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• **Conformal mapping in linear time:** In [17] I prove that if Ω is a simply connected n-gon, then in time $O(n \cdot p \log p)$ we can construct a $(1+\epsilon)$-quasiconformal (QC) map $f : \mathbb{D} = \{|z| < 1\} \to \Omega$, where $p = |\log \epsilon|$. The linearity in n is clearly optimal and improves on methods in use, such as Driscoll's `SC-Toolbox` which is $O(n^3)$. The map is stored by dividing the unit disk, $\mathbb{D}$, into $O(n)$ pieces and giving a p-term series on each piece; a partition of unity gives a $(1+\epsilon)$-QC map $\mathbb{D} \to \Omega$. Geometric convergence of power series requires $p \simeq |\log \epsilon|$ terms to give accuracy ϵ. Therefore the time bound allows only $O(1)$ operations per series (multiplications, FFT's, ...).

• **Optimal angles bounds for quadrilateral meshes:** In [20] I solve a problem of Marshall Bern and David Eppstein [8] by showing that every n-gon has a mesh with $O(n)$ quadrilaterals in which all (new) angles are bounded between $60°$ and $120°$. These bounds are sharp. The mesh can be found in time $O(n)$, improving an $O(n \log n)$ method of Bern and Eppstein, that only gives the upper bound on the angles.

• **Tree-like decompositions:** In [23] I give an algorithm that inserts circular arc crosscuts into a simply connected domain Ω, creating a Lipschitz decomposition whose total length is $O(\ell(\partial\Omega))$. This strengthens a result of Peter Jones. Because only crosscuts are used, the subdomains form a tree under the obvious adjacency relation; a property that is useful in some applications. In [22] I use these decompositions to construct an approximation for the conformal map from Ω to the unit disk.

• **Geometric bounds for harmonic conjugation:** In [18] I prove harmonic conjugation on a rectifiable, simply connected domain Ω is bounded on $L^2(\partial\Omega, ds)$ iff the $|f'|d\theta$ is an A_2 weight, where $f : \mathbb{D} \to \Omega$ is a conformal map (one direction is easy; the other requires some work). I also give a geometric characterization of the domains for which this happens. The result states that if several technical conditions hold, then conjugation is L^2 bounded iff the Poincaré inequality holds on a weighted tree corresponding to a decomposition of Ω into chord-arc pieces. This work was motivated by questions of Steve Vavasis about condition numbers arising in certain finite element methods. I also give a chord-arc domain for which $|f'| \notin A_2$ (also see [60] by Jones and Zinsmeister)).

• **Bounds for the CRDT algorithm:** In [16] I prove that the numerical conformal mapping algorithm CRDT of Driscoll and Vavasis [45] always gives an answer within a bounded distance of the true conformal map (convergence remains open; see discussion later). CRDT adds extra vertices to the target polygon to improve the accuracy; [16] also gives a more efficient way of adding these points, improving the running time.

• **Jacobians of quasiconformal mappings:** In [14] I construct an A_1 weight on $\mathbb{R}^2$ that is not comparable to the Jacobian of any planar QC map, solving a well known problem first stated by Semmes in [72], and repeated in [29], [56], [57], [58]. As a consequence, I construct an Ahlfors 2-regular and locally linearly connected surface in $\mathbb{R}^3$ that is not bi-Lipschitz equivalent to the plane, answering a question of Heinonen and improving an example of Laakso [64]. In [21], I construct a set $E \subset \mathbb{R}^2$ so that no weight that blows up on E is comparable to a K-QC Jacobian for K close to 1.

• **Conformal welding:** A circle homeomorphism h is called a generalized conformal welding on $E \subset \mathbb{T}$ (denoted $h \in \mathrm{GCW}(E)$) if $h = g^{-1} \circ f$, where f and g are univalent maps from $\mathbb{D}$, $\mathbb{D}^* = \{|z| > 1\}$ to disjoint domains Ω, Ω^*, and the composition exists for radial limits of f on E and g on $h(E)$ (this was invented by David Hamilton; see [51], [52], [53]). Moreover, h is a (standard) conformal welding (denoted $h \in \mathrm{CW}$) if $E = \mathbb{T}$ and Ω, Ω^* are the two sides of a closed Jordan curve Γ. Not every h is a conformal welding, but in [15] I prove that every orientation preserving homeomorphism h agrees with some $H \in \mathrm{CW}$ off a set E of arbitrarily small Lebesgue measure. Moreover, every h is in $\mathrm{GCW}(\mathbb{T} \setminus E_1 \cup E_2)$ for some sets with $\mathrm{cap}(E_1) = \mathrm{cap}(h(E_2)) = 0$ (cap stands for logarithmic capacity). [15] also contains a new, short proof that quasisymmetric homeomorphisms are conformal weldings using Koebe's circle domain theorem.

• **Miscellaneous:** In [11], [12] I show any conformal map $f : \mathbb{D} \to \Omega$ can be written as $f = g \circ h$ where h is an 8-QC self-map of the disk and $|g'| > \epsilon$ (this implies any simply connected domain can be mapped to the disk by a locally Lipschitz QC homeomorphism). In [24], Hakobyan and I construct a Lipschitz domain whose central set (centers of maximal subdisks) has dimension 2 (a question of Fremlin [50]). In [13] I disprove Rudin's orthogonality conjecture by constructing a large class of non-inner functions so that the sequence of powers $f, f^2, f^3, \ldots$ is orthogonal in H^2 (Carl Sundberg independently did this in [77]). In [9] I show Bowen's dichotomy holds for divergence type Fuchsian groups, (i.e., any QC deformation has a limit set that is either a circle or has dimension > 1), completing work of Bowen [32], Sullivan [75] and Astala and Zinsmeister [3], [4], [5].

PROJECT DESCRIPTION

We start with a description of the fast conformal mapping algorithm from [17] and discuss potential improvements to it and other conformal mapping techniques. We then discuss connections to Brennan's conjecture, conformal collapsing and other problems.

