WEIL-PETERSSON CURVES, CONFORMAL ENERGIES, β-NUMBERS, AND MINIMAL SURFACES

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Abstract. This paper gives geometric characterizations of the Weil-Petersson class of rectifiable quasicircles, i.e., the closure of the smooth planar curves in the Weil-Petersson metric on universal Teichmüller space defined by Takhtajan and Teo [125]. Although motivated by the planar case, many of our characterizations make sense for curves in $\mathbb{R}^n$ and remain equivalent in all dimensions. We prove that Γ is Weil-Petersson if and only if it is well approximated by polygons in a precise sense, has finite Möbius energy or has arclength parameterization in $H^{3/2}(\mathbb{T})$. Other results say that a curve is Weil-Petersson if and only if local curvature is square integrable over all locations and scales, where local curvature is measured using various quantities such as Peter Jones's β-numbers, conformal welding, Menger curvature, the "thickness" of the hyperbolic convex hull of Γ, and the total curvature of minimal surfaces in hyperbolic space. Finally, we prove that planar Weil-Petersson curves are exactly the asymptotic boundaries of minimal surfaces in $\mathbb{H}^3$ with finite renormalized area.

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1. Introduction

This paper gives several geometric characterizations of the Weil-Petersson class of rectifiable quasicircles. This collection of planar closed curves has close connections to geometric function theory, operator theory and certain random processes such as Schramm-Loewner evolutions (SLE). Our new characterizations will also link it to various ideas in harmonic analysis and geometric measure theory (e.g., Sobolev spaces, knot energies, β-numbers, biLipschitz involutions, Menger curvature) and hyperbolic geometry (e.g., convex domes, minimal surfaces, isoperimetric inequalities, renormalized area). Moreover, most of our characterizations extend to curves in $\mathbb{R}^n$ and remain equivalent there, defining new classes of curves that may be of interest in analysis and geometry. The name "Weil-Petersson class" comes from work of Takhtajan and Teo [125] defining a Weil-Petersson metric on universal Teichmüller space. The same collection of curves was earlier studied by Guo [66] and Cui [36] using the terms "integrable Teichmüller space of degree 2" and "integrably asymptotic affine maps" respectively.

A quasicircle is the image of the unit circle $\mathbb{T}$ under a quasiconformal mapping f of the plane, e.g., a homeomorphism of the plane that is conformal outside the unit disk $\mathbb{D}$, and whose dilatation $\mu = f_{\overline{z}}/f_z$ belongs to $\mathbb{B}_1^\infty$, the open unit ball in $L^\infty(\mathbb{D})$. The collection of planar quasicircles corresponds to universal Teichmüller space $T(1)$ and the usual metric is defined in terms of $\|\mu\|_\infty$. Motivated by problems arising in string theory (e.g. [23], [24]), Takhtajan and Teo [125] defined a Weil-Petersson metric on universal Teichmüler space $T(1)$ that makes it into a Hilbert manifold. This topology on $T(1)$ has uncountably many connected components, but one of these components, denoted $T_0(1)$, is exactly the closure of the smooth curves; this is the Weil-Petersson class. These curves are precisely the images of $\mathbb{T}$ under quasiconformal maps with dilatation $\mu \in L^2(d\mathrm{A}_\rho) \cap \mathbb{B}_1^\infty$, where A_ρ is hyperbolic area on $\mathbb{D}$. Thus, roughly speaking, Weil-Petersson curves are to L^2 as quasicircles are to L^∞.

An alternate characterization is in terms of the conformal mapping $f : \mathbb{D} \to \Omega$, the domain bounded by Γ. Then Γ is Weil-Petersson if and only if $\log f'$ is in the Dirichlet space, i.e., $(\log f')' = f''/f' \in L^2(\mathbb{D}, dxdy)$. Takhtajan and Teo [125] showed this condition is the same as

$$\frac{1}{\pi}\iint_{\mathbb{D}}\left|\frac{f''(z)}{f'(z)}\right|^2 dxdy + \frac{1}{\pi}\iint_{\mathbb{D}^*}\left|\frac{g''(z)}{g'(z)}\right|^2 dxdy + 4\log\frac{|f'(0)|}{|g'(\infty)|} < \infty.$$

where g is a conformal map from $\mathbb{D}^* = \{|z| > 1\}$ to $\mathbb{C} \setminus \overline{\Omega}$ so that $g(\infty) = \infty$. They called this quantity the universal Liouville action, and showed that it is Möbius invariant. More recently, Yilin Wang [129] proved it equals the Loewner energy of Γ, as defined by her and Steffen Rohde in [110]; we will use this term and denote it by $\mathcal{LE}(\Gamma)$. This characterization of Weil-Petersson curves implies:

Theorem 1.1. *Γ is Weil-Petersson iff it is chord-arc and the arclength parameterization is in the Sobolev space $H^{3/2}(\mathbb{T})$.*

It is known that $\mathcal{LE}(\Gamma) < \infty$ implies that Γ is a chord-arc curve, i.e., $\ell(\gamma) = O(|x-y|)$ where $\gamma \subset \Gamma$ is the shortest sub-arc with endpoints x, y. The definition of $H^{3/2}(\mathbb{T})$ will be given in Section 2, as will the easy proof of necessity (although easy, it still seems to be a new observation). Sufficiency appears harder; a function theoretic proof is given in [16], but it also follows from other geometric characterizations given in this paper. This was first observed by David Mumford, who formulated Theorem 1.1 based on an earlier draft of this paper. The tangent space at a point of $T_0(1)$ is naturally identified with $H^{3/2}$ (see [125]), but it was not previously known how to identity $T_0(1)$ itself with a subset of $H^{3/2}(\mathbb{T})$. Theorem 1.1 identifies $H^{3/2}$ with the topological group $T_0(1)$, giving the endpoint case of results for H^s, $s > 3/2$ in [30], [71]; see the remarks following Definition 6.

It has been an open problem to give a "geometric" characterization of the Weil-Petersson class, e.g., Remark II.1.2 of [125]. Although Theorem 1.1 does not explicitly mention conformal mappings, its proof still has a strong function theoretic flavor and it is not clear how to check this condition just by "looking at" a curve. However, it will serve as a stepping stone to a variety of more purely geometric characterizations. Like being an $H^{3/2}$ curve, most of these conditions also make sense for curves in $\mathbb{R}^n$, $n \geq 2$ and we will prove that they remain equivalent in higher dimensions. For example, one immediate consequence of Theorem 1.1 is:

Theorem 1.2. *Γ is Weil-Petersson iff it has finite Möbius energy, i.e.,*

$$\text{Möb}(\Gamma) = \int_\Gamma\int_\Gamma \left(\frac{1}{|x-y|^2} - \frac{1}{\ell(x,y)^2}\right) dxdy < \infty. \tag{1.1}$$

Möbius energy is one of several "knot energies" introduced by O'Hara [94], [96], [95]. It blows up when the curve is close to self-intersecting, so continuously deforming a curve in $\mathbb{R}^3$ to minimize the Möbius energy should lead to a canonical "nice" representative of each knot type. This was proven for irreducible knots by Freedman, He and Wang [67], who also showed that Möb(Γ) is Möbius invariant (hence the name), that Möb(Γ) attains its minimal value 4 only for circles, that finite energy curves are chord-arc, and in $\mathbb{R}^3$ they are topologically tame (there is an ambient isotopy to a smooth embedding). Theorem 1.2 follows from Theorem 1.1 by a result of Blatt [20] (see Section 3).

Theorem 1.2 has several interesting reformulations. For example, it is essentially the same as the "Jones Conjecture" stated in [57] and we will explain the connection in Section 3. Another reformulation is rather elementary, using only the definition of arclength. If a closed Jordan curve Γ has finite length $\ell(\Gamma)$, choose a base point $z_1^0 \in \Gamma$ and for each $n \geq 1$, let $\{z_j^n\}$, $j = 1, \dots, 2^n$ be the unique set of ordered points with $z_1^n = z_1^0$ that divides Γ into 2^n equal length intervals (called the nth generation dyadic subintervals of Γ). Let Γ_n be the inscribed 2^n-gon with these vertices. Clearly $\ell(\Gamma_n) \nearrow \ell(\Gamma)$ and the Weil-Petersson class is characterized by the rate of this convergence:

Theorem 1.3. *With notation as above, a curve Γ is Weil-Petersson if and only if*

$$\sum_{n=1}^{\infty} 2^n \left[\ell(\Gamma) - \ell(\Gamma_n)\right] < \infty \tag{1.2}$$

with a bound that is independent of the choice of the base point.

A more technical looking consequence of Theorem 1.1 involves Peter Jones's β-numbers: given a curve $\Gamma \subset \mathbb{R}^2$, $x \in \mathbb{R}^2$ and $t > 0$, define

$$\beta_\Gamma(x,t) = \inf_L \sup_{z \in D(x,t)} \frac{\mathrm{dist}(z, L)}{t},$$

where the infimum is over all lines hitting $D(x,t)$ (we take the infimum to be zero if the disk does not hit Γ). We shall prove

Theorem 1.4. *A Jordan curve Γ is Weil-Petersson if and only if*

$$\int_\Gamma \int_0^\infty \beta_\Gamma^2(x,t) \frac{dtdx}{t^2} < \infty. \tag{1.3}$$

This is our fundamental "Γ is Weil-Petersson iff curvature is square integrable over all locations and scales" result. All of our other criteria can be formulated in an analogous way, using different measures of local curvature (even Theorems 1.2 and 1.3, although they do not immediately look like L^2 curvature conditions). We will also see versions that involve Schwarzian derivatives of conformal maps, Menger curvature of sets, and the Gauss curvatures of minimal surfaces; these all measure deviation from flatness in different, but closely related, ways. Our results characterize $H^{3/2}$ curves in terms of β-numbers; related results for graphs of Besov functions (which include H^s as a special case) are given in [42]. I thank Xavier Tolsa for pointing out this reference to me.

Theorem 1.4 shows that Weil-Petersson curves need not be C^1. The curve $\gamma(t) = t \cdot \exp(i/|\log 1/t|)$ is smooth except at the origin where $\beta(0,t) \simeq 1/|\log|t||$ and one can check that (1.3) is satisfied even though γ has an infinite spiral at 0.

