SUMMARY OF RESULTS FROM PRIOR NSF SUPPORT

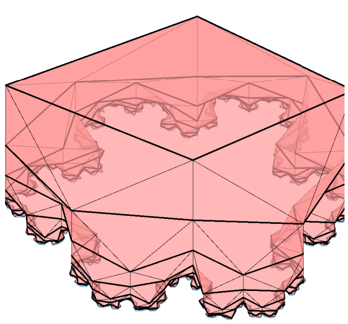
Quasiconformal Constructions in Analysis and Dynamics

DMS-1906259, 9/1/19 - 9/1/22, $269,102

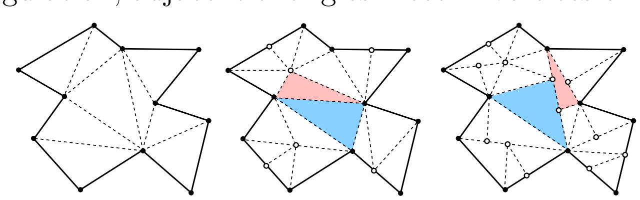
INTELLECTUAL MERIT OF PREVIOUS WORK

My recent work has focused on understanding the Weil-Petersson class of closed curves, optimal meshing algorithms for planar domains, constructing equilateral triangulations of Riemann surfaces, and a variety of results involving holomorphic dynamics and quasiconformal analysis. Some highlights are described below.

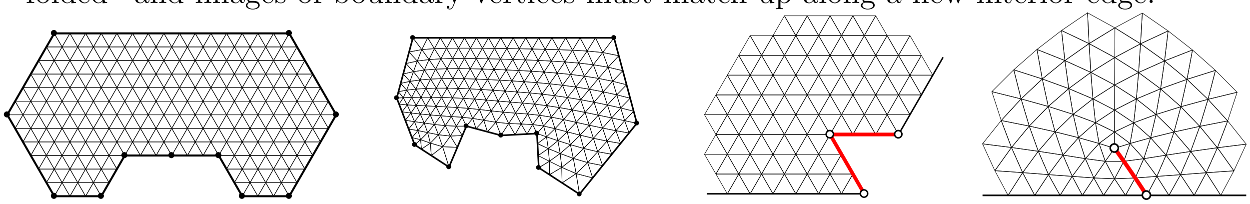
• **Weil-Petersson curves [28],[29]:** Motivated by questions in string theory, Takhtajan and Teo defined a Weil-Petersson metric on universal Teichmüler $T(1)$ (a space of closed curves in plane, moduli similarities), generalizing the usual definition for finite dimensional Teichmüller spaces associated to Riemann surfaces. The resulting topology is disconnected, and the Weil-Petersson class is the connected component of T(1) that contains the smooth curves (it is also the closure of the smooth curves). Besides string theory, these curves arise naturally in computer vision (see broader impacts) and are closely connected to Schramm-Loewner evolutions (SLE). No geometric characterization of Weil-Petersson curves was previously known, but my paper [29] gives 20 equivalent definitions in terms of various quantities such as Peter Jones's β-numbers, Sobolev parameterizations, lengths of inscribed curves, and knot energies. Particularly interesting are the connections to hyperbolic geometry and minimal surfaces: I prove in [29] that $\Gamma \subset \mathbb{R}^2$ is Weil-Petersson iff it is the boundary of a minimal surface S in the hyperbolic upper half-space $\mathbb{R}^3_+$ that has finite total curvature (sectional curvatures κ_j are in L^2 for hyperbolic area on S). Using this, I show the asymptotic boundary Γ of a finite total curvature surface need not be C^1, answering a question of de Oliveira [50]. I prove that Γ is Weil-Petersson iff the minimal surface S has finite renormalized area (introduced by Graham and Witten [65]), and give a formula for renormalized area, valid for all Jordan curves Γ, extending a result of Alexakis and Mazzeo [3], [4] for $C^{3,\alpha}$ curves. One of the novel ideas introduced in [29] is the "dyadic dome" (see picture at left). This is a polyhedral surface with horizontal cross sections that project vertically to inscribed polygons in the curve Γ, and that has a "horizontal"' projection onto the minimal surface corresponding to Γ, thus providing the connection between the Euclidean and hyperbolic characterizations. The dyadic dome should be useful for both theoretical investigations and numerical calculations of minimal surfaces in hyperbolic space.



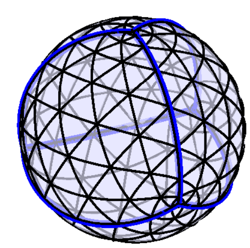
• **Optimal triangulation of polygons [31]:** Efficient meshing and triangulation of polygons is one of the fundamental problems of computational geometry with numerous applications (see broader impacts). In a triangulation, adjacent triangles meet in vertices or full edges; but in a dissection, triangles can meet along subsets of edges (compare the center and right figures). Triangulation vertices other than the polygon's vertices are called Steiner points (left versus center picture). Charles Lawson [81] proved in 1977 that the famous Delaunay triangulation maximizes the minimum angle (MaxMin) over the finitely many triangulations of a polygon when no Steiner points are allowed, and algorithms for minimizing the maximum angle (MinMax) in this case are given in [15], [54].



In [31] I solve both the MaxMin and MinMax problems when Steiner points are permitted (the most common case in applications). Since the space of such triangulations is infinite dimensional, it is not obvious that extremal triangulations even exist, but I prove that they do (except in one special case, explained later). Moreover, I give a linear-time algorithm for computing the optimal angle bound $\Phi(P)$ for any given polygon P. A consequence is that a polygon with minimal interior angle θ has a triangulation with all angles in $[\theta, \max(72°, 90° - \theta/2)]$. This sharpens a 1960 result of Burago and Zalgaller [46] that every polygon has an acute triangulation (angles $< 90°$). Surprisingly, the optimal bounds for polygonal triangulations are the same as for triangular dissections, proving, in a stronger form, a 1984 conjecture of Gerver. My proof in [31] involves conformally mapping an equilateral triangulation from a model domain (sometimes a Riemann surface) to the target polygon. The two left pictures show a simple model/target pair. The two right pictures illustrate a conformal welding problem arising in some cases: the model is conformally "folded" and images of boundary vertices must match up along a new interior edge.



- **Traveling salesman for Jordan curves [32]:** Peter Jones's traveling salesman theorem (TST) estimates the length of the shortest curve Γ passing through a given set $E \subset \mathbb{R}^2$ in terms of "β-numbers", $\beta_E(Q)$, that measure the "flatness" of the set E inside a dyadic square Q (a precise definition will be given later). In the special case when $E = \Gamma \subset \mathbb{R}^n$ is a Jordan curve, I prove that $\ell(\Gamma) = |z - w| + O[\sum_Q \beta_E^2(Q)\mathrm{diam}(Q)]$, where z, w are the endpoints of Γ. This sharpens Jones's original theorem, and is used in the paper on WP-curves above.
- **Equilateral triangulations of Riemann surfaces [40]:** A Belyi function on a Riemann surface R is a holomorphic map from R to the Riemann sphere S^2 that is branched over only three points (usually $0, 1, \infty$). A theorem of Voevodsky and Shabat [118] says R has a Belyi function iff it can be constructed by gluing together equilateral triangles. An equilateral triangulation of the 2-sphere is shown at right. There are only countably many ways to glue together a finite number of equilateral triangles, so at most countably many compact surfaces have this property. A famous theorem of Belyi [13] says these are exactly the compact surfaces defined over number fields; this result inspired Grothendieck's theory of *dessins d'enfants*. However, Lasse Rempe and I used my quasiconformal folding method [24] to show that