1. Conformal mapping

• **The fast mapping algorithm:** When computing a conformal map onto Ω numerically, there are (at least) two possible strategies. First, build approximations that are conformal, but only map $\mathbb{D}$ onto an approximation of Ω. Second, build maps directly onto Ω, but which are only approximately conformal. Some methods, such as Davis's method and CRDT (both to be described later), implement the first strategy using the

Schwarz-Christoffel formula, but convergence is hard to prove because it's difficult to predict how changes in the parameters will effect the image domain. The fast mapping algorithm (FMA) from [17] uses the second approach by building quasiconformal (QC) maps onto Ω and reducing the QC constant on each iteration. A $(1+\epsilon)$-quasiconformal map (called an ϵ-map below) has tangent maps that send circles to ellipses of eccentricity at most $1+\epsilon$. An iteration of FMA takes an ϵ-map $f : \mathbb{D} \to \Omega$ and approximately solves a Beltrami equation, i.e., I find $g : \mathbb{D} \to \mathbb{D}$ so that $g_{\bar{z}}/g_z \approx f_{\bar{z}}/f_z$ and prove $f \circ g^{-1}$ is an $O(\epsilon^2)$-map onto Ω (an exact solution would give a conformal map). This gives local quadratic convergence for FMA. The hard part is to prove the radius of convergence, $\epsilon_0 > 0$, is independent of n and Ω and to bound the time needed to perform an iteration.

To make FMA globally convergent, we need an ϵ_0-map onto Ω to start the iteration. We build it using ideas from hyperbolic geometry: iota maps and angle scaling. First approximate the domain by a finite union of disks Ω; rewrite this as the union of a single disk and a number of crescents; collapse the crescents by multiplying their angles by t for $0 \leq t \leq 1$. See Figure 1. This defines $\{\Omega_t\}, 0 \leq t \leq 1$ with $\Omega_1 = \Omega$ and $\Omega_0 = \mathbb{D}$. The

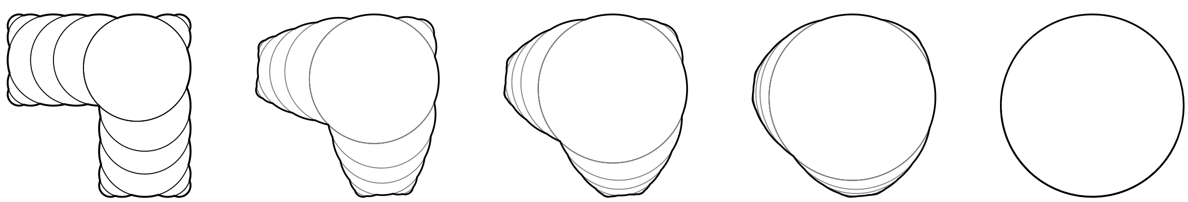

FIGURE 1. Angle scaling gives a QC-Lipschitz path from Ω to $\mathbb{D}$.

resulting map $\iota : \partial\Omega \to \partial D$ is called iota and has a K-QC extension to the interiors with K independent of n and Ω (see [17] and the papers of Sullivan [74] and Epstein-Marden [47]). We know $2.1 < K < 7.82$ ([12], [46]) but have no estimate for ϵ_0.

Problem 1. *Give explicit estimates for ϵ_0 and better estimates for K. Is $1+\epsilon_0 > K$?*

If $1+\epsilon_0 > K$, then iota could be the initial map of the iteration, which would greatly simplify FMA. Since we don't know this, the current algorithm takes a finite subcollection $\{\Omega_k\}_0^N \subset \{\Omega_t\}$ so that each domain is a $< \epsilon_0/2$ QC distortion of the previous domain. We can do this since the intermediate iota maps $\iota_{s,t} : \partial\Omega_t \to \partial\Omega_s$ extend to $O(|s-t|)$-maps of the interiors (see [17]) and we call $\{\Omega_t\}$ a QC-Lipschitz path connecting Ω to $\mathbb{D}$. Because of the Lipschitz estimate, we can take $N = O(K/\epsilon_0)$ independent of n or Ω.

We start with the identity $\mathbb{D} \to \Omega_0$ and iterate FMA to get a nearly conformal map onto Ω_1. Compose this with the iota map $\Omega_1 \to \Omega_2$ to get a ϵ_0-map onto Ω_2 and iterate FMA until we get a nearly conformal map onto Ω_2. Continue by induction until we get an ϵ_0-map onto Ω_N. The work to this point is only $O(n)$, since we compute everything to a fixed accuracy (depending on ϵ_0). The final step is to iterate to the desired accuracy ϵ, and this takes $O(n \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

The intermediate domains $\{\Omega_t\}$ need not be planar; sometimes they have self-intersections. This does not effect the algorithm, but can it be avoided?

Problem 2. *Can Ω be connected to a disk by a QC-Lipschitz path of planar domains?*

What about QC-continuous or QC-Hölder instead of Lipschitz? Problem 2 is closely connected to other problems about "convexifying" domains such as whether the space of chord-arc curves is connected in the BMO topology and the carpenter's rule problem (CRP): a closed polygonal path can be deformed to a convex polygon without self-intersections. Perhaps by combining angle scaling with ideas from the recent solution of CRP in [38] and [73], we can prove the space of chord-arc curves is, indeed, connected.

- **Schwarz-Christoffel iterations:** Suppose Ω is polygonal with vertices $\mathbf{v} = \{v_1, \dots v_n\}$ and $f : \mathbb{D} \to \Omega$ is conformal. The Schwarz-Christoffel formula ("SC-formula" below) says

$$f(z) = A + C \int_0^z \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw,$$

where $\alpha\pi = \{\alpha_1\pi, \dots, \alpha_n\pi\}$, are the interior angles of the polygon and $\mathbf{z} = \{z_1, \dots, z_n\} \subset \mathbb{T}$ map to the corresponding vertices (these are called the SC-parameters). For fixed angles, evaluating the SC-formula defines a map S from n-tuples in $\mathbb{T}$ to n-tuples in $\mathbb{C}$. In fact, S is a well defined map from $\mathbb{T}^n_*$ (n-tuples of distinct points on $\mathbb{T}$ modulo Möbius transformations) into $\mathbb{C}^n_*$ (complex n-tuples modulo Euclidean similarities). Moreover, we can identify $\mathbb{T}^n_*$ with $\mathbb{R}^{n-3}$ as follows: fix a triangulation of the n points on $\mathbb{T}$, and for each pair of adjacent triangles record the logarithm of the cross ratio of the four vertices (the cross ratio is positive if we take the correct ordering of the four points). The original n-tuple can be recovered, up to Möbius transformations, from these $n-3$ real values so $\mathbb{T}^n_* = \mathbb{R}^{n-3}$. This allows us to apply linear algebra to n-tuples on $\mathbb{T}$.

