuchsian groups these always exist, but Kleinian groups can be both finitely generated and geometrically infinite: this is one reason why hyperbolic 3-manifolds are much more interesting (and difficult) than Riemann surfaces.

Given a Kleinian or Fuchsian group G we can study orbit of a point z and count the number of orbit points within distance n of z. This number grows exponentially and the exponential rate of growth is called the critical exponent of G. The orbit of z accumulates on the sphere at infinity of hyperbolic space; the accumulation set is called the limit set. Hyperbolic geodesics ending at limit points project to geodesics in the quotient manifold and can either remain in some compact set forever, leave every compact set, or oscillate (do neither). Peter Jones and I proved several fundamental facts about these objects in [21].

Theorem 10. *The critical exponent of a non-elementary Kleinian group equals the Hausdorff dimension of the set of bounded geodesic rays.*

Theorem 11. *A finitely generated Kleinian group is geometrically finite iff its limit set has Hausdorff dimension* < 2.

Theorem 12. *The Hausdorff dimension of the limit set of a Kleinian group is an upper semi-continuous function of its generators.*

We also have a number of other more technical results about the geometry of Kleinian limit sets. One of the most exotic is the exact computation of the "size" of the limit set of a geometrically infinite group. Assuming there is a positive lower bound on the associated 3-manifold's injectivity radius we showed in [22] that the limit set has positive, finite Hausdorff measure with respect to the gauge function

$$\varphi(t) = t^2 \sqrt{\log \frac{1}{t} \log \log \log \frac{1}{t}}.$$

As one might suspect from the this formula, there are close connections to harmonic functions, Brownian motion, and the classical law of the iterated logarithm for random walks. The proof also establishes bounds for the Green's function on a hyperbolic 3-manifold that are of independent interest.

Kleinian groups also motivated at least one result that does not directly involve groups at all. Recall the β-numbers defined earlier.

Theorem 13. [23] *A connected set that has β-numbers bounded uniformly below by $\beta_0 > 0$ (a "uniformly wiggly" set) has Hausdorff dimension $> 1 + C\beta_0^2 > 1$.*

Despite the simple statement, our proof of this is surprisingly difficult (it uses Jones's characterization of rectifiable sets), but no shorter argument has yet been discovered. A special case concerns Fuchsian groups where the limit set is a Jordan curve and the quotient Riemann surface is compact: Rufus Bowen proved the limit set is either a circle or has Hausdorff dimension > 1. We call this Bowen's dichotomy. Dennis Sullivan extended it to finite area surfaces and Astala and Zinsmeister showed it fails whenever $R = \mathbb{D}/G$ has a Green's function. I completed the picture by proving:

Theorem 14. [13] *Bowen's dichotomy holds iff $R = \mathbb{D}/G$ has no Green's function.*

This is now one of several characterizations of a class of Fuchsian groups called divergence groups. Hyperbolic convex hulls play a fundamental role in the proof of this result: the hulls give a precise way of quantifying that deformations of a group expand the limit set more than they contract it, i.e., hyperbolic hulls encode the hyperbolic behavior of the group.

• **Iteration of entire functions:** Transcendental dynamics refers to the iteration theory of non-polynomial entire functions. As usual, the Fatou set $\mathcal{F}$ is the maximal open set where the iterates of f form a normal family and the Julia set $\mathcal{J}$ is its complement. While similar to polynomial dynamics in many respects, there are several significant differences: wandering domains can exist, Fatou components of any finite or infinite multiplicity may occur, the escaping set $I(f) = \{z : f(z) \to \infty\}$ plays a more prominent role (as do subsets based on rates of escape), $\mathcal{J}$ always contains a non-trivial continuum, and it is generally harder to build "small" Julia sets than "large" ones, in the sense of fractal dimensions.

The singular set of an entire function f is the closure of its critical values and finite asymptotic values (limits of f along curves to ∞); the complement of the singular set is the largest open set where f acts as a covering map. In transcendental dynamics much attention is devoted to the Eremenko-Lyubich class (bounded singular set) and the Speiser sub-class (finite singular set). I adapted the quasiconformal constructions used to prove the density of true trees to approximate certain unbounded, connected planar sets by infinite trees $T = f^{-1}([-1, 1])$ associated to entire functions f with two critical values. This method is known as quasiconformal folding, and is well adapted to constructing examples in the Eremenko-Lyubich and Speiser classes with specified geometry. QC-folding has already solved several longstanding problems, such as the following.

