SUMMARY OF RESULTS FROM PRIOR NSF SUPPORT
Quasiconformal methods in analysis, geometry and dynamics
DMS 1305233, 6/15/13 - 5/31/16, $176,100
INTELLECTUAL MERIT

Computational geometry: A planar straight line graph (PSLG) is any finite union of segments and points. A PSLG can be a polygon, a point cloud, a triangulation, a tree; almost anything we can draw. A conforming mesh for a PSLG Γ fills the convex hull of Γ with simple polygons whose edges and vertices cover Γ; we will only consider meshes involving all triangles or all quadrilaterals. Below, n will denote the number of vertices of a PSLG. A triangulation of a point set V is Delaunay if each edge is the chord of an open disk that misses V; such triangulations are fundamental objects of computational geometry.

• **NOTs for PSLGs** [21]**:** I proved any PSLG with n vertices has a $O(n^{2.5})$ non-obtuse triangulation (all angles $\leq 90°$; called a NOT for brevity). Giving any polynomial bound was a long standing open problem. This result implies that every PSLG with n vertices has a conforming Delaunay triangulation with $O(n^{2.5})$ elements, improving a famous $O(n^3)$ bound of Eldesbrunner and Tan [50] from 1993. Given a PSLG, I also construct a point set V of size $O(n^{2.5})$, so that the Voronoi diagram of V covers Γ (the Voronoi diagram consists of points in $\mathbb{R}^2$ that are closest to two or more points of V). The covering problem arises from machine learning [88] and this is the first solution with polynomial complexity.

• **Refining triangulations** [21]**:** Any triangulation of a simple n-gon has a $O(n^2)$ non-obtuse refinement; this improves an $O(n^4)$ bound of Bern and Eppstein from 1995 [15].

• **Almost non-obtuse triangulations** [21]**:** For any $\epsilon > 0$, every PSLG has a conforming triangulation with $O(n^2/\epsilon^2)$ elements and all angles $\leq 90° + \epsilon$. This improves a 1993 result of S. Mitchell [76] with $\epsilon = \frac{3}{8}\pi = 67.5°$ and a 1996 result of Tan [96] with $\epsilon = \frac{7}{30}\pi = 42°$.

• **Optimal quad-meshes** [23]**:** I prove any PSLG has a conforming $O(n^2)$ quadrilateral mesh with all angles $\leq 120°$ and all new angles $\geq 60°$. This is the first polynomial time quad-meshing algorithm for PSLGs with a positive lower angle bound; moreover, both angle bounds and the complexity bound are sharp. See [16], [31] for previous results on polygons.

QC distortion: Briefly, quasiconformal (QC) maps are homeomorphisms of the plane that distort angles by a bounded amount (more precisely, they are absolutely continuous on a.e. line, and the dilatation satisfies $|\mu_f| \equiv |f_{\bar{z}}/f_z| \leq k$ a.e. for some $k < 1$, i.e., tangents maps are affine with bounded eccentricity a.e.). Conversely, the measurable Riemann mapping theorem (MRMT) says that for any $\mu \in L^\infty(\mathbb{R}^2)$ with $\|\mu\|_\infty \leq k < 1$ there is a QC map with $\mu_f = \mu$. QC maps need not be smooth and can even change dimension, e.g., can map a line to a fractal curve. Hausdorff, upper Minkowski and packing dimension are defined as

$$\mathrm{Hdim}(K) = \inf\{s : \inf\{\textstyle\sum_j r_j^s : K \subset \cup_j D(x_j, r_j)\} = 0\},$$
$$\overline{\mathrm{Mdim}}(K) = \inf\{s : \limsup_{r\to 0} \inf_N N r^s = 0 : K \subset \cup_{j=1}^N D(x_j, r)\}\},$$
$$\mathrm{Pdim}(K) = \inf\{s : K \subset \cup_{j=1}^\infty K_j : \overline{\mathrm{Mdim}}(K_j) \leq s \text{ for all } j\}.$$

• **Frequency of dimension distortion** [36]**:** H. Hakobyan, M. Williams and I show that if $E \subset \mathbb{R}^n$ is Ahlfors regular and $f : \mathbb{R}^n \to \mathbb{R}^n$ is QC then $\mathrm{Hdim} f(y + E) = \mathrm{Hdim}(E)$ for a.e. $y \in \mathbb{R}^n$ (this also holds in Carnot groups). For a QC $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $S \subset \mathbb{R}$, we prove

$\inf_{y\in S} \mathrm{Hdim}(f(\mathbb{R}\times\{y\})) \le 2/(d+1)$ and $\inf_{x\in\mathbb{R}} \mathrm{Hdim}(f(\{x\}\times S)) \le 2d/(d+1)$ and prove sharpness, extending work of Balogh, Monti and Tyson and answering questions in [9], [10].

• **QC maps destroying rectifiability** [36]**:** Hakobyan, Williams and I construct $E \subset \mathbb{R}$, $\mathrm{Hdim}(E) = 1$ and a QC map $f : \mathbb{C} \to \mathbb{C}$ so that $f(E \times [0,1])$ contains no rectifiable sub-arcs; this is the first uncountable example of such a set E.

Transcendental dynamics: Let $\mathcal{E}$ denote the class of entire functions and $\mathcal{T} \subset \mathcal{E}$ be the transcendental functions (non-polynomials). For $f \in \mathcal{E}$, the singular set $S(f)$ is the closure of its critical values and finite asymptotic values (limits of f along curves to ∞). We define the Speiser class as $\mathcal{S} = \{f \in \mathcal{T} : S(f) \text{ is finite}\}$, and the Eremenko-Lyubich class as $\mathcal{B} = \{f \in \mathcal{T} : S(f) \text{ is bounded}\}$. The Fatou set of f, $\mathcal{F}(f)$, is the open set where the iterates $\{f^n\}$ form a normal family. The Julia set, $\mathcal{J}(f)$, is the complement of $\mathcal{F}(f)$. It is usually a "fractal" and determining its dimension and structure are basic problems.

• **The smallest transcendental Julia set** [24]**:** I give the first example of a $f \in \mathcal{T}$ whose Julia set $\mathcal{J}(f)$ has Hausdorff dimension 1. This was open since 1975 when Baker [7] proved that $\mathrm{Hdim}(\mathcal{J}(f)) \ge 1$ for all $f \in \mathcal{T}$. Moreover, $\mathrm{Pdim}(\mathcal{J}(f)) = 1$; it is the first example in $\mathcal{T}$ with $\mathrm{Pdim}(\mathcal{J}) < 2$, and the first where components of $\mathcal{F}(f)$ are bounded by C^1 curves.