Suppose we have a explicit way of guessing the SC-parameters for a polygon, i.e., a map $G : \mathbb{C}^n_* \to \mathbb{T}^n_*$. Then $F = G \circ S$ gives a map $\mathbb{R}^{n-3} \to \mathbb{R}^{n-3}$ and the correct SC-parameters for Ω solve $F(\mathbf{z}) = \mathbf{z}_0$ (where $\mathbf{z}_0 = G(\mathbf{v}) \in \mathbb{T}^n_*$) and hence are fixed by

$$\mathbf{z}_{k+1} = \mathbf{z}_k - A^{-1}(F(\mathbf{z}_k) - \mathbf{z}_0). \tag{1}$$

where A is an $(n-3) \times (n-3)$ matrix. Taking the derivative matrix $A = DF$ gives Newton's method (or take a discrete approximation to DF). If F is close to the identity, it may be much faster to just take $A = I$ (the identity). A compromise is to use Broyden updates, i.e., start with $A = I$ and multiply A by a rank one matrix at each step, chosen to optimize the approximation to DF given the evaluations of F made so far (see [43]). This is often fastest in practice. If $\mathbf{w} = \{w_1, \dots, w_n\}, \mathbf{z} = \{z_1, \dots, z_n\} \in \mathbb{T}$, define

$$d_{QC}(\mathbf{w}, \mathbf{z}) = \inf\{\log K : \exists\ K\text{-QC } h : \mathbb{D} \to \mathbb{D} \text{ such that } h(\mathbf{z}) = \mathbf{w}.\}$$

FMA is locally convergent with respect to d_{QC}. Is this true for some SC-iteration, i.e., can we choose G and A so that (1) always converges to the SC-parameters? For example, what about the following SC-iterations that are used in pratice?

• **Davis's method:** Define G by $\arg(z_{k+1}) - \arg(z_k) = 2\pi|v_{k+1} - v_k|/\ell(\partial\Omega)$, i.e., parameter spacing is proportional to the edge lengths of the polygon. Davis's method is (1) with this G and $A = I$. See [7], [40]. The idea behind Davis' method is that for fixed angles, longer edges should have larger harmonic measure. This is false in general (see Figure 2). Howell [59] showed Davis's method in $\mathbb{T}^n$ locally diverges for this polygon, but my own experiments indicate this example converges in $\mathbb{T}^n_*$, i.e., the n-tuples diverge unless we renormalize at each step by Möbius transformations. Is this always the case?

Problem 3. *Does Davis's method converge? Does it converge locally w.r.t. d_{QC}?*

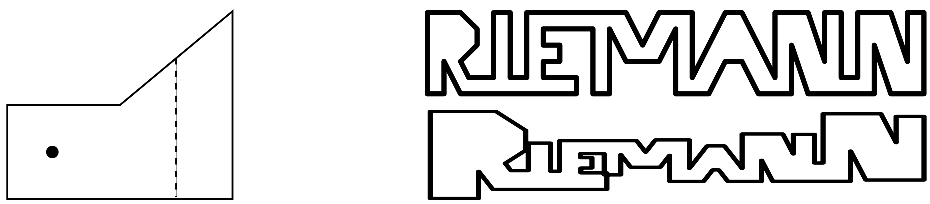

FIGURE 2. Left: longer edges can have less harmonic measure. Right: a polygon and the SC-image using the CRDT guessed parameters.

• **Cross Ratios and Delaunay Triangulations (CRDT):** This algorithm is due to Driscoll and Vavasis [45]. To define G, triangulate the polygon, form the $n-3$ quadrilaterals of adjacent triangles and record the log absolute value of the corresponding cross ratios, giving an element of $\mathbb{T}^n_* = \mathbb{R}^{n-3}$. CRDT is (1) with this map G and $A = DF$.

Problem 4. *Does CRDT converge? Converge locally w.r.t. d_{QC}?*

In [16] I prove G maps polygons to within a uniform d_{QC} distance of the correct parameters, i.e., F is almost the identity at large scales (this requires CRDT to add extra vertices so the triangulation on Ω is "nice"). If I can prove corresponding estimates at small scales (e.g., bounds on DF), these should imply the convergence of CRDT.

• **The iota iteration:** The iota map $\iota : \partial\Omega \to \mathbb{T}$ was defined earlier by angle scaling finite unions of disks, but it can be defined explicitly for n-gons, giving a map G from the vertices to n-tuples on $\mathbb{T}$. The iota-iteration is (1) with this G.

Problem 5. *Is the iota-iteration convergent? Locally convergent for d_{QC}?*

Computing iota reduces to finding the medial axis, i.e., the centers of all open disks in Ω whose boundary hits $\partial\Omega$ in at least two points. It can be computed in $O(n)$ time by an algorithm of Chin, Snoeyink and Wang, [37]. To visualize iota, fix a medial axis disk D and foliate $\Omega \setminus D$ by boundary arcs of medial axis disks, then flow orthogonally to this foliation. This gives the map $\partial\Omega \to \partial D$. See Figure 3. The flow defines Möbius maps between medial axis disks; I explicitly compute these maps in [19] and use them show iota can be applied to all n vertices in time $O(n)$. I am working on implementing the iteration using the program `VORNI` by Martin Held to compute the medial axis.

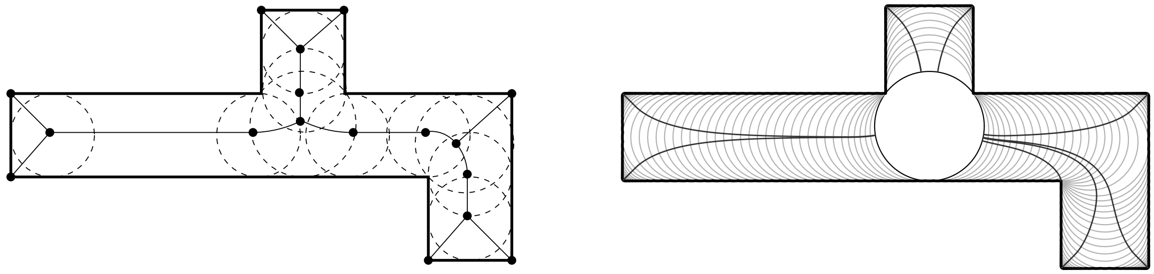

FIGURE 3. Left: the medial axis of a polygon. Right: the medial axis foliation and orthogonal flow lines from the vertices that define iota.