Theorem 15. [15] *There is an Eremenko-Lyubich function with a wandering domain.*

Theorem 16. [15] *There is a Speiser class function whose Julia set contains no non-trivial path components (a counter-example to the Strong Eremenko Conjecture in the Speiser class).*

Theorem 17. [15] *There are two Speiser class functions that are quasiconformally equivalent, but have different orders of growth (a counterexample to the Order Conjecture).*

Theorem 18. [12] *Given any discrete set $X \subset \mathbb{C}$, any $\epsilon > 0$, and any map $h : X \to X$, there is a meromorphic f whose postcritical set $P(f)$ approximates X to within ϵ and so that $f|_P$ approximates $h|_X$.*

For finite sets and rational maps, the latter result is due to Laura DeMarco, Sarah Koch and Curt McMullen [28]. Interestingly, both their proof, based on iterations on Teichmüller space, and our proof, based on quasiconformal folding, make essential use of fixed point theorems (an infinite dimensional version in our proof).

• **The smallest transcendental Julia set:** In [17] I construct a transcendental entire function f whose Julia set has Hausdorff dimension 1. This problem had been open since 1975 when Baker [4] proved that $\text{Hdim}(\mathcal{J}(f)) \geq 1$ for all such f (the Julia set always contains a continuum). My example has finite spherical 1-measure and packing dimension 1 (the first with $\text{Pdim}(\mathcal{J}) < 2$), but the following remains open:

Question 2. *Can a transcendental Julia set lie on a rectifiable curve on the sphere?*

The Julia set of $\tan(z)$ is $\mathbb{R}$, so this can occur for meromorphic functions. My "dim =1" example is the transcendental analog to Shishikura's construction [42] of quadratic Julia sets of dimension 2. Similarly, finding a rectifiable example would be analogous to Buff and Cheritat's construction [25] of a positive area polynomial Julia set.

The "dim = 1" example above has an unbounded singular set. Indeed, this must hold since Gwyneth Stallard [43], [44] proved Julia sets for the Eremenko-Lyubich class have Hausdorff dimensions strictly bigger than 1 and that all values in $(1, 2]$ can occur. Stallard's examples are not in the more restrictive the Speiser class, but using a refinement of QC folding, Simon Albrecht and I [1] have shown that

Theorem 19. $\inf\{\dim(\mathcal{J}(f)) : f \in \mathcal{S}\} = 1$.

These are the first Speiser class examples with dimension < 2. Do all dimensions occur?

Question 3. *Is* $\{\text{Hdim}(\mathcal{J}(f)) : f \in \mathcal{S}\} = (1, 2]$*?*

Given an entire function f we let M_f denote the class of quasiconformal deformations of f, i.e., entire functions of the form $g = \psi \circ f \circ \varphi$, where ψ, φ are quasiconformal. For Speiser class functions this is a finite dimensional manifold. The Hausdorff dimension of the Julia set is continuous on M_f, so Question 3 would follow from:

Conjecture 4. *If* $f \in \mathcal{S}$*, then* $\sup\{\text{Hdim}(\mathcal{J}(g)) : g \in M_f\} = 2$.

This is an analog of Shishikura's result [42] about dimensions of quadratic Julia sets tending to 2 near generic points in the boundary of the Mandelbrot set (also analogous to my theorem with Peter Jones that Kleinian limit sets have dimension tending to 2 near most boundary points of Teichmüller space [21]). Possibly Shishikura's proof can be adapted to this case. In the other direction,

Question 5. *Is there an* $f \in \mathcal{S}$ *with* $\inf\{\text{Hdim}(\mathcal{J}(g)) : g \in M_f\} = 1$*?*

However, we do not currently even know any example of a Speiser class f where $\text{Hdim}(\mathcal{J})$ is non-constant on the moduli space M_f, so the last two questions are somewhat ambitious.

Computational Geometry and Optimal Meshing

My work on hyperbolic 3-manifolds motivated me to look at numerical conformal mapping in order to test certain conjectures. This led me to consider rigorous bounds for the work needed to compute conformal maps, which resulted in the fast mapping theorem described earlier. Its proof required various ideas from computational geometry, such as the medial axis (a cousin of the Voronoi diagram). Once the connection was made, it seemed natural that ideas from analysis and hyperbolic geometry might be useful in resolving open problems from discrete and computational geometry.

For example, it is a problem of longstanding theoretical and practical interest to triangulate a polygon with the best possible bounds on the angles used. Many algorithms, such as finite element methods, work best when the associated meshes have well formed elements. The constrained Delaunay triangulation famously maximizes the minimal angle if no additional vertices (called Steiner points) are allowed [35], [36], and algorithms for minimizing the maximum angle (again without Steiner points) are given in [7] and [30]. If Steiner points are allowed, Burago and Zalgaller [26] proved in 1960 that every planar polygon P has an acute triangulation (all angles $< 90°$). This is best possible if we want an angle bound independent of P, and there is now a large collection of theorems, heuristics and applications involving acute triangulations, but several fundamental questions have remained open. What are the optimal angle bounds for triangulating a given polygon P with Steiner points? Are these bounds attained or can they only be approximated?

