RESULTS FROM PRIOR NSF SUPPORT
Geometry of conformal and quasiconformal mappings
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My recent work concerns conformal and quasiconformal mappings. The main results are summarized below; preprints are at `www.math.sunysb.edu/~bishop/papers/papers.html`.

• **Conformal mapping in linear time:** One of the major goals of my previous proposal was accomplished by proving [24]: *If Ω is a simply connected n-gon, then in time $O(n \cdot p \log p)$ we can construct a $(1+\epsilon)$-QC map from $\mathbb{D} \to \Omega$, where $p = |\log \epsilon|$.*

The linearity in n is clearly optimal and improves on methods in use, such as Toby Driscoll's `SC-Toolbox` which is $O(n^3)$. My representation of conformal maps involves a partition of $\mathbb{D}$ into $O(n)$ pieces and a p-term power series on each piece. Geometric convergence of the series requires $p \simeq |\log \epsilon|$ terms to give accuracy ϵ. The time bound $O(p \log p)$ therefore allows only $O(1)$ operations per series (multiplications, Fast Fourier Transforms, ...). The proof is long (about 100 pages) but contains various intermediate results which may be of independent interest (e.g., a thick/thin decomposition of polygons, linear time construction of bending laminations, piecewise Möbius approximations to conformal mappings, using the medial axis to organize data for optimal use of the fast multipole method).

• **Jacobians of quasiconformal mappings:** In [20] I construct an A_1 weight on $\mathbb{R}^2$ which is not comparable to the Jacobian of any planar quasiconformal map, solving a problem of Stephen Semmes. As a consequence, I obtain an Ahlfors 2-regular and locally linearly connected surface in $\mathbb{R}^3$ which is not bi-Lipschitz equivalent to the plane, answering a question of Heinonen and improving an example of Laakso [50]. In [23], I construct a set $E \subset \mathbb{R}^2$ so that no weight that blows up on E is comparable to a K-QC Jacobian for K close to 1.

• **Quadrilateral meshes with no small angles:** The linear time Riemann mapping theorem is strong enough to prove new results in computational geometry using conformal mappings. In [26] I solve a problem of Marshall Bern and David Eppstein [6] by showing that every n-gon has a mesh with $O(n)$ quadrilaterals in which all (new) angles are bounded between $45-\epsilon$ and $135+\epsilon$, any $\epsilon > 0$. The mesh can be found in time $O(n)$, improving an $O(n \log n)$ method in [6] for finding a quadrilateral mesh with an upper bound only on the angles.

• **Medial axis and central sets:** The medial axis of $\Omega \subset \mathbb{R}^2$ is the set of $x \in \Omega$ with two or more closest points on $\partial\Omega$. This is a subset of the central set, the set of centers of maximal disks in Ω. The two sets agree for polygons, but differ in general (e.g., an ellipse contains maximal disks tangent at only one point). In 1945 Erdös [39] showed the medial axis always has dimension 1 and in 1997 Fremlin [40] showed the central set has zero area. In [28], Hakobyan and I show the gap exists by constructing a central set of dimension 2.

• **A conjecture of Driscoll and Vavasis:** In [34], Driscoll and Vavasis give an iterative method for conformal mapping. Based on numerical experiments, they conjectured that their method always gives a uniformly good starting point, in a precise sense, and that their iteration converges linearly to the correct answer. In [17] I show their starting point estimate is false as stated, but can be modified to be true. The convergence of their method is still open. [17] also contains a linear time algorithm for computing the iota-map for a polygon (see proposal).

- **Distortion of disks:** Part of Astala's celebrated work on sharp estimates for QC mappings is the estimate $\sum \operatorname{diam}(D_k)^2 \sim \sum \operatorname{diam}(f(D_k))^2$ whenever f is a normalized QC mapping which is conformal off the finite collection of disjoint closed disks $\{D_k\}$. In [1] Astala and his coauthors ask if this is still true for $0 < d < 2$, but in [25] I provide a counterexample.
- **Dilations of dilatations:** If μ is the dilatation of a QC map f_μ which is conformal outside $\mathbb{D}$ and which maps $\partial\mathbb{D}$ to a chord-arc curve, Cui and Zinsmeister [33] asked if the same must be true for $t \cdot \mu$ for all $t \in [0, 1]$. In [21] I disprove this by showing that $\dim(f_{t\mu}(\mathbb{T})) > 1$ is possible for some $t < 1$.
- **Conformal welding:** A circle homeomorphism h is called a generalized conformal welding on $E \subset \mathbb{T}$ (denoted $h \in \mathrm{GCW}(E)$) if $h = g^{-1} \circ f$, where f and g are univalent maps from $\mathbb{D}$, $\mathbb{D}^*$ to disjoint domains Ω, Ω^*, and the composition exists for radial limits of f on E and g on $h(E)$ (this was invented by David Hamilton in [42]; see also [43], [44]). Moreover, h is a (standard) conformal welding (denoted $h \in \mathrm{CW}$) if $E = \mathbb{T}$ and Ω, Ω^* are the two sides of a closed Jordan curve Γ. Not every h is a conformal welding, but in [8] I prove that every h agrees with some $H \in \mathrm{CW}$ off a set E of arbitrarily small Lebesgue measure. Other results in [8] include: every h is in $\mathrm{GCW}(\mathbb{T} \setminus E_1 \cup E_2)$ for some sets with $\operatorname{cap}(E_1) = \operatorname{cap}(h(E_2)) = 0$ (cap stands for logarithmic capacity); if h is log-singular (i.e., $\mathbb{T} = E_1 \cup E_2$ with $\operatorname{cap}(E_1) = \operatorname{cap}(h(E_2)) = 0$) then $h \in \mathrm{CW}$ (this is quite different from the usual quasisymmetric (QS) condition); a new, short proof that QS homeomorphisms are conformal weldings using Koebe's circle domain theorem.
- **Factoring conformal maps:** In [13], [18] I show any conformal map $f : \mathbb{D} \to \Omega$ can be written as $f = g \circ h$ where h is a 8-QC self-map of the disk and $|g'| > \epsilon$ is bounded away from zero uniformly. Among the geometric consequences: any simply connected domain can be mapped to the disk by a locally Lipschitz homeomorphism and any quasidisk can be mapped to a disk by a Lipschitz homeomorphism of the plane.
- **Miscellaneous:** In [22] I disprove Rudin's orthogonality conjecture by constructing a large class of non-inner functions so that the sequence of powers $f, f^2, f^3, \ldots$ is orthogonal in H^2 (Carl Sundberg independently disproved the conjecture). In [10] I show that a compact $E \subset \mathbb{T}$ is a boundary interpolation set for conformal maps iff it has zero logarithmic capacity. In [11] I show Bowen's dichotomy holds for divergence type Fuchsian groups, i.e., a QC deformation of such a group has a limit set which is either a circle or had dimension > 1, extending work of Bowen [31], Sullivan [57] and Astala and Zinsmeister [2], [3], [4]. Further results on the failure of the dichotomy for divergence groups are given in [12], [14]. Results on the failure of Ruelle's property (analytic dependence of dimension on parameters) are given in [9], [15], and further results on dimensions of limit sets are given in [19] and [16]. [27] is a review of "Harmonic Measure" by Garnett and Marshall and a survey of recent results in geometric function theory.

