RESEARCH PLANS
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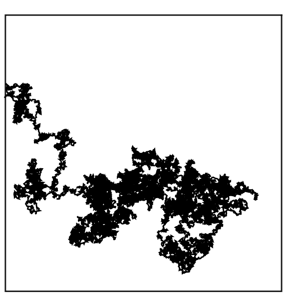
My current research interests include various topics in geometric function theory, holomorphic dynamics, computational geometry and probability theory. Below I describe some specific problems I am thinking about. More detailed descriptions can be found in my past and current NSF proposals, and the unifying role of harmonic measure in all these problems is described in my 2018 ICM article “Harmonic measure: algorithms and applications”. Links to these are given at `http://www.math.stonybrook.edu/~bishop/vita`.

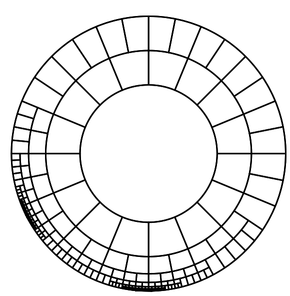
Harmonic measure, trees and triangulations

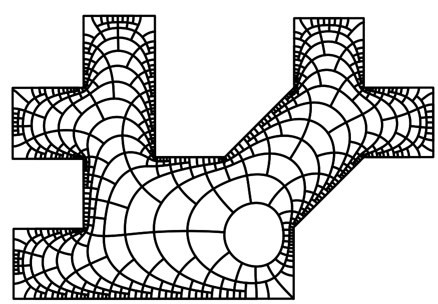
• **Harmonic measure:** The most intuitive definition of harmonic measure is as the boundary hitting distribution of Brownian motion. More precisely, suppose $\Omega \subset \mathbb{R}^n$ is a domain (open and connected) and $z \in \Omega$. We start a random particle at z and let it run until the first time it hits $\partial\Omega$. See left figure below. We will assume this happens almost surely; this is true for all bounded domains in $\mathbb{R}^n$ and many, but not all, unbounded domains. Then the first hit defines a probability measure on $\partial\Omega$. The measure of $E \subset \partial\Omega$ is usually denoted $\omega(z, E, \Omega)$ or $\omega_z(E)$. For E fixed, $\omega(z, E, \Omega)$ is a harmonic function of z on Ω, hence the name “harmonic measure”. Therefore harmonic measure can also be defined using the Riesz representation theorem: if f is continuous on $\partial\Omega$ and u is its harmonic extension to Ω then $f \to u(z)$ is a bounded linear functional on $C(\partial\Omega)$, so there is a measure ω_z so that

$$u(z) = \int_{\partial\Omega} f(x) d\omega_z(x).$$

This is harmonic measure again. Finally, for simply connected planar domains, harmonic measure for z is the also image of normalized length measure on the unit circle under a conformal map from the disk to Ω taking 0 to z (because Brownian motion is conformally invariant). This allows allows much of complex function theory to be applied, at least in 2 dimensions. The two right figures below illustrate a conformal map from the disk to a polygon; harmonic measures of various edges can be estimated by the number of dyadic grid boxes needed to cover them.







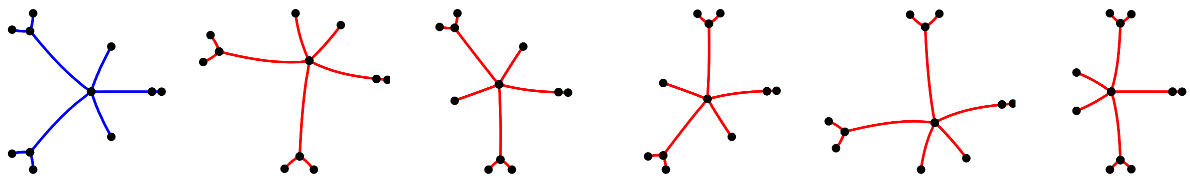
Harnack’s inequality implies that $\omega(z, E, \Omega)$ is either the constant 0 or 1 or is strictly between 0 and 1 on all of Ω. Thus the null sets for harmonic measure are independent of z. Much of my previous work centers around the geometric properties of harmonic measure, such as rectifiability, and its connections to dynamics and probability. Such results from my previous work are described in the survey mentioned above; here I will discuss some problems I am currently thinking about.

• **True trees:** It is not hard to prove that if the harmonic measures for two points on opposite sides of a closed Jordan curve are exactly the same measure, then the curve is a circle, and the points are reflections of each other. A more interesting version is to ask if any finite planar tree T can be drawn so that harmonic measure is equal on "both sides" of each edge? More precisely, with base point equal to infinity, can we draw T so that

(1) every edge has equal harmonic measure,

(2) any subset of any edge has equal harmonic measure from both sides?

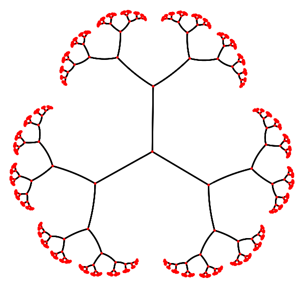
Perhaps surprisingly, the answer is yes, every finite planar tree T has such drawing, unique up to similarities, called the "true form of the tree" (or a "true tree" for short). Here are a few examples with 10 edges.

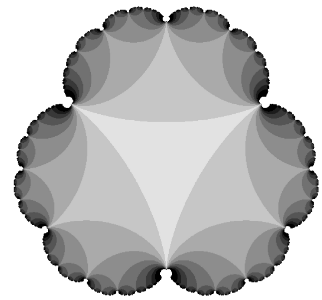


The fact that every tree can be drawn in this way is well known, and closely related to the uniformization theorem: given a planar tree T, we can connect each vertex to infinity, so as to obtain a topological triangulation of the sphere where each side of the tree occurs as one side of a triangle. If we identify each topological triangle with an equilateral triangle, this gives the sphere a conformal structure, which must agree with the usual one by the uniformization theorem, and the true form of T is its image under identification with the usual Riemann sphere.

This is a special case of Grothendieck's theory of *dessins d'enfants*, in which a finite graph on a topological surface imparts a conformal structure to the surface (in our case, a tree on the sphere). Associated to each true tree is a polynomial that has critical values exactly ± 1. This polynomial has algebraic coefficients ad gives rise to an action of the absolute Galois group on planar trees, i.e., an action that permutes finite trees, although a general description of the orbits is unknown e.g., two trees in the same orbit must have the same vertex degree sequences, but this is not sufficient. The blue tree (leftmost) above is a fixed point of the action, and the red ones (the rest) form a single orbit.