• **Optimal triangulation of polygons:** Suppose P is a polygon and $\mathcal{T}$ is a triangulation of P. Let V_P be the vertex set of P, $V_{\mathcal{T}}$ the vertex set of $\mathcal{T}$, $\partial\mathcal{T} = V_{\mathcal{T}} \cap P$ the boundary vertices of $\mathcal{T}$, and let $\mathrm{int}(\mathcal{T}) = V_{\mathcal{T}} \setminus \partial\mathcal{T}$ denote the interior vertices. Label each $v \in V_{\mathcal{T}}$ with the number, $L(v)$, of triangles in $\mathcal{T}$ that have v as a vertex. For $v \in \partial\mathcal{T}$, we define its discrete curvature as $\kappa(v) = 3 - L(v)$, and for an interior vertex we set $\kappa(v) = 6 - L(v)$. Using these definitions, Euler's formula applied to a triangulation can be rewritten to look like the Gauss-Bonnet formula:

$$\sum_{v\in\mathrm{int}(\mathcal{T})} \kappa(v) = 6 - \sum_{v\in\partial\mathcal{T}} \kappa(v). \tag{4}$$

We define this common value to be $\kappa(\mathcal{T})$, the curvature of the triangulation.

For $\phi > 0$, a ϕ-triangulation of P is one with all angles at most ϕ. For $\phi \in [60°, 90°]$ define the interval $I(\phi) = [180 - 2\phi, \phi]$; any ϕ-triangulation must have all of its angles in $I(\phi)$. Let $|V_P|$ be the number of vertices in P, and for $v \in V_P$, let θ_v denote the interior angle of P at v. A labeling $L : V_P \to \mathbb{N} = \{1, 2, \dots\}$ is called ϕ-admissible if $\theta_v \in L(v) \cdot I(\phi)$ for every $v \in V_P$. The curvature of a labeling L is $\kappa(L) = 6 - \sum_{v\in V_P} \kappa(v)$.

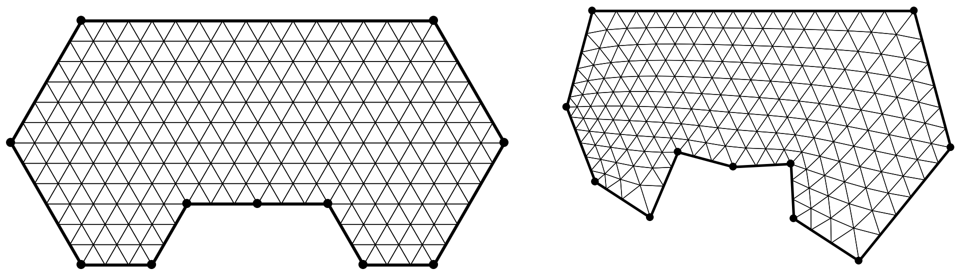
Theorem 20. *For $60° < \phi \leq 90°$, a polygon P has a ϕ-triangulation iff*

(1) $72° \leq \phi < 90°$ *and there is some ϕ-admissible labeling L of V_P,*
(2) $\frac{5}{7} \cdot 90° \leq \phi < 72°$, *and there is a ϕ-admissible labeling with $\kappa(L) \leq 0$,*
(3) $60° < \phi < \frac{5}{7} \cdot 90°$, *and there is a ϕ-admissible labeling with $\kappa(L) = 0$.*

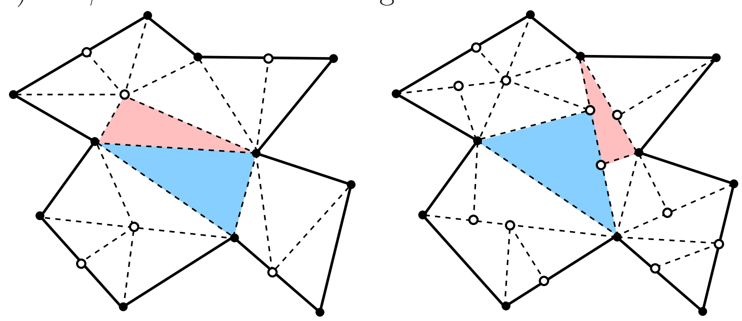
Necessity of these conditions is easy to check using Euler's formula and was observed by Joseph Gerver in 1984 [31]; sufficiency is the hard and surprising part. Using this result, the optimal upper angle bound for any polygon can be computed in time in the number of vertices; this is faster than the best known $O(n^2 \log n)$ algorithm for minimizing the maximum angle without Steiner points in [7]. This is surprising, since usually Steiner points make problems more difficult, or even intractable.

The basic idea is to associate to each polygon P another polygon P' whose angles are all multiples of 60°. Such a P' has a nearly equilateral triangulation and we transfer this triangulation to P via a conformal map. See the figure below. However, in many situations a variety of "tricks" are needed to build P', e.g., P' may need to me a Riemann surface. and P may need to be cut by slits arising from conformal welding problems. One corollary is a strengthening of the 1960 Burago-Zalgaller result.