PROJECT DESCRIPTION

First we review fast conformal mapping and discuss related open questions. Then we move to other connections between conformal analysis and computational geometry and end with some problems about QC mappings and conformal welding. Problems of various difficulty will be discussed; some seem close to solution, while others remain long term goals.

1. Fast conformal mapping

• **The algorithm:** Roughly, I give a quadratically convergent iteration and a starting point from which it is guaranteed to converge. I will briefly describe some of the main ideas.

We start with the "iota-map" $\iota : \partial\Omega \to \partial\mathbb{D}$. This is simplest when Ω is a finite unions of disks. In that case, Ω is a union of one disk (which we may assume is $\mathbb{D}$) and a finite number of crescents. Each crescent has a foliation by circular arcs orthogonal to its boundaries and ι simply follows the foliation from $\partial\Omega$ to $\partial\mathbb{D}$. See Figure 1. For general domains, one can approximate by disks and pass to a limit. For polygons, I give an explicit formula in [17].

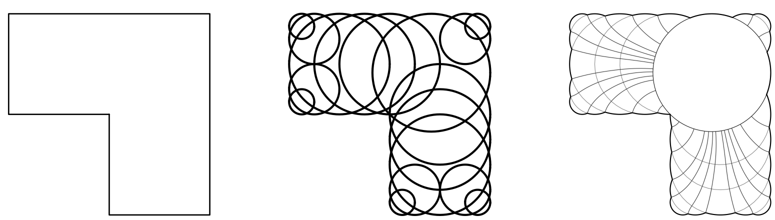

FIGURE 1. A polygon, an approximating finite union of disks, and the foliated crescents.

Alternatively, given a domain written as a union of a disk and crescents, we can multiply the angle of each crescent by a factor $t \in [0, 1]$ to get an "angle scaling family" of domains which connects $\mathbb{D} = \Omega_0$ to $\Omega = \Omega_1$. Moreover, for $0 \le s < t \le 1$, Ω_s and Ω_t differ by an explicit $1 + O(|s - t|)$-QC mapping. For $s = 0, t = 1$ this map is $\iota : \partial\Omega \to \mathbb{D}$, so we see it has a uniformly K-QC extension to the interiors (we can take $K = 8$ by [18]). This follows from the Sullivan-Epstein-Marden convex hull theorem [56], [36] in 3-dimensional hyperbolic geometry, as explained in [13], [24]. Observing this connection between planar maps and 3-dimensional hyperbolic geometry was the main new insight leading to the fast mapping theorem. Given any $\epsilon > 0$, the angle scaling family allows us to build a chain from Ω to $\mathbb{D}$ with at most $O(1/\epsilon)$ domains $\{\Omega_k\}$, each of which is a $(1 + \epsilon)$-QC image of its predecessor. See Figure 2.

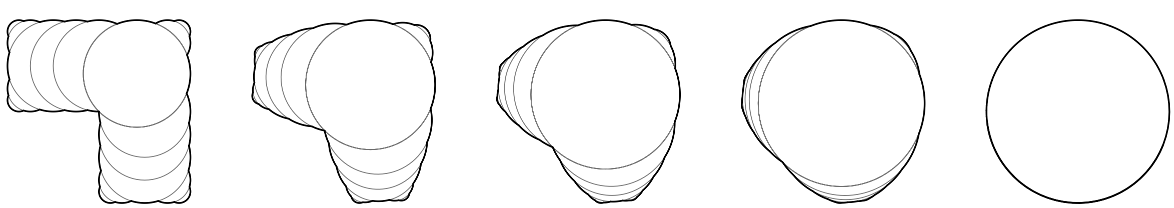

FIGURE 2. An angle scaling chain from Ω to $\mathbb{D}$

Given a finite set $E \subset \mathbb{R}$, cover the hyperbolic convex hull of E by $O(n)$ Whitney boxes and "arches" and cover the rest of $\mathbb{H}$ by $O(n)$ Carleson squares. An arch is a Carleson square with a smaller Carleson square removed; these cover parts of the convex hull which are very "thin". See Figure 3. An ϵ-representation of an n-gon Ω consists of n points $E \subset \mathbb{R}$, the associated Carleson-Whitney decomposition of $\mathbb{H}$, and an analytic map of each piece into Ω which sends E to the vertices of $\partial\Omega$, the components of $\mathbb{R} \setminus E$ to the edges of $\partial\Omega$, and so that the maps for adjacent pieces agree to within ϵ (with respect to a certain metric on Ω). Smoothing an ϵ-representation with a partition of unity gives a $(1 + O(\epsilon))$-QC map $F : \mathbb{H} \to \Omega$.