• **The Ahlfors iteration:** Schwarz-Christoffel iterations solve for the unknown parameters by treating the SC-formula as a generic non-linear equation and applying Newton's method (or a simplification). One advantage of FMA is that it uses specific information about conformal maps and the domain Ω to find the parameters. Can we take a similar approach in a simple iteration? Suppose Ω is polygonal, $\mathbf{z}_k \in \mathbb{T}^n_*$ is our current guess for the SC-parameters and $\Omega_k = S(\mathbf{z}_k)$ is the domain corresponding to our guess.

Let $\{T_j\}$ be a triangulation of Ω_k. Map the vertices of Ω_k to the corresponding vertices of Ω and extend to a map f that is affine on each triangle. Let $\mu_f = f_{\bar{z}}/f_z$ be the dilatation of f (which is a constant μ_j on each T_j). Suppose that φ is a QC map of the plane that preserves the real line, fixes the points $0, 1, \infty$ and has dilatation ν. Then (see [1], [10]),

$$\varphi(w) = w - \frac{1}{\pi}\int_{\mathbb{C}} \nu(z)\frac{w(w-1)}{z(z-1)(z-w)}dxdy + O(\|\nu\|^2_\infty) \tag{2}$$

for all $|w| \leq 1$. This suggests we replace the iteration (1) by

$$\mathbf{z}_{k+1} = \mathbf{z}_k + \frac{1}{\pi}\int_{\mathbb{C}} \mu_k(z) R(z, \mathbf{z}_k) dxdy, \qquad R(z,w) = \frac{w(w-1)}{z(z-1)(z-w)} \tag{3}$$

where $\mu_k = \mu_{f\circ g_k}$ and $g_k : \mathbb{D} \to \Omega_k$ is the Schwarz-Christoffel map. Note that μ_k has the form $\mu_j \exp(i \arg(g_k'))$ on the preimage of T_j where μ_j is a known constant and $\arg(g_k')$ has a simple, explicit expression in terms of $\mathbf{z}_k$ coming from the SC-formula.

Problem 6. *Show (3) converges quadratically in $\mathbb{T}^n_*$.*

Problem 7. *How fast can an iteration of (3) be computed (to within $O(\|\mu\|^2_\infty)$)?*

Because of the special form of μ_k, the integral can probably be computed quickly by fast multipole methods. This iteration represents a compromise between the proven quadratic convergence of the fast mapping algorithm and the simplicity and speed of SC-iterations such as Davis's method and CRDT. Among the methods discussed above, I believe it has the best chance of being both fast in practice and provable convergent.

• **Multiply connected domains:** It is important for applications to consider conformal mapping of multiply connected domains, but there are several ways to interpret the problem in this case. For example, a Fuchsian group is a discrete group G of Möbius transformations acting on $\mathbb{D}$. By the uniformization theorem, for any bounded Ω there is such a group so that $\mathbb{D}/G = \Omega$ (the quotient is a locally conformal covering map).

Problem 8. *Given a multiply connected n-gon, can we compute the generators of the covering Fuchsian group G in time $O(n \log n)$?*

The constant may depend on the desired accuracy. The $n \log n$ is the best known bound for computing the medial axis in the multiply connected case (and hence to compute the iota map). The iota map easily gives generators of a Fuchsian group G' that is K-quasiconformally equivalent to G, but to force G' towards G, we have to solve a Beltrami equation where the data is supported on infinite orbits of G'. We could either try to truncate this orbit and proceed as before (computing Beurling transforms to approximate a solution), or replace the Beurling transform by its average over orbits. The latter seems more feasible, since the averaged operator might have a nice formula, or be computable without actually summing over orbits. Another approach is to map general domains into a special class, e.g., domains bounded by points and circles (Koebe's theorem).

Problem 9. *Can we map a multiply connected n-gon to a circle domain in $O(n \log n)$?*

Problem 10. *Given a planar Fuchsian group G, compute a circle domain covered by G.*

Is there an analog of SC-iterations for Fuchsian groups (perhaps using the SC-formulas for multiply connected domains in [39], [42], [41])? Yet another generalization is to compute the hyperbolic metric on Ω (i.e., the image of the hyperbolic metric, $d\rho = |dz|/(1-|z|^2)$, on the disk under the covering map).

Problem 11. *Given Ω, compute λ so that $d\rho = \lambda |dz|$ is the hyperbolic metric on Ω. Alternatively, given two points in Ω, compute the hyperbolic distance between them.*

David Mumford has asked (personal communication) for an efficient solution to this in the case when $\partial\Omega$ is a finite set. This question arises in his work applying hyperbolic geometry and Teichmüller theory to computer vision and pattern recognition.

2. Conformal Numerology

The "numerology" refers to various conjectures about best constants in geometric function theory that have been the focus of intensive investigation. We start with (see [33])

- **Brennan's Conjecture:** *If $f : \Omega \to \mathbb{D}$ is conformal then $\int_\Omega |f'|^p dxdy < \infty, \forall p < 4$.*

A slit disk shows 4 is sharp. The iota map $\iota : \partial\Omega \to \partial\mathbb{D}$ we discussed earlier solves Brennan's conjecture in a certain sense; I showed in [11] that it has an extension, f, to the interior that is K-QC (for a K independent of Ω) and that is also locally Lipschitz, i.e., $f' \in L^\infty$. Thus a stronger version of Brennan's conjecture is true for a map $f : \Omega \to \mathbb{D}$ that is "almost" conformal in a uniform sense. Can we deduce the original version? We can solve a Beltrami equation to find a K-QC map $h : \mathbb{D} \to \mathbb{D}$ so that $f = h \circ g$ is conformal. By a celebrated result of Kari Astala [2], $|h'| \in L^p$ for $p < 2K/(K-1)$, so $|f'| = |h'| \cdot |g'| \in L^p$ as well. So Brennan's conjecture follows from

Problem 12. *If Ω is simply connected, is there a 2-QC, locally Lipschitz map $f : \Omega \to \mathbb{D}$?*

• **The modified iota map:** Unfortunately the iota map itself does not work. Epstein and Markovic [46] showed that $K > 2$ for certain spiral domains. However, these domains are not counterexamples to Problem 12, because we can easily build alternate maps that do work. In [19] I observed that the iota map makes certain consistent "errors" and introduced a modified iota map for polygons to reduce these errors.