Peter Jones [74] introduced the β-numbers in his traveling salesman theorem that characterizes subsets of rectifiable curves in the plane. In the special case of a Jordan curve Γ, his result says that

$$\ell(\Gamma) \simeq \mathrm{diam}(\Gamma) + \int_{\mathbb{R}^2}\int_0^\infty \beta_\Gamma^2(x,t)\frac{dtdx}{t}. \tag{1.4}$$

Thus the condition in Theorem 1.4 is a strengthening of Jones's condition (1.4). Analogs of Jones's theorem are known in $\mathbb{R}^n$ [97], Hilbert space [116], and some other metric spaces [39], [53], [80], [81]. The fact that (1.3) holds for $H^{3/2}$ curves is fairly straightforward, but the reverse implication seems less so. For curves in $\mathbb{R}^2$, we can prove this direction though a chain of function theoretic characterizations that eventually lead back to the $H^{3/2}$ condition. For curves in $\mathbb{R}^n$, $n \geq 3$, we will prove that (1.3) is equivalent to the conditions in Theorems 1.1, 1.2 and 1.3, using a refinement of Jones's theorem from [17]:

Theorem 1.5. *If $\Gamma \subset \mathbb{R}^n$ is a Jordan arc, then*

$$\ell(\Gamma) = \mathrm{crd}(\Gamma) + O\left(\int_\Gamma\int_0^\infty \beta_\Gamma^2(x,t)\frac{dtdx}{t}\right). \tag{1.5}$$

Here $\mathrm{crd}(\Gamma) = |z-w|$ denotes the distance between the endpoints z, w of Γ. The point of Theorem 1.5 is that the $\mathrm{diam}(\Gamma)$ term in (1.4) can be replaced by the smaller value $\mathrm{crd}(\Gamma)$, and that this term is only multiplied by "1" in the estimate (1.5).

For the rest of the introduction we return to the planar case $n = 2$ where the statements are simplest; in higher dimensions some changes are needed due to technicalities that arise, e.g., a minimal surface in $\mathbb{H}^3$ must be replaced by a minimal current or flat chain in $\mathbb{H}^{n+1}$. These changes will be discussed in Section 6.

The hyperbolic upper half-space is defined as $\mathbb{H}^3 = \mathbb{R}^3_+ = \{(x,t) : x \in \mathbb{R}^2, t > 0\}$, with the hyperbolic metric $d\rho = ds/t$ is chosen so that $\mathbb{H}^3$ has constant Gauss curvature -1. The hyperbolic convex hull of $\Gamma \subset \mathbb{R}^2$, denoted $\mathrm{CH}(\Gamma)$, is the smallest convex set in $\mathbb{H}^3$ that contains all (infinite) hyperbolic geodesics with both endpoints in Γ. See Figure 1. Except when Γ is a circle, $\mathrm{CH}(\Gamma)$ has non-empty interior and two boundary surfaces (both with asymptotic boundary Γ). We define $\delta(z)$ to be the maximum of the hyperbolic distances from z to the two boundary components of $\mathrm{CH}(\Gamma)$. This function serves as our Möbius invariant version of the β-numbers. Instead of integrating over all points x in the plane and all scales $t > 0$, our hyperbolic Weil-Petersson criteria will involve integrating over points (x,t) on some surface $S \subset \mathbb{H}^3$ that has $\Gamma \subset \mathbb{R}^2$ as its asymptotic boundary; usually S will be one of the two connected components of $\partial\mathrm{CH}(\Gamma)$, the cylinder $\Gamma \times (0,1]$, or a minimal surface contained in $\mathrm{CH}(\Gamma)$.

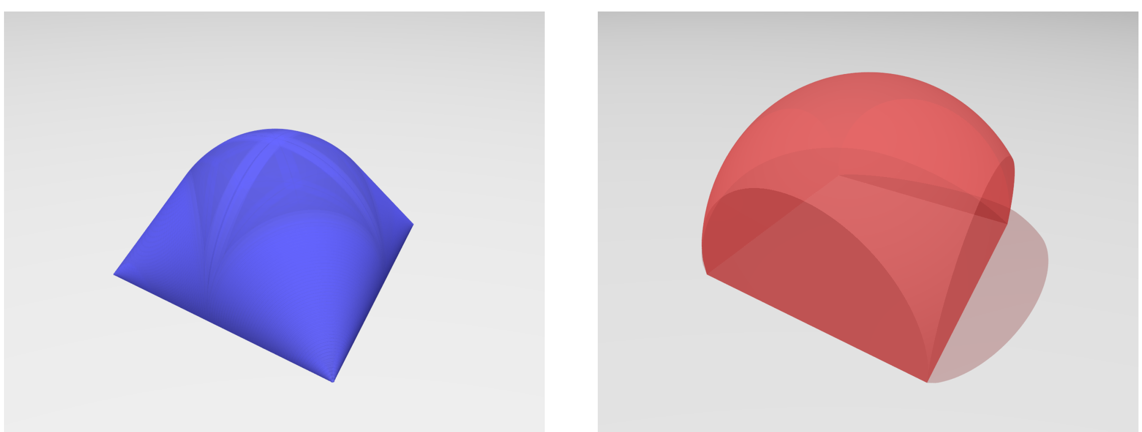

FIGURE 1. The domes of a square and its (spherical) complement. The convex hull is the region bounded between these two surfaces.

Suppose $S \subset \mathbb{H}^3$ is a 2-dimensional, properly embedded sub-manifold that has an asymptotic boundary that is a closed Jordan curve in $\mathbb{R}^2$. The Euler characteristic of S will be denoted $\chi(S)$, and equals $2 - 2g - h$ if S is a surface of genus g with h holes.

We let $K(z)$ denote the Gauss curvature of S at z. The hyperbolic metric $d\rho = ds/2t$ is chosen so that $\mathbb{H}^3$ has constant Gauss curvature -1. If the principle curvatures of S at z are $\kappa_1(z), \kappa_2(z)$, then $K(z) = -1 + \kappa_1(z)\kappa_2(z)$ (this is the Gauss equation, e.g., Equation (2.1) of [113]). The norm of the second fundamental form is given by $|\mathcal{K}(z)|^2 = \kappa_1(z)^2 + \kappa_2(z)^2$. The surface S is called a minimal surface if $\kappa_1 = -\kappa_2$ (the mean curvature is zero). In this case we will write $\kappa = |\kappa_j|$, $j = 1, 2$ and so $K(z) = -1 - \kappa^2(z)$. The surface S is called area minimizing if any compact Jordan region $\Omega \subset S$ has minimal area among all compact surfaces in $\mathbb{H}^3$ with the same boundary. All such surfaces are minimal, but not conversely. Michael Anderson [7] has shown that every closed Jordan curve on $\mathbb{R}^2$ bounds a simply connected minimal surface in $\mathbb{H}^3$, but there may be other minimal surfaces with boundary Γ that are not disks (see Figure 14). However, every such S is contained in $\mathrm{CH}(\Gamma)$ and the principle curvatures of S at a point z can be controlled by the function $\delta(z)$ introduced above (see Lemma 19.1). Let A_ρ denote hyperbolic area and L_ρ hyperbolic length.

Theorem 1.6. *For a closed curve $\Gamma \subset \mathbb{R}^2$, the following are equivalent:*

(1) $\Gamma \subset \mathbb{R}^2$ *is a Weil-Petersson curve.*

(2) Γ *asymptotically bounds a surface $S \subset \mathbb{H}^3$ so that*
$$\int_S |\delta(z)|^2 d\mathrm{A}_\rho(z) < \infty.$$

(3) Γ *asymptotically bounds a surface $S \subset \mathbb{H}^3$ so that $|\mathcal{K}(z)| \to 0$ as $z \to \mathbb{R}^2 = \partial\mathbb{H}^3$ and*
$$\int_S |\mathcal{K}(z)|^2 d\mathrm{A}_\rho(z) < \infty.$$

(4) *Every minimal surface S asymptotically bounded by Γ has finite Euler characteristic and finite total curvature, i.e.,*
$$\int_S |\kappa(z)|^2 d\mathrm{A}_\rho(z) = \int_S |K(z)+1| d\mathrm{A}_\rho(z) < \infty.$$

(5) *There is some minimal surface S with finite Euler characteristic and asymptotic boundary Γ so that S is the union of a nested sequence of compact Jordan subdomains $\Omega_1 \subset \Omega_2 \subset \dots$ with*
$$\limsup_{n\to\infty} \left[L_\rho(\partial\Omega_n) - \mathrm{A}_\rho(\Omega_n)\right] < \infty.$$

In [40] Geraldo de Oliveira Filho showed that a complete, immersed minimal disk in $\mathbb{H}^n$ having finite total curvature has an asymptotic boundary Γ that is rectifiable and he asked if Γ is C^1. Theorem 1.6 shows this fails since $H^{3/2} \not\subset C^1$, but is close to the truth since $H^s \subset C^{1,s-3/2} \subset C^1$ for all $s > 3/2$ (e.g., Lemma 8.2 of [41]).

The isoperimetric difference in part (5) of Theorem 1.6 is also known as the renormalized area of S, at least in the special case that Ω is the truncation of $S \subset \mathbb{H}^3$ at a fixed height above the boundary. More precisely, set

$$S_t = S \cap \{(x, y, s) \in \mathbb{H}^3 : s > t\}, \quad \partial S_t = S \cap \{(x, y, s) \in \mathbb{H}^3 : s = t\}$$

and define the renormalized area of S to be

$$\mathcal{RA}(S) = \lim_{t \searrow 0} [\mathrm{A}_\rho(S_t) - L_\rho(\partial S_t)]$$

when this limit exists and is finite. We will show that these truncations satisfy the last part of Theorem 1.6 and hence

Corollary 1.7. *For any closed curve $\Gamma \subset \mathbb{R}^2$ and for any minimal surface $S \subset \mathbb{H}^3$ with finite Euler characteristic and asymptotic boundary Γ, we have*

$$\mathcal{RA}(S) = -2\pi\chi(S) - \int_S \kappa^2(z) d\mathrm{A}_\rho, \tag{1.6}$$

In other words, either Γ is Weil-Petersson and both sides are finite and equal, or Γ is not Weil-Petersson and both sides are $-\infty$.

Proposition 3.1 of Alexakis and Mazzeo's paper [5] gives a version of (1.6) for surfaces in the setting of n-dimensional Poincaré-Einstein manifolds (that formula also contains a term involving the Weyl curvature), but they use the additional assumption that Γ is $C^{3,\alpha}$. However, as noted earlier, Weil-Petersson curves need not be even C^1, in general. Corollary 1.7 shows that the Alexakis-Mazzeo result holds without any conditions on Γ, at least in the case of $\mathbb{H}^3$. In Section 8 of [5] Alexakis and Mazzeo also connect the problem of maximizing the renormalized area of S to minimizing the Willmore area of the double M of S, obtained by reflecting across $\mathbb{R}^2$. Our proof of Corollary 1.7 will show that the exact method of truncation in the definition of renormalized area is not important, and that it can be defined intrinsically on S, without explicit reference to the boundary:

Corollary 1.8. *Suppose $S \cup_n K_n \subset \mathbb{H}^3$ is a minimal surface where $K_1 \subset K_2 \subset \dots$ are nested compact sets such that $S \setminus K_n$ is a topological annulus for all n. Then*

$$-2\pi\chi(S) - \int_S \kappa^2(z) d\mathrm{A}_\rho, = \mathcal{RA}(S) = \lim_{n\to\infty} \sup_{\Omega \supset K_n} [\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega)]$$

where the supremum is over compact domains $K_n \subset \Omega \subset S$ bounded by a single Jordan curve. As above, either all terms are finite and equal, or all are $-\infty$.