• **True trees are dense** [32]**:** I show any compact, connected set $K \subset \mathbb{R}^2$ can be approximated in the Hausdorff metric by the critical points of a Shabat polynomial p (polynomials with only ± 1 as critical values). Then $T = p^{-1}([-1,1])$ is a finite tree; it is a "true tree" in the sense of Grothendieck's *dessins d'enfants*, and this shows "true trees" are dense in all planar continua (for the Hausdorff metric), answering a question of Alex Eremenko.

• **Dynamical dessins are dense** [39]**:** Using the result above, Kevin Pilgrim and I prove that Julia sets of post-critically finite polynomials are dense in all planar continua.

• **Quasiconformal folding** [33]**:** This extends the "true trees are dense" result to entire functions: given a locally finite, unbounded planar tree T satisfying some natural conditions, I construct $f \in \mathcal{S}$ with $S(f) = \{\pm 1\}$ so that $T' = f^{-1}([-1,1])$ approximates T. This gives a simple "machine" for constructing entire functions such as:

- A $f \in \mathcal{B}$ with a wandering domain (open since 1985; false for $\mathcal{S}$, see [52], [60]).
- A $f \in \mathcal{S}$ so that $\mathrm{area}(\{z : |f(z)| > \epsilon\}) < \infty$ for all ϵ (this is a strong counterexample to the area conjecture of Eremenko and Lyubich [52]).
- A $f \in \mathcal{S}$ whose escaping set has no non-trivial path components (a counterexample to the strong Eremenko conjecture in $\mathcal{S}$; improves the example in [87] for $\mathcal{B}$).
- A $f \in \mathcal{S}$ so that $\limsup_{r\to\infty} \log m(r,f)/\log M(r,f) = -\infty$ (improves Hayman's [63] counterexample to Wiman's conjecture; m, M are the min, max of $|f|$ on $\{|z| = r\}$).
- Finite type maps (after Adam Epstein) $Y \to X$, where Y is a pair-of-pants inside a compact surface X (first example with non-trivial topology).

• **The order conjecture** [22]**:** I disprove A. Epstein's conjecture that if $f, g \in \mathcal{S}$ are QC equivalent (i.e., $\psi \circ f = g \circ \phi$ for some QC maps ψ, ϕ) then $\rho(f) = \rho(g)$ where $\rho(f) = \limsup_{r\to\infty} \log\log M(r,f)/\log r$ is the order of growth. See [51] for an example in $\mathcal{B}$.

• **Models for $\mathcal{B}$ and $\mathcal{S}$** [34]**,** [20]**:** If Ω is a disjoint union of smooth, unbounded Jordan domains (called tracts) and $F : \Omega \to \{|z| > R\}$ is a holomorphic covering map on each tract, then (Ω, F) is called a model. For example, Eremenko and Lyubich observed [52] that if $f \in \mathcal{B}$ then $(\Omega = \{|f| > R\}, f|_\Omega)$ is a model if R is large enough; we call these EL-models. In [34] I prove that every model can be extended from $\Omega(\rho) = \{|F| > R + \rho\}$ to a quasiregular map on the whole plane; this shows that every model is can be QC approximated by an EL-model with the same number of tracts, answering a question of Rempe-Gillen [84]. In [20] I prove an analogous result for $\mathcal{S}$ (but now the extension may have twice as many tracts as the model, and this bound is sharp).

BROADER IMPACT

• **Developing infrastructure for academic and industrial computing:** My meshing results enhance the suite of available automatic meshing algorithms available for research and industry, and improve known methods in various ways. For example, condition numbers for certain matrices associated with general triangulations grow exponentially with the size of the mesh, but only linearly for NOTs, [100]; the finite element method on a NOT leads to a matrix that is symmetric, positive definite and negative off the diagonal, giving a linear system that is easier to solve [90]. Other advantages are described in [46] (maximum principles for discrete PDE's), [13] (Hamilton-Jacobi equations), [68], [89] (finding geodesics on a triangulated surface), [1], [97], [98] (meshing space-time), [17], [90] (dual triangulations).

• **Building interdisciplinary connections:** The interdisciplinary character of the problems in the proposal can serve as a bridge between researchers with common interests but different backgrounds. For example, my papers on fast conformal mapping [30] and meshing [21], [23], [31] have appeared in (or been submitted to) a premier computer science journal and are intended to be accessible to non-mathematicians. My work on a easily computed, combinatorial version of the Riemann map (the so-called iota map, e.g., [26], [27], [30], [31]) has been cited in papers by applied mathematicians (e.g., [11], [53], [54]). I spoke at the 2010 Fall Workshop on Computational Geometry, and gave a keynote address at the 2012 Symposium on Computational Geometry, where I also organized a workshop on the interactions between computational geometry and analysis; this was a sequel to a similar workshop Steve Vavasis and I hosted at Stony Brook in 2007. I maintain websites for both.

My recent work has included a paper [35] with Eugene Feinberg (an applied mathematician at Stony Brook, an expert on Markov decision processes and efficiency in power grids) and on-going work with Andrew Mullhaupt (also in Applied Mathematics at Stony Brook) on the information geometry of time invariant linear systems; the motivation comes from quantitative finance, but the results apply to any type of signal processing such as arises in radar tracking, medical imaging, subterranean imaging, ...

• **Educational impact:** The results obtained have been the basis of a series of graduate courses; a set of lecture notes on dynamics and quasiconformal analysis, and another set on conformal mapping and meshing. Some of the results will appear in a forthcoming book with Yuval Peres giving an introduction to fractals occurring in analysis and probability. I have mentored a number of undergraduate research projects on related topics: Ahmed

Rafiqi used accelerated random walks to calculate conformal maps (he is now a Ph.D. student at Cornell); Kevin Sackel worked on geometric problems related to removability (he won a Churchchill fellowship to Cambridge and is now at MIT); Shalin Parekh numerically estimated percolation dimension of random walks on a grid (he is now in a masters program run by Stas Smirnov and Wendelin Werner); I am currently working with Christopher Dular (nominated for a Churchill) to implement my $O(n^2)$ triangulation refinement algorithm.

I have supervised 5 Ph.D. dissertations on topics related to my previous and current proposal: Zsuzanne Gö:nye (geodesics in hyperbolic manifolds), Karyn Lundberg (boundary convergence of conformal maps), Hrant Hakobyan (dimension distortion under QC maps) and Chris Green (numerical conformal mapping), Kirill Lazebnik (transcendental dynamics and numerical QC mapping). Most of my students have had some computational aspect to their thesis and have had to study topics such as numerical linear algebra, optimization and probability in addition to complex analysis; this makes them better suited to both academic and non-academic jobs (Green went into finance and Lundberg works at Lincoln Labs at MIT). Producing mathematicians who can talk to and work with applied mathematicians (or even non-mathematicians) is a form of infrastructure enhancement that makes it easier to transfer decades (centuries?) of mathematical progress into practical solutions of important problems. As computers approach the physical limits of performance, the interaction of mathematics and computation will become more critical and these skills more important.