every non-compact Riemann surface has uncountably many combinatorial distinct Belyi functions. In particular, this proves that every Riemann surface is a holomorphic cover of the 2-sphere branched over finitely many points (for compact surfaces this follows from the Riemann-Roch theorem). Our construction also produces finite type dynamical systems (as defined by Epstein [55]) on hyperbolic surfaces. All previously known examples were defined on parabolic surfaces, e.g., rational maps on S^2, or Speiser class entire functions on $\mathbb{C}$.
- **Post-singular orbits of meromorphic functions [37], [38]:** Kirill Lazebnik and I show that any dynamics on any discrete planar sequence can be approximated by the postsingular dynamics of some transcendental meromorphic function. This generalizes work of DeMarco, Koch and McMullen [51] for finite sets and rational functions. Subsequently, with Mariusz Urbanski, we extended this to any discrete set in any planar domain D. A

lemma of independent interest is that for any continuous $f > 0$ on a planar domain D, there is an equilateral triangulation of D dominated by f ($z \in T$ implies $\mathrm{diam}(T) \leq f(z)$).

• **Small transcendental Julia sets [2], [27]:** In 1975 Baker proved that if f is a transcendental (i.e., non-polynomial) entire function, then its Julia set contains a continuum and hence has Hausdorff dimension $d \geq 1$. Examples with all dimensions $1 < d < 2$ are due to Gwyneth Stallard, who also proved $d > 1$ if f has bounded singular set. Simon Albrecht and I proved [2] that dimensions arbitrarily close to 1 occur for functions with only three singular values. In [27], I gave the first example of a transcendental Julia set attaining dimension $d = 1$. This was also the first example with packing dimension < 2, and the first with a multiply connected Fatou component where the dynamics can be completely described.

• **Conformal removability is hard [30]:** I prove that compact planar sets that are removable for conformal homeomorphisms are not a Borel subset of the space of all compact subsets with the Hausdorff metric. By contrast, removable sets for bounded holomorphic functions are a G_δ set (and were famously characterized by Tolsa [110], [111]).

• **Falconer's distance set conjecture for general norms [33]:** Hindy Drillick, Dimitrios Ntalampekos and I show that for any norm $\|\cdot\|$ on $\mathbb{R}^d$ with countably many extreme points, there is a set $E \subset \mathbb{R}^d$ whose distance set $\Delta(E) = \{\|x - y\| : x, y \in E\}$ has zero linear measure. This says Falconer's distance set conjecture (e.g., [59], [69]) fails for such norms. Falconer proved, for the Euclidean norm on $\mathbb{R}^d$, that $\Delta(E)$ has positive length if $\dim(E) > (d+1)/2$ and conjectured this holds if $\dim(E) > d/2$. This problem has attracted much attention, e.g., [42], [52], [56], [66], [122]. Failure of the conjecture was previously known only for certain polygonal norms in $\mathbb{R}^2$, [76], [77]. We also show it fails for some norms that are strictly convex and C^1; the conjecture holds for C^2 norms [70].

• **Exotic 4-manifolds [39]:** This joint paper with Claude LeBrun constructs a co-compact quasi-Fuchsian hyperbolic 3-manifold M so that the harmonic measure of one end at infinity defines a harmonic function on M that has a critical point. This implies the existence of a compact Riemannian 4-manifold that is anti-self-dual but not almost-Kahler. The almost-Kahler metrics form a non-empty, open subset of the anti-self-dual metrics on such a manifold, and this example shows, for this first time, that this can be a proper open subset. Many interesting questions remain open, e.g., how few generators can the fundamental group of M have? Are such examples "generic" near the boundary of Teichmüller space?

BROADER IMPACT OF PREVIOUS WORK

• **Physics and pattern recognition:** The impact of my Weil-Petersson results on these areas will be discussed at the end of the proposal.

• **Optimal meshing:** My meshing results enhance the suite of available automatic meshing algorithms available for research and industry, and improve practical computational methods in various ways. The optimal triangulation result gives an easy-to-compute angle bound that can be used to benchmark a wide variety of meshing algorithms in use, showing how close they are to optimal, and it provides a stopping criterion for methods that iteratively improve the quality of a mesh. See e.g., [57], [64], [67], [102].

• **Other scientific impacts:** My work on non-obtuse triangulation is cited in [10], a paper dealing with the optimal placement of heat sinks on integrated circuits (removing excess heat is one of the primary bottlenecks in circuit design). The related problem of efficient packing by Voronoi cells also occurs in biological growth models [106], geographic information systems [123], and facility location problems [84], [121]. My work on optimal meshing depends on my earlier work on numerical conformal mapping [22], which also has applications include automated face recognition (which enhances privacy and security), medical imaging, obstacle avoidance for robots (or self-driving cars), among others. My work on numerical conformal mapping has been cited in papers by applied mathematicians, e.g., [11], [60], [61]. My paper [34] with Feinberg and Zhang has been cited in the economics literature, [72], [73].

• **Building interdisciplinary connections:** The interdisciplinary character of the problems in the proposal serve as a bridge between researchers with common interests but different backgrounds. I have spoken at various computer science conferences: the annual Fall Workshop on Computational Geometry, SoCG (Symposium on Computational Geometry) and most recently at SODA 2022 (Symposium on Discrete Algorithms). I co-hosted a graduate workshop on computational geometry which included a mixture of "pure" and "applied" topics, e.g., mini-courses by Scott Sheffield, Esther Ezra, David Mount, and Yusu Wang.

• **Educational impact:** The results obtained under previous grants have been the basis of a series of graduate courses and lecture notes on dynamics, quasiconformal analysis, and conformal mapping. In 2020 I gave an online course on Weil-Petersson curves attended by about 40 students and postdocs around the world, and in Spring 2022 gave another such course on DLA (diffusion limited aggregation) and Brownian motion. Problems raised in the 2020 course on Weil-Petersson curves were answered in the PhD theses of Jared Krandel (Stony Brook) and Tim Mesikepp (Univ. of Washington). My recent students Kirill Lazebnik and Jack Burkart wrote theses on transcendental dynamics problems from past proposals, and I expect to have another student working on these next year. My paper on removable sets was motivated by questions of Guillaume Baverez (a PhD student at Cambridge) during discussions of his thesis on SLE. My most recent work on optimal triangulation answered a question of Florestan Brunck, needed to complete his master's thesis [44] at McGill (currently he is a PhD student of Edelsbrunner at IAS in Austria). Topics in past proposals have been investigated by several Stony Brook post-docs: Simon Albrecht, Matthew Romney, Peter Lin, James Waterman and Dimitrios Ntalampekos (now tenure track at Stony Brook). Dimitrios recently published a paper in *Inventiones* solving a problem (conformal removability of gaskets) stated in my 2016 proposal.

Past NSF proposals have also contributed topics for undergraduate research projects: Ahmed Rafiqi used random walks to calculate conformal maps (Cornell Ph.D.); Kevin Sackel worked on QC removability (MIT PhD); Shalin Parekh numerically estimated percolation dimension of random walks (PhD program at Columbia); Christopher Dular implemented my $O(n^2)$ triangulation refinement algorithm (grad student at Georgia Tech); Ray Zhang wrote his honors thesis with on functional analysis and machine learning (working in the financial sector); Joe Suk implemented my "Trues trees are dense" theorem numerically (Ph.D. program at Columbia). The paper on Falconer's conjecture grew out of Hindy Drillick's undergraduate thesis: she found an incorrect theorem in Falconer's 30-year-old textbook *Fractal Geometry* [58] (Falconer was shocked no one had noticed it before), and our paper gives the counterexample and fixes an application from [80] of the faulty theorem. Drillick is now a PhD student at Columbia. My former student Jack Burkart used problems on the carpenter's rule problem from a previous proposal as the basis for an 2022 REU, and I met with the students via Zoom. Currently I am working with two undergraduates, Thant Win Saw and Luke Russo, studying Riemann surfaces and *dessins d'enfants*.