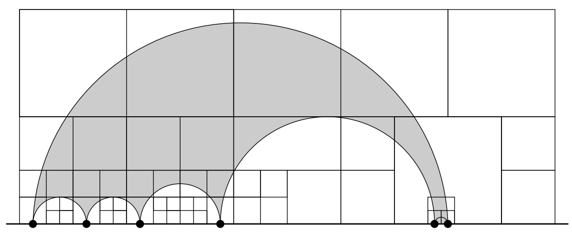

FIGURE 3. A Carleson-Whitney decomposition of $\mathbb{H}$ into $O(n)$ pieces. There is one arch on right which covers a thin part of the convex hull.

Our iteration takes an ϵ-representation and approximately solves a Beltrami equation to find map $h : \mathbb{H} \to \mathbb{H}$ whose Beltrami dilatation $\mu_h = h_{\bar{z}}/h_z$ satisfies $\mu_h = \mu_F + O(\|\mu_F\|_\infty^2)$. Then $F \circ h^{-1}$ gives an $O(\epsilon^2)$-representation of Ω. Moreover, this works as long as $\epsilon < \epsilon_0$ for some ϵ_0 which is independent of n and Ω. The Beltrami equation is solved using fast multipole methods to evaluate the Beurling transform. To obtain time estimates $O(p \log p)$, $p = |\log \epsilon|$, independent of the geometry, the prevertices are organized in a tree structure using the medial axis of a sawtooth domain associated to E. Similar methods should work in higher dimensions.

To construct the conformal map in linear time, first construct the angle scaling chain from $\mathbb{D}$ to Ω with "gaps" that have QC size $< \frac{1}{2}\epsilon_0$. Use the identity map on $\mathbb{D}$ as the starting point and iterate towards the conformal map onto Ω_1. Once we get an $\frac{1}{2}\epsilon_0$-representation of Ω_1, use it to iterate to an $\frac{1}{2}\epsilon_0$-representation of Ω_2. Continue for $O(1/\epsilon_0)$ steps until we get an ϵ_0-representation of Ω. Since ϵ_0 is bounded below independent of Ω, this only takes time $O(n)$. Then continue the iteration for Ω until we reach the desired accuracy ϵ. The length of the power series needed in our representations at each step of the iteration grows exponentially, so the total work is dominated by the final step, $O(n \cdot |\log \epsilon| \cdot \log |\log \epsilon|)$, as claimed.

Several of the steps already exist as code (computing the medial axis, computing angle scaling families, computing Beurling transforms by fast multipole methods, ...), so I hope to find a student or collaborator with the skills to pull these together and write an implementation of the algorithm. Even if it turns out to be impractical to implement the method exactly, I expect that some parts of it will be useful supplements to existing methods (e.g., using the angle scaling family to lead any iterative method towards the correct solution). In this proposal, however, I will only discuss more theoretical problems that remain open.

• **Complexity of conformal mapping:** In the fast mapping theorem, the linearity in n can't be improved, but what about the dependence on ϵ?

Problem 1. *Is the current algorithm optimal, i.e., can the $O(p \log p)$ term be improved?*

Since we represent the map using $O(np)$ numbers ($O(n)$ power series of length p), and the current method uses only $O(1)$ operations per series, improving this constant will require either a faster method of doing Fast Fourier Transforms (FFT) or a different representation of the maps that requires fewer numbers. The former might be possible since we are computing an approximation and so only approximate FFT's are needed. Can these be done faster? To represent the maps with fewer numbers, consider that the Schwartz-Christoffel formula represents a map onto an n-gon with $2n$ numbers: the angles and the prevertices. Can we construct a faster method using it? The CRDT (= Cross Ratios and Delaunay Triangulation) method of Driscoll and Vavasis uses this representation, as does Driscoll's `SC-Toolbox`, but both methods are asymptotically slower and neither is known to converge in all cases.

The current theorem assumes infinite precision arithmetic, and it would be useful to understand the bit-complexity, i.e., given an n-gon with k-bit numbers as vertices, how much work is needed to find all n conformal prevertices with k bits of accuracy? Arches in the decomposition of $\mathbb{H}$ cause problems because they can combine numbers of vastly different scales. Let N be the number of Whitney squares needed to cover all the arches.

Conjecture 2. *The bit-complexity of finding the prevertices is $O((n+N)k \log k)$.*

• **More general domains:** The current proof of the fast mapping theorem uses both polygons and finite unions of disks, switching between them using approximation lemmas. This is awkward both in theory and in practice. The class of circular arc polygons contains both types of domains and is a more natural class for the proof (e.g. is closed under angle scaling).

Problem 3. *Extend the algorithm to circular arc polygons.*

In general, the algorithm should extend to any class of domains where we can compute (or estimate) the medial axis quickly and can explicitly "straighten" any boundary point (i.e., map a neighborhood of any boundary point to a neighborhood of 0 in $\mathbb{H}$ by an explicit map or fast approximation). Another problem is to find fast methods for mapping multiply connected domain to canonical domains, e.g. Koebe domains (all boundary components points or circles).

Problem 4. *Given a polygonal domain Ω with n vertices and k boundary components, can we compute a $(1+\epsilon)$-QC map to a Koebe domain in time $C(\epsilon)k^\alpha n$? $C(\epsilon)kn$? $C(\epsilon)n$?*

This is interesting even for $\epsilon \sim 1$, i.e., find a uniformly K-QC map to a Koebe domain in linear time. Assuming this, the main difficulty in Problem 4 is solving the Beltrami problem with a map that sends the Koebe domain Ω to another Koebe domain. This can be done exactly by extending a dilatation supported on Ω to be invariant under the (infinite) reflection group generated by the bounding circles. In practice we need to truncate the extension. For an annulus this gives good estimates, but higher connectivity leads to exponential growth of the reflection group; can we approximate the invariant dilatation accurately but efficiently? Are other types of canonical domains easier? Can the method for planar domains be modified to find maps onto surfaces in $\mathbb{R}^3$ (important for various applications in computer graphics)? Finally, we mention the important problem of doing all of this in higher dimensions. There are no non-Möbius conformal maps in this case, but the following variations make sense:

Problem 5. *Is there a linear time algorithm to solve the Dirichlet problem on a 3-dimensional polyhedron with piecewise linear data with given accuracy?*

Problem 6. *Is there polynomial time algorithm to find a quasiconformal map of a 3-dimensional polyhedron to a ball when one exists (or even to decide if one does exist).*

This seems very difficult, except that David Hamilton has announced a characterization of quasi-balls in higher dimensions [45]. Perhaps his results will be helpful here.