PUBLICATIONS SUPPORTED UNDER DMS-1305233

- True trees are dense. *Invent. Math.*, 197(2):433–452, 2014.
- Constructing entire functions by quasiconformal folding. *Acta Math.*, 214(1):1–60, 2015.
- Models for the Eremenko–Lyubich class. *J. Lond. Math. Soc. (2)*, 92(1):202–221, 2015.
- Examples concerning Abel and Cesàro limits. *J. Math. Anal. Appl.*, 420(2):1654–1661, 2014, with E.A. Feinberg and J. Zhang.
- Dynamical dessins are dense. to appear in *Revista Mat. Iberoamericana*, with K.Pilgrim
- The order conjecture fails in class $\mathcal{S}$. to appear in *J. d'Analyse.*
- Models for the Speiser class. under revision for *P. Lond. Math. Soc.*
- Nonobtuse triangulations of PSLGs. under revision for *Discrete Comput. Geom.*
- Quadrilateral meshes for PSLGs. under revision for *Discrete Comput. Geom.*
- Frequency of dimension distortion under quasisymmetric mappings. under revision for *GAFA*, with H. Hakobyan, and M. Williams.
- A transcendental Julia set of dimension 1. submitted to *Acta. Math.*
- Another Besicovitch-Kakeya set. preprint 2014.
- Conformal images of Carleson curves. preprint 2014.

EVIDENCE OF RESEARCH PRODUCTS AND THEIR AVAILABILITY

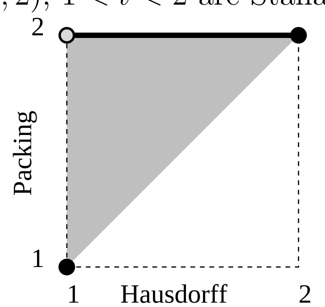
All preprints are posted on `www.math.sunysb.edu/~bishop/papers`. My webpage also contains lecture notes and slides of lectures related to my research (also links to videos of my lectures when they exist), as well as abstracts of my papers, descriptions of my research and links to related work of other mathematicians.

PROJECT DESCRIPTION

The proposal contains a wide range of problems grouped into three general areas: (1) geometric questions about the Julia and Fatou sets of transcendental (non-polynomial) entire functions; (2) removability for conformal homeomorphisms with an emphasis on the removability of Brownian motion in $\mathbb{R}^2$ and related geometric problems; (3) the interaction of conformal analysis with computational and discrete geometry. Despite the variety, certain themes persist throughout, e.g., in each category we will seek to understand infinite dimensional analogs of results known in a finite dimensional setting. The problems vary in difficulty and include both well known questions and some novel conjectures.

1. Transcendental Dynamics

• **Dimensions of Julia sets:** A striking 1975 theorem of Baker [7] states that the Julia set of a $f \in \mathcal{T}$, i.e., a transcendental entire function, must contain a non-trivial continuum and hence has Hausdorff dimension at least 1. In [24] I construct a $f \in \mathcal{T}$ with $\mathrm{Hdim}(\mathcal{J}(f)) = \mathrm{Pdim}(\mathcal{J}(f)) = 1$; this is the first example that attains Baker's lower bound and the first example that has packing dimension less than 2. The gray triangle below shows the possible pairs $(\mathrm{Hdim}, \mathrm{Pdim})$ for a transcendental Julia set (note that $\mathrm{Hdim} \le \mathrm{Pdim}$ for any set); the black shows all known examples (the dot at $(2,2)$ is due to McMullen [74]; the top edge $(t,2)$, $1 < t < 2$ are Stallard's examples [91], [92]; the dot $(1,1)$ is my example from [24]).



Question 1. *For each $1 < s < t < 2$ is there a $f \in \mathcal{T}$ with $\mathrm{Hdim}(\mathcal{J}) = s$ and $\mathrm{Pdim}(\mathcal{J}) = t$? (gray triangle)*

Question 2. *Is there a transcendental Julia set with $\mathrm{Hdim}(\mathcal{J}) = 1, \mathrm{Pdim}(\mathcal{J}) = 2$? (upper left corner)*

Question 3. *For $t \in (1,2)$, is there a $f \in \mathcal{T}$ with $\mathrm{Hdim}(\mathcal{J}(f)) = \mathrm{Pdim}(\mathcal{J}(f)) = t$? (diagonal edge)*

My example in [24] is of the form $f(z) = p(z)\prod_1^\infty(1 - \frac{1}{2}(z/R_k)^{m_k}$ where $\{R_k\}, \{m_k\} \nearrow \infty$ sufficiently quickly and p is a polynomial with $\mathrm{Hdim}(\mathcal{J}(p)) \ll 1$. Every component of the Fatou set is bounded by a countable union of C^1 curves; $\mathcal{J}(f)$ also contains perturbed copies of $\mathcal{J}(p)$; the rest of the Julia set has small dimension depending on the parameters $\{m_k\}, \{R_k\}$. Question 3 might be answered by first replacing p by a family p_λ of polynomials so that $\mathrm{Hdim}(\mathcal{J}(p_\lambda)) = \mathrm{Pdim}(\mathcal{J}(p_\lambda)) \nearrow 2$ continuously, and then proving the corresponding family of transcendental entire functions f_λ has Julia sets with the same structure as before: C^1 curves, perturbed copies of $\mathcal{J}(p_\lambda)$, and a set of small dimension. This would imply that the Hausdorff and packing dimensions of $\mathcal{J}(f)$ agree and move continuously towards 2.

My example is the first with multiply connected Fatou components bounded by Jordan curves and the first where the dynamics are completely understood; can the methods from this paper be used to analyze other such examples? What other types of boundary curves can occur? Can the boundaries of the Fatou components be smoother than C^1? Is this already the case for my example?

To deal with Question 2, instead of approximating a single polynomial p, we will want to construct an entire function f that mimics a sequence of polynomials $\{p_n\}$ on separated disks $\{D_n\}$, where $\mathcal{J}(p_n)$ has small dimension but approximates a disk in the Hausdorff metric to order $1/n$; constructing such polynomials is relatively easy, e.g., [64], [65], but it is not yet clear how to combine them into a single entire function. Most likely, methods for solving Problems 2 and 3 would lead to a solution of Problem 1 as well.