Because of the relation to balanced harmonic measure, conformal maps between the exteriors of true trees with the same number of vertices have analytic extensions across the interiors of every edge; this allows us to "glue" the exteriors in a natural way to form Riemann surfaces covering a punctured plane. Is the orbit structure of the trees be related to these surfaces and relations among them? There are also a variety of interesting problems about particular families of true trees. For example, it has been observed that taking the true forms of finite truncations of the infinite 3-regular tree gives a sequence of trees that seems to converge to the "deltoid fractal" (see right side below), a fractal set that arises in the iteration theory of anti-holomorphic dynamics and reflection maps. See below. Can we prove this convergence?





• **Triangulating surfaces by folding:** As noted above, true trees are related to equilateral triangulations of the sphere, and these triangulations are related to polynomials that have exactly three critical points (± 1 and ∞). Is this true for all Riemann surfaces? A holomorphic function $f : X \to \mathbb{S}^2$ on a Riemann surface X is called a Belyi function if f is branched only over 0, 1 and ∞, and f has no removable singularities at punctures of X. The latter condition implies that f cannot be holomorphically extended to a Riemann surface properly containing X. For example, the polynomials associated to true trees above are Belyi functions for the Riemann sphere.

Question 1. *Do all open Riemann surfaces have Belyi functions? All planar domains?*

If the surface X has a Belyi function f, then the preimages of the upper and lower half-planes are topological triangles in G which are conformally equivalent to equilateral triangles glued (according to arclength) along their edges. Thus the previous question really asks if every open Riemann surface can be constructed from a countable collection of equilateral triangles glued along their boundaries. If we also assume the Belyi functions have no asymptotic values (limits along curves tending to ∞ on the surface) then these triangles are compact, and we get a standard triangulation. More concisely:

Conjecture 2. *Every open Riemann surface has an equilateral triangulation.*

This is not true for compact surfaces: by a famous theorem of Belyi [7], a compact surface X has a Belyi function (= has an equilateral equilateral triangulation) iff it is algebraic. It is also easy to see directly that there are only countably many ways to glue equilateral triangles together to get a compact surface (compactness implies only finitely many triangles per surface). But for open surfaces, we can use countable triangulations and there are uncountable many ways to glue them together.

Above we saw that finite planar trees can be associated to polynomials with exactly two critical values. Recently I developed a method called quasiconformal folding, which associates entire functions with two critical values to certain infinite planar trees (an infinite version of *dessins d'enfants* which is still mostly unexplored).

Roughly speaking, a quasiconformal map is a homeomorphism that maps infinitesimal circles to ellipses of eccentricity at most K; taking $K = 1$ gives conformal maps. Quasiregular maps are defined similarly but need not be 1-1, i.e., quasiregular is to quasiconformal as holomorphic is to conformal. Indeed, every quasiregular map in two dimensions is a holomorphic map pre-composed with a quasiconformal one. The idea of QC folding is to explicitly build a quasiregular function with the desired properties and then make it holomorphic by solving a Beltrami equation; in many applications the holomorphic function can be taken to be as close to the quasiregular model as we wish. The QC-folding method is very flexible and can be used to construct quasiregular analogs of Belyi functions on any Riemann surface. However, solving the Beltrami equation to make this quasiregular function holomorphic may change the conformal structure of the surface, i.e., the holomorphic function is defined on a different Riemann surface than the quasiregular model was. Moreover, the case of compact surfaces shows this does, in fact, occur. For non-compact surfaces, however, it should be possible to alternate applications of the folding construction on compact sub-surfaces (which we can choose to alter the conformal structure only slightly), with conformal correction maps that "push" the perturbed sub-surface back into X. I am currently pursuing this approach with Lasse Rempe-Gillen.

• **Holomorphic embeddings:** Similar difficulties arise in constructions of minimal surfaces in $\mathbb{R}^3$ and holomorphic embeddings of Riemann surfaces, e.g., the famous and well studied Bell-Narasimhan Conjecture (e.g., page 20, [6]):

Conjecture 3. *Every open Riemann surface has a proper holomorphic embedding in* $\mathbb{C}^2$.

Certain approaches (e.g., [27]) use Runge's theorem, but have to "cut out" certain regions where the Runge approximation may be too large (Runge's theorem gives no control of the approximating function off the set where the approximation is made). QC-folding gives similar approximations by entire or meromorphic functions, but these also come with growth bounds everywhere on the plane in terms of the approximation set and the desired degree of approximation. Perhaps this "Quantitative Runge's Theorem", can be applied to some aspects of the embedding problem?

• **Transcendental dynamics:** Transcendental dynamics refers to the iteration theory of non-polynomial entire functions. We let $\mathcal{T}$ denote this class of functions. As usual, the Fatou set $\mathcal{F}$ is the maximal open set where the iterates of f form a normal family and the Julia set $\mathcal{J}$ is its complement (and is always non-empty). While similar to polynomial dynamics in many respects, there are several significant differences: wandering domains can exist, Fatou components of any finite or infinite multiplicity may occur, the escaping set $I(f) = \{z : f(z) \to \infty\}$ plays a more prominent role (and has interesting subsets based on rates of escape), the Julia set always contains a non-trivial continuum, and it is generally harder to build "small" Julia sets than "large" ones, in the sense of fractal dimensions.

The singular set of an entire function f is the closure of its critical values and finite asymptotic values (limits of f along curves to ∞); the complement of the singular set is the largest open set where f^{-1} is always well defined locally. In transcendental dynamics much attention is devoted to the Eremenko-Lyubich class (transcendental entire functions with bounded singular set) and the Speiser class (finite singular set). My quasiconformal folding method [12] is well adapted to constricting examples in these classes based on the desired geometry. QC-folding has already solved several longstanding problems, such as the existence of wandering domain in the Eremenko-Lyubich class, disproving Adam Epstein's order conjecture (that quasiconformally equivalent Speiser class functions have the same order or growth), and finding a Speiser class counterexample to the strong Eremenko conjecture (connected components of the escaping set are path connected).

Recall that Hausdorff, upper Minkowski and packing dimension are defined as

$$\mathrm{Hdim}(K) = \inf\{s : \inf\{\textstyle\sum_j r_j^s : K \subset \cup_j D(x_j, r_j)\} = 0\},$$
$$\overline{\mathrm{Mdim}}(K) = \inf\{s : \limsup_{r\to 0} \inf_N N r^s = 0 : K \subset \cup_{j=1}^N D(x_j, r)\},$$
$$\mathrm{Pdim}(K) = \inf\{s : K \subset \cup_{j=1}^\infty K_j : \overline{\mathrm{Mdim}}(K_j) \le s \text{ for all } j\}.$$

It is elementary that $\mathrm{Hdim} \le \mathrm{Pdim} \le \overline{\mathrm{Mdim}}$. We shall see below that these can differ for transcendental Julia sets.