PUBLICATIONS RESULTING FROM RECENT NSF SUPPORT

- A transcendental Julia set of dimension 1. *Invent. Math.*, 212(2) 407–460, 2018.
- Harmonic measure: algorithms and applications, *Proc. ICM. 2018*, Vol. 2, 1507–1534.
- Prescribing the postsingular dynamics of meromorphic functions, with K. Lazebnik, *Math. Annalen*, vol 365, no 3, 1761–1782, 2019.
- Anti-self-dual 4-manifolds, quasi-Fuchsian groups and almost-Kahler geometry, with C. LeBrun, *Comm. in Analysis and Geo.*, vol 28, no 4, 745–780, 2020 (volume in honor of K. Uhlenbeck).

- Speiser class Julia sets with dimension near one, with S. Albrecht. *J. d'Analyse* vol 141, no 1, 49–98, 2020 (volume in honor of L. Zalcman).
- Falconer's (K, d) distance set conjecture can fail for strictly convex sets K in R^d, with H. Drillick and D. Ntalampekos, *Revista Mat. Iberoamericana*, vol 37, no 5, 1953–1968, 2021.
- The traveling salesman theorem for Jordan curves, *Advances in Math.* vol 404, 2022.
- Conformal images of Carleson curves, *Proc. Amer. Math. Soc.*, vol. 9, 2022, 90–94.
- Quasiconformal maps with thin dilatations, to appear in *Publicacions Matematiques.*
- Uniformly acute triangulation of PSLGs, to appear *Discrete Comp. Geom.*
- Uniformly acute triangulation of polygons, to appear *Discrete Comp. Geom.*
- Weil-Petersson curves, β-numbers, and minimal surfaces, submitted to *Annals of Math.*
- Optimal triangulation of polygons, submitted to *J. Amer. Math. Soc.*
- Non-compact Riemann surfaces are equilaterally trianguable, with L. Rempe, submitted to *Inventiones.*
- Equilateral triangulations and the postcritical dynamics of meromorphic functions, with K. Lazebnik and M. Urbanski, submitted to *Math. Annalen.*
- Function theoretic characterizations of Weil-Petersson curves, submitted to *Revista Iberoamericana* (volume in honor of A. Cordoba and J.L. Fernandéz).
- Conformal removability is hard, preprint, 2021.

EVIDENCE OF RESEARCH PRODUCTS AND THEIR AVAILABILITY

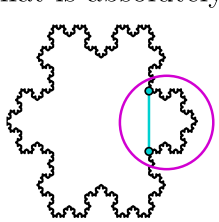
All preprints are posted on my Stony Brook webpage. My webpage also contains lecture notes and slides of lectures related to my research (also links to videos of my lectures), class notes with links to relevant literature, as well as abstracts of my papers, descriptions of my research, and links to workshops I have organized or attended. A survey of my recent work was published in the proceedings of the 2018 ICM. Two online graduate courses related to this proposal were recorded; videos and slides and are available from my website.

PROJECT DESCRIPTION

The proposal involves applications of conformal, quasiconformal and hyperbolic geometry to: (1) studying the Weil-Petersson class of closed curves, (2) finding triangulations of polygons and PSLGs, and (3) investigating the geometric properties of various dynamical and probabilistic fractal sets. The proposed questions range from long standing (and probably very difficult) conjectures to problems suitable for students.

1. THE WEIL-PETERSSON CLASS

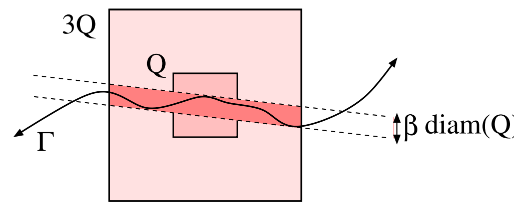
• **Quasicircles and β-numbers:** A quasiconformal map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism that is absolutely continuous on almost all lines and satisfies $|\mu_f| = |f_{\bar{z}}/f_z| \le k < 1$ almost everywhere. Quasicircles are quasiconformal images of circles. These can be characterized by the Ahlfors M-condition: if z is on the smaller diameter arc of Γ with endpoints x, y, then $|z - x| \le M|x - y|$ for some fixed $M < \infty$. Quasicircles can be fractals; the figure at left shows that the von Koch snowflake satisfies the M-condition. The set of quasicircles, modulo similarities, forms universal Teichmüller space, $T(1)$, and it was a long standing problem to put a Hilbert manifold structure on this space of loops, corresponding to the Weil-Petersson metric on finite dimensional Teichmüller spaces. In 2009 such a metric was found by Takhtajan and Teo [107], but this metric makes $T(1)$ disconnected, and they asked for a geometric characterization of the connected component that contains all smooth curves, the so called Weil-Petersson class. More precisely, is there an analog for WP curves of the M-condition for quasicircles? My paper [29] gives over twenty equivalent geometric definitions of such curves. Many of these new definitions make sense and remain equivalent for curves in $\mathbb{R}^n$, or even in Hilbert space or in a metric space.



Given a set E, Peter Jones's β-numbers measure the local deviation from a line:

$$\beta_E(Q) = \inf_L \sup\{\mathrm{dist}(z, L) : z \in 3Q \cap E\}/\mathrm{diam}(Q),$$

where the infimum is over all lines L hitting Q, and where Q is dyadic square (see figure). Jones's traveling salesman theorem (TST) says that a planar set E is contained in a curve Γ of length $\simeq \sum_Q \beta_\Gamma(Q)^2 \mathrm{diam}(Q)$ where the sum is over all dyadic squares in the plane; convergence of the sum for a set E characterizes subsets of rectifiable curves. I proved in [29] that Γ is Weil-Petersson iff



$$(1) \qquad \sum_Q \beta_\Gamma(Q)^2 < \infty.$$

This is Jones's criterion but with the "diam(Q)" term left out. Thus WP curves have "finite total curvature" in an L^2 sense, and all the conditions in [29] say, in some sense, that curvature is square summable over all locations and all scales.

Question 1. *Construct a section for $T(1)$, i.e., a collection of quasicircles containing one representative from each connected component of $T(1)$ in the WP metric.*

This was suggested by Leon Takhtajan. A good starting point might be Rohde's paper [98] giving a section for quasicircles modulo biLipschitz images. I characterized the connected component of $T(1)$ containing smooth functions. What about the other components?

Question 2. *Geometrically characterize other connected components of $T(1)$. Which curves are finite WP distance from a given polygon? From the von Koch snowflake?*

Question 3. *Characterize subsets of Weil-Petersson curves.*

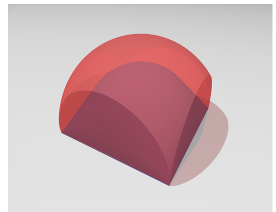
This is the direct analog of Jones's characterization of subsets of rectifiable curves. So far, Matthew Hyde and I have shown the obvious guess is wrong: the series in (1) can converge for a set E, but E is not a subset of any WP curve. A set E is uniformly perfect if for any $x \in E$ and $0 < r < \mathrm{diam}(E)$ there is a $y \in E$ with $|x - r| \simeq r$. Our counterexample is not uniformly perfect, and as we zoom into a point, the best approximating lines defining β_E rotate by $\pi/2$ between widely separated scales, even though the β-numbers for intermediate scales are all small. We are working on showing that (1) is sufficient if we also assume that the β's are all small, and that E is uniformly perfect. This should lead to a general criteria involving (1) summed over squares where E looks uniformly perfect, and the oscillation of the best approximating directions over scales where this fails.