Problem 13. *Is the modified iota map always a locally Lipschitz 2-QC map?*

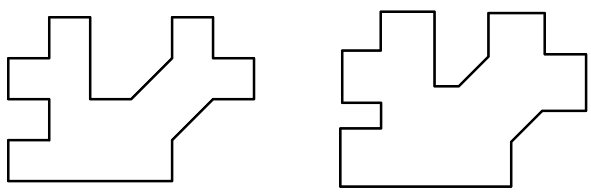

FIGURE 4. A polygon (left) and the SC images using parameters given by the iota map (center) and modified iota map (right).

Using triangulations, we can define a piecewise-affine map from our guessed polygon to the target polygon whose dilatation is an upper bound for the d_{QC} distance from our guess to the correct parameters. In Figure 4 this gives the bounds $K \leq 2.8955$ (iota), $K \leq 1.23762$ (modified iota). What are the actual distances, i.e.,

Problem 14. *Given two n-tuples $\mathbf{z}, \mathbf{w} \subset \mathbb{T}$, compute $d_{QC}(\mathbf{z}, \mathbf{w})$.*

Teichmüller theory says an optimal map sending $\mathbf{z} \to \mathbf{w}$ exists and can be written as a composition of three maps: a conformal map φ from $\mathbb{D}$ to a polygon Ω with all sides parallel to the coordinate axes and sending points of $\mathbf{z}$ to the vertices; an affine stretching $(x, y) \to (Kx, y)$; and a conformal map back to the disk. Moreover, $\varphi = \int \sqrt{\psi}$ where ψ is a quadratic differential with zeros and poles at the points of $\mathbf{z}$. Finding the correct K should be easy if we know the correct polygon. The real problem is to find the angles, i.e., the right "shape" of Ω. This is a discrete problem since the angles are all multiples of $\pi/2$, but testing all possibilities is infeasible. Since the angles are determined by whether ψ has a zero or pole at the corresponding point of $\mathbf{z}$, can we deduce the angles by a simple test on $\mathbf{z}, \mathbf{w}$? I have discussed this with Vlad Markovic who thinks it may be possible.

• **Conformal spectrum:** Harmonic measure, ω, on $\partial\Omega$ is the pullback of normalized Lebesgue measure on $\mathbb{T}$ under a conformal map $\varphi : \Omega \to \mathbb{D}$. Makarov's theorem says that (up to log factors), ω scales linearly at ω-a.e. point. However, ω may have different scaling on sets of zero harmonic measure, and the size of these sets it of great interest. Let $F_\alpha = \{z \in \partial\Omega : \omega(D(z, r)) \simeq r^\alpha \text{ as } r \searrow 0\}$ and let $f_\Omega(\alpha) = \dim(F_\alpha)$. If $\varphi : \Omega \to \mathbb{D}$ is conformal, then on a disk $D = D(z, r)$ we expect $|\varphi'| \sim \omega(D)/r$. If $p < 4$ and if $\{D_k\}$ is a cover of F_α by disks of radius r, then Brennan's conjecture implies

$$\infty > \int_\Omega |\varphi'|^p dxdy \geq \#(D_k)(\omega(D_k)/r)^p \text{area}(D_k) \sim r^{-f_\Omega(\alpha)} r^{p(\alpha-1)+2}.$$

taking $r \searrow 0$ and $p \nearrow 4$, shows we need $f_\Omega(\alpha) \leq 4\alpha - 2$.

Universal Spectrum Conjecture (USC): $f(\alpha) = \sup_\Omega f_\Omega(\alpha) = 2 - \frac{1}{\alpha}$.

This strengthens Brennan's conjecture and also contains conjectures of Carleson, Jones, Krätzer and Makarov (see [34], [36], [71]). If Γ is a closed Jordan curve with complementary components Ω_1, Ω_2, let $f_1 : \Omega_1 \to \mathbb{D}$ and $f_2 : \Omega_2 \to \mathbb{D}^* = \{|z| > 1\}$ be conformal. Kari Astala, Istvan Prause, Stas Smirnov have shown the following implies USC (personal communication).

Astala-Prause-Smirnov (APS) Conjecture: *Suppose Γ is a closed Jordan curve and f_1, f_2 are as above. Suppose ν is a measure supported on Γ and ν_k is the push forward of ν under f_k, $k = 1, 2$. Let $D = \dim(\nu)$ and $D_k = \dim(\nu_k)$, $k = 1, 2$. Then $(D-1)^2 \leq (1-D_1)(1-D_2)$.*

Problem 15. *With notation as above, is $(D-1)^2 \leq 1 - D_1$? Is $|D-1| \leq 1 - D_1$?*

The first follows trivially from conjecture (so should be easier to prove) and the second would trivially imply it (so should be easier to disprove). The APS conjecture is trivial if $D = 1$ and known to be true if $D_1 = 1$ (by Makarov's results in [66], [67]).

- **Hyperbolic 3-manifolds:** The APS conjecture is of particular interest when the curve Γ is the limit set of a discrete Möbius group G (a quasi-Fuchsian group) and ν is the Patterson-Sullivan measure (this is the "natural" measure on Γ with $\dim(\nu) = \dim(\Gamma)$). G extends to a group of isometries acting on the hyperbolic upper half-space, $\mathbb{R}^3_+$, and the properties of Γ and ν are closely connected to the geometry of the hyperbolic 3-manifold $M = \mathbb{R}^3_+/G$. For example, if $D = \dim(\nu)$, then $1 - \lambda_0 = (1-D)^2$, where λ_0 is the base eigenvalue for the Laplacian on M (see [76]). The number λ_0 is also the exponential decay rate of the heat kernel on M, i.e., the probability that a Brownian motion on M will be in fixed ball at time t is $O(\exp(-\lambda_0 t))$. This was the crucial observation in [26], a paper that resolved several important problems about Kleinian groups.

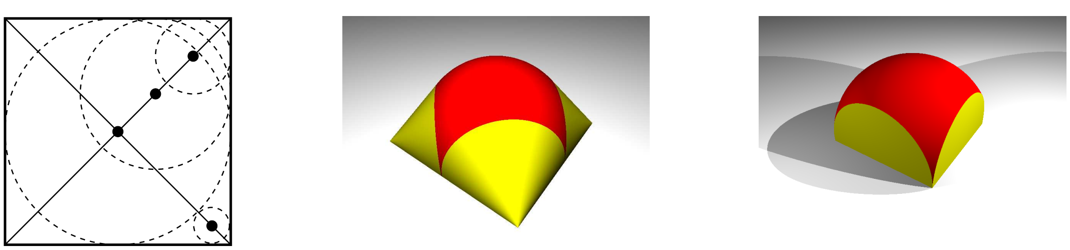

FIGURE 5. The medial axis of a square, and the domes of its the interior and exterior. The hyperbolic convex hull lies between the two domes.