• **The smallest transcendental Julia set** In [14] I construct a transcendental entire function f whose Julia set has Hausdorff dimension 1. This had been open since 1975 when Baker [4] proved that $\mathrm{Hdim}(\mathcal{J}(f)) \ge 1$ for all such f, by showing every such Julia set contains a non-degenerate continuum. This example has finite spherical 1-measure, and packing dimension 1 (the first with $\mathrm{Pdim}(\mathcal{J}) < 2$), but the following remains open:

Question 4. *Can a transcendental Julia set lie on a rectifiable curve on the sphere?*

The Julia set of $\tan(z)$ is $\mathbb{R}$, so this can occur for meromorphic functions. My "dim =1" example is the transcendental analog to Shishikura's construction [36] of quadratic Julia sets of dimension 2. Similarly, finding a rectifiable example would be analogous to Buff and Cheritat's construction [19] of a positive area polynomial Julia set. Question 4 seems very delicate, and I have ideas for both constructing such an example and for proving it can't exist. The Fatou components in my example are infinitely connected, which leads to the infinite packing measure and the impossibility of connecting the Julia set by a finite length curve. Kisaka and Shishikura [25] have constructed examples in $\mathcal{T}$ with annular Fatou components, and I believe similar examples can also be constructed by QC-folding. Can we combine all these ideas (perhaps with characterizations of rectifiability from [3], [23]) to produce a positive answer to Question 4?

• **Small Julia sets in the Speiser class:** The "dim = 1" example discussed above has an unbounded singular set. Indeed, this must hold for any such example: Gwyneth Stallard [37], [38] proved Julia sets for the Eremenko-Lyubich class have Hausdorff dimensions strictly bigger than 1 and all values in $(1, 2]$ can occur. However, Rippon and Stallard [35] proved the packing dimension for Eremenko-Lyubich Julia sets is always 2, so the two dimensions need not be the same, even in "nice" cases. Stallard's examples are not in the more restrictive the Speiser class (finite singular set), but using a refinement of QC folding, Simon Albrecht and I [16] have shown that $\inf\{\dim(\mathcal{J}(f)) : f \in \mathcal{S}\} = 1$. These are the first Speiser class examples with dimension < 2. Do all dimensions occur?

Question 5. *Is* $\{\mathrm{Hdim}(\mathcal{J}(f)) : f \in \mathcal{S}\} = (1, 2]$*?*

Given an entire function f we let M_f denote the class of quasiconformal deformations of f, i.e., entire functions of the form $g = \psi \circ f \circ \varphi$, where ψ, φ are quasiconformal. For Speiser class functions this is a finite dimensional manifold. The Hausdorff dimension of the Julia set is continuous on M_f, so Question 5 would follow from:

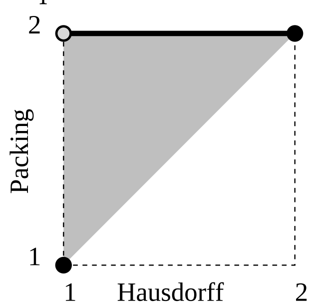
Conjecture 6. *If* $f \in \mathcal{S}$*, then* $\sup\{\text{Hdim}(\mathcal{J}(g)) : g \in M_f\} = 2$.

This is an analog of Shishikura's result [36] about dimensions of quadratic Julia sets tending to 2 near generic points in the boundary of the Mandelbrot set (also analogous my theorem with Peter Jones that Kleinian limit sets have dimension tending to 2 near most boundary points of Teichmüller space [17]). Possibly Shishikura's proof can be adapted to this case. In the other direction,

Question 7. *Is there an* $f \in \mathcal{S}$ *with* $\inf\{\text{Hdim}(\mathcal{J}(g)) : g \in M_f\} = 1$*?*

However, we do not currently even know any example of a Speiser class f where $\text{Hdim}(\mathcal{J})$ is non-constant on the moduli space M_f, so the last two questions are somewhat ambitious.

• **Packing dimension:** As noted above, the Hausdorff and packing dimensions of transcendental Julia sets can be different, but only a few examples of this are known. The gray triangle below shows the possible pairs $1 \le \text{Hdim} \le \text{Pdim} \le 2$ and the black parts denote all known transcendental examples: the vertex $(2, 2)$ is due to Misiurewicz [29] (see also McMullen [28]); the top edge $(t, 2)$, $1 < t < 2$ is due to Stallard [37], [38]; $(1, 1)$ is my example.



Question 8. *For each* $1 < s < t < 2$ *is there a* $f \in \mathcal{T}$ *with* $\text{Hdim}(\mathcal{J}) = s$ *and* $\text{Pdim}(\mathcal{J}) = t$*? (gray triangle)*

Question 9. *Is there a transcendental Julia set with* $\text{Hdim}(\mathcal{J}) = 1, \text{Pdim}(\mathcal{J}) = 2$*? (upper left corner)*

Question 10. *For* $t \in (1, 2)$*, is there a* $f \in \mathcal{T}$ *with* $\text{Hdim}(\mathcal{J}(f)) = \text{Pdim}(\mathcal{J}(f)) = t$*? (diagonal edge)*

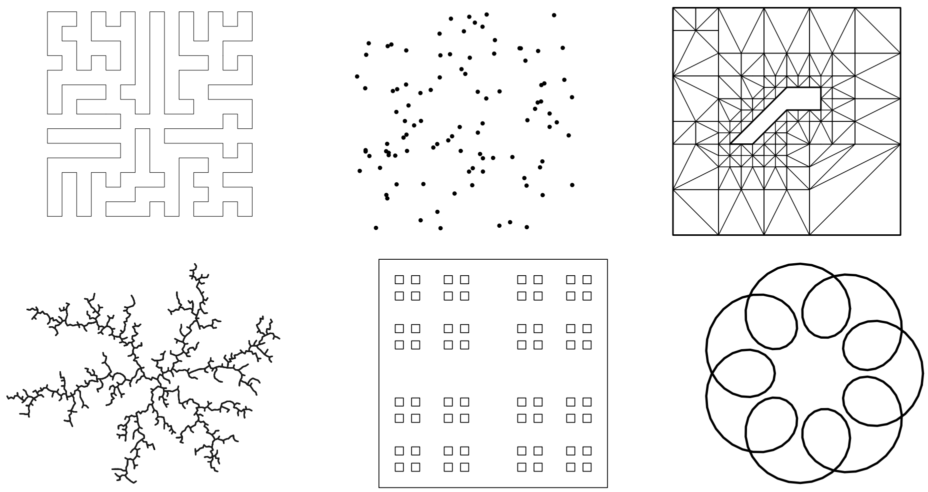
Jack Burkart, a current PhD student of mine, has shown that for any $1 \le s < t \le 2$ there is a transcendental Julia set with $s < \text{Hdim} \le \text{Pdim} < t$, so the diagonal edge in the diagram above is in the closure of occurring pairs. We do not currently know if Jack's examples have equal Hausdorff and packing dimension, or whether all packing dimensions between 1 and 2 occur. Surprisingly, the analogous questions for polynomials also seem to be open:

Question 11. *Are the Hausdorff and packing dimensions of a polynomial Julia set always equal? If not, what pairs* $0 < \text{Hdim} \le \text{Pdim} \le 2$ *can occur?*

The two dimensions are known to be equal in many cases (e.g., hyperbolic examples).