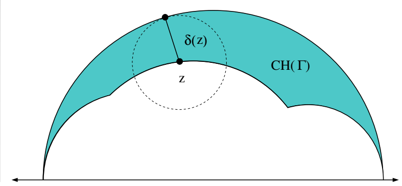
• **Higher dimensions:** What is a Weil-Petersson surface? Following Takhtajan and Teo for curves, we could consider the image of $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ under a quasiconformal mapping of $\mathbb{R}^n$ whose dilatation is in L^2 for $dx/|1-|x||$. Can such surfaces be characterized using the β-numbers? The sum $\sum_Q \beta^2(Q)$ diverges even for smooth surfaces, but both

$$\sum_Q \beta^2_{\Gamma,n-1}(Q)\mathrm{diam}(Q)^{n-1} < \infty \qquad \text{and} \qquad \sum_Q \beta^{n-1}_{\Gamma,n-1}(Q) < \infty,$$

are finite for spheres. The second condition implies the first, is scale invariant, and does not allow "corners". Is this the correct generalization of Weil-Petersson curves to surfaces? Do such surfaces have connections to physics as WP curves do? Is there a characterization of such surfaces in terms of the minimal hyperbolic varieties they bound?

• **Hyperbolic convex hulls:** Given a set Γ in $\mathbb{R}^2$ we can consider its hyperbolic convex hull $\mathrm{CH}(\Gamma)$ in the upper half-space $\mathbb{R}^3_+$ (take the smallest hyperbolically convex set containing all geodesics with endpoints in Γ). For circles, $\mathrm{CH}(\Gamma)$ is a hemi-sphere, but otherwise $\mathrm{CH}(\Gamma)$ has non-empty interior and it is bounded by two disjoint surfaces, each meeting $\mathbb{R}^2$ along Γ. The left figure shows the hyperbolic convex hull of a square. For $z \in \mathrm{CH}(\Gamma)$, let $\delta(z)$ be the hyperbolic distance to farther boundary component; this measures the "width" of the convex hull near z; see right figure. I prove in [29] that Γ is Weil-Petersson iff





$$\int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z)\, dA(z) < \infty, \tag{2}$$

where dA denotes hyperbolic surface area on the boundary of the convex hull. Condition (2) is the conformally invariant version of (1). In 1982, Mike Anderson [5], [6] proved that any Jordan curve $\Gamma \subset \mathbb{R}^2$ bounds at least one minimal surface S in the hyperbolic upper half-space, and this surface is contained inside $\mathrm{CH}(\Gamma)$. I showed in [29] that Γ is WP iff S has finite total curvature (i.e., the scalar curvatures κ_j are in L^2 for hyperbolic area). The total curvature of S is a Möbius invariant of Γ; is there a formula linking it to the Weil-Petersson distance of Γ to a circle, or other invariants of Γ? More generally, I expect δ and κ_j to be more natural for studying conformally invariant properties than the β-numbers.

Problem 4. *Connect estimates of $\delta(z)$ to the geometry Γ.*

Problem 5. *Connect curvature bounds for the minimal surface S_Γ to the geometry of Γ.*

These are open-ended questions and contain many concrete sub-problems. For example, Michel Zinsmeister asked if BMO-curves (i.e., the conformal map f to the interior domain satisfies $\log|f'| \in \mathrm{BMO}$) can be characterized in terms of δ or κ on S. These curves were studied by Zinsmeister and Astala [7] and by Jones and myself [35]. Another direction to pursue these questions is suggested by a result of Brock [43] which says that the usual Weil-Petersson distance between two Riemann surfaces X, Y is related to the hyperbolic volume of the convex hull of a related fractal curve (the limit set of the Kleinian group given by the Bers simultaneous uniformization theorem). Is there a precise connection between his result and mine? Both measure distances in terms of the "size" of a hyperbolic convex hull and perhaps Brock's result follows from an analogous result about $T(1)$.

• **Knot energies:** Using a result of Blatt [41], I proved that a curve Γ is WP iff it has arclength parameterization in the Sobolev space $H^{3/2}$, and iff it has finite Möbius energy:

$$\text{Möb}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|x-y|^2} - \frac{1}{\ell(x,y)^2} \right) dx dy < \infty.$$

This is exactly the Hadamard renormalization of the divergent energy integral $\int_\Gamma \int_\Gamma \frac{dxdy}{|x-y|^2}$, corresponding to placing a charge distributed as arclength on Γ under an inverse-cube repulsive force (e.g., the Newtonian kernel in $\mathbb{R}^4$). Möbius energy was introduced in knot theory by O'Hara [90],[92], [91]; it blows up if the curve self-intersects, and a gradient flow for this energy was studied as a means of finding canonical representation of knots in $\mathbb{R}^3$ [68].

Question 6. *How does this gradient flow act on the Weil-Petersson curves in $\mathbb{R}^n$? Is the limit always a circle if $n \neq 3$?*

Another class of knot energies considered in [105] are the Menger energies

$$\mathcal{M}_p(\Gamma) = \int_\Gamma \int_\Gamma \int_\Gamma c^p(x,y,z) |dx||dy||dz|,$$

where $c(x,y,z)$ is the curvature of the unique circle passing through x, y, z. It is known that $\mathcal{M}_2(\Gamma) < \infty$ is equivalent to rectifiability e.g., see [83]. For $p \geq 3$, can finite Menger energy curves be characterized in terms of β-numbers, hyperbolic convex hulls or minimal surfaces? The most interesting case is $p = 3$; this is the only scale-invariant Menger energy.

Problem 7. *Geometrically characterize curves of finite $\mathcal{M}_3$ energy. Other energies?*

In [29] I show that Weil-Petersson curves are characterized by the related condition

$$\int_\Gamma \int_\Gamma \int_\Gamma \frac{c^2(x,y,z)|dx||dy||dz|}{|x-y|+|y-z|+|z-x|} < \infty,$$

using β-numbers and my extension of Jones's TST, and I expect the techniques used there can be applied to describe other types of finite energy curves. Some of my techniques for WP curves have already been extended from $p = 2$ to general p by Tim Mesikepp [86].

• **SLE and random minimal surfaces:** Schramm-Loewner Evolutions, SLE(κ), are families random Jordan curves depending on a parameter $\kappa \in (0, \infty)$: for κ small the curves are nearly straight, and for large κ they become space filling. SLE was invented by Oded Schramm (calling them Stochastic Loewner Evolutions) using random conformal maps generated by Loewner's equation with Brownian motion as data, and they are now a major research topic. Recently, Wang and Viklund [119], [117] showed that Weil-Petersson curves are related to the large deviations theory of Schramm-Loewner evolutions (SLE) as the parameter κ tends to either zero or infinity. See also [99], [116], [120]. Is there some more direct connection? We noted above that Weil-Petersson curves are characterized by the Hadamard renormalization of a inverse-cube repulsive energy that prevents self-intersections. Do SLE(κ) curves have finite energy (or finite renormalized energy) for some repulsive force depending on κ? Can an SLE path be considered as a "uniformly random" choice of such finite energy curves? If so, this might suggest a way to extend the definition of SLE to $\mathbb{R}^n$. An SLE(κ) curve also has an associated hyperbolic convex hull and (at least one) associated (random) minimal surface S_κ.