We define $K(f) = \{z : f^n(z) \text{ is bounded}\}$, $I(f) = \{z : f^n(z) \to \infty\}$ (the escaping set) and $BU(f) = \mathbb{C} \setminus (K(f) \cup I(f))$ (points with unbounded orbits that return to some bounded set infinitely often; called the "**bu**ngee set" in [78]). There are numerous open questions about the geometry and dimension of these sets; the following are a few such problems related to my previous work (recall from above that f, g are QC-equivalent if $f \circ \psi = \varphi \circ g$ for some QC maps ψ, φ of $\mathbb{R}^2$ to itself):

Question 4 ([83])**.** *If $f, g \in \mathcal{B}$ are QC-equivalent is* $\mathrm{Hdim}(I(f)) = \mathrm{Hdim}(I(g))$*?*

I expect that my work on models for $\mathcal{B}$ [34] and on the order conjecture [22] will help resolve this. Stallard has shown that the corresponding result for Julia sets fails in $\mathcal{B}$, but it remains open in class $\mathcal{S}$; perhaps my QC-folding methods will give a counterexample:

Question 5. *If $f, g \in \mathcal{S}$ are QC equivalent is* $\mathrm{Hdim}(\mathcal{J}(f)) = \mathrm{Hdim}(\mathcal{J}(g))$*?*

• **Bounded Fatou orbits:** Baker proved [7], [8] that a multiply connected Fatou component (as in my example above) must be bounded and contained in $I(f)$, and hence it must be wandering (i.e., the orbit consists of distinct Fatou components). $BU(f)$ can also contain wandering components (e.g., [52], [33]), but the following is a major open problem:

Question 6. *Can $K(f)$ contain a wandering domain? Can we take $f \in \mathcal{B}$?*

This is probably very difficult, but is certainly very interesting. On the one hand, one can imagine perturbing a polynomial with a Cremer point (an irrational fixed point in the Julia set) into an entire function with a Fatou component that somehow "orbits around" the Cremer point. On the other hand, one can also imagine extending Dennis Sullivan's no-wandering-domains theorem for polynomials, to rule out bounded wandering orbits. Sullivan's proof is a dimension argument: using the MRMT, the collection of possible QC perturbations of a polynomial with a wandering domain contains a subset parameterized by the unit ball of L^∞, but the set of polynomials (of a fixed degree) is obviously finite dimensional and this gives a contradiction. Can we replace the space of polynomials by a space of entire functions restricted to the compact closure of bounded orbit, and show this is "small" in some sense that prevents a nice parameterization by L^∞?

Some exciting progress in this direction was given by Rempe-Gillen and Mihaljević-Brandt in [75]; they proved non-wandering for some infinite dimensional families using hyperbolic geometry and special assumptions on the singular set. More generally, they asked

Problem 7. *If $|f^n(s)| \nearrow +\infty$ uniformly on $S(f)$, can f have a wandering domain?.*

Every limit point of an orbit in a wandering domain is a non-isolated point of the post-singular set (page 370, [14]), so if the post-singular set has no finite limit points, there can't be a wandering domain with bounded orbits. In Problem 7, the only limit points of the post-singular set are the limit points of $S(f)$ itself (or their iterates). It should be possible to build a counterexample using my QC-folding technique from [33], but this will require extending the basic construction, not just applying the existing method.

• **Escaping Fatou orbits:** As noted earlier, multiply connected Fatou components must escape to ∞, and the proof of this shows their boundaries also escape. However, for simply connected Fatou components, $\Omega \subset I(f)$ does not imply $\partial\Omega \subset I(f)$, although we expect this is true of "most" boundary points; to make this precise we give a few definitions.

If Ω is simply connected, $z_0 \in \Omega$ and $E \subset \partial\Omega$, the harmonic measure of E in Ω with respect to z_0 is defined as $\omega(z_0, E, \Omega) = |f^{-1}(E)|/2\pi$, where $f : \mathbb{D} \to \Omega$ is conformal with $f(0) = z_0$, i.e., harmonic measure is the push-forward of normalized Lebesgue measure on $\partial\mathbb{D}$. Similarly we can push-forward logarithmic capacity by $\text{logcap}(z_0, E, \Omega) = \text{logcap}(f^{-1}(E))$ (the definitions make sense because f has well defined radial boundary values except on a set of zero capacity). The vanishing of both ω and logcap is independent of z_0 and we generally drop z_0 from our notation; we also drop Ω when it is clear from context. Rippon and Stallard [85] showed that $\Omega \subset I(f)$ iff $\omega(I(f) \cap \partial\Omega) > 0$. In fact, I expect that:

Conjecture 8. $\Omega \subset I(f)$ *iff* $\text{logcap}(\partial\Omega \cap I(f)) > 0$ *iff* $\text{logcap}(\partial\Omega \setminus I(f)) = 0$.

This holds if f is 1-to-1 on the Fatou components by an extremal length argument; in general, it will depend on the distortion of logarithmic capacity on the circle under certain inner functions. Results of Fernàndez, Pestana and Rodríguez [55], [56] may be crucial here.

2. Removable sets

• **Basic properties:** A closed set $E \subset \mathbb{S}^2$ ($= \mathbb{C} \cup \{\infty\}$, the Riemann sphere) is removable for a property P if whenever a function f on $\mathbb{S}^2$ has property P on $\mathbb{S}^2 \setminus E$, then it also has property P on all of $\mathbb{S}^2$. Our main examples will be "holomorphic" and "conformal" (= holomorphic and 1-to-1). More precisely, let $A(E)$ denote continuous functions on $\mathbb{S}^2$ that are holomorphic functions off E and let CH(E) be homeomorphisms of $\mathbb{S}^2$ that are conformal off E. We say E is CA-removable (**c**ontinuous **a**nalytic) if $A(E)$ consists only of constants and we say E is CH-removable if CH(E) consists only of the Möbius transformations. All sets of positive area are CH-non-removable and sets of σ-finite 1-dimensional measure are CH-removable, but little else is known, e.g.,

Question 9. *Is the union of two compact CH-removable sets CH-removable?*

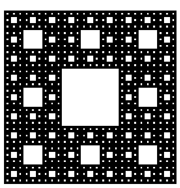
This is easy if the sets are disjoint, but is unknown even for Jordan arcs intersecting at one point (a case I plan to attack with Malik Younsi, currently a postdoc here at Stony Brook). In all the examples I know of, if CH(E) is non-trivial (i.e., contains a non-Möbius element) then it is "large". For example, if E has positive area then the measurable Riemann mapping theorem gives an element of CH(E) for each $\mu \in L^\infty(E, dxdy)$ with $\|\mu\|_\infty < 1$.