Computational geometry and optimal meshing

• **Optimal triangulation:** A planar straight line graph (PSLG) is any finite union of segments and points in the plane; a polygon is a special case when the segments meet end-to-end. Besides a polygon, a PSLG could be a point cloud, a triangulation, a tree, . . . ; almost anything we can draw. A few examples are given below:



Meshes for numerical PDE and other problems generally perform better if the angles are not too small or too large, so there is great interest in triangulating domains bounded by polygons or PSLGs with angle bounds strictly between 0° and 180°, and using a small number of triangles, e.g., only a polynomial number of them (as a function of the number n of boundary vertices). Easy examples show that polynomial complexity rules out any uniform lower bound on angles (consider a long narrow rectangle), and also any upper bound that is less than 90° (since the angles sum to 180°, an upper bound $< 90°$ implies a strictly positive lower bound).

Polynomial algorithms giving 90° for simply polygons and larger angle bounds for PSLGs were found in the 1990's (see [9], [30], [40]), but the first polynomial bound with optimal angles for PSLGs is more recent: I proved in [13] that any PSLG with n vertices has a $O(n^{2.5})$ conforming **n**on-**o**btuse **t**riangulation (called a NOT for brevity; non-obtuse means all angles $\leq 90°$, conforming means the edges of the triangulation cover the edges of the PSLG). The NOT theorem improves a famous $O(n^3)$ bound of Eldesbrunner and Tan [21] for conforming Delaunay triangulations, and also improves a variety of other optimal triangulation results, e.g., from [8]. The worst known PSLG requires $\simeq n^2$ for a non-obtuse triangulation and this is likely sharp:

Conjecture 12. *Every PSLG has a conforming NOT with $O(n^2)$ elements.*

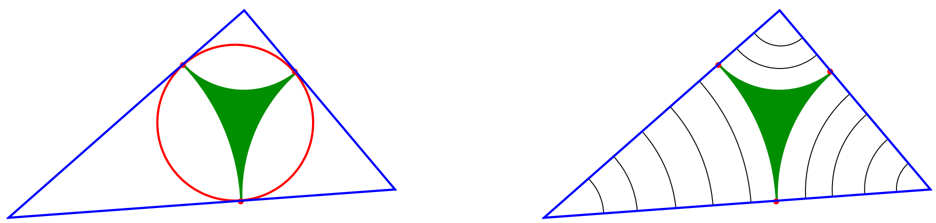
Conjecture 13. *Every PSLG has an $O(n^2)$ conforming Delaunay triangulation.*

Conjecture 14. *Every PSLG has an $O(n^2)$ conforming Voronoi diagram.*

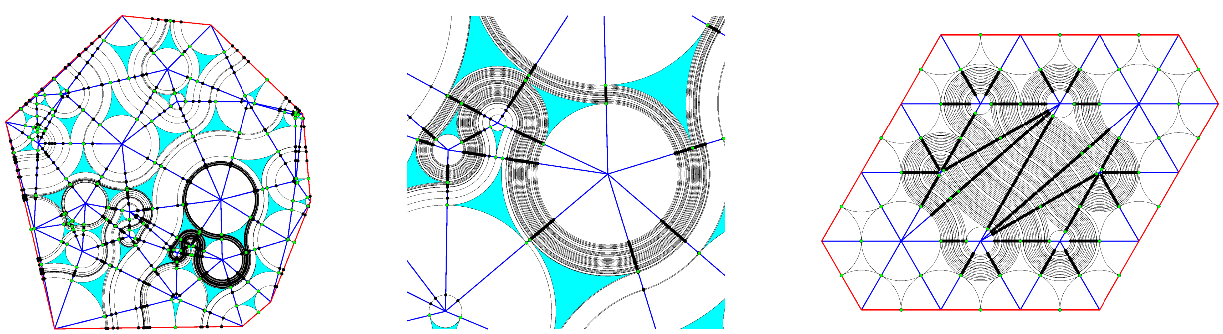
A Delaunay triangulation is defined by the property that any pair of triangles sharing an edge having opposite angles summing to $\leq \pi$. Given a point set V, the corresponding Voronoi digram is the collection of points that are nearest to two or more different points.

Conforming means that the edges of the triangulation or diagram covers the edges of the given PSLG. It is obvious that a NOT is also Delaunay, and it is easy to build a conforming Voronoi diagram for a NOT by placing six points in each triangle in a certain way. Thus the first conjecture is stronger than the second two, but I have never found any approach for the latter cases that simplified the proof, or gave a better estimate, than in the NOT problem. Perhaps all three problems are equivalent to each other: can we prove they have the same complexity (even if we can't determine exactly what that complexity is)?

• **Meshing and dynamics:** Given a PSLG, the NOT algorithm uses a natural flow associated to any triangulation. Given a triangle, take the in-circle as shown at left below. The three tangent points (cusp points) define three disjoint sectors, each of which is foliated by circular arcs centered at a vertex:



We can flow each cusp point along this foliation until it hits another cusp point, or exits the triangulation. The left figure below shows a flow on random triangulation, the center is an enlargement. The right is an example where some paths propagate forever.