Problem 8. *Is the minimal surface S_κ for* SLE(κ) *unique? Does this depend on κ?*

Problem 9. *Does S_κ have a well defined "average curvature"? If so, compute it.*

• **Inscribed polygons:** The most elementary description of Weil-Petersson curves involves approximation by inscribed polygons. Given a closed rectifiable curve Γ, let Γ_n be an inscribed polygon with nested sets of 2^n equally spaced vertices $\{z_j^n\}_{j=1}^{2^n}$ on Γ (there is a 1-parameter family of such polygons, depending on the choice of a base point $z_0 \in \Gamma$). Clearly $\ell(\Gamma_n) \nearrow \ell(\Gamma)$, and I proved Γ is WP iff $\sum_n 2^n(\ell(\Gamma) - \ell(\Gamma_n)) < \infty$ with a bound independent of the base point. Consider the angles of Γ_n: $\theta(n,k) = \arg\left(z_{j+1}^n - z_j^n)/(z_j^n - z_{j-1}^n\right)$.

Conjecture 10. *Γ is WP iff $\sum_{n=1}^\infty \sum_{k=1}^{2^n} \theta^2(n,k) < \infty$, with a bound independent of z_0.*

I can prove "WP $\Rightarrow$ convergence" using (1). The converse is harder because θ can be zero at a point, even if β is large, e.g., at the center of a symmetric spiral. The same thing happens in the longstanding ϵ^2-conjecture of Carleson [20], characterizing tangent points of a curve Γ using a quantity $\epsilon(x,t)$ defined in terms of the angle formed at x by $\Gamma \cap \{y : |y-x| = t\}$. Like θ above, $\epsilon(x,t)$ can vanish at points where the β-numbers are large. In a *tour-de-force*, Jaye, Tolsa and Villa [71] overcame the difficulties using a "smoothed" version of $\epsilon(x,t)$:

$$\alpha(x,r) = \left| \frac{\pi}{2} - \frac{1}{r^2}\int_\Omega e^{-|y-x|^2} dy \right|, \qquad \mathcal{A}(x)^2 = \int_0^1 \alpha(x,r)^2 \frac{dr}{r}.$$

This acts as an intermediary between $\epsilon(x,t)$ and $\beta(x,t)$, a disk-based variant of the square-based β-numbers defined earlier, and reduces the conjecture to a theorem of Peter Jones and myself. Because of the similarity between Conjecture 10 and Carleson's conjecture, I expect that the new α-numbers (or some discrete analog) will be the key to proving Conjecture 10.

2. OPTIMAL TRIANGULATIONS

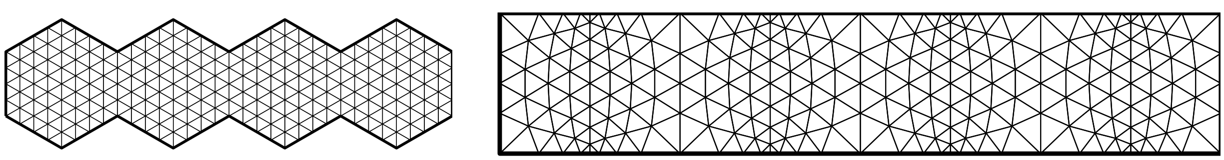
It is a problem of longstanding theoretical and practical[1] interest to triangulate a polygon with the best possible bounds on the angles used. Many algorithms, such as finite element methods, work best when the associated meshes have well formed elements (see discussion of broader impacts). The constrained Delaunay triangulation famously maximizes the minimal angle if no additional vertices (called Steiner points) are allowed [48] [81], [82], and algorithms for minimizing the maximum angle (again without Steiner points) are given in [15] and [54]. If Steiner points are allowed, Burago and Zalgaller [46] proved in 1960 that every planar polygon P has an acute triangulation (all angles $< 90°$). This is the best possible bound independent of P, but in [31] I compute the optimal bound $\Phi(P) = \inf\{\phi : P \text{ has a } \phi\text{-triangulation}\}$ for any given polygon. There are many open questions about efficiently constructing triangulations that have the optimal (or nearly optimal) angle bounds, and how to extend these ideas to $\mathbb{R}^3$.

- **Minimal weight triangulations:** When we allow Steiner points, the space of possible triangulations of P becomes infinite dimensional, and it is far from clear that the infimum defining $\Phi(P)$ is attained. In [31], I prove the optimal angle bounds are always attained except for some $60°$-polygons (e.g., polygons where all angles are multiples of $60°$). For $60°$-polygons, I prove $\Phi(P) = 60°$, but (up to similarity) only countably many polygons have equilateral triangulations, so "most" $60°$-polygons cannot have triangulations attaining the infimum $\Phi(P)$. The analogous problem of minimizing the total edge length remains open. In [21] I give a polygon with no minimal weight Steiner triangulation (MWST), but the example has three co-linear vertices. To computer scientists, this is cheating, and they ask:

Question 11. *Does a polygon in general position always have a MWST?*

Without Steiner points, finding a minimal weight triangulation is NP-hard [88], versus polynomial time for MinMax angles, [15], [54]. Thus proving even the existence of an optimal triangulation may be significantly harder for lengths than for angles.

- **Near optimal complexity:** My proof that angle-optimal triangulations exist does not attempt to optimize the number of triangles needed. Indeed, it uses an exponential number of triangles for a $1 \times n$ rectangle, when it is easy to see that $O(n)$ suffices by choosing a model $60°$-polygon that "looks rectangular", as shown below:

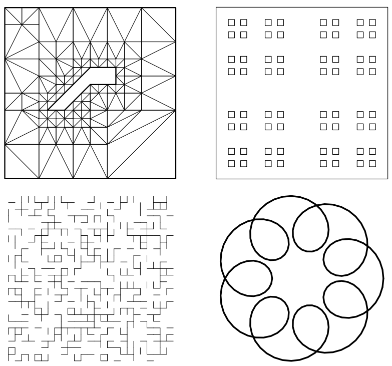


Question 12. *Estimate the number of triangles needed to attain the $\Phi(P)$ bound for a given polygon P. Is computing the exact minimal number of triangles needed NP hard?*

My current proof uses the Schwarz-Christoffel formula to construct P' from P, giving P' the same number of vertices as P. The rectangle example above shows we can do better by letting P' have more vertices, thus allowing it to mimic the overall shape of P better. The problem is to find a systematic way to do this that maintains the sharp angle bounds of the current proof, but also approximates "shape" of P as well as possible. My paper [22] on fast conformal mapping introduces the thick-thin decomposition of a polygon (analogous to the thick-thin decomposition of a Riemann surface) and [23] uses this decomposition to give a quadrilateral mesh of nearly optimal size for any polygon. Very likely, by combining approximating the thick-thin pieces of P by $60°$-polygons, we can construct angle-optimal triangulations with nearly optimal complexity.

[1] In theory, practice and theory are the same. In practice, they are not. – A. Einstein

• **Planar straight line graphs:** A planar straight line graph (PSLG) is any finite union of segments and points; a polygon is a special case when the segments meet end-to-end. A PSLG could also be a point cloud, a triangulation, a tree, ..., or almost anything we can draw. See some examples below. A mesh of a PSLG must be consistent across all the edges:



meshing each face separately is not enough, since boundary vertices for adjacent faces must match exactly. This makes triangulating PSLGs significantly harder than triangulating polygons, e.g., every n-gon has a $O(n)$ non-obtuse triangulation (a "NOT") [18], but non-obtuse triangulations for some PSLGs require at least n^2 elements. Polynomial algorithms giving angle bounds $< 180°$ for PSLGs were found in the 1990's (see [18], [87], [108]). My "NOT Theorem" [26] says every PSLG has a NOT of size $O(n^{2.5})$, but there is a gap between this and the n^2 example.