The iota map discussed earlier originally arose in the study of hyperbolic 3-manifolds. The dome of a planar domain Ω is a surface $S \subset \mathbb{R}^3_+$ given by the upper envelope of all hemispheres with base disk in Ω (also allow complements if $\infty \in \Omega$). S has a hyperbolic path metric and this metric makes it isometric to the hyperbolic disk. The iota map on $\partial\Omega = \partial S$ is the boundary restriction of this isometry. The extension of iota from $\partial\Omega$ to Ω can be constructed by precomposing iota $S \to \mathbb{D}$ with a uniformly K-QC map $\sigma : \Omega \to S$ (this exists by Sullivan's convex hull theorem [74], [47]). It was the observation that iota could be defined either by using hyperbolic domes or the medial axis that eventually led me to the fast mapping algorithm described earlier. When Γ is a closed curve, the region between the domes of its complementary components is $C(\Gamma)$, the hyperbolic convex hull of Γ. If Γ is the limit set of a quasi-Fuchsian group G, then $C(\Gamma)/G = C(M) \subset M$ is the convex core of M and has finite hyperbolic volume if G is finitely generated.

Thus the left side of the Astala-Prause-Smirnov inequality gives a bound on the probability of finding a Brownian path in $C(M)$ at time t. Do the two terms on the right hand side correspond to some sort of decay rates for random paths on the Riemann surfaces $R_1 = \Omega_1/G, R_2 = \Omega_2/G$? Perhaps Brownian paths in M that are conditioned to stay in the convex core behave like random walks on the boundary components of the convex core. These boundaries are mapped with bounded distortion to Ω_1, Ω_2 via Sullivan's σ maps, and perhaps this can be used to prove the APS conjecture for Kleinian limit sets.

• **Twisted Poincaré series:** Another tool we might apply is the twisted Poincaré series introduced by Steger and myself in [28]. Given a finitely generated quasi-Fuchsian group G with limit set Γ choose base points $z_1 \in \Omega_1, z_2 \in \Omega_2$ and let ρ_1, ρ_2 be the hyperbolic metrics on these components. Define

$$P_G(s,t) = \sum_{g \in G} \exp(-s\rho_1(g(z_1), z_1) - t\rho_2(g(z_2), z_2)), \quad s, t \geq 0.$$

If $s = 0$ this is a standard Poincaré series and converges iff $t > 1$. In [28] we showed that if G is not Fuchsian (so Γ is not a circle and hence $\dim(\Gamma) > 1$) then $P_G(s,t) < \infty$ whenever $s + t \geq 1$ and $s \neq 0, 1$. How small we can take (s,t) and still have convergence measures how mutually singular the harmonic measures for Ω_1, Ω_2 are. The boundary of the convex convergence region for P_G may encode much of the information in the APS conjecture. Can we make this explicit, e.g.,

Problem 16. *If $D = \dim(\Gamma)$, does $P_G(s,t)$ diverge if $s + t < 2 - D$?*

• **Diffusion limited aggregation (DLA):** Fix a disk in the plane. Start another disk far away, moving as Brownian motion, and stopped when it hits the first disk. Continue adding disks in this way and let D_n be the cluster's diameter at time n (see Figure 6).

Problem 17. *Compute $\gamma = \limsup_{n\to\infty} \frac{1}{n} \log D_n$, i.e., compute the growth rate of DLA.*

Trivially $\frac{1}{2} \leq \gamma \leq 1$ and experimentally $\gamma \approx .58$. As a graduate student I proved $\gamma \leq \frac{4}{5}$ (unpublished) and Kesten proved $\gamma \leq \frac{2}{3}$, [63], but there is no non-trivial lower bound (i.e., $\gamma > \frac{1}{2}$). Even $D_n/\sqrt{n} \nearrow \infty$ is open. This is a "numerology" problem because it requires we have better understanding of the scaling of harmonic measure on fractal sets.

3. Conformal rectifiability

• **Non-rectifiable sets:** The growth rate of DLA may be too hard to attack directly, but there are other geometric properties to investigate. For example, the "arms" of DLA in Figure 6 look fairly straight, but are they finite length, i.e., can we connect points $\approx D_n$ apart by a path in DLA of length $O(D_n)$? Alternatively,

Problem 18. *Does the scaled limit of DLA contain rectifiable arcs?*

If the arms are not rectifiable, they may have some "uniform wiggliness" (see [27]) that prevents straight paths in the cluster and this might also force the growth rate of the cluster to be $\gg \sqrt{n}$. The question of whether a particular connected set contains a rectifiable arc comes up in several part of conformal analysis. Another well known example is the range of a 2-dimensional Brownian motion.

Problem 19. *Does $R = B_2([0,1])$ contain a rectifiable arc?*

FIGURE 6. Does a DLA cluster or a Brownian path contain a rectifiable arc?.

Pemantle [70] showed that R contains no line segments almost surely; in fact, it hits no segment in positive length. Moreover, we expect something even stronger to be true.

Problem 20. *Show than any Jordan arc in R has dimension $\geq \tau > 1$.*

It is known that $\tau \leq \frac{4}{3}$; the boundaries of the complementary components have this dimension by a celebrated result of Lawler, Schramm and Werner [65]. A weaker version had been obtained earlier by Jones, Pemantle, Peres and myself [25].

• **QC rectifiability:** In [14] I constructed an A_1 weight not comparable to the Jacobian of any QC map (I omit the definitions since they are not essential to the discussion). A stronger result would be to build a set E of zero area so that no function that blows up as $z \to E$ can be comparable to a QC Jacobian. In [14] I show it suffices to:

Problem 21. *Construct E so that every QC image of E contains a rectifiable arc.*

A Kakeya set is a closed set of zero area that contains a line segment in every direction (these are important in harmonic analysis and have been intensively studied, e.g., [31], [62], [78]). Such a set may be a good candidate for solving this problem.

• **BiLipschitz planes:** A metric space is locally linearly connected (LLC) if any ball of radius r can be contracted to a point inside a ball of radius $O(r)$. The space is Ahlfors 2-regular if 2-dimensional Hausdorff measure satisfies $m_2(B(x,r)) \simeq r^2$. A theorem of Bonk and Kleiner [30] says these characterize quasisymmetric equivalence to the plane.