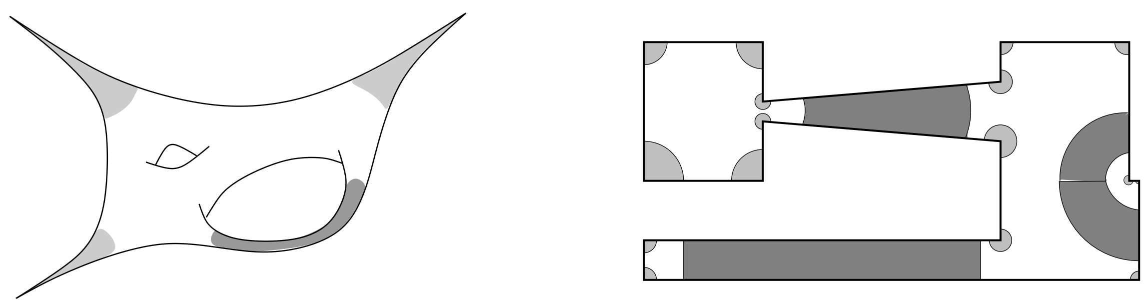


The proof of the NOT theorem involves showing that the flow associated to n triangles can be perturbed in a precise way so that the average number of triangles each flow line hits is only $O(n)$. To make this work, I currently have to add up to $O(\sqrt{n})$ new flow lines for each original one, and these are also propagated for $O(n)$ steps, giving the $O(n^{2.5})$ bound. In order to get this uniform complexity bound, the algorithm bends paths to terminate them faster; in order to maintain the desired angle bounds the bending is limited by constraints that closely resemble keeping a discrete second derivative of the propagation paths bounded. The fact that the bending process is constrained by something that looks like a discrete derivative bound is reminiscent of Pugh's closing lemma: every C^1 vector field has a small perturbation with a closed orbit [32], [33], [34]. This is still open for C^2 vector fields and Dennis Sullivan suggested there might be a connection:

Question 15. *Can a closing lemma help prove the $O(n^2)$ NOT-theorem? Can the NOT argument help prove a C^2-closing lemma (or suggest a counterexample)?*

The propagation paths described above define return maps on the triangle edges that preserve length, so perhaps the theory of interval exchange maps or billiards in polygons is also relevant to these problems. Possibly the theory of translations surfaces is also involved. So far as I know, these triangle flows have not been studied before, so essentially all reasonable questions are open and interesting.

• **Thick and thin parts:** I also have similar results for meshing with quadrilaterals rather than triangles [11]: every PSLG on size n can be meshed by $O(n^2)$ quadrilaterals with all angles between 60° and 120° (except for smaller angles in the original boundary, which are left unchanged). Both the complexity and angle bounds are sharp. One tool that I developed for optimal quad-meshing is a method for decomposing any polygon into disjoint thick and thin pieces that are analogous to the thick/thin pieces of a hyperbolic manifold (regions where the injectivity radius is larger/smaller than some ϵ). See the figure below. On the left are the thin parts of a surface and on the right the thin parts of a polygon:



For an n-gon, each thin piece is either a neighborhood of a vertex (parabolic thin parts), or corresponds to a pair of sides that have small extremal distance within Ω (hyperbolic thin parts); the thin parts are in 1-to-1 correspondence with the thin parts of the n-punctured Riemann sphere formed by gluing two copies of the polygon along its (open) edges. Despite there being $\simeq n^2$ pairs of edges, there are only $O(n)$ thin parts, and they can be found in time $O(n)$ using the iota-map, a fast quasiconformal approximation to the conformal map. See [10]. For the application to quad-meshing, ad hoc explicit constructions are used inside the thin parts, and in the thick parts approximate conformal maps are used to transfer meshes from the hyperbolic disk to the thick part. Thus the thick/thin decomposition of the polygon breaks it up into regions where we use Euclidean and hyperbolic geometry, respectively, to generate the mesh.

Polygons satisfy an analog of Mumford compactness: a family of n-gons is non-compact iff the thin parts degenerate; is this useful for anything, e.g., "soft" proofs of constructive results? It would be interesting if this could be exploited to give "compactness" proofs of various complexity results about polygons. The thick/thin decomposition is also a key component in my linear time conformal mapping algorithm [10], and I suspect it might have other numerical applications.

• **NOTs in 3 dimensions:** Solving Conjectures 12-14 would mostly complete the theory of optimal triangulation in $\mathbb{R}^2$, but the corresponding theory using tetrahedra in $\mathbb{R}^3$ (the really important case for applications) is wide open:

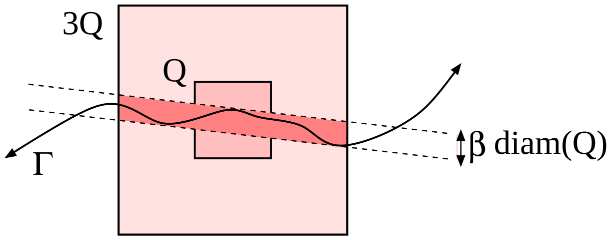
Question 16. *Do polyhedra in $\mathbb{R}^3$ have non-obtuse tetrahedralizations of polynomial size?*

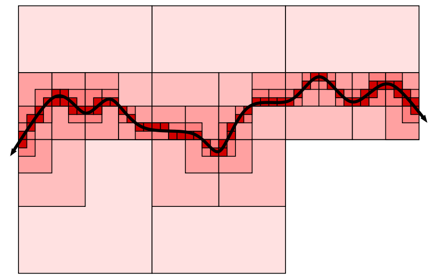
Even finding an acute tetrahedralization (all angles $< 90°$) of a cube in $\mathbb{R}^3$ was open until recently (the smallest known example uses 1,370 pieces [41]) and there is no acute decomposition for the cube in $\mathbb{R}^4$, [26]. As noted above, my work in 2-dimensional meshing uses a thick/thin decomposition of a polygon to partition the polygon into two types of regions where Euclidean and hyperbolic geometry respectively are used to create the mesh. Can we use analogous ideas in $\mathbb{R}^3$? Can one create a 3-manifold out of a polyhedron, run a Ricci flow on it (as in Perelman's proof of Thurston's geometrization conjecture) to decompose it into pieces with geometric structure and then utilize the "natural" geometries on the different pieces to define meshes? An intermediate problem between 2 and 3 dimensions is to find NOTs for triangulated surfaces in $\mathbb{R}^3$. The proof of the NOT theorem uses properties of planar geometry that may not hold on a general polyhedral surface. Is there always a uniform polynomial bound for a non-obtuse refinement of a triangulated surface, or does the size of a NOT necessarily depend on the geometry, e.g., the curvature properties of the surface?

Applications of hyperbolic geometry

Above I described the application of thick/thin decompositions and the iota map, ideas coming from hyperbolic manifolds, to 2 dimensional meshing and conformal mapping. Below I describe some other problems where 3-dimensional hyperbolic geometry makes an unexpected appearance.