2. Meshes and triangulations

• **Quadrilateral meshes:** The fast conformal mapping theorem uses ideas from computational geometry and, conversely, is strong enough to prove new results in computational geometry. In the summary of previous work, I described how this approach solved a problem of Bern and Eppstein by constructing quadrilateral meshes with angles between $45 - \epsilon$ and $135 + \epsilon$. These numbers are artifacts of the Whitney decomposition of $\mathbb{H}$ used in the proof. By introducing a decomposition based on hyperbolic polygons and Fuchsian groups I expect to prove:

Conjecture 7. *Any n-gon has a $O(n)$ quadrilateral mesh with angles between* 60 *and* 120.

Bern and Eppstein have proven that any quadrilateral mesh of a regular hexagon must have an angle ≥ 120 (Theorem 5 of [6]). Is the lower bound optimal as well? Does Conjecture 7 hold with multiple boundary components? (A solution to Problem 4 might help.)

• **The thick/thin decomposition:** A key idea in the proof of the fast mapping theorem is the decomposition of a polygon into thick and thin parts. This is analogous to the decomposition of a Riemann surface into thick and thin parts based on the size of the injectivity radius. For polygons, ϵ-thin parts correspond to pairs of edges which have extremal distance $< \epsilon$ in Ω. For adjacent edges (which have distance 0) we call this a parabolic thin part and for non-adjacent edges a hyperbolic thin part, borrowing terminology from the case of surfaces. A polygon is ϵ-thick if it has no hyperbolic ϵ-thin parts. See Figure 4 (dark=hyperbolic, light=parabolic).

For both surfaces and polygons the thin parts have a very simple structure and we can think of the surface (or polygon) as consisting of interesting thick parts, attached to annoying, but well understood thin parts. Is it possible to create a dictionary between hyperbolic surfaces and polygons which extends the analogy of the thick and thin parts, e.g., what is the analog of Mahler-Mumford-Bers compactness for polygons? Is it useful for applications to geometry?

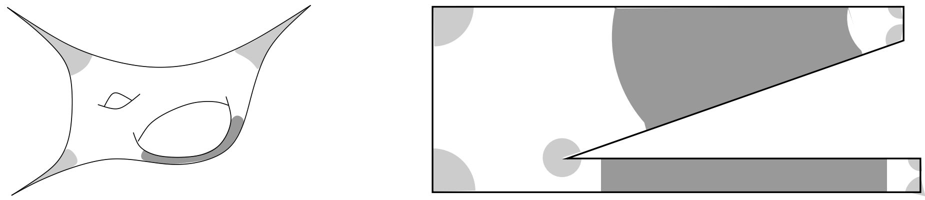

FIGURE 4. A surface and polygon with hyperbolic and parabolic thin parts.

In the fast mapping algorithm, the argument described earlier is really only applied to the thick parts; on the thin parts we have explicit formulas which approximate the map. For the meshing problem, we decompose the polygon into thick and thin parts, find explicit meshes on the thin parts, and use conformal mappings to push meshes from $\mathbb{H}$ to the thick part of Ω. I believe that many known results can be easily deduced using this approach, but it would be more interesting to solve some open problems.

• **Conforming Delaunay triangulations:** Given a polygon Γ with vertex set V and edges E we can either triangulate the interior of the polygon, or we can forget the edges, and triangulate the convex hull of V. The latter case is said to "conform" to Γ if the edges of the triangulation cover all the edges E of Γ. If the triangulation uses vertices in addition to points of V, these are called Steiner points. See Figure 5. It is often easier, in practice, to deal with point sets than polygons, so conforming triangulations are important for various applications.

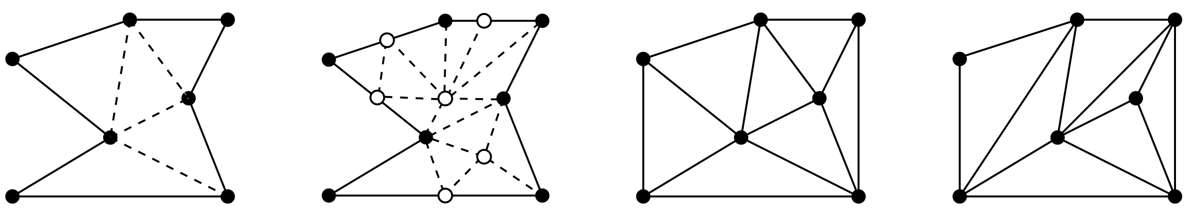

FIGURE 5. A triangulation of a polygon; another with Steiner points; a conforming and non-conforming triangulation of the vertices.

A triangulation of V is called Delaunay if the open disk defined by the three vertices of each triangle misses V. These always exist (and are essentially unique) and are important because they optimize various properties of triangulations. The Delaunay triangulation of the vertices of a polygon can always be made conforming by adding enough Steiner points to the edges. How many new points are needed? In 1993 Edelsbrunner and Tan [35] showed that $O(n^3)$ extra points suffice and sometimes $\sim n^2$ are needed, but this gap has remained open. A well known lemma (e.g., [5]) says a segment between two vertices is an edge of the Delaunay triangulation iff it is Delaunay, i.e., is the chord of some open disk disjoint from V. Hence,

Problem 8. *Can we subdivide the edges of an n-gon into $O(n^2)$ Delaunay sub-edges?*

If $O(n^2)$ is too hard, what about $O(n^\alpha)$, $\alpha < 3$? A stronger condition is to require every edge be the diameter (not just a chord) of a disk missing V. This is called a Gabriel edge.