Möbius energy is also an example of renormalization, namely the Hadamard regularization of a divergent integral. Given the divergent integral of a function that blows up on a set E, this is defined by integrating the function outside a t-neighborhood of E, writing the result as power series in t, and taking the constant term of this series as the renormalized value of the integral (of course, this depends on exactly how we choose the neighborhoods). To apply Hadamard renormalization to Möbius energy, note that the integral of the first term in (1.1) is infinite, but for smooth curves the truncated version equals

$$\iint_{\ell(x,y)>t} \frac{dxdy}{|x-y|^2} = \frac{2\ell(\Gamma)}{t} + C + O(t). \tag{1.7}$$

Regularizing the other term in (1.1) (e.g., Lemma 2.3 of [67]) gives

$$\iint_{\ell(x,y)>t} \frac{dxdy}{\ell(x,y)^2} = \frac{2\ell(\Gamma)}{t} - 4, \tag{1.8}$$

so that $\text{Möb}(\Gamma) = 4 + C$. The divergent integral in (1.7) is the energy of arclength measure with respect to a inverse cube law, e.g., electrostatics in four dimensions. It is infinite because Brownian motion in $\mathbb{R}^4$ almost surely misses any rectifiable curve, but Weil-Petersson curves are exactly those for which the electrostatic energy of arclength measure blows up as slowly as possible (up to an additive constant).

Incidentally, the Loewner energy of Γ can also be written as type of renormalization involving the Lawler-Werner Brownian loop measure of random closed curves hitting both sides of a neighborhood of Γ. This measure tends to infinity as the neighborhood shrinks, but subtracting the corresponding quantity for a circle gives a multiple of $\mathcal{LE}(\Gamma)$ in the limit. See Appendix A for the precise statement.

Since Loewner energy, Möbius energy and renormalized area are all Möbius invariant quantities that characterize Weil-Petersson curves, it seems natural to ask if they are essentially the same quantity, or at least comparable in size. In Section 23 we will give examples showing that for any large M we can have

(1) $\mathcal{LE}(\Gamma_1) \simeq \text{Möb}(\Gamma_1) \simeq \mathcal{RA}(\Gamma_1) \simeq M$.
(2) $\mathcal{LE}(\Gamma_2) \simeq \mathcal{RA}(\Gamma_2) \simeq M$ but $\text{Möb}(\Gamma_2) \simeq M \log M$.
(3) $|\mathcal{LE}(\Gamma_3) - \mathcal{LE}(\Gamma_4)| \simeq M \simeq |\mathcal{RA}(\Gamma_3)|$ but $|\mathcal{RA}(\Gamma_3) - \mathcal{RA}(\Gamma_4)| \simeq 1$.

Therefore it is unclear whether there is any simple relation among these quantities. In these estimates, $\mathcal{RA}(\Gamma)$ denotes $\mathcal{RA}(S)$ for some choice of minimal surface S with asymptotic boundary Γ; another example in Section 23 shows this need not be unique and that different choices can lead to very different values of the renormalized area.

Renormalized area has strong motivations arising from string theory. Maldacena [83] proposed that the expectation value of the Wilson loop operator (a precursor of string theory) should be the area of a minimal surface with asymptotic boundary Γ. It was pointed out by Hennington and Skenderis [69], and by Graham and Witten [65], that area should be renormalized area. More recently, it has been suggested that renormalized area be used to measure the entanglement entropy of regions in conformal field theory, in a way that is analogous to how the entropy of a black hole is measured by the area of its event horizon, e.g., [93], [112], [124]. See the introduction of [5] for further details and references. Also see [106], where the authors argue that Weil-Petersson curves are the correct setting for 2-dimensional conformal field theory.

The Weil-Petersson class also arises in computer vision: see the papers of Sharon and Mumford [118], Feiszli, Kushnarev and Leonard [51], and Feiszli and Narayan [52]. The second of these anticipates the connections between $H^{3/2}$, Weil-Petersson curves and β-numbers (though β is defined differently there than here), and the latter paper computes geodesics for the Weil-Petersson metric as optimal "morphing" paths between different shapes. Indeed, the problem of geometrically characterizing Weil-Petersson curves was originally suggested to me by David Mumford in December of 2017. However, I did not start to think effectively about this problem until attending a lecture of Yilin Wang at an IPAM workshop in January of 2019 that addressed connections between SLE and the Weil-Petersson class (see Appendix A). Her lecture caused me to think about the question in terms of Peter Jones's β-numbers and this intuition was strengthened by reading [57] by Gallardo-Gutiérrez, González, Pérez-González, Pommerenke and Rättyä. It states a conjecture of Peter Jones that

has a re-formulation in terms of β-numbers and is proven in this paper (essentially Theorem 1.2, but stated in a different way). I am grateful to Atul Shekhar for pointing me to this very interesting paper. I also thankfully acknowledge discussions with and corrections from Kari Astala, Martin Chuaqui Blaine Lawson, Pekka Koskela, Dragomir Saric, Raanan Schul, Leon Takhtajan, Dror Varolin, Rongwei Yang and Michel Zinsmeister. I thank Jack Burkart and María González for reading various early drafts and providing many helpful comments and corrections. I am deeply appreciative to Mike Anderson, Claude LeBrun, Rafe Mazzeo and Andrea Seppi for very enlightening discussions of curvature, minimal surfaces, renormalized area and Willmore energy, to John Morgan for explaining the Smith Conjecture, and especially to David Mumford for sharing his perspective on these problems and his thoughtful encouragement of this work.

The next five sections state various definitions of the Weil-Petersson class: known function theoretic ones, new criteria involving Sobolev smoothness, new conditions involving the β-numbers, and finally new characterizations using hyperbolic geometry, first in $\mathbb{H}^3$ and then in higher dimensions. For the convenience of the reader, Table 1 in Section 7 lists the definitions and Figure 7 shows a directed graph indicating the particular implications that are proven in this paper and where to find the corresponding proofs. Several appendices provide some additional proofs and describe some other known characterizations of the Weil-Petersson class.

2. Function theoretic characterizations

A quasiconformal (QC) map h of a planar domain Ω is a homeomorphism of Ω to another planar domain Ω' that is absolutely continuous on almost all lines and whose dilatation $\mu = h_{\overline{z}}/h_z$ is satisfies $\|\mu\|_\infty \leq k < 1$. See [3] or [78] for the basic properties of such maps. We say the h is a planar quasiconformal map if $\Omega = \Omega' = \mathbb{R}^2$. The measurable Riemann mapping theorem says that given such a μ, there is a planar quasiconformal map h with this dilatation. If μ is supported on the unit disk, $\mathbb{D}$, then there is a quasiconformal $h : \mathbb{D} \to \mathbb{D}$ with this dilatation. A quasiconformal map h is called K-quasiconformal if its dilatation satisfies $\|\mu\|_\infty \leq k = (K-1)/(K+1)$. More geometrically, at almost every point h is differentiable and its derivative (which is a real linear map) send circles to ellipses of eccentricity at most K (the eccentricity of an ellipse is the ratio of the major to minor axis).

A quasicircle $\Gamma = f(\mathbb{T})$ is the image of the unit circle $\mathbb{T}$ under a planar quasiconformal map. Such curves have a well known geometric characterization: Γ is a quasicircle if and only if for all subarcs $\gamma \subset \Gamma$ with $\operatorname{diam}(\gamma) \leq \operatorname{diam}(\Gamma)/2$ we have $\operatorname{diam}(\gamma) = O(|z-w|)$, where z, w are the endpoints of γ (this is one of about thirty equivalent conditions given in [63]). In general, quasicircles can be fractal curves, such as the von Koch snowflake or various polynomial Julia sets or Kleinian limit sets. Weil-Petersson curves are quasicircles by definition, so satisfy the condition above, but they are also rectifiable and satisfy even more stringent conditions.

Suppose Γ is a closed curve in the plane and let f be a conformal map from the unit disk $\mathbb{D} = \{z : |z| < 1\}$ to Ω, the bounded complementary component of Γ. If f is conformal on $\mathbb{D}$, then f' is never zero, so $\Phi = \log f'$ is a well defined holomorphic function on $\mathbb{D}$. Recall that the Dirichlet class is the Hilbert space of holomorphic functions F on the unit disk such that $|F(0)|^2 + \int_{\mathbb{D}} |F'(z)|^2 dxdy < \infty$. In other words, the Dirichlet space consists of the holomorphic functions in the Sobolev space $W^{1,2}(\mathbb{D})$ (functions with one derivative in $L^2(dxdy)$).

Definition 1. Γ *is a quasicircle and* $\Gamma = f(\mathbb{T})$, *where* f *is conformal on* $\mathbb{D}$ *and* $\log f'$ *is in the Dirichlet class.*

This definition immediately provides some geometric information about the curve Γ. For a Jordan arc γ, let $\ell(\gamma)$ denote its arclength and let $\operatorname{crd}(\gamma) = |z-w|$ where z, w are the endpoints of γ. If $\log f'$ is in the Dirichlet class, then $\log f' \in \mathrm{VMOA}$ (vanishing mean oscillation; see Chapter VI of [59]). The John-Nirenberg theorem (e.g., Theorem VI.2.1 of [59]) then implies f' is in the Hardy space $H^1(\mathbb{D})$, so Γ is rectifiable. Even stronger, a theorem of Pommerenke [104] implies that Γ is asymptotically smooth, i.e., $\ell(\gamma)/\operatorname{crd}(\gamma) \to 1$ as $\ell(\gamma) \to 0$, i.e., a Weil-Petersson curve has "no corners". Asymptotic smoothness implies Γ is chord-arc; a fact observed in [57] (see also Theorem 2.8 of [105], but there is a gap due to the non-standard definition of "quasicircle" in a result quoted from [48].) An estimate of Beurling [13] (simplified and extended by Chang and Marshall in [32] and [86]) says that $\log |f'|$ in the Dirichlet class implies $\int \exp(\alpha \log^2 |f'|^2) ds < \infty$ for all $\alpha \leq 1$; in particular $|f'| \in L^p(\mathbb{T})$ for every $p < \infty$ (but examples show $|f'|$ need not be bounded). Thus f is almost, but not quite, Lipschitz. We shall describe its precise smoothness later.