Corollary 21. *If P has minimum interior angle θ, then P has a triangulation with all angles $\leq 90° - \min(\theta, 36°)/2$, and this is sharp.*



Perhaps the most surprising consequences of Theorem 20 concerns dissections. A triangular dissection covers P and its interior by finitely many closed triangles with disjoint interiors. Adjacent triangles need not match up exactly; if they do, then we have a triangulation. The following figure illustrates the difference between a dissection (left) and a triangulation (right). A ϕ-dissection is a triangular dissection with maximum angle $\leq \phi$.



Corollary 22. *For a polygon P and $\phi \in (60°, 90°]$, the following are equivalent:*

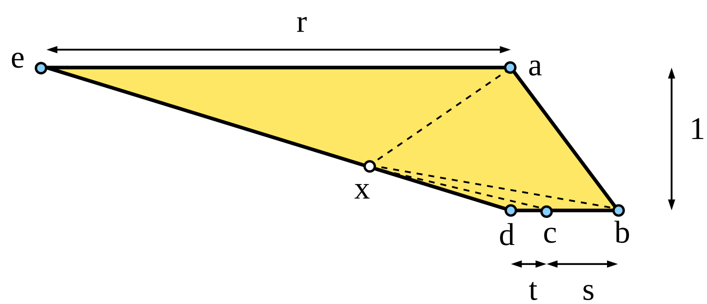
(1) *For every $\epsilon > 0$, P has a $(\phi + \epsilon)$-dissection,*
(2) *P has a ϕ-dissection,*
(3) *P has a ϕ-triangulation.*

Thus ϕ-dissections can be automatically improved to ϕ-triangulations. This is rather shocking and suggests a whole program of improving known results from dissections to triangulations. For example, the Wallace-Bolyai-Gerwien theorem states that any two equal area polygons P_1, P_2 are scissors-congruent, i.e., they can each be dissected by the same set of triangles.

Question 6. *Can two equal area polygons be triangulated by isometric sets of triangles?*

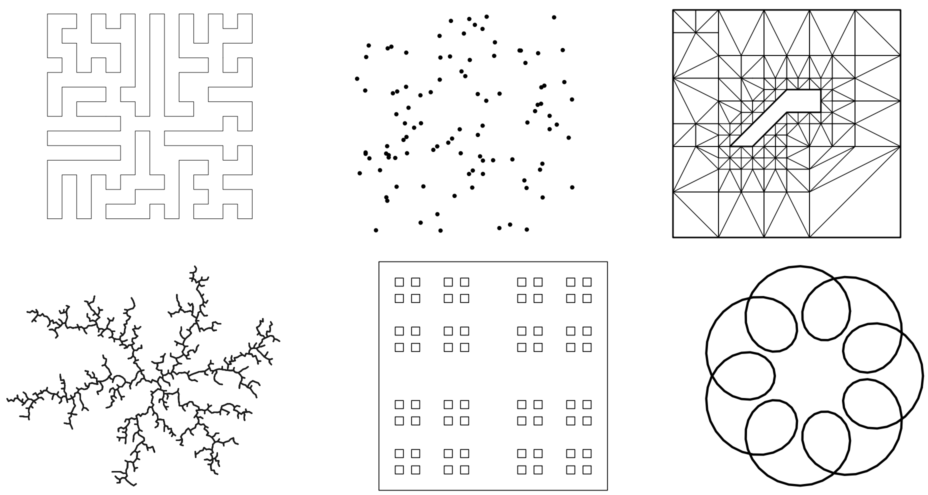
I believe this can be proven by studying a certain novel dynamical system obtained from the dissections of the two polygons. A related approach to meshing is discussed below.

Another consequence of Theorem 20 is that the optimal angle bound is usually achieved by some finite triangulation; the only exceptions are certain polygons whose interior angles are all multiples of 60°. The corresponding problem for minimizing total edge length over all Steiner triangulations is open. The following diagram shows a pentagon for which I can prove no minimal weight Steiner triangulation (MWST) exists:



However, this polygon contains three co-linear vertices, and the remaining question is whether a polygon in general position always has a MWST?

• **Planar straight line graphs:** A planar straight line graph (PSLG) is any finite union of segments and points in the plane; a polygon is a special case when the segments meet end-to-end. Besides a polygon, a PSLG could be a point cloud, a triangulation, a tree, ...; almost anything we can draw. A few examples are given below:



Meshes of PSLGs are important for applications (e.g., modeling an interface between two materials), but the extra constraints along internal edges make them much harder to construct. As with polygons, many applications involving PSLGs benefit from triangulations with good angle bounds and small size, e.g., only a polynomial number of triangles (in terms of the number n of boundary vertices). Easy examples show that polynomial complexity rules out any positive lower bound on angles, and any upper bound that is $< 90°$. Polynomial algorithms giving angle bounds $< 180°$ for PSLGs were found in the 1990's (see [9], [37], [46]), but such a method with the optimal angle bound $90°$ is more recent, [16].