Problem 9. *Can we subdivide the edges of an n-gon into $O(n^a)$ Gabriel sub-edges, $a < \infty$?*

I believe conformal maps and the thick/thin decomposition can help answer both questions. Suppose Γ is a n-gon and H is its convex hull. Then $H \setminus \Gamma$ consists of $O(n)$ polygons and for each we compute the thick parts and thin parts ($O(n)$ in total). Using conformal maps, we can subdivide the boundaries of the thick parts so each edge is the base of a halfdisk in a thick part. If such an edge bounds two adjacent thick parts then we are done. We can choose the thin parts to be circular arc quadrilaterals (as in Figure 4), which are foliated by circular arcs orthogonal to two opposite, straight edges. Thus the union of the thin parts, W, is a domain foliated by piecewise circular paths. Consider the leaves of the foliation which terminate at vertices of a thick part. These cut W into $O(n)$ "tubes". Each tube can be swept out by a disk of fixed size, and this implies that adding vertices everywhere a tube side meets Γ maintains the Gabriel condition. If W is simply connected, then each of the $O(n)$ tubes can meet at most $O(n)$ thin parts and hence at most $O(n^2)$ vertices are added, solving Problems 8 and 9.

Unfortunately, W can be multiply connected and the tubes can "spiral" hitting the same thins parts over and over again. However, there are at most $O(n)$ different spirals and if a spiral hits $k = O(n)$ different thin parts, it is easy to "block" it with $O(k^2)$ points, i.e., we can add points to Γ which end the tube and preserve the Delaunay condition. This gives a $O(n^3)$ bound on the number of vertices needed, i.e., a new proof of the Edelsbrunner-Tan result. We could improve $O(n^3)$ to $O(n^{1+a})$ if we could block spirals with only $O(k^a)$ points. For example, if the thin parts look like n parallel strips of equal width, then a blocking set can be built using a finite approximation of an Appolonian packing and the number of points needed is $O(n^a)$, where $a \approx 1.3058$ is the Minkowski dimension of the Apollonian residual set. This construction is too technical to present in detail here, but I mention it to give an example of how different ideas from analysis (dimension, circle packings, Kleinian limit sets) might find new applications.

If a spiral hits k thin parts and a tube wraps around the spiral at least $\sqrt{k}$ times, there is an alternate blocking method using $O(k^{1.5})$ vertices, which preserves the Gabriel condition. This implies a $O(n^{3.5})$ bound in Problem 9, if certain difficult technical points can be worked out.

- **Non-obtuse triangulation:** The conforming Delaunay triangulation problem is closely related to another elementary sounding (but unsolved) problem:

Problem 10. *Does an n-gon have a polynomial sized non-obtuse, conforming triangulation?*

More precisely, does the convex hull of a n-gon Γ have a triangulation with all angles $\leq 90°$ that conforms to Γ and uses only $O(n^a)$ triangles for some $a < \infty$? Any non-obtuse triangulation must be a Delaunay triangulation (e.g., Lemma 9 of [5]), so Problem 8 is a special case of this one. In 1994 Bern, Mitchell and Ruppert [7] proved one can always triangulate the interior of a n-gon with $O(n)$ non-obtuse triangles. For the conforming problem, we could try to triangulate one side first, but this adds Steiner points to the polygon which must be included when we triangulate the second side. This, in turn, adds new points to be added into the triangulation of the first side. Can we eventually stop this "back-and-forth" process? My idea is to use the thick/thin decomposition as with conforming triangulations. The spiral blocking that preserves the Gabriel condition also gives a polynomial sized non-obtuse triangulation of the thin parts. Thus the problem now is the thick parts. A special case would be to solve:

Problem 11. *Suppose Ω_1 and Ω_2 are disjoint thick polygons. Are there non-obtuse triangulations which use the same Steiner points on $\partial\Omega_1 \cap \partial\Omega_2$?*

If we weaken "non-obtuse" to "no angles $> 90 + \epsilon$" it suffices to solve following problem. Suppose f_1, f_2 are conformal maps of $\mathbb{H}$ onto the two polygons and let $h = f_2^{-1} \circ f_1$ (where the composition is defined). Given finite sets $S_2 = h(S_1)$, can we build Carleson-Whitney decompositions for each (see Figure 3) so that any additional boundary vertices also correspond under h? This is a discrete conformal welding problem. If h were quasisymmetric then it should be straightforward to build the desired pair of decompositions. We don't know h is quasisymmetric in our case, but perhaps the thickness assumption is enough (it roughly says h has bounded distortion at the scales of interest). We will return to conformal welding later, but first we will discuss some more interactions of computational geometry and analysis.

3. BMO, chord-arc curves and the carpenter's rule problem

A rectifiable curve $\Gamma \subset \mathbb{R}^2$ is called chord-arc if the arclength σ satisfies $\sigma(x, y) = O(|x - y|)$ for $x, y \in \Gamma$. The arclength parameterization of such a curve has tangent of the form $e^{if(t)}$ where $f \in$ BMO (= Bounded Mean Oscillation, see e.g. [41]). The BMO norm induces a topology on the space of chord-arc curves and it has been a long standing problem to prove

Conjecture 12. *The space of chord-arc curves is connected in the BMO topology.*

If so, any chord-arc path Γ of length 1 can be "straightened" to a unit line segment, i.e., there is a $\gamma : [0, 1]^2 \to \mathbb{R}^2$ so that the map $t \to \gamma_t = \gamma(\cdot, t)$ is a continuous path in the BMO topology, γ_1 is a line segment and $\gamma_0 = \Gamma$. We call this motion of the curve expansive if all distances increase, i.e., $s < t$ implies $|x_s - y_s| \leq |x_t - y_t|$ where $x_s = \gamma(u, s)$, $y_s = \gamma(v, s)$, $u, v \in [0, 1]$.

Conjecture 13. *A chord-arc curve can be straightened by an expansive motion.*

Conjecture 14. *We can take $\frac{d}{dt}|x_t - y_t| \simeq (\sigma(x_0, y_0) - |x_0 - y_0|) \geq 0, \forall x_0, y_0 \in \Gamma$.*

Since $x, y \in \Gamma$ start at distance $|x - y|$ and end at distance $\sigma(x, y)$, Conjecture 14 just asks for all distances between pairs of points to increase at their average rate.