Conjecture 13. *(NOT Conj.) Every PSLG has a conforming NOT with $O(n^2)$ elements.*

Below we will discuss the proof of the NOT theorem and how it might be improved to give the NOT conjecture. First we mention some variations and corollaries. A Delaunay triangulation is defined by the property that any pair of triangles sharing an edge having opposite angles summing to $\leq 180°$. Any NOT is also Delaunay, so the NOT theorem improves a famous $O(n^3)$ bound of Eldesbrunner and Tan [53] for conforming Delaunay triangulations, (and improves other optimal meshing results, e.g., [16] by Bern and Eppstein).

Conjecture 14. *Every PSLG has an $O(n^2)$ conforming Delaunay triangulation.*

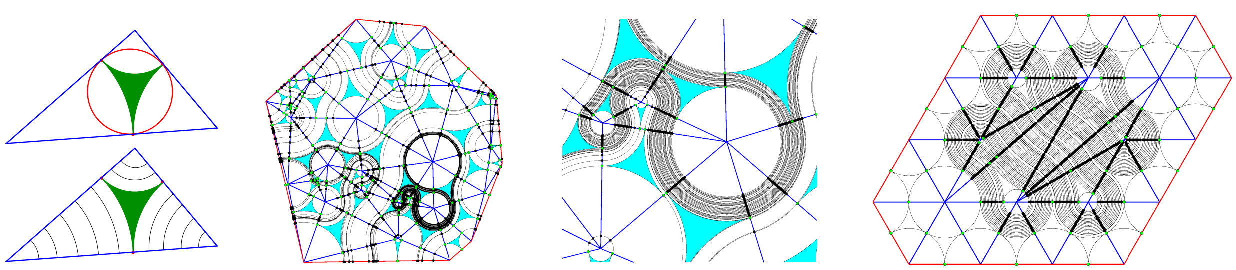
Conjecture 13 immediately implies Conjecture 14, but I suspect they are equivalent to each other. Can we prove the two problems have the same complexity, even if we can't determine exactly what that complexity is? Can my angle-optimal triangulation results for polygons, discussed earlier, be extended to PSLGs?

Problem 15. *Compute the optimal upper angle bound, $\Phi(\Gamma)$, for triangulating a PSLG Γ.*

Conjecture 16. *If a PSLG Γ has minimal angle θ, then $\Phi(\Gamma) \leq 90° - \min(36°, \theta)$.*

[31] shows the latter holds for polygons, and in [19] I prove it for PSLGs, but with $36°$ replaced by some $\theta_0 > 0$ given by a compactness argument. Elementary examples show that $36°$ is best possible (e.g., any triangulation of a square has an angle $\geq 72°$). Conjecturing $\theta_0 = 36°$ for PSLGs may be too optimistic, but any explicit bound would be very significant.

• **Meshing and dynamics:** Non-obtuse triangulation of PSLGs reduces to the study of certain dynamical systems. Given a triangle, take the in-circle as shown at left below. The three tangent points (cusp points) define three disjoint sectors, each of which is foliated by circular arcs centered at a vertex. We flow each cusp point along this foliation until it hits



another cusp point or exits the triangulation. The points where the flow hits triangle edges define the vertices of non-obtuse refinement of the original triangulation. The size of the

NOT is thus related to the number of times flow lines hit triangles. The pictures above show a random triangulation flow, an enlargement of it, and an example with infinite paths.

The proof of the NOT theorem perturbs the triangulation flow so that each flow line hits only $O(n)$ triangles, but for each original point, it adds $O(\sqrt{n})$ new points to be propagated, giving the $O(n^{2.5})$ bound. The "enemies" are certain spirals that cause flow lines to visit the same triangles many times (see enlarged figure above). The proof uses a worst case estimate that $\simeq n$ such spirals occur and that each involves $\simeq n$ triangles, but it seems impossible to draw triangulations where this actually occurs. I am trying to quantify how "bad" a spiral is in terms of how many new flow lines need to be added, and to show the "total badness" is bounded, not growing with n. This would prove Conjecture 13.

To limit how many triangles a flow line hits, curves are carefully "bent" to make them collide with other paths, reducing the number of paths until none remain. To show that the perturbed flows lines still generate the vertices of a NOT, we can only allow very small changes in path curvatures, i.e., there are constraints on a discrete second derivative. This is very reminiscent of Pugh's closing lemma in surface dynamics: every C^1 vector field has a small C^1 perturbation with a closed orbit [93], [94], [95]. This is still open for C^2 perturbations, and Dennis Sullivan suggested there might be a connection:

Question 17. *Can a closing lemma help prove the $O(n^2)$ NOT-theorem? Can the NOT argument help prove a C^2-closing lemma (or suggest a counterexample)?*

I plan to study the literature on continuous closing lemmas to see if the heuristic similarity described above can be made mathematically precise. So far as I know, these triangle flows have not been studied before, so essentially all reasonable questions are open and interesting.
• **NOTs in 3 dimensions:** Solving the conjectures above would essentially complete the theory of optimal triangulation in $\mathbb{R}^2$, but the corresponding theory using tetrahedra in $\mathbb{R}^3$ (the really important case for applications) is wide open:

Question 18. *Do polyhedra in $\mathbb{R}^3$ have non-obtuse tetrahedralizations of polynomial size?*

Even finding an acute tetrahedralization (all dihedral angles $< 90°$) of a cube in $\mathbb{R}^3$ was open until recently (the smallest known example uses 1,370 pieces, [114]) and there is no acute decomposition for the cube in $\mathbb{R}^4$, [78]. My work in 2-dimensional meshing uses a thick/thin decomposition of a polygon (analogous to the thick/thin decomposition of a hyperbolic manifold) to partition the polygon into two types of regions where Euclidean and hyperbolic geometry respectively are used to create the mesh. Can we use analogous ideas in $\mathbb{R}^3$? Can one create a 3-manifold out of a polyhedron, run a Ricci flow on it (as in Perelman's proof of Thurston's geometrization conjecture) to decompose it into pieces with geometric structure and then utilize the "natural" geometries on the different pieces to define meshes? An intermediate problem between dimensions 2 and 3 is to find NOTs for triangulated surfaces in $\mathbb{R}^3$.

Question 19. *Does a polyhedral surface with n faces have a non-obtuse triangulation with a polynomial number of triangles? With $O(n^2)$ triangles?*

A polyhedral surface is flat except at the vertices, where there is concentrated curvature due to the angle sums not equalling 2π. The basic idea of creating a flow from a triangulation and counting "bad spirals" on the surface should be the same as before, but we have to check the affect curvature has on our ability to bend curves. Almost certainly, a polynomial complexity bound depending on curvature bounds is possible. Whether a uniform complexity bound, independent of curvature, holds is less clear. If it does not, then solid regions in $\mathbb{R}^3$ bounded by such surfaces would give a negative answer to Question 18.