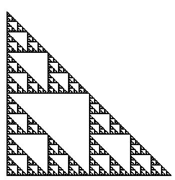
Question 10. *If* CH(E) *is non-trivial, is it infinite dimensional in some sense? Are there non-trivial elements converging to the identity?*

• **Flexible sets:** Another case that only gives "large" examples of CH(E) arises from my proof in [28] that a closed curve Γ is flexible iff its complementary components are log-singular (i.e., $\Gamma = E_1 \cup E_2$ with $\text{logcap}(E_1, \Omega_1) = \text{logcap}(E_2, \Omega_2) = 0$). Here flexible means that any homeomorphic image of Γ can be approximated in the Hausdorff metric by an image from CH(E). This is analogous to a result of Browder and Wermer [40], [41], that $A(\Gamma)$ is "large" iff the harmonic measures for the two sides are singular; here large means $A(\Gamma)$ is a Dirichlet algebra (real parts of functions in $A(\Gamma)$ are dense in all real, continuous functions on Γ). In [48] Davie generalized the Browder-Wermer theorem to general compact K (also see [25]): $A(K)$ is Dirichlet iff different complementary components of K have singular harmonic measures and each component is "nicely connected" (this means the Riemann map onto the component Ω has boundary values that are 1-to-1 Lebesgue a.e.; we will say Ω is "very nicely connected" if the boundary values are 1-to-1 except on a set of zero logarithmic capacity).

Conjecture 11. *K is CH-flexible iff it has log-singular complementary components and each component is very nicely connected.*

The proof in [28] that a log-singular curve Γ is flexible starts by mapping the two components of $\mathbb{S}^2 \setminus \Gamma$ to smooth domains with disjoint closures, and then iteratively "pushes" corresponding pairs of boundary points towards each other along hyperbolic geodesics in the annulus separating the smooth domains. Only one point from each pair has to move at each step, and the log-singularity implies that one point from every pair can be moved half-way towards its partner. To prove the general case, multiple points will have to move simultaneously along more topologically complex paths to avoid "collisions". Just proving K is CH-non-removable in Conjecture 11 would be interesting even in special cases:





Question 12. *Is the Sierpinski Carpet removable? (complementary components have disjoint closures)*

Question 13. *Is the Sierpinski Gasket removable? (component closures meet in finite sets)*

• **Proving removability:** The most general result for proving a set K is CH-removable is due to Jones and Smirnov [67]. They assume each $x \in K = \partial\Omega$ is the limit an infinite chain $\{Q_n(x)\}_0^\infty$ of adjacent Whitney squares for Ω such that $Q_0(x)$ has unit size. They define the shadow set of a Whitney square Q as $I(Q) = \{x \in K : Q = Q_n(x) \text{ for some } n\}$ and assume that $\mathrm{dist}(Q, I(Q)) + \mathrm{diam}(I(Q)) \to 0$ as $\mathrm{diam}(Q) \to 0$ and that $\sum_Q \mathrm{diam}(I(Q))^2 < \infty$. If these conditions hold, then K is CH-removable. Chains of adjacent squares are used so that consecutive squares have conformal images of comparable size.

Conjecture 14. *The Jones-Smirnov theorem still holds for $K = \partial\Omega$ (even if Ω is disconnected) if we only assume consecutive squares in a chain can be connected by a path family with conformal modulus $\simeq 1$ and with every path hitting K only countable often.*

This should be straightforward; the real challenge will be to formulate and prove versions that allow the uniform modulus estimate to be weakened to deal with sets with non-uniform geometry, such as Julia sets or random sets, e.g.,

Question 15. *Is 2-dimensional Brownian motion CH-removable?*

More precisely, we should ask if the set visited by Brownian motion during some compact interval, say $[0, 1]$, is removable almost surely (a.s.). We call this set the Brownian trace and denote it by $B([0, 1])$. The trace is obviously compact and connected, and Burdzy [43] proved its open complementary components are Jordan domains with tangents almost nowhere (for harmonic measure). A result of mine with Carleson, Garnett and Jones [19] then implies that harmonic measures on different components are mutually singular, and hence $B([0,1])$ is CA-non-removable (actually, $A(B([0,1]))$ is a Dirichlet algebra as discussed earlier).

However, I believe the Brownian trace is CH-removable. As evidence, we can show the complementary components are not log-singular. We call the boundary of a single component a "frontier", and the (non-empty) intersection of two frontiers a "crossing set". Jones, Pemantle, Peres, and I [38] proved that frontiers have dimension > 1. Later, Lawler, Schramm and Werner [71] proved frontiers have dimension $4/3$. Their methods should also show the crossing sets have dimension $3/4$ (crossing points are closely related to cut-points, i.e., points whose removal disconnects the trace; the cut-points have dimension $3/4$ by [70], [72]). The Riemann map from the disk to a complementary component of the trace is Hölder continuous; this implies the preimages on $\partial\mathbb{D}$ of crossing points will have positive dimension (hence positive capacity) and this contradicts log-singularity of the components. Hölder continuity of the Riemann map follows from a result of Rohde and Schramm [86] on Schramm-Loewner evolutions (SLE) and the fact that the Brownian frontier "is" SLE(8/3).

In order to apply Conjecture 14 to the Brownian trace, we need to know Whitney squares in different complementary components can be connected by paths that hit the trace only countably often (and satisfy statistical modulus estimates). Do such paths exist? We say that two components are adjacent if their frontiers intersect, and we say they are crossing-equivalent if they can be connected by a finite chain of adjacent components.

Conjecture 16. *Any two complementary components are crossing-equivalent.*

Conjecture 16 is listed on Burdzy's website [42], where he states that it implies

Conjecture 17. *The Brownian trace minus the union of frontiers is totally disconnected.*

• **Percolation dimension:** If Conjecture 17 fails, then $B([0,1])$ contains a continuum disjoint from every frontier. Since this continuum lies in a "very dense" part of trace, perhaps it contains a "nice" curve, at least in the sense of dimension. Burdzy [44], defines percolation dimension as $\text{Hdim}_{\text{perc}}(K) = \inf\{\dim(\gamma) : \gamma \subset K, \gamma \text{ a Jordan arc}\}$.

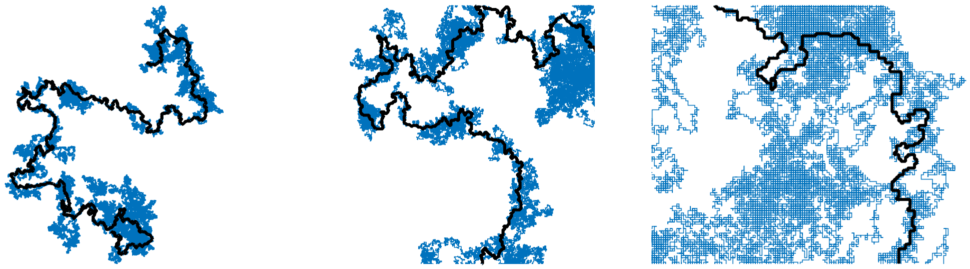
Question 18. *What is the percolation dimension of the Brownian trace?* $= 1$? > 1?