It is easy to prove using power series (e.g., Lemma 10.2 of [16]) that for any holomorphic function F on $\mathbb{D}$

$$|F(0)|^2 + \int_{\mathbb{D}} |F'(z)|^2 dxdy < \infty$$

if and only if

$$|F(0)|^2 + |F'(0)|^2 + \int_{\mathbb{D}} |F''(z)|^2 (1-|z|^2)^2 dxdy < \infty.$$

Applying this to $F = \log f'$, we see that

$$\int_{\mathbb{D}} |(\log f')'|^2 dxdy = \int_{\mathbb{D}} \left| \frac{f''}{f'} \right|^2 dxdy < \infty. \tag{2.1}$$

could be replaced by the condition

$$\int_{\mathbb{D}} \left| \left(\frac{f'''}{f'} \right) - \left(\frac{f''}{f'} \right)^2 \right|^2 (1-|z|^2)^2 dxdy < \infty. \tag{2.2}$$

This integrand is reminiscent of the Schwarzian derivative of f given by

$$S(f) = \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2 = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2. \tag{2.3}$$

The quantities in (2.2) and (2.3) are very similar, except that a factor of 1 has been changed to 3/2. However, this represents a non-linear change, and it is difficult to compare the two quantities directly, e.g., for a Möbius transformation sending $\mathbb{D}$ to a half-plane, the Schwarzian is constant zero, but the expression in (2.2) blows up to infinity at a boundary point. Nevertheless, for conformal maps into bounded quasidisks, the integrals of these two quantities are simultaneously finite or infinite:

Definition 2. *Γ is quasicircle and $\Gamma = f(\mathbb{T})$, where f is conformal on $\mathbb{D}$ and satisfies*

$$\int_{\mathbb{D}} |S(f)(z)|^2 (1-|z|^2)^2 dxdy < \infty. \tag{2.4}$$

Proposition 1 of Cui's paper [36] says that Definitions 2 and 1 are equivalent to each other. See also Theorem II.1.12 of Takhtajan and Teo's book [125] and Theorem 1 of [102] by Pérez-González and Rättyä. If f is univalent on $\mathbb{D}$ then

$$\sup_{z \in \mathbb{D}} |S(f)(z)|(1-|z|^2)^2 \leq 6. \tag{2.5}$$

See Chapter II of [79] for this and other properties of the Schwarzian. If f is holomorphic on the disk and satisfies (2.5) with 6 replaced by 2, then f is injective, i.e., a conformal map. If 2 is replaced by a value $t < 2$, then f also has a K-quasiconformal extension to the plane, where K depends only on t. This is due to Ahlfors and Weill [2], who gave a formula for the extension and its dilatation

$$f(w) = f(z) + \frac{(1-|z|^2)f'(z)}{\overline{z} - \frac{1}{2}(1-|z|^2)(f''(z)/f'(z))} \tag{2.6}$$

$$\mu(w) = -\frac{1}{2}(1-|z|^2)^2 S(f)(z) \tag{2.7}$$

where $w \in \mathbb{D}^*$ and $z = 1/\overline{w} \in \mathbb{D}$. See Section 4 of [35] for a lucid discussion of the Ahlfors-Weill extension and a proof that when $t = 2$, this extension gives a homeomorphism of the sphere except in one extremal case where the disk maps to a strip. See also Formula (3.33) of [99] and or Equation (9) of [107]. The Alhfors-Weill extension shows that Definition 2 implies:

Definition 3. $\Gamma = f(\mathbb{T})$ *where* f *is a quasiconformal map of the plane that is conformal on* $\mathbb{D}^*$ *and whose dilatation* μ *on* $\mathbb{D}$ *satisfies satisfies*

$$\int_{\mathbb{D}} \frac{|\mu(z)|^2}{(1-|z|^2)^2} dxdy < \infty. \tag{2.8}$$

This was shown to be equivalent to Definition 2 by Guizhen Cui; see Theorem 2 of [36]. The integral in (2.8) is the same as

$$\int_{\mathbb{D}} |\mu(z)|^2 d\mathrm{A}_\rho < \infty, \tag{2.9}$$

where $d\mathrm{A}_\rho$ denotes integration against hyperbolic area; this was one of the definitions of the Weil-Petersson class mentioned in the introduction.. Another variation on this theme is to consider the map $R(z) = f(1/\overline{f^{-1}(z)})$. This is an orientation reversing quasiconformal map of the sphere to itself that fixes Γ pointwise, exchanges the two complementary components of Γ and whose dilatation satisfies

$$\int_{\Omega \cup \Omega^*} |\mu(z)|^2 d\mathrm{A}_\rho(z) < \infty, \tag{2.10}$$

where $d\mathrm{A}_\rho$ is hyperbolic area on each of the domains Ω, Ω^*. This version is sometimes easier to check, and we will use it interchangeably with Definition 3. The map R is called a quasiconformal reflection across Γ (and constructing R is often the easiest way to check Γ is WP). Definition 13 gives a biLipschitz variation of Definition 3.

A circle homeomorphism $\varphi : \mathbb{T} \to \mathbb{T}$ is called a conformal welding if $\varphi = f^{-1} \circ g$ where f, g are conformal maps from the two sides of the unit circle to the two sides of a closed Jordan curve Γ. There are many weldings associated to each Γ, but they all differ from each other by compositions with Möbius transformations of $\mathbb{T}$. Not every circle homeomorphism is a conformal welding, but weldings are dense in the homeomorphisms in various senses; see [58].

A circle homeomorphism is called M-quasisymmetric if it maps adjacent arcs in $\mathbb{T}$ of the same length to arcs whose length differ my a factor of at most M; we call φ quasisymmetric if it is M-quasisymmetric for some M. The quasisymmetric maps are exactly the circle homeomorphisms that can be continuously extended to quasiconformal self-maps of the disk, and are also exactly the conformal weldings of quasicircles. See [3]. A quasisymmetric homeomorphism is called symmetric if the constant M tends to 1 on small scales (Pommerenke [104] proved such weldings characterize curves where $\log f'$ is in the little Bloch space; see also [58] by Gardiner and Sullivan and [122] by Strebel). We will prove that φ corresponds to a Weil-Petersson curve iff $M - 1$ is in L^2 over all scales and positions. If $I \subset \mathbb{T}$ is an arc, let $m(I)$ denote its midpoint. For a homeomorphism $\varphi : \mathbb{T} \to \mathbb{T}$ define

$$\mathrm{qs}(\varphi, I) = \frac{|\varphi(m(I)) - m(\varphi(I))|}{\ell(\varphi(I))}.$$

Definition 4. Γ *is closed Jordan curve whose welding map* φ *satisfies*

$$\sum_I \mathrm{qs}^2(\varphi, I) < \infty, \tag{2.11}$$

where the sum is over some dyadic decomposition of $\mathbb{T}$.

Weil-Petersson weldings were first characterized by Yuliang Shen [119] in terms of the Sobolev space $H^{1/2}$. We will describe his result in the next section.

3. Sobolev conditions

We start by recalling some standard notation. Given two quantities A, B that both depend on a parameter, we write $A \lesssim B$ if there is a constant C so that $A \leq CB$ holds independent of the parameter. We write $A \gtrsim B$ if $B \lesssim A$, and we write $A \simeq B$ if both $A \lesssim B$ and $A \gtrsim B$ hold. The notation $A \lesssim B$ means the same as the "big-Oh" notation $A = O(B)$.

Definition 1 can be interpreted in terms of Sobolev spaces. The space $H^{1/2}(\mathbb{T}) \subset L^2(\mathbb{T})$ is defined by the finiteness of the seminorm

$$\begin{aligned} D(f) &= \iint_{\mathbb{D}} |\nabla u(z)|^2 dxdy \\ &= \frac{1}{8\pi}\int_0^{2\pi}\int_0^{2\pi} \left|\frac{f(e^{is})-f(e^{it})}{\sin\frac{1}{2}(s-t)}\right|^2 dsdt \simeq \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|f(z)-f(w)|^2}{|z-w|^2}|dz||dw|. \end{aligned}$$

where u is the harmonic extension of f to $\mathbb{D}$. The equality of the first and second integrals is called the Douglas formula, after Jesse Douglas who introduced it in his solution of the Plateau problem [43]. See also Theorem 2.5 of [4] (for a proof of the Douglas formula) and [111] (for more information about the Dirichlet space). For $s \in (0,1)$ we define the space $H^s(\mathbb{T})$ using

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|f(z)-f(w)|^2}{|z-w|^{1+2s}}|dz||dw| < \infty.$$

See [1] and [41] for additional background on fractional Sobolev spaces. Also [92] by Nag and Sullivan; in the authors' words its "purpose is to survey from various different aspects the elegant role of $H^{1/2}$ in universal Techmüller theory" (a role we seek to explore in this paper too).

Shen [119] proved Γ is Weil-Petersson iff its welding map satisfies $\log\varphi' \in H^{1/2}$. To see necessity, observe that $\log f'$ is in the Dirichlet class on $\mathbb{D}$ if and only if its radial boundary values satisfy $\log f' \in H^{1/2}(\mathbb{T})$. Thus Definition 1 implies $\log f', \log g' \in H^{1/2}(\mathbb{T})$ and a simple computation shows $\log\varphi'(x) = -\log f'(\varphi(x)) + \log g'(x)$. Beurling and Ahlfors [12] proved $H^{1/2}(\mathbb{T})$ is invariant under pre-compositions with quasisymmetric circle homeomorphisms, so $\log\varphi' \in H^{1/2}(\mathbb{T})$. Thus Definition 1 implies Definition 4. The converse seems harder; we provide a geometric alternative to Shen's operator theoretic approach by showing that $\log\varphi' \in H^{1/2}$ implies Definition 4, and that this implies Definitions 1, 2 and 3.

As noted above, $\log f'(z)$ is in the Dirichlet class on $\mathbb{D}$ if and only if the radial limits $\log|f'|$ and $\arg(f')$ are in $H^{1/2}(\mathbb{T})$. Since $\arg(f')$ can be unbounded, it is, perhaps, surprising that this is equivalent to $f'/|f'| \in H^{1/2}$:

Definition 5. $\Gamma = f(\mathbb{T})$ *is chord-arc and* $\exp(i\arg f') = f'/|f'| \in H^{1/2}(\mathbb{T})$.

One direction is easy. Definition 1 implies $\log f' = \log|f'| + i\arg f'$ is in the Dirichlet class, so $\arg f' \in H^{1/2}(\mathbb{T})$. Using $|e^{ix} - e^{iy}| \leq |x-y|$ and the Douglas formula we get

$$\int_{\mathbb{T}}\int_{\mathbb{T}}\left|\frac{e^{i\arg f'(x)}-e^{i\arg f'(y)}}{x-y}\right|^2 dxdy \leq \int_{\mathbb{T}}\int_{\mathbb{T}}\left|\frac{\arg f'(x)-\arg f'(y)}{x-y}\right|^2 dxdy < \infty.$$

Thus $\exp(i\arg f') \in H^{1/2}(\mathbb{T})$. The converse direction seems harder. A direct function theoretic proof is given in [16] and it also follows from chain of geometric characterizations givin in this paper.

Let $a : \mathbb{T} \to \Gamma$ be an orientation preserving arclength parameterization (i.e., a multiplies the arclength of every set by $\ell(\Gamma)/2\pi$). For $z \in \Gamma$, let $\tau(z)$ be the unit tangent direction to Γ with its usual counterclockwise orientation. Then $\tau(a(x)) = a'(x)2\pi/\ell(\Gamma)$, where $a' = \frac{da}{d\theta}$ on $\mathbb{T}$. Thus $a' = \exp(i\arg f') \circ \varphi$ where $\varphi = a^{-1} \circ f$ is a circle homeomorphism. We shall prove in Section 8 that this map φ is quasisymmetric (and hence so is its inverse). It is a result of Beurling and Ahlfors [12] that pre-composing with such maps preserves $H^{1/2}(\mathbb{T})$, so Definition 5 is equivalent to saying $a' \in H^{1/2}(\mathbb{T})$. Every arclength parameterization is Lipschitz hence absolutely continuous, and therefore the distributional derivative of a equals its pointwise derivative a'. Thus, for arclength parameterizations, $a' \in H^{1/2}(\mathbb{T})$ is the same as $a \in H^{3/2}(\mathbb{T})$. Therefore Definition 5 is equivalent to

Definition 6. Γ *is chord-arc and the arclength parameterization* $a : \mathbb{T} \to \Gamma$ *is in the Sobolev space* $H^{3/2}(\mathbb{T})$.