Problem 22. *Characterize metric spaces that are biLipschitz equivalent to the plane.*

The construction of the A_1 weight in [14] also gives a surface in $\mathbb{R}^3$ that is LLC and Ahlfors 2-regular, but not biLipschitz equivalent to the plane, answering a question of Heinonen. Thus we need some additional condition. Peter Jones's traveling salesman theorem (TST) [61] gives a characterization of rectifiable curves in the plane. When we examine the proof that the surface constructed in [14] is not biLipschitz to the plane, it boils down to a calculation that a certain curve on S has infinite length, but the image under a biLipschitz map to the plane would have finite length by Jones's theorem. This shows the surface is not biLipschitz to the plane and suggests that one answer to Problem 22 is to find a condition that makes Jones' TST work on the surface. I suspect something like "curvature is a Carleson measure" is the correct answer, assuming we can find the correct formulation. Raanan Schul, an expert on TST in metric spaces, has just come to Stony Brook as an assistant professor and we will work on this together.

4. Conformal collapsing

• **The generalized Koebe conjecture:** A decomposition $\mathcal{C}$ of $\mathbb{C}$ is a collection of pairwise disjoint closed sets whose union is all of $\mathbb{C}$. $\mathcal{C}$ is a Koebe decomposition if every set is either a disk or a point. $\mathcal{C}$ is called upper semi-continuous if a sequence of elements of $\mathcal{C}$ that converges in the Hausdorff metric must converge to a subset of an element of $\mathcal{C}$. If all elements of $\mathcal{C}$ are dendrites (continua that don't separate the plane), we call $\mathcal{C}$ a Moore decomposition. R.L. Moore proved in [69] that $\mathbb{C}/\mathcal{C} = \mathbb{C}$, i.e., identifying such sets to points and using the quotient topology gives the plane again. We say decompositions $\mathcal{M}$ and $\mathcal{K}$ are conformally equivalent if there is a bijection $f : \mathcal{M} \to \mathcal{K}$ such that (1) f is conformal on the interior of the singletons and (2) if $\{E_n\} \subset \mathcal{M}$ converges (in the Hausdorff metric) to $E \in \mathcal{M}$ then $f(E_n) \to f(E) \in \mathcal{K}$.

Problem 23. *Is every Moore decomposition conformally equivalent to a Koebe decomposition? Which Moore decompositions are conformally equivalent to the plane (no disks)?*

If Ω is a domain, take singletons inside Ω and decompose Ω^c into its connected components. This gives a Moore decomposition of the plane, so to Problem 23 contains

Koebe's Conjecture: *Every domain is conformally equivalent to a circle domain.*

The best results so far are by He and Schramm [54], [55]. If $h : \mathbb{T} \to \mathbb{T}$ is an orientation preserving (o.p.) homeomorphism we can decompose the annulus $A = \{1 \le |z| \le 2\}$ with curves connecting z to $2h(z)$ and take singletons outside A. See the left of Figure 7. If this decomposition of the plane is equivalent to the plane, then A collapses to a closed curve that makes h a conformal welding. Thus the second part of Problem 23 contains the problem of characterizing which circle homomorphisms are conformal weldings, so is probably extremely difficult. If the first part of Problem 23 is true, then only countable many of our curves do not collapse to points (since there can only be countable many disjoint disks in the image), so there should only be countably many obstructions to any o.p. homeomorphism being a conformal welding. More precisely,

Problem 24. *Every circle homeomorphism $h \in \mathrm{GCW}(\mathbb{T} \setminus E)$ for some countable E.*

(See summary of previous work for definition of GCW.) For any o.p. circle homeomorphism h, I showed in [15] there are conformal maps $\{f_n\}$, $\{g_n\}$ onto disjoint domains so that $|f_n - g_n \circ h| \to 0$ except on a countable set. However, passing to a limit $f_n \to f$, $g_n \to g$, I can currently only deduce that $f = g \circ h$ off a set of zero logarithmic capacity.

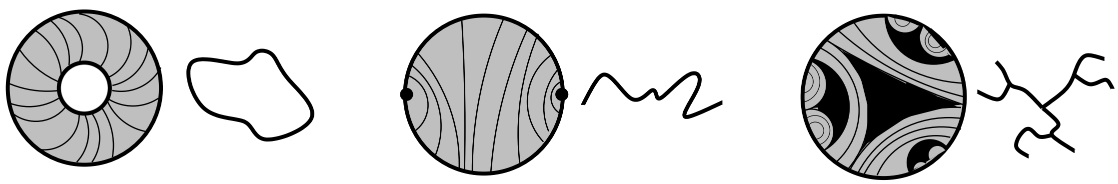

FIGURE 7. Collapsings giving a closed curve, an arc and a dendrite.

• **John domains:** We can extend a decomposition of $\mathbb{T}$ to a decomposition of $\mathbb{C}$ by taking the hyperbolic convex hull of each set inside $\mathbb{D}$, and taking singletons outside $\overline{\mathbb{D}}$. For example, if h is a quasisymmetric involution of $\mathbb{T}$ that fixes 1 and -1, consider sets of the form $\{x, h(x)\}$. The resulting decomposition is conformally equivalent to the plane and $\mathbb{T}$ collapses to a quasi-arc (this just restates a well known result in new terminology). See the center of Figure 7. Can we extend this from arcs to dendrites?

Given an decomposition $\mathcal{C}$, we write $x \sim y$ if they are in the same set of $\mathcal{C}$ and write $E \sim F$ if for every $x \in E$ there is a $y \in F$ with $x \sim y$ and conversely. The decomposition is called quasisymmetric if there are $N, C < \infty$ so that given any interval $I \subset \mathbb{T}$ there is a collection of $n \leq N$ disjoint intervals $I_0 = I, I_1, I_2, \dots I_n$ so that

(1) $\cup\{E \in \mathcal{C} : E \cap I \neq \emptyset\} \subset \cup_{k=0}^{n} I_n$
(2) There are sets $E_k, F_k \subset I_k$ with $\mathrm{cap}(E_k) \simeq \mathrm{cap}(F_k) \simeq \mathrm{cap}(I_k)$ such that $\mathrm{cap}(E) \simeq \mathrm{cap}(F)$ for any $E \subset E_k$ and $F \subset F_{k+1}$ such that $E \sim F$ (this holds for all $k = 0, \dots, n$ with indices considered modulo n).