• **Weil-Petersson curves and hyperbolic convex hulls:** Quasicircles are closed Jordan curves Γ in the plane that are quasiconformal images of circles. These can be characterized by the Ahlfors M-condition: if z is on the smaller diameter arc of Γ with endpoints x, y then $|z - x| \leq M|x - y|$ for some $M < \infty$. In general, quasicircles can be fractals, such as the von Koch snowflake. Quasicircles can be identified with points of universal Teichmüller space (usually identified as quasi-symmetric homeomorphisms of the unit circle to itself), and it was a long standing problem to put a metric on this space that corresponds to the Weil-Petersson metric on finite dimensional Teichmüller spaces of Riemann surface; this was of interest for applications in, e.g., string theory and computer vision. Such a metric was found by Takhtajan and Teo [39]; it gives universal Teichmüller space the structure of a Hilbert manifold, and the finite dimensional metrics can be derived from it. However, the Takhtajan-Teo topology is disconnected and it was an open problem to give a geometric characterization of the connected component that contains all smooth curves: this sub-class of quasicircles is called the Weil-Petersson class.





For a curve Γ, Peter Jones' β-numbers $\beta_\Gamma(Q)$ measures the deviation of $\Gamma \cap 3Q$ from a straight line where Q is a dyadic square in the plane. See left-hand figure above. Jones used them to characterize rectifiable curves: his "traveling salesman theorem" implies that Γ has finite length iff

$$\sum_Q \beta_\Gamma(Q)^2 \mathrm{diam}(Q) < \infty,$$

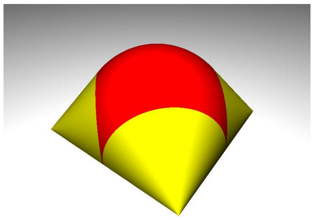
where the sum is over all dyadic squares in the plane that hit Γ (see right-hand figure above). I proved in [15] in [15] that Γ is Weil-Petersson iff

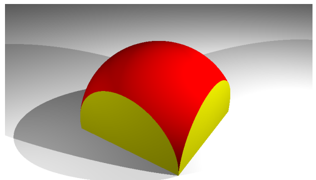
$$\sum_Q \beta_\Gamma(Q)^2 < \infty,$$

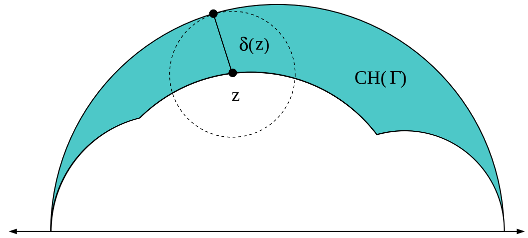
and I give a number of natural alternative conditions that are equivalent to this one. One of the most surprising involves the hyperbolic convex hull of Γ in the upper half-space $\mathbb{R}^3_+ = \mathbb{R}^3_+$. This is the union of all hyperbolic geodesics (half-circles perpendicular to $\mathbb{R}^2$ with endpoints in Γ. If Γ is a circle then its hyperbolic convex hull, $\mathrm{CH}(\Gamma)$ is a hemi-sphere of zero hyperbolic volume, but otherwise it has non-empty interior and is bounded by two surfaces, each of which meets $\mathbb{R}^2$ along Γ, e.g., the left and center figures below show the lower and upper boundaries of the hyperbolic convex hull of a square. For z in one of these two boundary surfaces we let $\delta(z)$ be the hyperbolic distance to other boundary component; this measures the "width" of the convex hull near z. See rightmost figure below. I prove in [15] that Γ is Weil-Petersson iff

$$\int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z) dA(z) < \infty,$$

where dA denotes hyperbolic surface area on the boundary of the convex hull. This is very reminiscent of a result of Brock [18] that says the Weil-Petersson distance between two Riemann surfaces X, Y (the same topological surface Σ, but with different conformal structures) is approximately the volume of the convex core of an associated 3-manifold with X and Y as boundaries at infinity. In this case, the volume of the convex core is comparable to the L^2 norm of the function δ defined above, so it would be interesting to see if the two results can be unified.







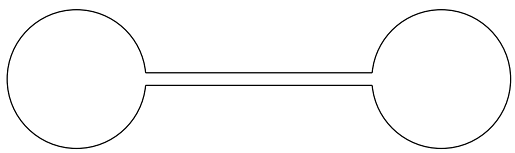
• **Minimal surfaces and renormalized area:** One consequence of the hyperbolic convex hull characterization of Weil-Petersson curves is that $\Gamma \subset \mathbb{R}^2$ is Weil-Petersson if and only if it is the boundary of a minimal surface $S \subset \mathbb{R}^3_+$ with finite total curvature, i.e., $\int_S |K|^2 dA$ where K is the norm of the second fundamental form of S. This leads to a number of questions that I am currently investigating, one of which concerns the idea of renormalized area. A surface $S \subset \mathbb{R}^3_+$ that has a non-degenerate curve $\Gamma \subset \mathbb{R}^2$ as its asymptotic boundary must have infinite hyperbolic area. However Robin Graham and Ed Witten [22] have introduced the idea of a renormalized area that is finite if the boundary curve is sufficiently smooth. Alexakis and Mazzeo [1], [2] have shown $C^{3,\alpha}$ is sufficient. I show that if Γ is Weil-Petersson (and such curves need not be even C^1) then the associated minimal surface has finite renormalized still area. Conversely,

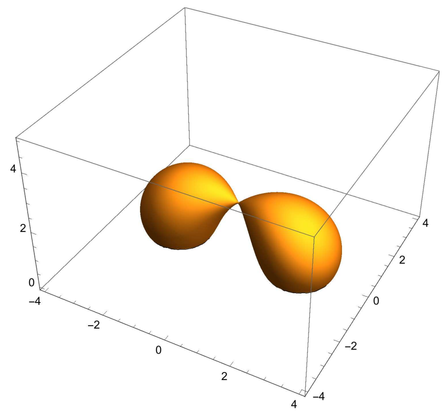
Conjecture 17. *A curve $\Gamma \subset \mathbb{R}^2$ is Weil-Petersson iff it is the asymptotic boundary of a minimal surface in hyperbolic space that has finite renormalized area.*

The main idea of my proof is a "discrete approximation" of the minimal surface. I show the renormalized area of this object is finite iff the boundary curve is Weil-Petersson, and finiteness for the discrete approximation implies it for the actual minimal surface. The opposite implication is what remains open. Renormalized area is relevant conformal field theory and quantum entanglement, and I would also like to understand the connections, if any, of Weil-Petersson curves to physics, and see whether my results in 2 and 3 dimensions extend to the settings relevant to these topics.