Why should we expect the stronger versions of Conjecture 12 to hold? If Γ is a polygon and our motion is through polygons, then BMO continuity implies that angles change continuously and edge lengths are constant. Straightening a polygon in this way is the "carpenter's rule" problem. Recently, Connelly, Demaine and Rote [32], and independently Streinu [55], proved there is always an expansive solution. So to prove Conjecture 13 we might try to approximate by polygons, straighten them and pass to a limit. This requires two opposing estimates: a lower bound on how fast distances increase (to make sure the limiting motion actually straightens the curve in finite time) and an upper bound (to make sure the motions of the polygons are equicontinuous in the BMO topology). To see if this reasonable, we need to briefly review the solution for polygons in [32], which is based on duality in linear programming.

Consider a polygonal curve Γ as a graph. The edges of Γ are called "bars" (whose length is fixed) and we add edges called "struts" (whose length will be allowed to increase) between all remaining non-adjacent vertices. Add new vertices wherever edges cross. See Figure 6.

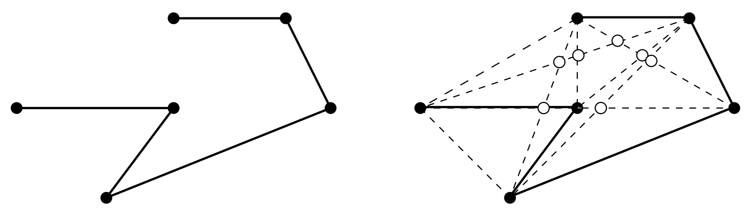

FIGURE 6. Solid lines are bars, dashed lines are struts, open dots are new vertices.

Let $p_i(t)$ be the position of the ith vertex at time t and $v_i(t) = \frac{d}{dt}p_i(t)$ its velocity. The authors of [32] reduce to proving that given the p_i's, there are v_i's so that

$$\frac{1}{2}\frac{d}{dt}(p_i - p_j)^2 = (v_j - v_i) \cdot (p_j - p_i) \geq 0, \text{ for every edge } (i, j), \tag{1}$$

with equality for bars and strict inequality for struts. The dual system of linear inequalities assigns non-negative variable $w_{i,j}$ to each strut (called the stress) and requires that for each vertex i, $\sum_j w_{ij}(p_j - p_i) = 0$ (w is an "equilibrium stress"). The principle of complementary slackness implies strict inequality holds for an edge in (1) iff any solution of the dual system has $w_{i,j} = 0$ for that edge. Thus a strictly expansive infinitesimal motion exists if the only equilibrium stress on the graph is the zero stress. Briefly, stresses correspond to the Laplacians of piecewise linear functions on plane supported on the convex hull of the graph. The condition $w_{i,j} \geq 0$ implies this function is subharmonic. If there is a strut on the boundary of the convex hull, this function takes a positive value, which violates the maximum principle. Thus either the polygon is convex (i.e., a line segment), or a strictly expanding motion exists.

To make this work for chord-arc curves, we have to incorporate bounds on the rate at which distances change, i.e., generalize (1) to a linear system of the form

$$c_{i,j} \leq (v_j - v_i) \cdot (p_j - p_i) \leq C_{i,j}. \tag{2}$$

First, we would like $c_{i,j} \simeq |p_i(0) - p_j(0)| \cdot [\sigma(p_i(0), p_j(0)) - |p_i(0) - p_j(0)|]$, so the distances increase quickly enough for the curve to straighten in bounded time. Second, a result of Semmes [54] shows the motion is continuous in BMO if it extends to a quasiconformal motion of the plane where the dilatation $\mu_t(z)$ satisfies a Carleson condition, i.e., $\int_{B(x,r)} |\mu(z)|^2 \text{dist}(z, \Gamma)^{-1} dxdy = O(r)$. Define the "beta-numbers" $\beta_\Gamma(x, r) = \inf_L \sup_{z \in \Gamma \cap D(x,r)} \text{dist}(z, L)$, where the infimum is over all lines L. These measure deviation from "flatness" and are a kind of curvature measurement. For a chord-arc curve, Peter Jones' "traveling salesman theorem" (TST) [49], implies

$$\sigma(x,y) - |x-y| \simeq \int_0^{|x-y|} \int_{[x,y]\subset\Gamma} \beta(x,y)^2 \frac{dxdt}{t}, \tag{3}$$

and that the β's satisfy a square Carleson condition. Thus the idea is to pass from a Carleson condition on the β's, to a Carleson condition on the bounds in the linear program, to a Carleson condition on a QC extension of the motion. For snowflake type curves with small chord-arc constant (where connectedness is already known) this argument works correctly.

The final step is show the generalized linear program has a solution, i.e., show a certain dual system has only the zero solution. Skipping over the calculations, it comes down to showing that a non-positive function of compact support with certain estimates on its Laplacian must be the zero function (the estimates roughly say the positive part dominates the negative part when we integrate against certain Carleson measures, so is a weak form of subharmonicity). This seems quite plausible, but various (probably difficult) technical problems remain.

4. Distortion by quasiconformal maps

In 1993 Stephen Semmes proved that the Jacobian J_f of any quasiconformal map f is a strong A_∞ weight, and asked in [53] if every A_1 weight must be comparable to a QC-Jacobian. In [20] I construct a counterexample: an A_1 weight w which blows up on a set E, so that if f is QC and $J_f \simeq w$ then $f(E)$ contains a rectifiable curve γ. For such a f, $J_{f^{-1}}$ would vanish on γ and $f^{-1}(\gamma)$ would have zero length, i.e., be a point, which contradicts f being QC.