Problem 25. *Show a decomposition of $\mathbb{T}$ collapses to the boundary of a simply connected John domain iff it is quasisymmetric.*

This arose in discussions with Peter Jones. Ω is John domain if any two points $z, w \in \Omega$ can be joined by an interior path γ so that $\mathrm{dist}(x, \partial\Omega) \simeq \mathrm{dist}(x, \{z, w\})$ for all $x \in \gamma$. John domains occur naturally in polynomial dynamics and Kleinian groups, e.g., as in the papers of Carleson-Jones-Yoccoz [35], and McMullen [68]. Solving Problem 25 would say that certain combinatorial properties of the dynamics force nice geometric properties. Perhaps the results of [35] and [68] (or new ones) can be proved in this way.

5. Educational and broader impact of the proposal

• **Improved mapping and meshing:** Conformal mapping is currently the fastest way to solve certain problems in 2-dimensional potential theory that have various applications (e.g., computing electrical resistance, capacitance, Green's functions; designing microwave guides, integrated circuits, magnetic disk drive heads; investigating crack detection). See [44] and its references. Conformal mapping is used by David Mumford in his work on pattern recognition and computer vision. The meshing algorithm in [20] gives meshes with the best possible angles, which is important for the convergence of various finite element methods. Thus several of the problems proposed here can directly impact the effectiveness of algorithms used in practice. One referee of my paper [17] put it this way: "Algorithmic conformal mapping is a small topic – one cannot pretend that thousands of people pay attention to it. What it does have going for it is durability. These problems have been around since 1869 and they have proved of lasting interest and importance."
• **Building interdisciplinary connections:** Many of the problems posed here have a strong interdisciplinary character, connecting classical analysis, hyperbolic geometry, computational geometry and numerical analysis and they serve as a bridge between researchers with common interests but different backgrounds. For example, my paper on fast conformal mapping [17] was specifically written to be accessible to both mathematicians and computer scientists and was submitted to a premier computer science journal (although I hope it would also have been acceptable for Annals or Acta). The iota map and its connections to conformal mapping (introduced in [11], [12], [17], [20]) have already started to appear in the work of some applied mathematicians (e.g., [6], [48], [49]). This work also resulted in Steve Vavasis and I hosting a workshop at Stony Brook on conformal and computational geometry in April 2007. Speakers included David Mumford, Marshall Bern, David Gu, Lehel Banjai, Toby Driscoll, Joe Mitchell, Alan Saalfield, Vlad Markovic, Don Marshall and Ken Stephenson; a mixture of about one third each of pure and applied mathematicians and computer scientists. (The lecture slides are available at `www.math.sunysb.edu/~ccg2007`.) Raanan Schul and I have discussed plans for a sequel. In addition, I have spoken about my own results at Microsoft research and in various applied mathematics seminars. This fall I will give a lecture at FWCG-09 (an annual workshop on computational geometry). I have been invited to by Paul Weigmann to visit the physics department at University of Chicago and invited to speak at the Simons Center for Geometry and Physics. These various invitations (in addition to the usual mathematics colloquia and conferences) show that the proposed work will have some impact beyond geometric function theory.

• **Impact on education:** The problems described in the proposal generally have simple statements, a significant geometric and computational component and interesting applications. This makes them attractive to students and I have supervised 3 Ph.D. dissertations on these topics: Zsuzanne Gonye (geodesics in hyperbolic manifolds), Karyn Lundberg (boundary convergence of conformal maps) and Hrant Hakobyan (dimension distortion under QC maps). These account for three of eight PhD's in analysis at Stony Brook over the last ten years. I have a new student, Chris Green (who started in algebraic geometry, but was attracted to more concrete kinds of geometry), and expect to have another next year, Maxime Fortier Bourque, who is coming to Stony Brook to work with me. I could certainly take on more students and as we continue to strengthen the analysis group here, I hope we will also attract more students and postdocs interested in analysis.

I am currently teaching an undergraduate seminar on conformal mapping where several of the problems described in the proposal will be discussed. The lecture notes are available on my webpage (under MAT 401, Fall 2009 and MAT 626, Fall 2008). Yuval Peres and I are also close to completing a text about analysis on fractals that includes chapters on Brownian motion and harmonic measure. Many of the problems discussed in the proposal could be explored computationally and used for research projects (recently I had an undergraduate write a honors thesis on Davis's method). For example:

• implement the optimal angle meshing algorithm,
• implement algorithm to estimate L^2 norm of harmonic conjugation on a domain,
• compute the region of convergence of some twisted Poincaré series,
• estimate QC distance between polygons, search for polygons with worst iota estimates,
• estimating length of connected arcs in the trace of a random walk or DLA
• implement the Ahlfors iteration, the iota iteration, improve CRDT
• experimentally seek better modifications of the iota map,
• implement the tree-like decomposition algorithm in [23],
• test the APS conjecture numerically on various fractals,
• modify angle scaling to scale different crescents at different rates. Can we preserve planarity this way and still have a QC-continuous path?
• When $n = 5$, $\mathbb{T}^n_* = \mathbb{R}^2$, so SC-iterations act on a plane. Draw the actions and describe their behavior (see Figure 8). Can we prove convergence for small n in this way?

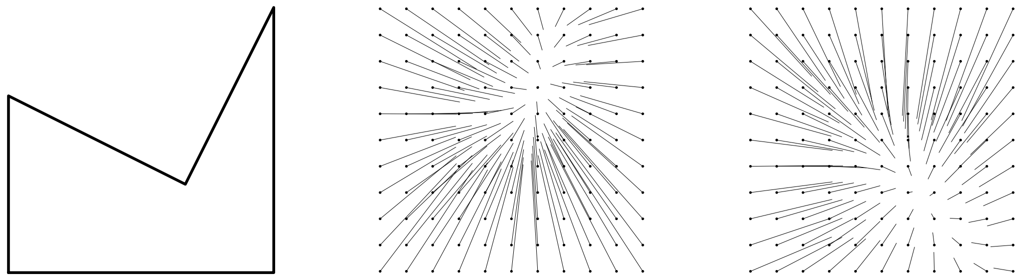

FIGURE 8. The CRDT iteration for the polygon on the left. Each point of $\mathbb{R}^2$ represents an element of $\mathbb{T}^5_*$ (SC-parameters modulo Möbius transformations) for the given angles. The middle picture shows $\mathbf{z} \to F(\mathbf{z})$ and the right one shows the simple SC-iteration $\mathbf{z} \to \mathbf{z} - (F(\mathbf{z}) - \mathbf{z}_0)$. Convergence seems obvious; can we give a proof motivated by the picture?

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