• **Some exotic 4-manifolds:** Claude LeBrun and I recently constructed the first examples of anti-self-dual 4-manifolds N, so that the almost-Kähler metrics on N form an non-empty but proper subset of the moduli space. The simplest case is to start with a hyperbolic 3-manifold M that is homeomorphic to $\mathbb{R} \times \Sigma$ where Σ is a compact surface. Such a 3-manifold has a harmonic function u that tends to 0 in one end of M and tends to 1 in the other end (u is called the tunnel vision function, or the harmonic measure of one end, since it measures the probability that a Brownian particle will tend to infinity down one end or the other). On the universal cover u lifts to the harmonic measure for one side of the limit set Γ of the Kleinian group associated to M (in this setting, this limit set is a Jordan curve, indeed, a quasicircle). To get a 4-manifold, we collapse the two ends of $M \times \mathbb{T}$ to two points; this gives a conformally flat 4-manifold N (but a hierarchy of topologically distinct non-flat examples also exists). One can show that this conformal metric is conformal to an almost-Kähler metric if and only if the tunnel vision function doesn't have any critical points.

Thus the construction is reduced to building quasi-Fuchsian Kleinian groups whose limit set is a quasicircle in the plane with the property that the harmonic function in the hyperbolic upper half-space with boundary values 0 on one side of the curve and 1 on the other side, has a critical point. Our example uses a group with a large number of generators and a limit set chosen to approximate a "dogbone" contour (see left below). At a certain height, the harmonic measure for the dogbone approximates the harmonic measure for two disjoint disks, whose level sets go from being disconnected to connected (see right below), indicating the existence of a critical point.





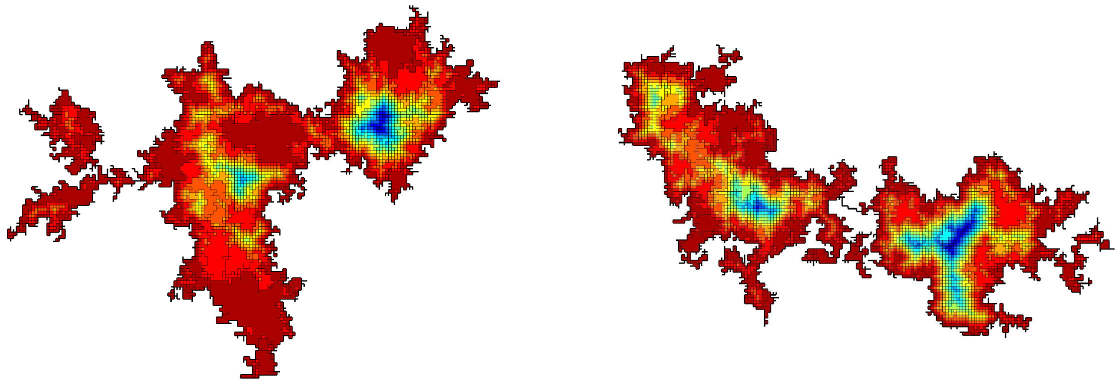
However, many questions remain. For which planar domains Ω does $\omega(z, \Omega, \mathbb{R}^3_+)$ have a critical point? Does it matter whether we consider Euclidean or hyperbolic harmonic functions? How few generators are needed to create an example? Can we locate "small" examples using numerical experiments? Are critical points common for groups near the boundary of Teichmüller space for any large G, e.g., near degenerate limit sets in the boundary? Are such examples "common" or "'rare" in Teichmüller space, e.g., finite or infinite volume?

Geometry of random sets

• **Werner's conjecture:** Brownian motion has been intensely studied for over a century, but some of its basic geometric properties, even in the plane, remain unknown. One of my favorite such problems was formulated by Wendelin Werner. Consider the Brownian trace in $\mathbb{R}^2$, i.e., the image of $[0, 1]$ under Brownian motion. This is a compact random set with infinitely many complementary components.

Conjecture 18. *Can any two complementary components be connected by a path that hits the trace only finitely often?*

Werner's problem is illustrated by the following figures. They show 2 random walks on a square grid, where the number of steps from each complementary component to the unique unbounded component is color coded: red is close to the outer component and blue is far. A counterexample would correspond to there being components with diameter bounded away from zero, but arbitrarily many "steps" from the unbounded component, i.e., a "big blue" component. Numerical simulations indicate that component diameters decrease like a negative power of the step distance to the unbounded component, supporting the conjecture.



An alternative version of Werner's question asks if every point of the Brownian trace can be surrounded by arbitrarily small closed curves that each only hit the trace finitely often. This would imply that the topological Hausdorff dimension is $\text{tHdim}(B([0,1])) = 1$ (see [5]; $\text{tHdim}(K) \leq 1 + \alpha$ if K has a neighborhood basis whose elements all have boundaries of Hausdorff dimension $\leq \alpha$).

• **Jordan curves inside the trace:** If Werner's conjecture fails, then $B([0,1])$ contains a continuum disjoint from every frontier. Since this continuum lies in a "very dense" part of trace, perhaps it contains a "fairly straight" curve, at least in the sense of dimension. See the figures below showing shortest paths in the trace of a random walk on a grid. Burdzy [20], defines percolation dimension as $\text{Hdim}_{\text{perc}}(K) = \inf\{\dim(\gamma) : \gamma \subset K, \gamma \text{ a Jordan arc}\}$.

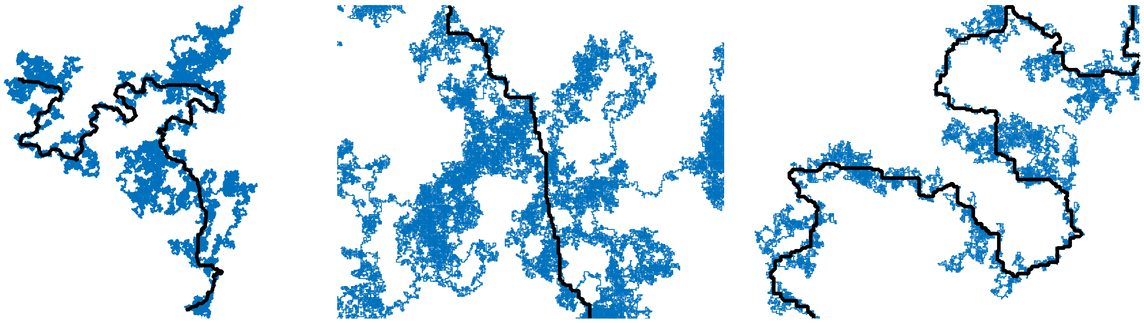
Question 19. *Is the percolation dimension of the Brownian trace strictly bigger than* 1*?*

Is the percolation dimension attained by some curve? Can any two points of the trace be connected by a minimal dimension curve? A result of Lawler and Werner shows Brownian frontiers (boundaries of the complementary components) have dimension $3/4$. These frontiers are Jordan curves, so the percolation dimension of Brownian motion is $\leq 3/4$. This has recently been improved to $5/4$ by Dapeng Zhan [42]. If the answer is 1, we can ask a stronger question:

Question 20. *Does* $B([0,1])$ *almost surely contain a rectifiable curve? Can any two points of* $B([0,1])$ *be connected by a rectifiable curve in the trace?*

Pemantle [31] showed that the Brownian trace almost surely contains no line segments (or even any positive length subset of a segment).