Theorem 23. *Any PSLG with n vertices has a $O(n^{2.5})$ conforming NOT.*

Here NOT stands for **n**on-**o**btuse **t**riangulation; non-obtuse means all angles $\leq 90°$, and conforming means the edges of the triangulation cover the edges of the PSLG. The NOT theorem improves a famous $O(n^3)$ bound of Eldesbrunner and Tan [29] for conforming Delaunay triangulations, and also improves a variety of other optimal triangulation results, e.g., from [8]. The worst known PSLG requires $\simeq n^2$ triangles, and this is likely sharp:

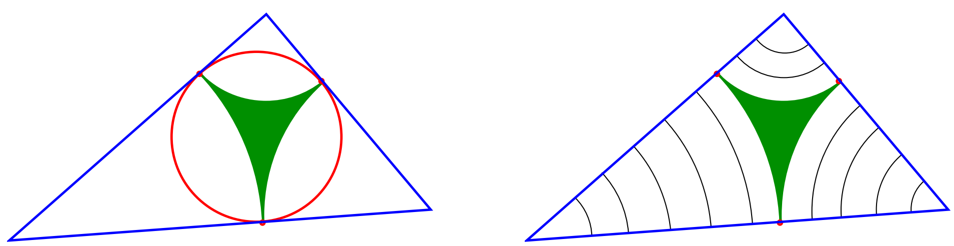
Conjecture 7. *Every PSLG has a conforming NOT with $O(n^2)$ elements.*

Conjecture 8. *Every PSLG has an $O(n^2)$ conforming Delaunay triangulation.*

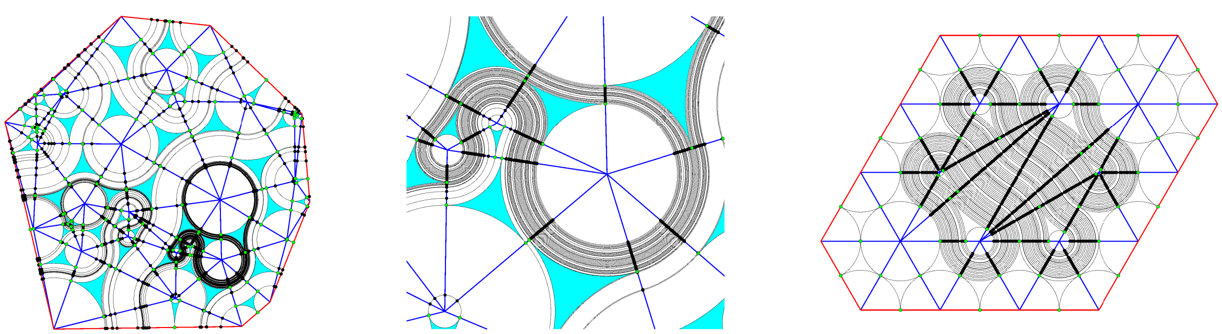
Conjecture 9. *Every PSLG has an $O(n^2)$ conforming Voronoi diagram.*

A Delaunay triangulation is defined by the property that any pair of triangles sharing an edge having opposite angles summing to $\leq \pi$. Given a point set V, the corresponding Voronoi digram is the collection of points that are nearest to two or more different points. Conforming means that the edges of the triangulation or diagram covers the edges of the given PSLG. Any NOT is also Delaunay, and it is easy to build a conforming Voronoi diagram for a NOT. Thus the first conjecture implies the second two, but perhaps all three problems are equivalent: can we prove they have the same complexity (even if we can't determine exactly what that complexity is)?

• **Meshing and dynamics:** Given a PSLG, the NOT algorithm uses a natural flow associated to any triangulation. Given a triangle, take the in-circle as shown at left below. The three tangent points (cusp points) define three disjoint sectors, each of which is foliated by circular arcs centered at a vertex:



We can flow each cusp point along this foliation until it hits another cusp point, or exits the triangulation. The left figure below shows a flow on random triangulation, the center is an enlargement. The right is an example where some paths propagate forever.



The proof of the NOT theorem involves showing that the flow associated to n triangles can be perturbed in a precise way so that the average number of triangles each flow line hits is only $O(n)$. To make this work, I currently have to add up to $O(\sqrt{n})$ new flow lines for each original one, and these are also propagated for $O(n)$ steps, giving the $O(n^{2.5})$ bound. In order to get this uniform complexity bound, the algorithm bends paths to terminate them faster; in order to maintain the desired angle bounds the bending is limited by constraints that closely resemble keeping a discrete second derivative of the propagation paths bounded. The fact that the bending process is constrained by something that looks like a discrete derivative bound is reminiscent of Pugh's closing lemma: every C^1 vector field has a small perturbation with a closed orbit [39], [40], [41]. This is still open for C^2 vector fields and Dennis Sullivan suggested there might be a connection:

Question 10. *Can a closing lemma help prove the $O(n^2)$ NOT-theorem? Can the NOT argument help prove a C^2-closing lemma (or suggest a counterexample)?*

So far as I know, these triangle flows have not been studied before, so essentially all reasonable questions are open and interesting.