Brownian frontiers show the percolation dimension is $\leq 3/4$, but I don't know of any better estimate. If the answer is 1, we can ask a stronger question:

Question 19. *Does* $B([0,1])$ *almost surely contain a rectifiable curve? Can any two points of* $B([0,1])$ *be connected by a rectifiable curve in the trace?*

Pemantle [79] showed that the Brownian trace almost surely contains no line segments (or even any positive length subset of a segment). His proof estimates the probability of Brownian motion ϵ-approximating a segment and crucially uses the fact that the set of lines is finite dimensional; Question 19 is an infinite dimensional version of Pemantle's result, so the situation is reminiscent of Sullivan's no-wandering-domains theorem, where we also wished to replace a dimension argument by a geometric one.

Last year an undergraduate, Shalin Parekh, wrote an honors thesis with me using random walks on a square grid to estimate the percolation dimension of $B([0,1])$ as ≈ 1.02. The sizes of his random walks were fairly small and the numerical result is too close to 1 to be decisive, but the pictures generated are very suggestive. Here is a random walk with 10^7 steps, a shortest path between two points and two blow-ups of different portions of the path:



In the "thick" portions of the trace the shortest path seems quite straight, but when the path runs between two adjacent complementary components, it is forced to travel through the corresponding crossing set, which suggests we ask

Question 20. *Is the crossing set of two components contained in a rectifiable curve a.s.?*

Peter Jones' traveling salesman theorem (TST) [66] characterizes subsets of rectifiable curves in terms of sums of "β-numbers" (β's measure the local deviation of a set from a line segment); does Pemantle's proof, mentioned above, gives a useful statistical lower bound on β-numbers? I recently discovered a new proof of the TST where the β-numbers are interpreted as the probabilities that random lines hit a set in a certain way; this may be better suited for applying TST to random sets (applying the TST directly to random sets can be challenging, as illustrated by the dimension estimates for frontiers in [38]). Exciting new results of Badger and Schul [5], [6] on rectifiable measures may also be relevant.

3. Computational geometry and conformal analysis

• **Conformal geometry gives good meshes.** As discussed earlier, I recently proved the existence of polynomial sized conforming non-obtuse triangulations (NOTs; all angles $\leq 90°$) for any planar straight line graph (PSLG) [21]. If the PSLG has n vertices, the construction gives $O(n^{2.5})$ elements; the best lower bound is $\simeq n^2$, so a gap exists:

Conjecture 21. *Every PSLG has a conforming NOT with $O(n^2)$ elements.*

The $O(n^2)$ is a worst case estimate, and it would be very interesting (and important for applications) to know if the algorithm can do better in better cases:

Problem 22. *Find an algorithm that, given a PSLG Γ, produces a $O(N)$ sized conforming NOT, where N is the size of a minimal conforming NOT for that Γ.*

The same problem is open for quadrilateral meshes; I proved in [23] that every PSLG with n vertices has, at worst, a $O(n^2)$ quad-mesh with all angles between $60°$ and $120°$ (except for existing angles in the PSLG of measure $< 60°$). Can we produce a near-optimal quad-mesh for each particular input? Furthermore, my method does not bound the eccentricity of the quadrilaterals used (the ratio of the longest to shortest sides); sometimes long, narrow elements are unavoidable in order to achieve the $O(n^2)$ complexity bound. Can we efficiently build angle-optimal meshes that also satisfy a uniform bound on eccentricity?

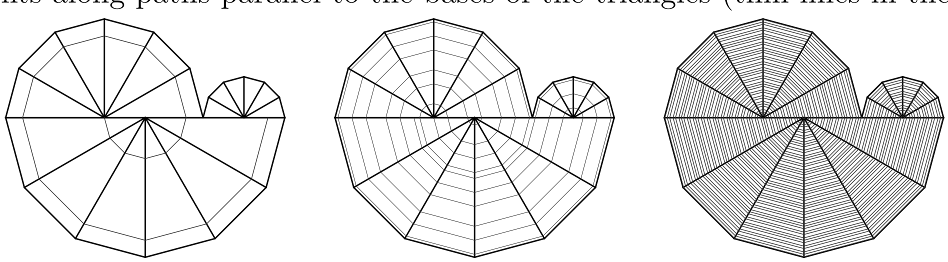
The $O(n^2)$ mesh constructed in [23] actually consists of $O(n)$ sub-meshes that each have the combinatorial structure of a rectangular grid. It would be extremely interesting to see whether various numerical methods that use quad-meshes might have faster implementations for rectangular grids, and whether these methods can be adapted to make use of rectangular structure within more general meshes (e.g., like block structure in linear algebra).

The theory of optimal meshing in $\mathbb{R}^3$ (the really important case) is wide open; there are a multitude of examples, heuristics and implementations, but few rigorous results.

Question 23. *Do polyhedra in $\mathbb{R}^3$ have non-obtuse tetrahedralizations of polynomial size?*

Does this hold for any dihedral angle bound $< 180°$? Even finding an acute tetrahedralization of a cube in $\mathbb{R}^3$ was open until recently (the smallest known example uses 1,370 pieces [99]) and there is no acute decomposition for the cube in $\mathbb{R}^4$, [69]. The breakthrough in the 2-dimensional case was to introduce the idea of a thick/thin decomposition of a polygon that is analogous to the thick/thin decomposition of a Riemann surface; in the thin parts of the polygon, Euclidean geometry is used to create the mesh and in the thick parts hyperbolic geometry is used. Fast, approximate conformal mapping gives the decomposition into thick and thin parts, and the use of the alternate geometries gives the optimal angles bounds. Can we use analogous ideas in $\mathbb{R}^3$? Can one create a 3-manifold out of a polyhedron, run a Ricci flow on it (as in Perelman's proof of Thurston's geometrization conjecture) to decompose it into pieces with geometric structure and then utilize the "natural" geometry on the different pieces to define meshes? Any progress would have a significant impact.