Proving this is equivalent to Definition 1 gives Theorem 1.1. Previous to Shen's result described earlier, Gay-Balmaz and Ratiu [61] had proved that if Γ is Weil-Petersson, then $\varphi \in H^s(\mathbb{T})$ for all $s < 3/2$, but Shen [119] gave examples not in $H^{3/2}(\mathbb{T})$ or Lipschitz. Thus Theorem 1.1 implies that having an $H^{3/2}$ arclength parameterization is not equivalent to having an $H^{3/2}$ conformal welding. These are equivalent conditions for $s > 3/2$: for such weldings the Sobolev embedding theorem implies that φ' is Hölder continuous, which implies that the conformal mappings f, g have non-vanishing, Hölder continuous derivatives (e.g.,[76]), and therefore φ is biLipschitz. This implies Γ has an H^s arclength parameterization (copy the argument following Definition 5, using the fact that biLipschitz circle homeomorphisms preserve $H^s(\mathbb{T})$ for $1/2 < s < 1$, e.g., [22]).

When identified with quasisymmetric circle homeomorphisms, elements of the universal Teichmüller space $T(1)$ form a group under composition. It is not a topological group under the usual topology because left multiplication is not continuous (e.g., Theorem 3.3 in [79] or Remark 6.9 in [73]). However, the subgroup $T_0(1)$ is a topological group with its Weil-Petersson topology. Circle diffeomorphisms in $H^s(\mathbb{T})$ with $s > 3/2$ also form a group, e.g., [71], [119], and by the previous paragraph this means H^s curves are identified with a topological group via conformal welding. Even though $H^{3/2}$-diffeomorphisms of the circle are not a group, Theorem 1.1 shows the set of $H^{3/2}$ curves can also be identified with a group via conformal welding, namely $T_0(1)$. See also [11], [61], [88], [89] for relevant discussions of groups, weldings, Sobolev embeddings and immersions.

Next we consider some consequences of Definition 6. Since Γ is chord-arc,

$$\frac{1}{C} \leq \frac{|a(x) - a(y)|}{|x - y|} \leq 1, \quad x, y \in \mathbb{T},$$

so setting $z = a(x), w = a(y)$, we have

$$\begin{aligned} \int_\Gamma \int_\Gamma \left| \frac{\tau(z) - \tau(w)}{z - w} \right|^2 |dz||dw| &= \int_\mathbb{T} \int_\mathbb{T} \left| \frac{a'(x) - a'(y)}{a(x) - a(y)} \right|^2 dxdy \\ &= \int_\mathbb{T} \int_\mathbb{T} \left| \frac{a'(x) - a'(y)}{x - y} \cdot \frac{x - y}{a(x) - a(y)} \right|^2 dxdy \\ &\simeq \int_\mathbb{T} \int_\mathbb{T} \left| \frac{a'(x) - a'(y)}{x - y} \right|^2 dxdy \end{aligned}$$

Thus Definition 6 is equivalent to:

Definition 7. Γ *is chord-arc and*

$$\int_\Gamma \int_\Gamma \left| \frac{\tau(z) - \tau(w)}{z - w} \right|^2 |dz||dw| < \infty.$$

This characterization of the Weil-Petersson class was independently discovered by Shen and Wu [120]. They prove that Γ is a Weil-Petersson curve iff $\tau(a(x)) = a'(x) = \exp(ib(x))$ for some $b \in H^{1/2}(\mathbb{T})$. Since

$$|\tau(a(x)) - \tau(a(y))| = O(|b(x) - b(y)|),$$

it is easy to check that $\tau \circ a \in H^{1/2}$, which gives Definition 7.

In Section 9 we will prove Definition 7 is equivalent to:

Definition 8. Γ *has finite Möbius energy, i.e,*

$$\text{Möb}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|z - w|^2} - \frac{1}{\ell(z, w)^2} \right) dzdw < \infty.$$

As mentioned in the introduction, Blatt [20] proved directly that Definition 6 is equivalent to Definition 8 (but there is a typo in Theorem 1.1 of [20]: it is stated that $s = (jp-2)/(2p)$, but this should be $s = (jp-1)/(2p)$, as given in the proof).

A Jordan curve with a $H^{3/2}$ arclength parameterization is chord-arc (Lemma 2.1 of [20], because this assumption prevents bending on small scales, but there is no quantitative bound on the chord-arc constant: Jordan curves with $H^{3/2}$ parameterizations can come arbitrarily close to self-intersecting (think of a smooth, Jordan approximation to a figure "8"). However, such a bound is possible in terms of $\text{Möb}(\Gamma)$. This is Lemma 1.2 of [67], but for the reader's convenience, we sketch a proof here. If $|z-w| \le \epsilon$, but $\ell(z,w) \ge M\epsilon$, let $\sigma_k, \sigma_k' \subset \gamma(z,w)$ be arcs of length $2^k\epsilon$ that are path distance (on Γ) $2^k\epsilon$ from z and w respectively, for $k = 1, \dots, K = \lfloor \log_2(M) \rfloor - 4$. Then $\sigma_k \cup \sigma_k'$ has diameter at most $\epsilon(1+2^{k+1})$ in $\mathbb{R}^n$, but these two arcs are at least distance $(M - 2^{k+2})\epsilon \ge M\epsilon/2$ apart on Γ. Thus

$$\begin{aligned}
\int_{\sigma_k}\int_{\sigma_k'} \left(\frac{1}{|z-w|^2} - \frac{1}{\ell(v,w)^2} \right) dzdw &\ge \left[\frac{1}{(2^{k+2}\epsilon)^2} - \frac{1}{(M/2)^2} \right] (2^k\epsilon)(2^k\epsilon) \\
&\ge \frac{1}{16} - \frac{2^{2K+2}}{M^2} \\
&\ge \frac{1}{16} - 2^{-6} > \frac{1}{32}
\end{aligned}$$

Summing over k shows $\text{Möb}(\Gamma) \ge K/32 \gtrsim \log M$, so $\text{Möb}(\Gamma) < \infty$ implies Γ is chord-arc. Using the fact that Γ is chord-arc, we get

$$\begin{aligned}
\text{Möb}(\Gamma) &= \int_\Gamma\int_\Gamma \frac{\ell(z,w)^2 - |z-w|^2}{|z-w|^2\ell(z,w)^2} dzdw \\
&= \int_\Gamma\int_\Gamma \frac{(\ell(z,w) - |z-w|)(\ell(z,w) + |z-w|)}{|z-w|^2\ell(z,w)^2} dzdw \\
&\simeq \int_\Gamma\int_\Gamma \frac{\ell(z,w) - |z-w|}{|z-w|^3}.
\end{aligned}$$

Thus Definition 8 holds iff

Definition 9. Γ *is chord-arc and satisfies*

$$\int_\Gamma\int_\Gamma \frac{\ell(z,w) - |z-w|}{|z-w|^3} |dz||dw| < \infty. \tag{3.1}$$

In [57], Gallardo-Gutiérrez, González, Pérez-González, Pommerenke and Rättyä claim that (3.1) follows from Definition 1, but their proof contains a small error. They state the converse as a conjecture of Peter Jones; our results prove both directions.

This definition does not immediately look like a "curvature is square integrable" criterion, but it can easily be put in this form. Set

$$\kappa(z,w) = \sqrt{24} \cdot \sqrt{\frac{\ell(z,w) - |z-w|}{|z-w|^3}}.$$

If Γ is smooth, then it is easy to check that $\kappa(x) = \lim_{y\to x} \kappa(x,y)$, is the usual Euclidean curvature of Γ at x. Thus (3.1) can be rewritten as

$$\int_\Gamma \int_\Gamma \kappa^2(z,w)|dz||dw| < \infty, \tag{3.2}$$

and this has much more of a "L^2-curvature" flavor.

4. β-NUMBERS

A dyadic interval I in $\mathbb{R}$ is one of the form $(2^{-n}j, 2^{-n}(j+1)]$. A dyadic cube in $\mathbb{R}^n$ is the product of n dyadic intervals of the same length. This length is called the side length of Q and is denoted $\ell(Q)$. Note that $\mathrm{diam}(Q) = \sqrt{n}\ell(Q)$. For a positive number $\lambda > 0$, we let λQ denote the cube concentric with Q but with diameter $\lambda \mathrm{diam}(Q)$, e.g., $3Q$ is the "triple" of Q, a union of Q and $3^n - 1$ adjacent copies of itself. We let $Q^\uparrow$ denote the parent of Q; the unique dyadic cube containing Q and having twice the side length. Q is one of the 2^n children of $Q^\uparrow$.

A multi-resolution family in a metric space X is a collection of sets $\{X_j\}$ in X such that there is are $N, M < \infty$ so that

1. For each $r > 0$, the sets with diameter between r and Mr cover X,
2. each bounded subset of X hits at most N of the sets X_k with $\mathrm{diam}(X)/M \le \mathrm{diam}(X_k) \le M\mathrm{diam}(X)$.
3. any subset of X with positive, finite diameter is contained in at least one X_j with $\mathrm{diam}(X_j) \le M\mathrm{diam}(X)$.

Dyadic intervals are not multi-resolution family, e.g., $X = [-1, 1] \subset \mathbb{R}$ is not contained in any dyadic interval, violating (3). However, the family of triples of all dyadic intervals (or cubes) do form a multi-resolution family. Similarly, if we add all translates of dyadic intervals by $\pm 1/3$, we get a multi-resolution family (this is sometimes called the "$\frac{1}{3}$-trick", [97]). The analogous construction for dyadic squares in $\mathbb{R}^n$ is to take all translates by elements of $\{-\frac{1}{3}, 0, \frac{1}{3}\}^n$.

During the course of this paper, we will deal with functions α that map a collection of sets into the non-negative reals, and will wish to decide if the sum $\sum_j \alpha(X_j)$ over some multi-resolution family converges or diverges. We will frequently use the following observation to switch between various multi-resolution families without comment.