• **NOTs in 3 dimensions:** Solving the conjectures above would essentially complete the theory of optimal triangulation in $\mathbb{R}^2$, but the corresponding theory using tetrahedra in $\mathbb{R}^3$ (the really important case for applications) is wide open:

Question 11. *Do polyhedra in $\mathbb{R}^3$ have non-obtuse tetrahedralizations of polynomial size?*

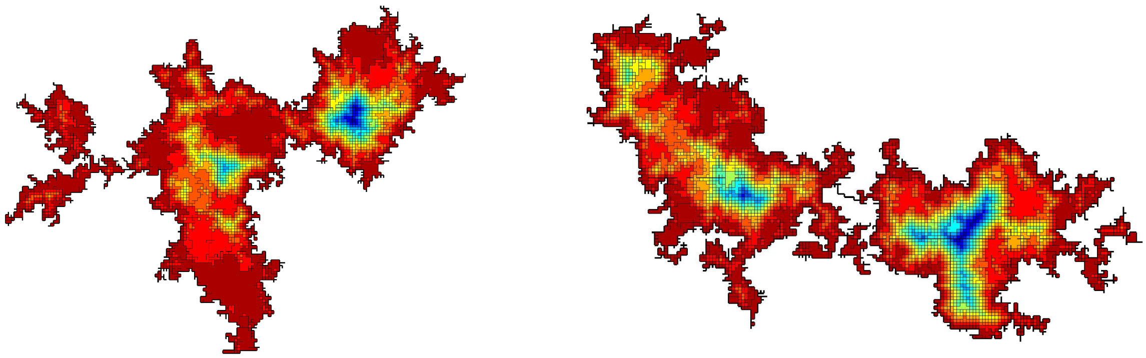
Even finding an acute tetrahedralization (all angles $< 90°$) of a cube in $\mathbb{R}^3$ was open until recently (the smallest known example uses 1,370 pieces [47]) and there is no acute decomposition for the cube in $\mathbb{R}^4$, [34]. My work in 2-dimensional meshing uses a thick/thin decomposition of a polygon (analogous to the thick/thin decomposition of a hyperbolic manifold) to partition the polygon into two types of regions where Euclidean and hyperbolic geometry respectively are used to create the mesh. Can we use analogous ideas in $\mathbb{R}^3$? Can one create a 3-manifold out of a polyhedron, run a Ricci flow on it (as in Perelman's proof of Thurston's geometrization conjecture) to decompose it into pieces with geometric structure and then utilize the "natural" geometries on the different pieces to define meshes? An intermediate problem between 2 and 3 dimensions is to find NOTs for triangulated surfaces in $\mathbb{R}^3$. The proof of the NOT theorem uses properties of planar geometry that may not hold on a general polyhedral surface. Is there always a uniform polynomial bound for a non-obtuse refinement of a triangulated surface, or does the size of a NOT necessarily depend on the geometry, e.g., the curvature of the surface?

Geometry of Random Sets

• **Werner's conjecture:** Brownian motion has been intensely studied for over a century, but some of its basic geometric properties remain unknown. One of my favorite such problems was formulated by Wendelin Werner. Consider the Brownian trace in $\mathbb{R}^2$, i.e., the image of $[0, 1]$ under Brownian motion. This is a compact random set with infinitely many complementary components.

Conjecture 12. *Can any two complementary components be connected by a path that hits the trace only finitely often?*

Werner's problem is illustrated by the following figures. They show 2 random walks on a square grid, where the number of steps from each complementary component to the unique unbounded component is color coded: red is close to the outer component and blue is far. A counterexample would correspond to there being components with diameter bounded away from zero, but arbitrarily many "steps" from the unbounded component, i.e., a "big blue" component. Numerical simulations indicate that component diameters decrease like a negative power of the step distance to the unbounded component, supporting the conjecture.



An alternative version of Werner's question asks if every point of the Brownian trace can be surrounded by arbitrarily small closed curves that each only hit the trace finitely often. This would imply that the topological Hausdorff dimension is $\mathrm{tHdim}(B([0,1])) = 1$ (see [5]; $\mathrm{tHdim}(K) \leq 1 + \alpha$ if K has a neighborhood basis whose elements all have boundaries of Hausdorff dimension $\leq \alpha$).

• **Jordan curves inside the trace:** If Werner's conjecture fails, then $B([0,1])$ contains a continuum disjoint from every frontier. Since this continuum lies in a "very dense" part of trace, perhaps it contains a "fairly straight" curve, at least in the sense of dimension. See the figures below showing shortest paths in the trace of a random walk on a grid. Burdzy [27], defines percolation dimension as $\mathrm{Hdim}_{\mathrm{perc}}(K) = \inf\{\dim(\gamma) : \gamma \subset K, \gamma \text{ a Jordan arc}\}$.