3. DYNAMIC AND RANDOM FRACTALS

The dynamic fractals we consider are Julia sets of transcendental entire functions, i.e., non-polynomial entire functions. We let $\mathcal{T}$ denote this class of functions. As usual, the Fatou set, $\mathcal{F}$, is the maximal open set where the iterates of f form a normal family and the Julia set, $\mathcal{J}$, is its complement (and is always non-empty). While similar to polynomial dynamics in many respects, transcendental dynamics is different in several ways: wandering domains can exist, Fatou components of any finite or infinite multiplicity may occur, and the escaping set $I(f) = \{z : f(z) \to \infty\}$ plays a more prominent role.

• **Rectifiable Julia sets:** Recall that Hausdorff dimension is defined as $\mathrm{Hdim}(K) = \inf\{s : \inf\{\sum_j r_j^s : K \subset \cup_j D(x_j, r_j)\} = 0\}$. In polynomial dynamics, the "exotic" examples have dimension 2 or positive area, but in transcendental dynamics, "easy" examples have these properties and the difficulty is constructing Julia sets with small dimension. In 1975, Baker [8] proved that transcendental Julia sets always contain a non-trivial continuum, and hence they have Hausdorff dimension at least 1. It was not until 2018 that I constructed the first example attaining dimension 1, [27]. It has finite 1-dimensional spherical measure, but I proved it is not a subset of any rectifiable curve on S^2.

Question 20. *Can a transcendental Julia set lie on a rectifiable curve on the sphere?*

My "dim =1" example is the transcendental analog to Shishikura's construction [103] of polynomial Julia sets of dimension 2, and a rectifiable example would be analogous to Buff and Cheritat's construction [45] of a positive area polynomial Julia set. My 2018 example has infinitely connected Fatou components; my idea is to adapt my quasiconformal folding technique to build a similar example but with finitely connected Fatou components, which will have rectifiable boundaries, and then prove these all lie on a single rectifiable curve.

• **The Speiser class $\mathcal{S}$:** The Speiser class consists of transcendental functions f that have a finite number of singular values (critical values or asymptotic values, i.e., limits of f along curves tending to ∞). Rippon and Stallard [97] showed that Speiser class Julia sets must have dimension > 1, and using my quasiconformal folding technique, Simon Albrecht and I [2] constructed a sequence in $\mathcal{S}$ with $\mathrm{Hdim}(\mathcal{J}_n) \searrow 1$. Is every value in $(1, 2]$ taken?

Question 21. *Is* $\{\mathrm{Hdim}(\mathcal{J}(f)) : f \in \mathcal{S}\} = (1, 2]$*?*

It would be surprising (but fascinating) if this failed. One possible approach is to try to quasiconformally deform my examples with Albrecht to see if the perturbed Julia sets sweep out an interval of dimensions. Each Speiser function f has a finite dimensional family of quasiconformal deformations $M_f = \{g = \psi \circ f \circ \varphi\}$ such that g is entire and and ψ, φ are both QC. But the behavior of these moduli spaces is still quite mysterious. We know the Hausdorff dimension of the Julia set changes continuously over M_f, so Question 21 would follow from my result with Albrecht, and the following conjecture of Lasse Rempe.

Conjecture 22. *If $f \in \mathcal{S}$, then* $\sup\{\mathrm{Hdim}(\mathcal{J}(g)) : g \in M_f\} = 2$.

This is an analog of Shishikura's result [103] about dimensions of quadratic Julia sets tending to 2 near generic points in the boundary of the Mandelbrot set (also to the fact that Kleinian limit sets have dimension tending to 2 near most boundary points of Teichmüller space [36]). Possibly Shishikura's proof can be adapted to this case. Another possible attack is to use a result of Bergweiler, Karpińska and Stallard [14] that gives a lower bound on the dimension of the Julia set in terms of the growth rate of the function f. In [25] I showed that the order of growth on an entire function need not be constant over M_f; possibly a similar example can be used to show $\mathrm{Hdim}(\mathcal{J})$ approaches 2 strictly from below on some moduli space (if not on all moduli spaces). Opposite to Conjecture 22 we can ask

Question 23. *Is there an $f \in \mathcal{S}$ with* $\inf\{\mathrm{Hdim}(\mathcal{J}(g)) : g \in M_f\} = 1$*?*

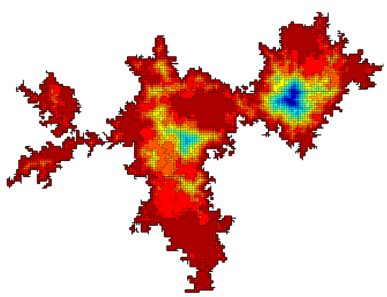
I suspect this is false. As noted above, Stallard showed that $\mathrm{Hdim}(\mathcal{J}(g)) > 1$ for any fixed Speiser class function. To do this, she constructed a Sullivan-type measure on the Julia set by taking a limit of point masses over inverse orbits of certain carefully chosen points. The idea here is to follow her proof, but show that the necessary estimates hold with bounds that are uniform over all of M_g, giving a lower dimension bound for $\mathrm{Hdim}(J) > 1+\epsilon$ over the whole family, not just one function. Note that both the last two problems pre-suppose that the dimension of the Julia set need not be constant on each moduli space, but even this fundamental fact is unknown:

Question 24. *If $f \in \mathcal{S}$ and $g \in M_f$ is $\mathrm{Hdim}(\mathcal{J}(f)) = \mathrm{Hdim}(\mathcal{J}(g))$?*

Before my examples with Albrecht, every known Speiser function f gave constant dimension 2 on M_f, so our examples are the first where this question can even be tested.

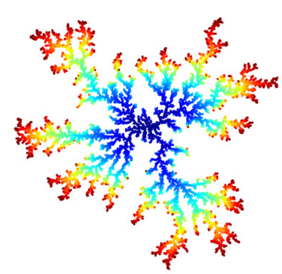
• **Werner's conjecture:** We now turn to some random fractals. Brownian motion has been intensely studied for over a century, but some of its basic geometric properties remain unknown. One of my favorite such problems was formulated by Wendelin Werner. Consider the Brownian trace $B_2([0,1]) \subset \mathbb{R}^2$, i.e., the image of $[0,1]$ under 2-dimensional Brownian motion. This is a compact random set with infinitely many complementary components.

Conjecture 25. *Can any two complementary components be connected by a path that hits the trace only finitely often?*



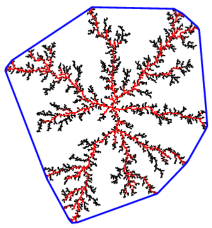
Werner's problem is illustrated by the figure at left, which shows a random walk on a square grid, and the number of steps from each complementary component to the unbounded component is color coded: red is close to the outer component and blue is far from it. A counterexample would correspond to components with diameter bounded away from zero, but arbitrarily many steps away from the unbounded component, i.e., a "big blue" component. Numerical simulations indicate that the component diameters decrease like a negative power of the step distance to the unbounded component, supporting the conjecture. Can every point of the Brownian trace can be surrounded by arbitrarily small closed curves that each only hit the trace finitely often? This would imply that the topological Hausdorff dimension is $\mathrm{tHdim}(B([0,1])) = 1$ (see [9]; $\mathrm{tHdim}(K) \leq 1+\alpha$ if K has a neighborhood basis whose elements all have boundaries of Hausdorff dimension $\leq \alpha$).