Lemma 4.1. *Suppose $\{X_j\}$, $\{Y_k\}$ are two multi-resolution families on a space X and that α is a function mapping subsets of X to $[0,\infty)$ that satisfies $\alpha(E) \lesssim \alpha(F)$, whenever $E \subset F$ and $\mathrm{diam}(F) \lesssim \mathrm{diam}(E)$. Then*

$$\sum_j \alpha(X_j) \simeq \sum_k \alpha(Y_k).$$

Proof. By Condition (3) above, each X_j is contained in some set $Y_{k(j)}$ of comparable diameter. Hence $\alpha(X_j) \lesssim \alpha(Y_{k(j)})$ by assumption. Each Y_k is contained in a comparably sized X_m, and X_m can contain at most a bounded number of comparably sized subsets X_j. Thus each Y_k is only chosen boundedly often as a $Y_{k(j)}$. Thus $\sum_j \alpha(X_j) \lesssim \sum_k \alpha(Y_k)$. The opposite direction follows by reversing the roles of the two families. □

For a Jordan arc γ with endpoints z, w recall $\mathrm{crd}(\gamma) = |z-w|$ and define $\Delta(\gamma) = \ell(\gamma) - \mathrm{crd}(\gamma)$. In Section 10 we will prove Definition 9 is equivalent to:

Definition 10. Γ *is chord-arc and*

$$\sum_j \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)} < \infty \tag{4.1}$$

for some multi-resolution family $\{\Gamma_j\}$ of arcs on Γ.

Condition (4.1) is just a reformulation of (1.2), since if $\{\Gamma_j\}$ corresponds to a dyadic decomposition of Γ we have

$$\sum_n 2^n[\ell(\Gamma) - \ell(\Gamma_n)] = \sum_j \Delta(\gamma_j)/\ell(\gamma_j). \tag{4.2}$$

Thus proving that Definition 10 is equivalent to being Weil-Petersson essentially proves Theorem 1.3. There is a slight gap here because Definition 10 uses a sum over a multi-resolution family and Theorem 1.3 is in terms of dyadic intervals. However, the theorem assumes a bound that is uniform over all dyadic decompositions, and this includes the $\frac{1}{3}$-translates of a single dyadic family, and these form another multi-resolution family (recall the "$\frac{1}{3}$-trick" from above). Conversely, Corollary 10.3 will show that $\Delta(\gamma) \leq \Delta(3\gamma)$, so the dyadic sum can be bounded by the sum over dyadic triples, a multi-resolution family. Thus (4.1) for any multi-resolution family is equivalent to (4.2) with a uniform bound over all dyadic decompositions of Γ.

Given a set $E \subset \mathbb{R}^n$ and a dyadic cube Q, define Peter Jones's β-number as

$$\beta(Q) = \beta_E(Q) = \frac{1}{\operatorname{diam}(Q)} \inf_L \sup\{\operatorname{dist}(z, L) : z \in 3Q \cap E\},$$

where the infimum is over all lines L that hit $3Q$. See the left side of Figure 2. Peter Jones invented the β-numbers as part of his traveling salesman theorem [74]. One consequence of his theorem is that for a Jordan curve Γ,

$$\ell(\Gamma) \simeq \operatorname{diam}(\Gamma) + \sum_Q \beta_\Gamma(Q)^2 \operatorname{diam}(Q), \tag{4.3}$$

where the sum is over all dyadic cubes Q in $\mathbb{R}^n$. Our main geometric characterization of Weil-Petersson curves is to simply the "$\operatorname{diam}(Q)$" terms from (4.3).

Definition 11. Γ *is a closed Jordan curve that satisfies*

$$\sum_Q \beta_\Gamma(Q)^2 < \infty, \tag{4.4}$$

where the sum is over all dyadic cubes.

This is not terribly surprising (in retrospect). Peter Jones and I proved (Lemma 3.9 of [18], or Theorem X.6.2 of [60]) that if Γ is a M-quasicircle, then

$$\ell(\Gamma) \simeq \operatorname{diam}(\Gamma) + \iint |f'(z)||S(f)(z)|^2(1-|z|^2)^3 dxdy \tag{4.5}$$

with constants depending only on M. By Koebe's distortion theorem

$$|f'(z)|(1-|z|^2) \simeq \operatorname{dist}(f(z), \partial\Omega),$$

and thus the factor on the left is analogous to the $\operatorname{diam}(Q)$ in Jones's β^2-sum. Dropping this term from (4.5) gives exactly the integral in Definition 2. Thus Definition 11 in the plane is a direct geometric analog of Definition 2.

It will be convenient to consider several equivalent formulations of condition (4.4). For $x \in \mathbb{R}^2$ and $t > 0$, define

$$\beta_\Gamma(x,t) = \frac{1}{t} \inf_L \max\{\operatorname{dist}(z, L) : z \in \Gamma, |x - z| \leq t\},$$

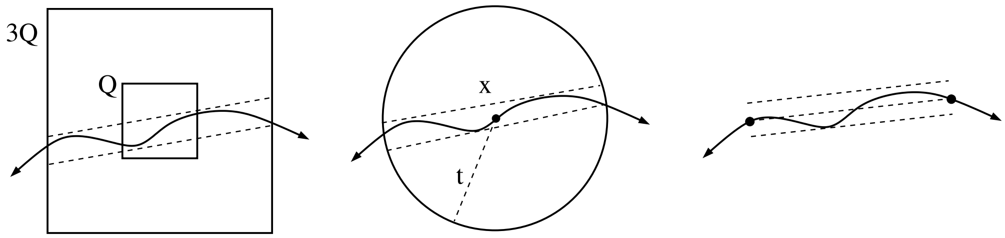

FIGURE 2. Three equivalent versions of the β-numbers.

where the infimum is over all lines hitting the disk $D = D(x,t)$ and let $\widetilde{\beta}_\Gamma(x,t)$ be the same, but where the infimum is only taken over lines L hitting x. Since this is a smaller collection, clearly $\beta(x,t) \le \widetilde{\beta}(x,t)$ and it is not hard to prove that $\widetilde{\beta}(x,t) \le 2\beta(x,t)$ if $x \in \Gamma$. See the center picture in Figure 2.

Given a Jordan arc γ with endpoints z, w we let

$$\beta(\gamma) = \frac{\max\{\mathrm{dist}(z, L) : z \in \gamma\}}{|z - w|},$$

where L is the line passing through z and w. See the right side of Figure 2.

Lemma 4.2. *If Γ is a closed Jordan curve or a Jordan arc in $\mathbb{R}^n$ such that (4.4) holds, then Γ is a chord-arc curve. Moreover, (4.4) holds if and only if any of the following conditions holds:*

$$\int_0^\infty \iint_{\mathbb{R}^n} \beta^2(x,t) \frac{dxdt}{t^{n+1}} < \infty, \tag{4.6}$$

$$\int_0^\infty \int_\Gamma \widetilde{\beta}^2(x,t) \frac{dsdt}{t^2} < \infty, \tag{4.7}$$

$$\sum_j \beta^2(\Gamma_j) < \infty, \tag{4.8}$$

where dx is volume measure on $\mathbb{R}^n$, ds is arclength measure on Γ, and the sum in (4.8) is over a multi-resolution family $\{\Gamma_j\}$ for Γ. Convergence or divergence in (4.6) and (4.7) is not changed if $\int_0^\infty$ is replaced by $\int_0^M$ for any $M > 0$.

The equivalence of these conditions is fairly standard, and a proof can be found Lemma B.2 [17]. Since $\beta(x,t) \simeq \widetilde{\beta}(x,t)$ if $x \in \Gamma$, the integral in (4.7) is finite iff it is finite with β replacing $\widetilde{\beta}$. However, putting $\widetilde{\beta}$ into (4.6) gives a divergent integral for every closed Jordan curve Γ.

The Menger curvature of three points $x, y, z \in \mathbb{R}^n$ is $c(x, y, z) = 1/R$ where R is the radius of the circle passing through these points. Equivalently,

$$c(x, y, z) = \frac{2\mathrm{dist}(x, L_{yz})}{|x-y||x-z|}, \tag{4.9}$$

where L_{yz} is the line passing through y and z, or

$$c(x, y, z) = 2\frac{\sin\theta}{|x-y|}, \tag{4.10}$$

where θ is the angle opposite $[x, y]$ in the triangle with vertices $\{x, y, z\}$. The perimeter of this triangle is denoted by $\ell(x, y, x) = |x-y| + |y-z| + |z-x|$.

Definition 12. Γ *is chord-arc and satisfies*

$$\int_\Gamma \int_\Gamma \int_\Gamma \frac{c(x, y, z)^2}{\ell(x, y, z)} |dx||dy||dz| < \infty. \tag{4.11}$$

Again, this is not unexpected. It is known that the conditions

$$\int_\Gamma \int_\Gamma \int_\Gamma c(x, y, z)^2 |dx||dy||dz| < \infty. \tag{4.12}$$

$$\sum_Q \beta_\Gamma^2(Q)\ell(Q) < \infty, \tag{4.13}$$

are equivalent, and the analog of dropping the length term from (4.13), would be to divide by a term that scales like length in (4.12), which gives (4.11). Indeed, to prove that Definitions 11 and 12 are equivalent, we will simply indicate how to modify the proof of the equivalence of (4.12) and (4.13) in Pajot's book [101].

Recall that a Whitney decomposition of an open set $W \subset \mathbb{R}^n$ is a collection of dyadic cubes Q with disjoint interiors, whose closures cover W and which satisfy

$$\mathrm{diam}(Q) \simeq \mathrm{dist}(Q, \partial W).$$

The existence of such decompositions is a standard fact (e.g., for each $z \in W$, take the maximal dyadic cube Q so that $z \in Q \subset 3Q \subset W$, see Section I.4 of [60]).

Suppose U is a neighborhood of $\Gamma \subset \mathbb{R}^n$ and $R : U \to U' \subset \mathbb{R}^n$ is a homeomorphism fixing each point of Γ. For each Whitney cube Q for $W = \mathbb{R}^n \setminus \Gamma$, with $Q \subset U$, define $\rho(Q)$ to be the infimum of values $\rho > 0$ so that R is $(1+\rho)$-biLipschitz on Q and $\mathrm{dist}(\frac{z+R(z)}{2}, \Gamma) \leq \rho \cdot \mathrm{diam}(Q)$ for $z \in Q$ (the latter condition ensures $R(z)$ is on the "opposite" side of Γ from z). R is called an involution if $R(R(z)) = z$.

Definition 13. *There is homeomorphic involution R defined on a neighborhood of Γ that fixes Γ pointwise, and so that*

$$\sum_Q \rho^2(Q) < \infty. \tag{4.14}$$

The sum is over all cubes of a Whitney decomposition of $\mathbb{R}^n \setminus \Gamma$ that lie inside U.

We will prove later (Lemma 14.1) that a map R satisfying Definition 13 is biLipschitz in U. We can also extend R to be a biLipschitz involution on the sphere $\mathbb{S}^n$, except in the case when Γ is knotted in $\mathbb{R}^3$; the solution of the Smith conjecture implies the fixed set of an orientation preserving diffeomorphic involution of $\mathbb{S}^3$ is an unknotted closed curve. See [91]. So, except for knotted curves in $\mathbb{R}^3$, we can say that Weil-Petersson curves are exactly the fixed point sets of biLipschitz involutions of $\mathbb{S}^n$ that satisfy (4.14). Although the Smith conjecture was stated for diffeomorphisms, John Morgan explains on page 4 of [91] that its proof extends to homeomorphisms when the fixed point set is locally flat (locally ambiently homeomorphic to a segment). This holds in our case by Theorem 4.1 of [67] (finite Möbius energy implies tamely embedded), and the fact that Definition 13 implies Definition 8.