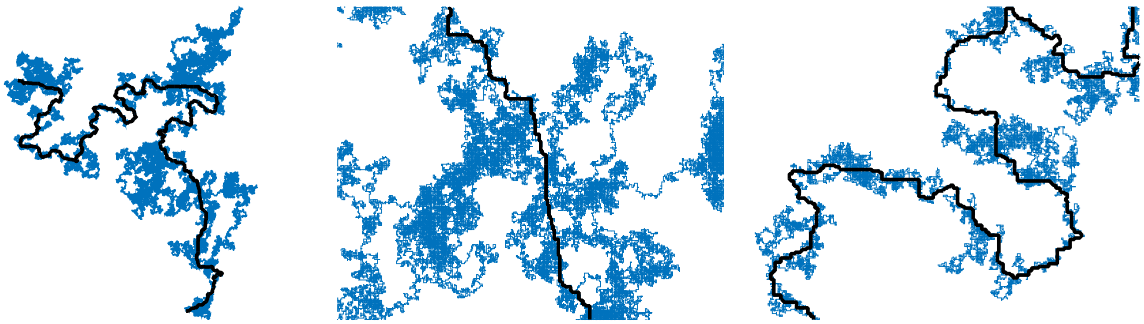
Question 13. *Is the percolation dimension of the Brownian trace strictly bigger than* 1*?*

Is the percolation dimension attained by some curve? Can any two points of the trace be connected by a minimal dimension curve? A result of Lawler and Werner shows Brownian frontiers (boundaries of the complementary components) have dimension 3/4. These frontiers are Jordan curves, so the percolation dimension of Brownian motion is $\leq 3/4$. This has recently been improved to 5/4 by Dapeng Zhan [48]. If the answer is 1, we can ask a stronger question:

Question 14. *Does* $B([0,1])$ *almost surely contain a rectifiable curve? Can any two points of* $B([0,1])$ *be connected by a rectifiable curve in the trace?*

Pemantle [38] showed that the Brownian trace almost surely contains no line segments (or even any positive length subset of a segment).

A few years ago an undergraduate student of mine, Shalin Parekh (currently in the PhD program at Columbia), wrote an honors thesis with me using random walks on a square grid to estimate the percolation dimension of $B([0,1])$ as ≈ 1.02. The sizes of his random walks were fairly small and the numerical result is too close to 1 to be decisive, but the pictures generated are very suggestive. Here is a random walk with 10^7 steps, a shortest path between two points and two blow-ups of different portions of the path:

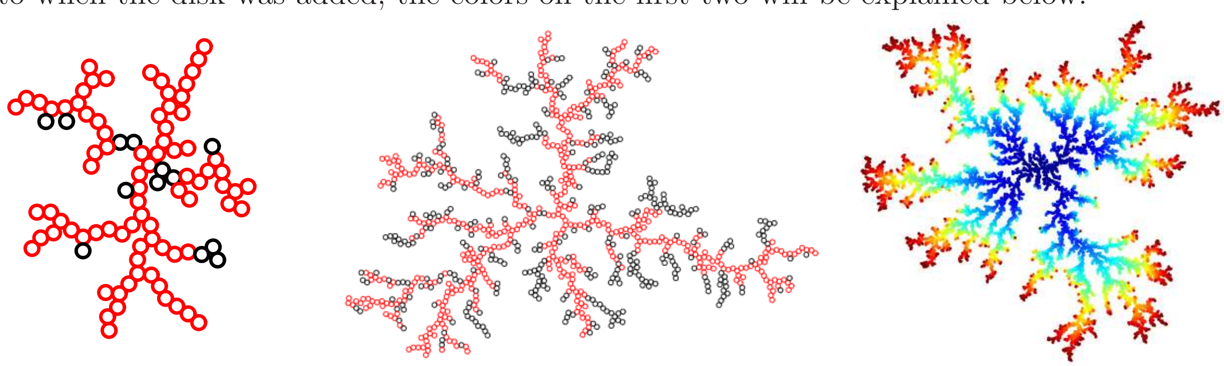


In the "thick" portions of the trace the shortest path seems quite straight, but when the path runs between two adjacent complementary components, Ω_1, Ω_2, it is forced to travel through the corresponding crossing set, $\partial\Omega_1 \cap \partial\Omega_2$, which suggests we ask

Question 15. *Is the intersection of two frontiers a.s. contained in a rectifiable curve?*

The intersection of two frontiers has dimension 3/4 almost surely, so it is small enough to lie on a rectifiable curve, if not too much length is needed to connect it. Perhaps Jones's traveling salesman theorem can be used to answer this, by computing estimated sizes of β-numbers and the distribution of dyadic squares hitting two frontiers.