Problem 15. *Is there a compact null set $E \subset \mathbb{R}^2$ so that no function that blows up on E is a QC-Jacobian? Can such a set have dimension < 2?*

Problem 16. *Is there a null set so that every QC-image of it contains a rectifiable curve?*

Does the standard Sierpinski carpet have a QC deformation with no rectifiable subarcs? In [23] I build a set E so this holds for all K-QC images for K near 1, but not for all K. The set E is a type of Sierpinski carpet and the nth generation "holes" are separated by annuli with large conformal modulus. This implies that a path can be made to avoid a hole (and hence stay in E) with only small extra length. We can use Peter Jones's TST (as in (3)) and estimates on the annuli to show the set contains rectifiable curves. The proof still works for small QC deformations of E, but there are large QC deformations of the set which do not contain any rectifiable arcs. To make this happen, the "holes" are placed in a special pattern called an "ϵ-forest" which can't be avoided by long line segments. Thus our set can't contain line segments (in a quantified way). The opposite would be to have a set which contains many line segments, e.g., a Kakeya set is a null set containing a unit line segment in every direction (e.g., [58]). Does every QC image of a Kakeya set have a rectifiable subarc?

A metric space is locally linearly connected (LLC) if any ball of radius r can be contracted to a point inside a ball of radius $O(r)$. It is Ahlfors 2-regular if 2-dimensional Hausdorff measure satisfies $m_2(B(x,r)) \simeq r^2$. A theorem of Bonk and Kliener [30] says that a surface which is LLC and Ahlfors 2-regular is a quasisymmetric (QS) image of the plane.

Problem 17. *Find a condition which, in addition to LLC and Ahlfors 2-regular, implies a surface in* $\mathbb{R}^n$ *is biLipschitz (BL) equivalent to the plane.*

Although the A_1 weight w constructed in [20] is not the Jacobian of planar QC map, it is the Jacobian of a QS map $\mathbb{R}^2 \to S \subset \mathbb{R}^3$ where S is LLC, Ahlfors 2-regular but not BL-equivalent to $\mathbb{R}^2$. Also see [29]. Thus the proof that w is not a QC-Jacobian must use some property of the plane not shared by S. When we examine the proof, the only step that breaks down is the application of Peter Jones' TST. Thus one answer to Problem 17 is to find a condition which makes Jones' TST work on the surface. In the plane, given points x, y with $|x-y| = 1$ and a disk D of radius ϵ, we can easily make a curve from x to y which avoids D and has length $\leq 1 + O(\epsilon^2)$, but this is false for general surfaces. The amount of extra length needed for a path to avoid a small ball is a measure of the curvature in some sense; we want a condition that says this is small at "most places and most scales". I suspect something of the form "curvature is a Carleson measure" is the correct answer. Indeed, Jones' TST implies biLipschitz images of $\mathbb{R}$ (i.e., chord-arc curves) have just such a characterization, so, in some sense, Problem 17 seeks a generalization of Jones's TST from 1 to 2 dimensions.

There are several other well known problems that are related to these ideas. I mention two; the first was stated by Semmes and Heinonen in [48] (Question 15), and the second conjectured by Astala, Clop, Mateu, Orobity and Uriarte-Tuero in [1] (Conjecture 2.3).

Problem 18. *Can a QS map from a surface to the plane send positive area to zero area?*

Conjecture 19. *Suppose* $d \in (1,2)$ *and* f *is a* K*-QC map of* $\mathbb{R}^2$ *to itself which is conformal off a compact set* E *and normalized so* $f(z) = z + o(1)$ *near* ∞*. If* $d \in (1,2)$ *then* $\mathcal{M}^d(f(E)) \simeq \mathcal{M}^d(E)$ *where* $\mathcal{M}^d$ *denotes* d*-dimensional content. This is already known when* $d = 1, 2$*.*

In [25] I have already disproven a stronger version of Conjecture 19. If it is false, perhaps a similar construction will work. Incidentally, the QC-Jacobian problem turns up in computer science as well. A cartogram is a map in which areas represent some statistic but which otherwise has small distortion (e.g., a map of the United States where states have area proportional to population). This is essentially constructing a map with specified Jacobian and small QC constant. There are several methods in the computer science literature for constructing cartograms. How do these behave with respect to QC constants? What happens when the data approximates a non-QC-Jacobian w?

5. Conformal welding

• **The generalized Moore-Koebe conjecture:** A decomposition $\mathcal{C}$ of a closed set K is a collection of pairwise disjoint closed sets whose union is all of K. It is called upper semi-continuous if a sequence of elements of $\mathcal{C}$ which converges in the Hausdorff metric must converge to a subset of an element of $\mathcal{C}$. If, in addition, $K = \mathbb{C}$ and all elements of $\mathcal{C}$ are continua which don't separate the plane, we call $\mathcal{C}$ a Moore decomposition after R.L. Moore. He proved in [51] that quotienting the plane by such a decomposition (i.e. identify each set to a point) gives the plane again. We say $\mathcal{M}$ is conformally equivalent to $\mathcal{K}$ if there is a bijection $f : \mathcal{M} \to \mathcal{K}$ such that (1) f is conformal on the interior of the singletons and (2) if $\{E_n\} \subset \mathcal{M}$ converges (in the Hausdorff metric) to $E \in \mathcal{M}$ then $f(E_n) \to f(E) \in \mathcal{K}$. A Koebe decomposition is one where every set is either a disk or a point.

Problem 20. *Is every Moore decomposition conformally equivalent to a Koebe decomposition?*

In particular, only countable many elements would not be collapsed to points, so (in some sense) there should only be countably many obstructions to any Moore decomposition being conformal, i.e., conformally equivalent to $\mathbb{R}^2$ (by Moore's theorem it is always topologically equivalent). Knowing which decompositions were conformal would be extremely helpful for various problems, such as building Julia sets or Kleinian limit sets with specified combinatorics. A positive answer to Problem 20 would also imply (recall GCW from summmary of pevious work):

Conjecture 21. *Every circle homeomorphism $h \in \mathrm{GCW}(\mathbb{T} \setminus E)$ for some countable set E.*

I proved a weaker version of this in [8]: given any h there are sequences of conformal maps $\{f_n\}, \{g_n\}$ so that $\lim_n |f_n(x) - g_n(h(x))| \to 0$ except on a countable set. This almost gives Conjecture 21, but when we pass to the limit, we lose control of the boundary values on a set of logarithmic capacity zero, instead of a countable set. The basic idea of the proof is easy. Build a n-connected domain by taking the union of $\{|z| < 1\} \cup \{|z| > 2\}$ with n "channels" that connect points $x_k \in \mathbb{T}$ to $2h(x_k) \in 2\mathbb{T}$. By Koebe's circle theorem this domain can be conformally mapped to the complement of n disks. Making the channels thinner gives limiting maps onto two sides of a chain of n tangent disks. As $n \to \infty$, only finitely many disks can remain larger than any given ϵ, from which we can deduce the limit is zero except on a countable set. The next step towards Conjecture 21 is to figure out how to choose the points $\{x_k\}$ so the necessary limits exist except on a countable set.