Next, we give a variation of the β-numbers that uses solid tori instead of cylinders and provides a stepping stone to the hyperbolic conditions discussed in the next section. We start with the definition in the plane. Given a dyadic square Q let $\varepsilon_\Gamma(Q)$ be the infimum of the $\epsilon \in (0,1]$ so that $3Q$ hits a line L, a point z and a disk D so that D has radius $\ell(Q)/\epsilon$, z is the closest point of D to L and neither D nor its reflection across L hits Γ. See Figure 3. If no such line, point and disk exist, we set $\varepsilon_\Gamma(Q) = 1$. It is easy to see that $\beta_\Gamma(Q) = O(\epsilon_\Gamma(Q))$, but the opposite direction can certainly fail for a single square Q. Nevertheless, we will see that that the corresponding sums over all dyadic squares are simultaneously convergent or divergent.

Definition 14. Γ *is chord-arc and satisfies*

$$\sum_Q \varepsilon_\Gamma^2(Q) < \infty \tag{4.15}$$

where the sum is over all dyadic squares Q that hit Γ and satisfy $\operatorname{diam}(Q) \le \operatorname{diam}(\Gamma)$.

In higher dimensions the disk D is replaced by a ball B of radius $\operatorname{diam}(Q)/\epsilon$ that attains its distance ϵ from L at $z \in Q$, and that the full rotation of B around L does

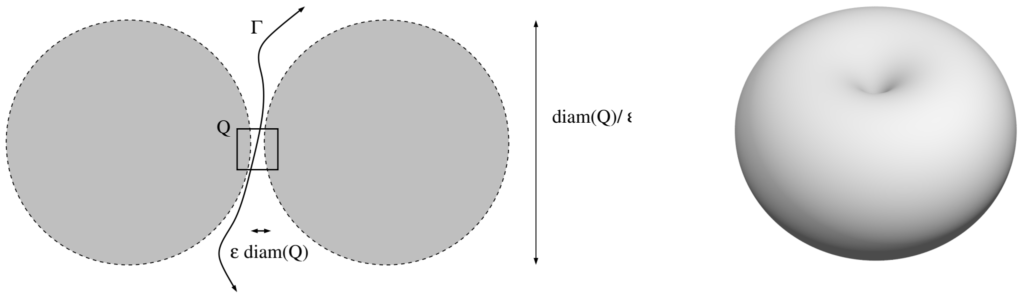

FIGURE 3. The left side illustrates the definition of $\varepsilon_\Gamma(Q)$ in the plane. In $\mathbb{R}^3$ the definition says Γ is surrounded by "thick tori" like the one shown on the right.

not intersect Γ. Thus Γ is surround by a "fat torus". The centers of the balls form a $(n-2)$-sphere that lies in a $(n-1)$-hyperplane perpendicular to L.

5. Hyperbolic conditions for planar curves

Many of our previous conditions involve sums or integrals over points $x \in \Gamma$ and scales $0 < t \leq \mathrm{diam}(\Gamma)$. Thinking of t as a height instead of scale, we could interpret these as integrals over the cylinder $\Gamma \times (0, \mathrm{diam}(\Gamma)] \subset \mathbb{H}^{n+1} = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$. Many of the conditions in this section will be integrals over various other surfaces in the upper half-space whose boundary on $\mathbb{R}^n$ is Γ. We start by recalling the basic definitions and then discuss Weil-Petersson curves in the plane. In the next section we describe the changes that have to be made for curves in $\mathbb{R}^n$, $n \geq 3$.

The hyperbolic length of a (Euclidean) rectifiable curve in the unit disk $\mathbb{D}$ or in the n-dimensional ball $\mathbb{B}^n$ is given by integrating

$$\frac{ds}{1-|z|^2},$$

along the curve. In the upper half-space $\mathbb{H}^n$ we integrate ds/t. Note that this definition differs by a factor of 2 from that given in some sources; we have made our choice so that hyperbolic space has constant Gauss curvature -1. The hyperbolic distance between two points is given by taking the infimum of all hyperbolic lengths of paths connecting the points. In the ball, hyperbolic geodesics are either diameters or subarcs of circles perpendicular to the boundary. In half-space model $\mathbb{H}^{n+1}$, hyperbolic geodesics are either vertical rays or semi-circles centered on the boundary.

Given a closed curve Γ (or more generally, a compact set E) the hyperbolic convex hull, denoted $\mathrm{CH}(\Gamma)$, is the convex hull in $\mathbb{H}^{n+1}$ of all infinite geodesics that have both endpoints in Γ. The complement of the convex hull is a union of hyperbolic half-spaces. Each such half-space intersects $\mathbb{R}^n$ in am open Euclidean ball (or half-space or exterior of a ball) that does not hit Γ. Conversely, each ball in $\mathbb{R}^n$ that does not intersect Γ corresponds to a hyperbolic half-space in the complement of $\mathrm{CH}(\Gamma)$.

In the planar case it is only necessary to consider medial axis disks, i.e., those that hit the boundary in at least two points. See Figure 4. A planar curve Γ divides $\mathbb{R}^2$ into two components and the boundary of $\mathrm{CH}(\Gamma)$ has two corresponding connected components (unless Γ is a circle) called the domes of the two sides of Γ. In higher dimensions, the complement of a Jordan curve is connected and $\partial\mathrm{CH}(\Gamma)$ has a single component.

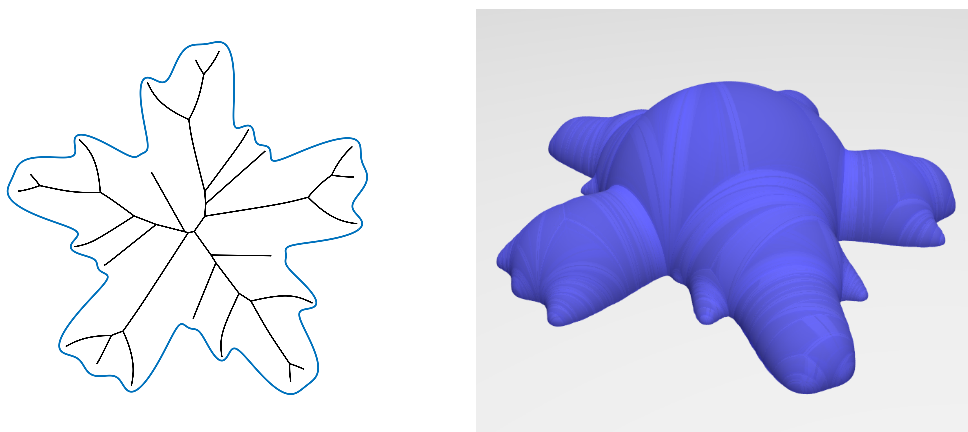

FIGURE 4. A smooth domain, its medial axis and its dome.

We describe the case $n = 2$ in more detail, since this case is of particular interest. As noted above, the boundary $\mathrm{CH}(\Gamma) \subset \mathbb{H}^3$ has two connected components, S_1, S_2. Each of these surfaces meets $\mathbb{R}^2$ exactly along Γ and each is isomorphic to the hyperbolic unit disk when given its hyperbolic path metric. Each of these surfaces is also a pleated surface. This means that it is a disjoint union of non-intersecting infinite geodesics for $\mathbb{B}^3$ (possibly uncountably many) and at most countably many regions lying on hyperbolic planes, each region bounded by disjoint hyperbolic geodesics. Roughly speaking, each surface is a copy of the hyperbolic disk that has been "bent" along a collection of disjoint geodesics, and there is an associated bending measure that gives the amount of bending on each geodesic. The bending measure actually measures arcs that are transverse to the bending geodesics and in general it may have both atoms and continuous parts. For more about convex hulls and pleated surfaces, see [45] by Epstein and Marden (or the revised version [46]). For an overview of domes and convex hulls see Marden's paper [85]; also his book [84] for a discussion related to hyperbolic 3-manifolds.

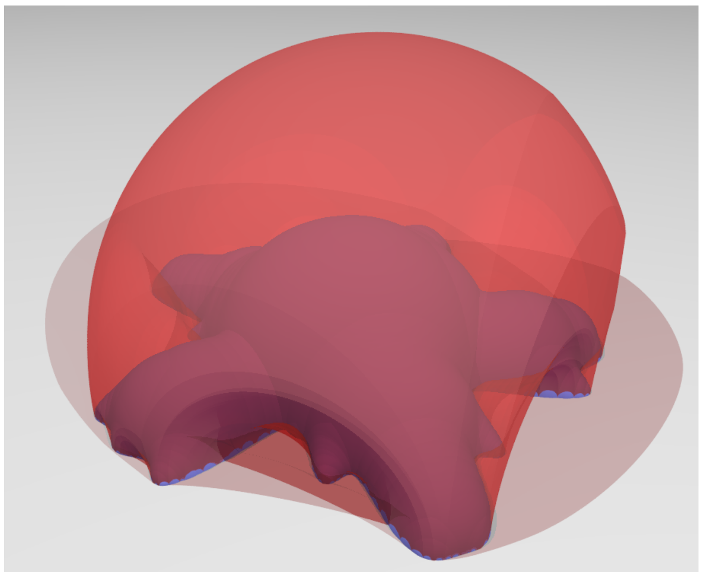

FIGURE 5. Domes for both sides of the curve in Figure 4; the convex hull of Γ is the region between the two surfaces.

For a point $z \in \mathrm{CH}(\Gamma)$ we define $\delta(z) = \max(\mathrm{dist}_\rho(z, S_1), \mathrm{dist}_\rho(z, S_2))$, i.e., $\delta(z)$ is the hyperbolic distance to the farther of the two boundary components of $\mathrm{CH}(\Gamma)$. For z inside the convex hull, $\delta(z)$ measures the "thickness" of the convex hull of Γ near z. We will show that Definition 14 implies

Definition 15.

$$\int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z) d\mathrm{A}_\rho(z) < \infty, \tag{5.1}$$

where $d\mathrm{A}_\rho$ *denotes hyperbolic surface area on* $\partial\mathrm{CH}(\Gamma)$.

We have integrated over all of $\partial\mathrm{CH}(\Gamma)$, but the proof will show that if the integral over one component is finite, then so is the integral over the other one.

If Γ is a quasicircle, then each point z of one boundary component is within a uniformly bounded hyperbolic distance $\delta(z)$ of the other boundary component, i.e., if Γ is a quasicircle then $\delta(z) \in L^\infty(\partial \mathrm{CH}(\Gamma), d\mathrm{A}_\rho)$. This holds because both complementary components of a quasicircle are uniform domains [87], and thus for every $x \in \Gamma$ and $0 < r \leq \mathrm{diam}(\Gamma)$, both complementary components contain disks of diameter $\simeq r$ inside $D(x, r)$. The converse is not true, since non-quasicircles may also have $\delta(z) \in L^\infty$. Definition 15 says that the Weil-Petersson class corresponds to $\delta(z) \in L^2(\partial \mathrm{CH}(\Gamma), d\mathrm{A}_\rho)$. The condition $\delta(z) \in L^1(\partial \mathrm{CH}(\Gamma), d\mathrm{A}_\rho)$ is equivalent to $\mathrm{CH}(\Gamma)$ having finite hyperbolic volume. However, for a closed curve, this is always either zero (for lines and circles) or infinite (everything else); we leave this as an exercise.