• **Diffusion limited aggregation:** One of the most interesting problems involving harmonic measure is the growth rate of diffusion limited aggregation. DLA is defined by fixing a unit disk at the origin and sending in a second unit disk moving by Brownian motion from infinity until it touches the first disk. Successive disks are added in the same way. The main problem is to determine the almost sure growth rate $\alpha = \limsup_n \frac{1}{n} \log \operatorname{diam}(\mathrm{DLA}(n))$. Some DLA clusters with $n = 100, 1000, 10000$ are shown below. The last one is colored according to when the disk was added; the colors on the first two will be explained below.



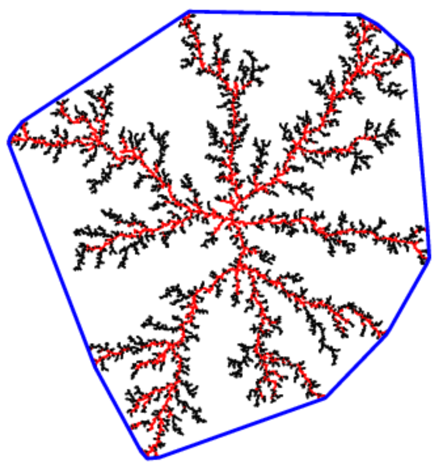
Obviously $\operatorname{diam}(\mathrm{DLA}(n)) \leq 2n$, but Harry Kesten [33] improved this to $O(n^{2/3})$ almost surely; this remains the best known upper bound even 30 years later. The trivial lower bound is $\gtrsim \sqrt{n}$ (consider the areas), and shockingly, this is still the best known:

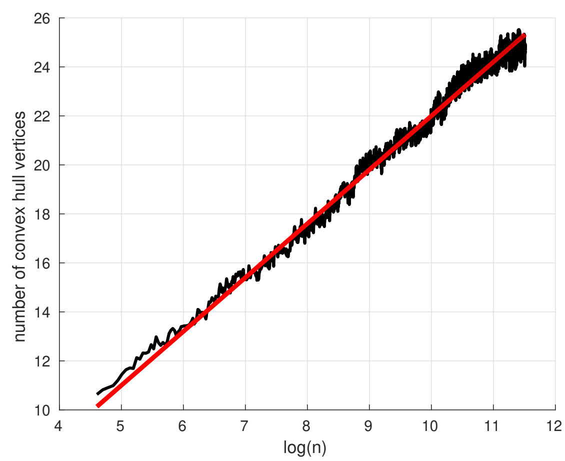
Conjecture 16. $\lim_n \operatorname{diam}(\mathrm{DLA}(n))/\sqrt{n} = \infty$ *almost surely.*

To prove this, we need to quantify that the "tips" of DLA have larger harmonic measure than the trivial $1/n$ estimate. One way to do this is to show there are few such tips. It seems reasonable to think of vertices of the convex hull of DLA to be the set of these tips:

Question 17. *Is the number of convex hull vertices $O(\log n)$ almost surely? If this is true, can we deduce Conjecture 16? A growth rate $\alpha > 1/2$?*

In the figures on the left and center above, disks are colored red if they were vertices of the convex hull when added. The pictures indicate this is fairly common, and numerical simulations strongly support the logarithmic growth rate of such points. See below for a plot of a DLA cluster and its convex hull. Also shown is a plot of the number of vertices in the convex hull as a function on $\log n$ (averaged over 100 random trials). The plot clearly looks linear as a function of $\log n$.





Further Information

Here are some links to documents that also discuss some some of topics above (and some others that I am interested in). If this statement is viewed in a web browser or PDF reader, clicking each link should open the corresponding document.

- A list of my publications with a brief description of each paper:
  http://www.math.stonybrook.edu/ bishop/vita/pubs23long.pdf
- Links to my publications and preprints are posted here:
  http://www.math.stonybrook.edu/ bishop/papers
- An expository account of how some of the topics in this statement tie together (written for the collection "All that math: portraits of mathematicians as young readers" celebrating the centennial of the Royal Spanish Mathematical Society in 2011):
  http://www.math.stonybrook.edu/ bishop/papers/rmi.pdf
- My contribution the 2018 ICM proceedings:
  http://www.math.stonybrook.edu/ bishop/vita/icm.pdf
- Links to my recent NSF proposals and their reviews are posted on:
  http://www.math.stonybrook.edu/ bishop/vita/
- My book "Fractals in Probability and Analysis" with Peres (2017, CUP):

http://www.math.stonybrook.edu/ bishop/fractalbook.pdf

- Lecture notes on transcendental dynamics (incomplete): https://www.math.stonybrook.edu/ bishop/classes/math627.S13/itd.pdf
- Lecture notes on numerical conformal mapping (incomplete): https://www.math.stonybrook.edu/ bishop/classes/math626.F08/rmt.pdf

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