My construction of NOTs of size $O(n^{2.5})$ in [21] can probably be improved. Given a PSLG, the algorithm first adds edges to form a dissection by isosceles triangles with good angles (a dissection is like a mesh, except that edges may overlap without being equal; thus some "bad" points are vertices of some triangles but interior to edges of others, as shown in the figure below). Next, the algorithm converts a dissection into a mesh by propagating the "bad" points along paths parallel to the bases of the triangles (thin lines in the figure):



If the paths terminate (by leaving the dissection or hitting another vertex), they cut the dissection into a mesh using triangles and quadrilaterals (and the latter can be made into triangles by adding diagonals). However, as shown above, a propagation path may cross the same isosceles triangle repeatedly. In order to get the uniform complexity bound, the algorithm bends paths to terminate them faster; to maintain the angle bounds the bending is limited by constraints that closely resemble keeping a discrete second derivative of the propagation paths bounded. The $O(n)$ bad vertices propagate for $O(n)$ steps before they terminate, but $O(\sqrt{n})$ new vertices might be added for each original one. These are also propagated for $O(n)$ steps, giving the $O(n^{2.5})$ bound. The bound on the number of new points assumes numerous worst cases occur simultaneously and I expect that a more careful analysis will show that only $O(n)$ new points are really needed. The fact that the bending process is constrained by something that looks like a discrete derivative bound is reminiscent of Pugh's closing lemma: every C^1 vector field has a small perturbation with a closed orbit [80],[81], [82]; this is open for C^2 vector fields. Dennis Sullivan asked if this could be made precise:

Question 24. *Can a closing lemma help prove the $O(n^2)$ NOT-theorem? Can the NOT argument help prove a C^2-closing lemma (or suggest a counterexample)?*

The propagation paths described above define return maps on the dissection edges that preserve length, so perhaps the theory of interval exchange maps or billiards in polygons is also relevant to these problems.

• **The discrete measurable Riemann mapping theorem.** Optimal meshing is intimately connected with numerical conformal mapping [30], [31], but many important applications require numerical quasiconformal mapping, an area still in its infancy. A natural starting point is to consider piecewise linear (PL) maps between planar triangulations.

Such a map is automatically QC (a PLQC map) since the dilatation μ is piecewise constant (PC), and hence only takes finitely many values. We would like the converse to be true: that every PC μ is the dilatation of a PLQC map; however this discrete version of the measurable Riemann mapping theorem (MRMT) is clearly false. To see this, note that a PC μ defines an image of each triangle up to Euclidean similarities, hence defines a QC map to a triangulated surface, but this surface is not planar unless the curvature at each vertex is zero (curvature $= 2\pi$ minus the angle sum at the vertex). We can lengthen or shorten the edges incident to a vertex to make this curvature zero, and this "redistributes" the curvature to neighboring vertices. Iterating the process causes curvature to "diffuse" from interior to boundary vertices, giving a planar surface in the limit. This "discrete Ricci flow" has been studied numerically [62], [73], no one has addressed the evolution of μ:

Conjecture 25. *(Discrete MRMT) Given a PC dilatation μ, the curvature diffusion process gives a PLQC dilatation μ_0 with $|\mu - \mu_0| = O(|\nabla \mu|)$.*

The usual MRMT follows easily from this. Here, $|\nabla \mu|$ refers to the maximum difference of a PC μ on adjacent triangles. The idea is that the total change in an edge's length is controlled by the amount of curvature that "diffuses through" that edge over time and hence is analogous to a Green's function on the triangulation (recall that the usual Green's function is the integral of the heat kernel over time). The change in μ however, depends only on relative length changes in adjacent edges and corresponds to the discrete gradient of this "Green's function"; this should be uniformly bounded in terms of the gradient of the data because of the smoothing of the diffusion. The situation is very reminiscent of heat kernel and Green's function estimates that Peter Jones and I proved for Riemann surfaces in [37]. Similar ideas should work here too.

• **Chord-arc curves and the carpenter's rule.** A rectifiable curve $\Gamma \subset \mathbb{R}^2$ is called chord-arc if the arclength σ between $x, y \in \Gamma$ satisfies $\sigma(x,y) \le M|x-y|$ for some $M < \infty$. The arclength parameterization γ_0 satisfies $\gamma_0'(t) = e^{if(t)}$ where $f \in \mathrm{BMO}$, see e.g. [59]. A famous question asks if the space of chord-arc curves is connected in the BMO topology. If so, any chord-arc path Γ can be "straightened" to a line segment, i.e., there is a homotopy

$\gamma : [0,1]^2 \to \mathbb{R}^2$ so that the map $t \to \gamma_t = \gamma(\cdot, t)$ is continuous in the BMO topology, γ_1 is a line segment and $\gamma_0 = \Gamma$. We call γ expansive if all distances increase, i.e., $s < t$ implies $|\gamma(u,s) - \gamma(v,s)| \leq |\gamma(u,t) - \gamma(v,t)|$, $\forall u, v \in [0,1]$. An expansive motion cannot increase the chord-arc constant M, so a natural strengthening of the connectedness conjecture is

Conjecture 26. *A chord-arc path can be straightened by an expansive motion.*

The good news is that Conjecture 26 has already been solved for polygons (the finite dimensional analog) by Connelly, Demaine and Rote [47], and independently by Streinu [93] (this discrete version is called the carpenter's rule problem after the hinged yardstick used in wood-working). The solution in [47] is based on linear programming; given the current position, a set of linear constraints determines what motions are expanding and using duality, one shows a strict expansion is always possible unless the arc is already straight. To prove Conjecture 26, we "just" have to pass to the limit, but this requires estimating how fast points move in the linear programming solution, in order to ensure the limiting motion is continuous in the BMO topology. One way to do this is to impose additional Lipschitz "speed limit" constraints such as $|\gamma(u,s) - \gamma(u,t)| \leq |s-t|$ and then ask if the time needed to straighten γ_0 can be bounded in terms of the chord-arc constant of γ_0. Can we prove a differential inequality for the chord-arc constant as a function of time? K. Astala and J.M. González [4] gave an operator theoretic characterization of chord-arc curves; perhaps this can be combined with the linear programming ideas above to prove such an inequality.

• **Factoring bi-Lipschitz maps.** The MRMT implies that every K-QC map on $\mathbb{R}^2$ can be written as a composition of $(1+\epsilon)$-QC maps for any $\epsilon > 0$, but the analogous statement for K-bi-Lipschitz (i.e., $K^{-1} \leq |f(x) - f(y)|/|x-y| \leq K$) maps remains open:

Conjecture 27. *If f is bi-Lipschitz, then $f = f_1 \circ \cdots \circ f_n$ where each f_k is $(1+\epsilon)$-biLipschitz.*

Markovic and Fletcher [57] verified this if $f \in C^1$ and He and Freedman [58] gave a lower bound for n in a special case (a map sending $\mathbb{R}$ to a logarithmic spiral). The obvious factorization into small QC factors need not give bi-Lipschitz factors (see e.g., [29]). Chord-arc paths are precisely the images of segments under bi-Lipschitz maps, so this problem is closely connected to the previous problem of straightening chord-arc curves. Can it also be attacked by solving a discrete problem and passing to a limit?