A few years ago an undergraduate student of mine, Shalin Parekh (currently in the PhD program at Columbia), wrote an honors thesis with me using random walks on a square grid to estimate the percolation dimension of $B([0,1])$ as ≈ 1.02. The sizes of his random walks were fairly small and the numerical result is too close to 1 to be decisive, but the pictures generated are very suggestive. Here is a random walk with 10^7 steps, a shortest path between two points and two blow-ups of different portions of the path:

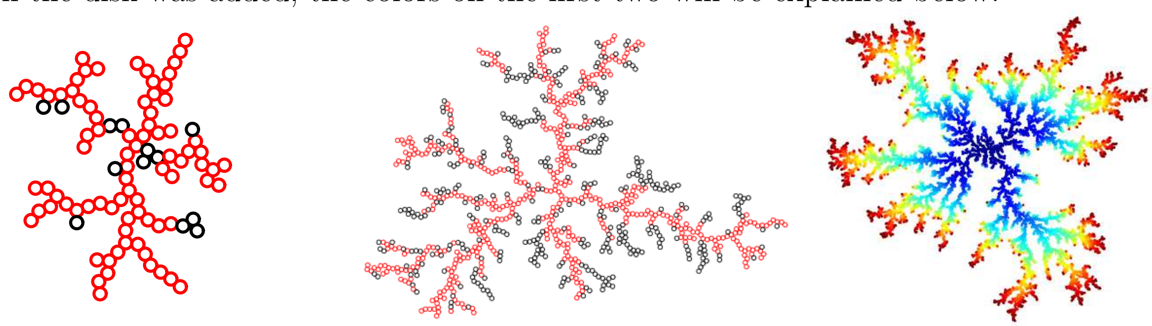


In the "thick" portions of the trace the shortest path seems quite straight, but when the path runs between two adjacent complementary components, Ω_1, Ω_2, it is forced to travel through the corresponding crossing set, $\partial\Omega_1 \cap \partial\Omega_2$, which suggests we ask

Question 21. *Is the intersection of two frontiers a.s. contained in a rectifiable curve?*

The intersection of two frontiers has dimension $3/4$ almost surely, so it is small enough to lie on a rectifiable curve, if not too much length is needed to connect it. Perhaps Jones' traveling salesman theorem can be used to answer this, by computing estimated sizes of β-numbers and the distribution of dyadic squares hitting two frontiers.

• **Diffusion limited aggregation:** One of the most interesting problems involving harmonic measure is the growth rate of diffusion limited aggregation. DLA is defined by fixing a unit disk at the origin and sending in a second unit disk moving by Brownian motion from infinity until it touches the first disk. Successive disks are added in the same way. The main problem is to determine the almost sure growth rate $\alpha = \limsup_n \frac{1}{n} \log \operatorname{diam}(\mathrm{DLA}(n))$. Some DLA clusters with $n = 100, 1000, 10000$ are shown below. The last one is colored according to when the disk was added; the colors on the first two will be explained below.



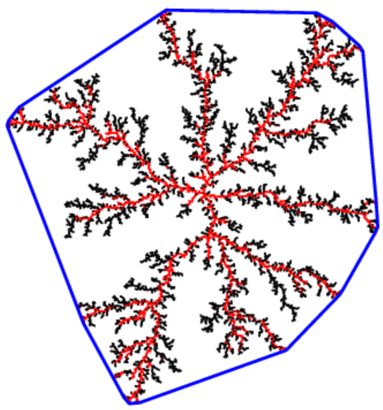
Obviously $\operatorname{diam}(\mathrm{DLA}(n)) \leq 2n$, but Harry Kesten [24] improved this to $O(n^{2/3})$ almost surely; this remains the best known upper bound even 30 years later. The trivial lower bound is $\gtrsim \sqrt{n}$ (consider the areas), and shockingly, this is still the best known:

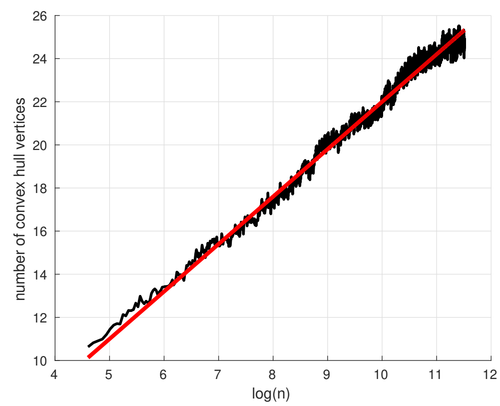
Conjecture 22. $\lim_n \operatorname{diam}(\mathrm{DLA}(n))/\sqrt{n} = \infty$ *almost surely.*

To prove this, we need to quantify that the "tips" of DLA have larger harmonic measure than the trivial $1/n$ estimate. One way to do this is to show there are few such tips. It seems reasonable to think of vertices of the convex hull of DLA to be the set of these tips:

Question 23. *Is the number of convex hull vertices $O(\log n)$ almost surely? If this is true, can we deduce Conjecture 22? A growth rate $\alpha > 1/2$?*

In the figures on the left and center above, disks are colored red if they were vertices of the convex hull when added. The pictures indicate this is fairly common, and numerical simulations strongly support the logarithmic growth rate of such points. See below for a plot of a DLA cluster and its convex hull. Also shown is a plot of the number of vertices in the convex hull as a function on $\log n$ (averaged over 100 random trials). The plot clearly looks linear as a function of $\log n$.





Books

I recently published "Fractals in probability and analysis" with Yuval Peres, a graduate level introduction to fractals, dimension, Brownian motion and other related topics. I am currently working on several book projects: "Conformal Fractals" with Yuval Peres (dealing with harmonic measure, Julia sets, Kleinian limit sets, DLA, SLE,...), "Quasiconformal Maps and Applications" (an introduction to QC mappings and their applications to function theory and dynamics, including a self-contained account of the folding theorem), and "The Riemann Mapping Theorem" (an introduction to conformal mapping and the various numerical methods for computing Riemann maps).

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