Given a domain Ω, the decomposition of Ω into singletons and Ω^c into its connected components is a Moore decomposition of the plane, so a positive answer to Problem 20 also implies Koebe's conjecture:

Conjecture 22. *Every domain is conformally equivalent to a circle domain.*

The best results on this so far are due to He and Schramm [46], [47]. In [8] a sketch is given to show that Conjecture 21 implies any planar domain is conformally equivalent to one whose complement has only countable many components which are not points. Thus conformal welding problems are closely related to Koebe's conjecture.

Conformal welding has many connections to computational problems. We have already seen how it comes up naturally in the non-obtuse triangulation problem. In a different direction, David Mumford has used conformal weldings as a "fingerprint" of the shape of a curve for use in computer vision, [52]. Finally, as noted above, recovering a curve from its conformal welding at n points on $\mathbb{T}$ is equivalent to mapping an n-connected $O(n)$-gon to a Koebe domain. Thus the "computational complexity" of conformal welding is tightly connected to the problem of conformal mapping of multiply connected domains, discussed earlier.

4. Educational and broader impact of the proposal

The principal broader impact of the proposal is to encourage interdisciplinary efforts within pure mathematics and develop new interactions of classical analysis with applied mathematics and computer science. We have already discussed connections between hyperbolic geometry, conformal mappings, computational geometry and numerical analysis. Such connections not only increase the number of ideas that can be used to attack a particular problem, but successful examples encourage researchers to seek more connections. For example, my work on hyperbolic geometry and conformal mappings is cited as motivation for some of the recent work of Epstein, Marden and Markovic [37], [38] on convex hulls and hyperbolic 3-manifolds.

I have seen some interest in my results outside the "pure math" community. I have discussed my initial conformal mappings results with Nick Trefethen who called them a "substantial contribution" and with David Mumford who uses conformal mappings in his work on pattern recognition and computer vision. Marshall Bern of the Xerox Palo Alto Research Center, suggested several problems that might be related. One of these was the quadrilateral meshing problem described in the summary of previous work. Moreover, both Bern and Mumford have agreed to speak at a workshop I am hosting, with Steve Vavasis, in April 2007 at Stony Brook. This will focus on the interactions of computer science and mathematics with an emphasis on the connections between computational, hyperbolic and conformal geometry. Other invited speakers include experts in meshing, computational topology, quasiconformal maps, circle packings, numerical conformal mapping, complexity theory and even geography. This workshop is a direct outgrowth of my work under previous NSF support and I hope will contribute to the solution of some problems in this proposal as well as generating many new ones. In addition to bringing computer scientists to the workshop, I have been invited to speak at various applied math and computer science departments and to visit the theory group at Microsoft research.

Recently, I have taught a graduate course on numerical conformal mappings and hyperbolic geometry and will teach new ones that incorporate the ideas discussed in this proposal. This teaches students to look for problems and ideas from a variety of sources and (I hope) encourages them to explore beyond their immediate expertise. Not only is this a benefit to their individual careers, but, in the long term, will help to disseminate mathematical ideas to non-mathematicians and to introduce mathematicians to new problems deserving their attention. Moreover, relating mathematical research to computational problems which are sometimes easier to state or have more obvious applications can make it more accessible to experts and non-experts alike. This is key to the dissemination of new ideas to other research communities, motivating students to enter mathematics, providing accessible problems for undergraduate and graduate research and communicating with non-technical audiences.

A former student, Karyn Lundberg, wrote her thesis on the sharpest estimate for boundary convergence of a sequence of conformal maps (this is related to my attempts to prove Conjecture 21). She also wrote computer code to implement an algorithm of mine for computing conformal weldings inspired by Koebe's circle theorem. Another, Hrant Hakobyan, will finish this year with a thesis on the distortion of Hausdorff dimension under QC mappings. Anirban Dutta is a student in applied math I have been directing in an effort to implement the conformal mapping algorithm into code. An undergraduate, J. Kim, wrote an honors thesis on the Schwartz-Christoffel formula and Davis' iterative method for computing conformal maps.

Many other problems related to the proposal could be suitable for graduate or even undergraduate research problems. Here are a few examples:

- Implement the fast mapping algorithm.
- Implement the thick/thin decomposition for polygons.
- Implement the angle scaling families for polygons. Use it to "lead" other methods, such as Driscoll's `SC-Toolbox` to the correct answer in cases where it "gets stuck".
- Implement the medial axis method for grouping data for use with fast multipole methods. Does this show any practical improvement over current methods? Extend to higher dimensions.
- Numerically estimate the QC-extension constant of the iota-map for different polygons. The best theoretical upper bound is $K \leq 7.82$, but largest known example is only about 2.1.
- A closely related problem is to search for counterexamples of Brennan's conjecture. Estimate L^p norms of f' for various snowflake domains. ($K = 2$ implies Brennan's conjecture, [13]).
- Simple variations of the iota-map can give improved approximation to conformal maps. Experiment with polygons to find "rules of thumb" which give the best variation.
- Find the formula for the iota-map for piecewise circular domains (in terms of medial axis).
- Set up linear programs as described in the section on chord-arc curves and use existing software to see if solutions exist. It so, plot the resulting expansive, BMO-continuous motions.
- Use computer graphics to draw the surface in $\mathbb{R}^3$ which is not BL equivalent to the plane.
- Investigate existing programs for plotting cartograms. What happens when we give them data which approximates something which can't be a QC-Jacobian?

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