One such discrete version is to ask if every pair of combinatorially equivalent planar triangulations where corresponding edge lengths differ by at most a factor of K can be connected by a chain of $N = N(K, \epsilon)$ triangulations where edges change by at most a factor of $1 + \epsilon$ at each step. There is an extensive literature on continuously morphing one triangulation to another (see [2], [3], [12], [18], [45], [61], [94], [95], but the quantitative estimates needed for the bi-Lipschitz factorization problem seem completely unexplored. Can we impose a stronger conditions, such as requiring edge lengths to change monotonely (e.g., short edges only get longer and long edges only get shorter)? Can this be formulated as a linear program as in the carpenter's rule problem?

In terms of PLQC maps (defined above) we are asking if any PLQC dilatation μ can be connected to 0 by a chain of PLQC dilatations $\{\mu_n\}_1^N$ so that corresponding edge lengths change at most by a factor of $1 + \epsilon$ at each step. Earlier, we conjectured that when we use curvature diffusion to “flatten” a PC dilatation to a PLQC dilatation, an edge’s length is multiplied by a factor that is bounded in terms of the total flow of discrete curvature across that edge. Moreover, this total flow corresponds to a kind of Green’s function on the triangulation and Conjecture 25 corresponded to bounding a discrete gradient of this function. Now we want to perturb a PLQC μ to a nearby PLQC dilatation, which is “smaller” than μ but so that the two Green’s functions are close i.e, we want to bound a variational derivative of the Green’s function in the direction of the perturbation. In addition to ideas from the morphing literature, the problem is open to tools including linear programming, discrete Ricci flow, Markov chains, potential theory and the calculus of variations. It is an outstanding example of a contact point between classical analysis and computational geometry where ideas might flow in either direction.

4. Broader impacts of the proposed work

Enhancing computational infrastructure: All of the broader impacts discussed in the summary of previous work (enhanced computational infrastructure, encouraging interdisciplinary research and enhancing STEM education) also apply to the current proposal. Solutions to the 2 and 3 dimensional meshing problems could have a dramatic impact on various aspects of modeling surfaces and 3-dimensional bodies, which in turn have numerous implications for computing in research and manufacturing. The increasing use of finite element methods increases the incentive to improve automatic meshing algorithms. However, many known algorithms can create distorted and even unusable grid elements, so automatic meshing methods with geometric guarantees are essential. As well as improving known theoretical results, I will work to implement my previous algorithms to demonstrate their utility and make them more accessible to potential users. Numerical QC mapping is only in its infancy, but has many potential benefits. For example, thinking of a human face a surface with marked features (mouth, eyes,. . .) it has been suggested that conformal invariants of and Teichmüller distance between such surfaces would be an effective, efficient way to do facial recognition [77] (automatic face recognition one of the major problems of computer vision with applications to enhancing individual privacy and national security). Conformal analysis can also lead to unanticipated applications, e.g., Torbjörn Lundh (Chalmers University), a former postdoc of mine, is spending this year at Stanford, participating in research relating to biomechanical interactions between medical devices and the vascular system (his work in conformal geometry led him to study the deformation of organism shapes).

Educational impact: In the summary of previous work, I described courses, lecture notes, graduate and undergraduate projects related to my work, and the current proposal has similar impacts on the infrastructure of research and education. Graduate students working on these problems receive training in aspects of both pure and applied mathematics, participate in seminars in both departments, and become more open to such collaborations; this improves the likelihood they will participate in interdisciplinary and academic/industrial collaborations and improves their ability to motivate and train their own students in the future. Moreover, the problems in this proposal suggest numerous projects that are accessible and appealing to undergraduates or even high school students; such problems can motivate them to the further study of mathematics, or at least give then a greater appreciation for the potential of mathematics in their own field. A few examples include:

• **Crossing-equivalence of Brownian components:** Run a random walk on a grid and compute the adjacency graph of its complementary components. Investigate its diameter and other graph theoretic properties.

• **Implement the NOT algorithm.** Implement the $O(n^{2.5})$ NOT algorithm and its variations for conforming Delaunay triangulations and Voronoi coverings.

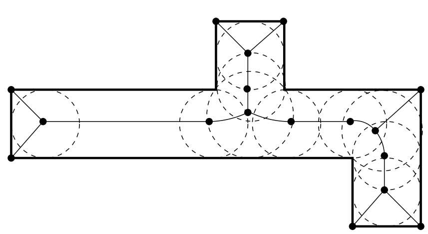
• **Diffusion and dilatation.** Implement the curvature diffusion method and investigate its effect on dilatation. Use it to construct PLQC maps from PC dilatations. Study the morphed triangulations.

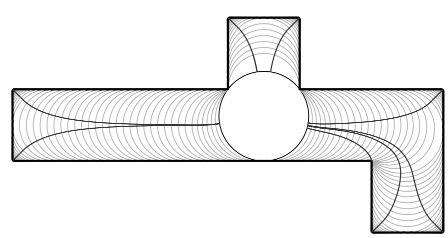
• **Chord-arc curves.** Numerically study the decay of the chord-arc constant under existing solutions of the carpenter's rule problem. Investigate "speed limits" in the linear programs.

• **Removability of the gasket:** The interiors of any two triangles are conformally equivalent, uniquely if vertices map to vertices. This implies that conformally mapping the exterior of the Sierpinski Gasket to the exterior of some other triangle and requiring that every triangular component map conformally to some triangle will uniquely determine a conformal map on every such component. Numerically compute the map using software such as Marshall's `ZIPPER` or Driscoll's `SC-Toolbox`. Is it uniformly continuous?

• **Non-commutative dynamics:** The map $(x, y, z) \to ([x, y], [y, z], [z, x])$ (where $[x, y] = xyx^{-1}y^{-1}$) is the commutator) sends $G^3 = G \times G \times G$ to itself and turns a group into a dynamical system. One of the simplest examples is SU(2) which acts by rotations on the 2-sphere and hence is a sort of non-commutative conformal dynamical system (this example arises from the Solovay-Kitaev theorem in quantum computation (e.g. [49]), but has not yet been studied for its own sake). The element Id^3 is an attracting fixed point whose attracting basin seems (experimentally) to be proper but full measure. Prove this. Draw a "picture" of its boundary (in nine dimensions? draw slices?). Is it fractal?

• **The iota map:** The medial axis of a planar domain consists of the centers of all internal disks that hit the boundary at least twice (see left below). Fixing one such disk and flowing orthogonally to the boundaries of others (see right below) gives a map from $\partial\Omega$ to a circle. This is the iota map. This map plays a crucial role my conformal mapping and meshing algorithms and investigations related to Brennan's conjecture. Implement my fast algorithm for computing iota. Numerically estimate its best QC extension to the interior.





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