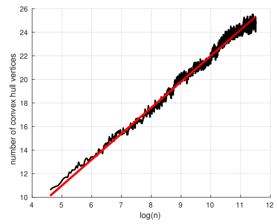
• **Diffusion limited aggregation:** DLA is defined by fixing a unit disk at the origin and sending in a second unit disk moving by Brownian motion from infinity until it touches the first disk. Successive disks are added in the same way (see the figure at right; colors represent arrival time). The main problem is to determine the almost sure growth rate $\alpha = \limsup_n \frac{1}{n} \log \mathrm{diam}(\mathrm{DLA}(n))$. Obviously $\mathrm{diam}(\mathrm{DLA}(n)) \leq 2n$, but Harry Kesten [74] improved this to $O(n^{2/3})$ almost surely; this remains the best known upper bound even 35 years later. The trivial lower bound is $\gtrsim \sqrt{n}$ (consider the area of n disjoint unit disks), and shockingly, this is still the best known:



Conjecture 26. $\lim_n \mathrm{diam}(\mathrm{DLA}(n))/\sqrt{n} = \infty$ *almost surely.*

To prove this, we need to quantify that the "tips" of DLA have larger harmonic measure than the trivial $1/n$ estimate. One way to do this is to show there are few such tips, e.g.,





the convex hull has few vertices (hence a vertex with a "large" angle). In the figure at left, disks are colored red if they were vertices of the convex hull when added. Also shown is a plot of the number of vertices in the convex hull as a function of $\log n$ (averaged over 100 random trials). The plot clearly looks linear as a function of $\log n$.

Question 27. *Is the number of convex hull vertices $O(\log n)$ almost surely? If this is true, can we deduce Conjecture 26? Can we deduce a growth rate n^α for some $\alpha > 1/2$?*

BROADER IMPACTS

- **WP and physics:** In [96], the authors state that "Weil-Petersson class boundary parameterizations provide the correct analytic setting for conformal field theory". My paper [29] also identifies WP curves as the boundaries of hyperbolic minimal surfaces with finite renormalized area, a concept introduced by Graham and Witten and related to the quantum entanglement and the AdS/CFT correspondence. Furthermore, WP curves are exactly the curves with finite renormalized energy under an inverse cube law, linking them to electrostatics in 4 dimensions, and they are connected to Schramm-Loewner curves in statistical mechanics. How can these connections be explained from a physical point of view?
- **WP and computer vision:** My interest in WP curves was motivated by a conversation with David Mumford related to his work on pattern recognition. In numerical computations, the Hilbert structure of the Weil-Petersson class allows efficient algorithms for manipulating and clustering shapes, but since the smooth functions are not closed in the WP metric, equations with smooth data can still lead to non-smooth solutions. My work identifies the exact degree of smoothness (Sobolev $H^{3/2}$) that WP curves have, which allows numerical methods to be designed appropriately. See the papers of Sharon and Mumford [101], Feiszli, Kushnarev and Leonard [62], and Feiszli and Narayan [63].
- **DLA and biology:** DLA models the growth of certain biological phenomena such as lichen colonies or cancer tumors. Question 27 provides a new way to compare this model to experimental data. See [112] for these and further applications to soot deposit in engines, mineral aggregation in rocks, and the propagation of electrical discharges, e.g., lighting bolts.
- **Meshing and triangulation:** All of the broader impacts discussed in the summary of previous work also apply to the current proposal. Solutions to the meshing problems could have a dramatic impact on various aspects of modeling surfaces and 3-dimensional bodies, which are widely used in research and manufacturing. Aerospace engineer Joe F. Thompson, head of a multi-institutional mesh generation effort called the National Grid Project, wrote that "An essential element of the numerical solution of partial differential equations (PDEs) on general regions is the construction of a grid (mesh) on which to represent the equations in finite form . . . it can take orders of magnitude more man-hours to construct the grid than it does to perform and analyze the PDE solution on the grid." (quoted in the introduction to [48]; they note that motion pictures are now the most economically significant consumers of high quality meshing algorithms). Triangulations with good angle bounds improve the efficiency of many numerical methods. For example, Vavasis [115] has shown that matrices associated to discretizing certain differential equations have condition numbers that grow exponentially with mesh size in general, but grow only linearly for acute triangulations. The finite element method on a NOT leads to a matrix that is symmetric, positive definite and negative off the diagonal, giving a linear system that is easier to solve [104]. Other practical advantages of NOTs are described in [49] (maximum principles for discrete PDE's), [12]

(Hamilton-Jacobi equations), [75], [100] (finding geodesics on a triangulated surface), [1], [109], [113] (meshing space-time), [17], [104] (dual triangulations).

• **Applications of QC maps:** Numerical QC mapping is still in its infancy, but has many potential benefits. For example, thinking of a human face a surface with marked features (mouth, eyes,...) it has been suggested that conformal invariants and Teichmüller distance between such surfaces would be an effective, efficient way to do facial recognition [89] (automatic face recognition one of the major problems of computer vision with applications to enhancing individual privacy and national security). Some recent papers in machine learning have utilized quasiconformality, e.g., see [47], [79], [85]. Among the tools are deep neural nets that implement the measurable Riemann mapping theorem: solving the Beltrami problem given a dilatation. Images, e.g, medical scans, can then be matched with images from a library of images by minimizing the quasiconformal distortion needed to map one onto the other. My investigations of the Weil-Petersson class and quasiconformal geometry would support this type of work.

• **Educational impact:** In the summary of previous work, I described courses, lecture notes, graduate and undergraduate projects related to my work, and the current proposal has similar impacts on the infrastructure of research and education. In particular, I plan to follow-up the 2017 graduate workshop on computational and random geometry with another such workshop, and to teach another online graduate class on the connections between hyperbolic and computational geometry and the applications to optimal meshing. Graduate students working on problems related to this proposal receive training in aspects of both pure and applied mathematics, participate in seminars in both departments, and become more open to such collaborations; this improves the likelihood they will participate in interdisciplinary and academic/industrial collaborations and improves their ability to motivate and train their own students in the future.

Moreover, the geometric and interdisciplinary nature of the problems in this proposal suggest numerous projects that are accessible and appealing to undergraduates or even high school students; such problems can motivate them to the further study of mathematics, or at least give then a greater appreciation for the potential of mathematics in their own field. A few problems related to the proposal that are suitable for undergraduate projects include:

• **Discrete Werner's conjecture:** Run a random walk on a grid and compute the adjacency graph of its complementary components. Investigate its diameter and other graph theoretic properties as a function of n, the number of steps.

• **Triangulations of random polygons:** If we consider planar n-gons as points in $\mathbb{R}^{2n}$, my paper [31] shows that polygons with optimal angle bound $\Phi(P) = \theta$ are at least codimension 1, except when θ is $72°$ or $\frac{4}{5} \cdot 90°$; these sets have non-empty interior in $\mathbb{R}^{2n}$. What is the probability that a "random polygon" has one of these special optimal bounds?

• **Simultaneous optimality:** Use the criteria in [31] to characterize which polynomials have a triangulation that attains both the optimal upper and optimal lower angle bounds (there are polygons for which at most one can be attained by any single triangulation).

• **Implement the triangulation algorithms.** Implement the $O(n^{2.5})$ NOT algorithm and its variations for conforming Delaunay triangulations and Voronoi coverings. Numerically study the "triangle flow" for triangulated surfaces, e.g., the Platonic solids.

• **Critical points of harmonic measure:** Compute limit sets of "small" quasi-Fuchsian groups and look for a critical point of the harmonic measure function of one side. This could give a "small" example of the 4-manifold as described in the summary of previous work.

• **Draw transcendental Julia sets:** Unlike polynomials, this requires "case-by-case" tricks. Can one draw a good picture of multiply connected transcendental Fatou component?

• **Draw minimal surfaces:** Compute the hyperbolic minimal surface S corresponding to a curve Γ. Compute the total curvature of S. Compute S when Γ is an SLE path.

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