Domes of simply connected domains need not be even C^1, e.g., the dome of two overlapping disks has a definite angle along an infinite geodesic where two hemispheres meet. But for Weil-Petersson curves the transverse bending measure cannot have an atom; this would violate Definition 15 by producing an infinite sequence of unit disks along the geodesic that all have a fixed, positive amount of bending. For the dome of a planar domain bounded by a Weil-Petersson curve, the amount of bending, $B(z)$, that lies within unit distance of $z \in S_1$ is $O(\delta(z))$. Indeed,

$$\int_S B^2(z) d\mathrm{A}_\rho < \infty \tag{5.2}$$

gives another characterization of Weil-Petersson curves. By smoothing the convex hull boundary, we can obtain a surface S with finite total curvature:

Definition 16. *$\Gamma \subset \mathbb{R}^2$ is the boundary of a smooth surface $S \subset \mathbb{H}^3$ such that $\mathcal{K}(z) \to 0$ as z tends to the boundary of hyperbolic space and*

$$\int_S |\mathcal{K}(z)|^2 d\mathrm{A}_\rho(z) < \infty, \tag{5.3}$$

where $\mathcal{K}$ is the second fundamental form of S.

By a result of Charles Epstein [44], this implies that Γ has a quasiconformal reflection satisfying (2.10) and hence Γ satisfies Definition 3. See Section 18. An alternate way to prove that Definition 15 implies Definition 16 is to show that a minimal surface in $\mathbb{H}^3$ with boundary Γ must have finite total curvature. The existence of such a minimal surface S for any closed Jordan curve in $\mathbb{R}^2$, is due to Michael Anderson

[7] and we have the estimate

$$\max(|\kappa_1(z)|, |\kappa_2(z)|) = O(\delta(z)), \tag{5.4}$$

for $z \in S \subset \mathrm{CH}(\Gamma)$ due to Andrea Seppi (see Lemma 19.1). Because of this we see that Γ is Weil-Petersson if and only if it satisfies

Definition 17. Γ *is a quasicircle that is the asymptotic boundary of an embedded minimal surface that is topologically a disk and satisfies* $\int_S \kappa^2 d\mathrm{A}_\rho < \infty$.

As noted earlier, the Gauss curvature of S satisfies $K(z) = -1 - \kappa^2(z) \leq -1$, so for a compact Jordan sub-domain Ω of S with area A and boundary length L, the isoperimetric equality for such surfaces (e.g., (4.30) of [100]) implies

$$L^2 \geq 4\pi A\chi + A^2,$$

where $\chi = \chi(\Omega)$ is the Euler characteristic of Ω. A short manipulation gives

$$L - A \geq \frac{4\pi A\chi}{L + A} \geq 4\pi\chi.$$

We shall prove $L - A1$ is bounded above iff Γ is Weil-Petersson:

Definition 18. Γ *is a closed Jordan curve and is the asymptotic boundary of a minimal surface* $S \subset \mathbb{H}^3$ *that has finite Euler characteristic and can be written as the nested unions of compact subsets* $\Omega_1 \subset \Omega_2 \subset \dots$ *such that*

$$\limsup_n [L_\rho(\partial\Omega_n) - \mathrm{A}_\rho(\Omega_n)] < \infty.$$

A special case of such compact nested subdomains is given by simply truncating the surface S at Euclidean height t above the boundary of $\mathbb{H}^3$. Define

$$S_t = S \cap \{(x, y, s) \in \mathbb{H}^3 : s > t\}, \text{ and } \partial S_t = S \cap \{(x, y, s) \in \mathbb{H}^3 : s = t\}.$$

The renormalized area of S is defined as

$$\mathcal{RA}(S) = \lim_{t \searrow 0} \left[\mathrm{A}_\rho(S_t) - L_\rho(\partial S_t)\right],$$

and we shall prove the limit always exists (possibly $-\infty$):

Definition 19. Γ *is a closed Jordan curve that is the asymptotic boundary of a minimal surface* $S \subset \mathbb{H}^3$ *with finite Euler characteristic and finite renormalized area.*

There is a discrete version of this that illustrates the connection between our Euclidean and hyperbolic conditions. Define the "dyadic cylinder"

$$X = \bigcup_{n=1}^{\infty} \Gamma_n \times [2^{-n}, 2^{-n+1}),$$

where $\{\Gamma_n\}$ are the dyadic polygonal approximations to Γ, as in Theorem 1.3. See Figures 6 and 9. X has holes, but we shall describe in Section 12 how to fill them to form a triangulated, simply connected "dyadic dome", that will work too. Theorem 1.3 is equivalent to finite renormalized area for these discrete surfaces:

Definition 20. Γ *is a closed Jordan curve and the corresponding dyadic cylinder* X *has finite renormalized area.*

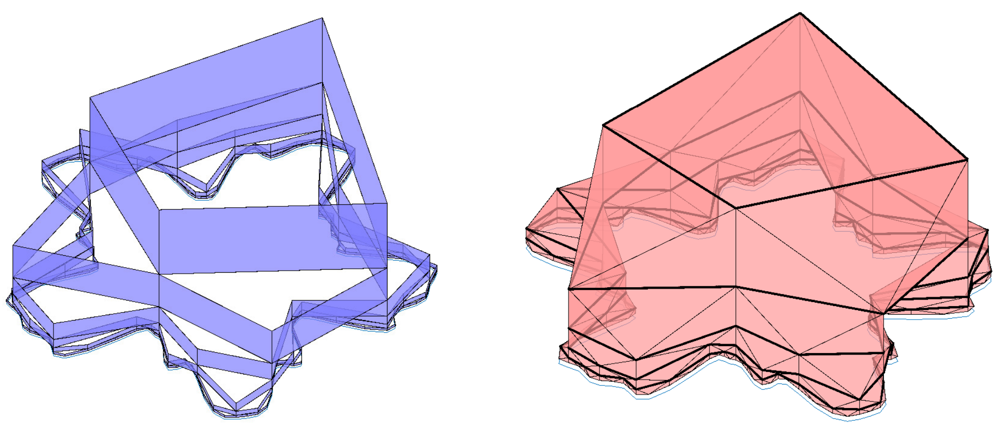

Figure 6. The dyadic cylinder and dome (same curve as Figure 4).

Although the statement of the definition is in terms of hyperbolic quantities, the proof will be entirely Euclidean: this definition is equivalent to Theorem 1.3.

6. Hyperbolic conditions in higher dimensions

Next we discuss how the definitions presented in the previous section have to be changed for curves $\Gamma \subset \mathbb{R}^n$. Definition 20 needs no change; the construction of the dyadic cylinder and dome is exactly the same and the proof that they have finite renormalized area iff Γ satisfies Definition 11 is valid in all dimensions. Definition 16 is also unchanged. Instead of smoothing a boundary component of the convex hull, we can smooth the dyadic dome instead. The proof that it implies Definition 3 using a theorem of Charles Epstein on quasiconformality of Gauss maps will be replaced by a construction of a biLipschitz involution fixing Γ and implying Definition 13.

If $\Gamma \subset \mathbb{R}^n$ is the asymptotic boundary of a minimal 2-surface in $\mathbb{H}^{n+1}$, then Definitions 17, 18 and 19 remain the same as before. In general, this need not be the case, and they each require a change of terminology, but not of concept. Anderson's result for $\mathbb{H}^3$ in [7] is replaced by his result from [6] for smooth curves in $\mathbb{R}^n$ giving the existence of a minimal 2-current. This is extended to the existence of minimal 2-chain by Fang-Hua Lin in [82] for C^1 curves, and his proof extends to $H^{3/2}$ curves. For definitions of currents see Federer's comprehensive text [49] or the more accessible [121] by Leon Simon. Brian White's paper [130] summarizes the basic definitions and results, and [90] by Frank Morgan starts with a very informative example.

In general, one cannot control the global topology of a minimal current or chain, but in [82] Lin proves that if Γ is C^1 then the there is a minimal 2-chain with asymptotic boundary Γ that agrees with a smooth surface in a neighborhood of the boundary. His proof of this only uses the C^1 assumption to deduce that near the boundary, the 2-chain is close to a vertical 2-plane, and this holds also for the curves $\Gamma \subset \mathbb{R}^n$ satisfying Definition 14. He proves that this surface is locally a Lipschitz graph with small norm with respect to this plane, and this also holds under our assumptions. Moreover, this surface is topologically an annulus and is asymptotic to the dyadic dome of Γ. Note that Lin's proof only gives that Γ is the boundary of some such 2-chain, not that every chain with asymptotic boundary Γ has this property.

Definition 17 will thus be replaced by: $\Gamma \subset \mathbb{R}^n$ is a closed Jordan curve that is the asymptotic boundary of a minimal 2-chain so that in $\{(x,t) \in \mathbb{H}^{n+1} : 0 < t < t_0\}$ agrees with an annular surface that has finite total curvature.

Definitions 18 and 19 are both changed in the obvious way, replacing the minimal surface by a minimal 2-chain or current which agrees with a surface S near the boundary. In Definition 18, we take Ω_n to be a smooth topological annulus contained in $S_t^* = S \cap \{(x,s) \in \mathbb{H}^{n+1} : s < t\}$ for $t > 0$ small enough. Definition 19 is unchanged, except that it suffices to consider the area of S between heights $0 < s < t$ for t fixed and s tending to zero and show the corresponding limit exists.

Finally, Definition 15 needs to be changed, because in higher dimensions the hyperbolic convex hull of Γ has a single boundary component and so it does not make sense to measure the thickness of the convex hull by the hyperbolic distance between the two boundary components. However, given a point $z \in \mathrm{CH}(\Gamma)$ and a tangent vector v at z it does make sense to ask how far it is from z to the boundary of $\mathrm{CH}(\Gamma)$ following a geodesic in direction w. We let $\delta(z, w)$ denote this distance. The thickness of $\mathrm{CH}(\Gamma)$ will be the supremum of these distances over different directions, but we need to avoid moving vertically or parallel to Γ. Therefore we want

$$\delta(z) = \inf_P \sup_{w \perp P} \delta(z, w),$$

where the infimum is over all tangent 2-planes at z generated by the vertical direction and one horizontal direction; the infimum will be attained when the horizontal direction is approximately parallel to Γ. With this definition, we have $\delta(z) = O(\varepsilon_\Gamma(Q))$ where $z = (x, t)$ and Q is a dyadic cube in $\mathbb{R}^n$ with $x \in Q$ and $\mathrm{diam}(Q) \simeq t$.

In Definition 15 we no longer integrate $\delta^2(z)$ over the boundary of the convex hull (the hyperbolic $(n-1)$-measure of a unit ball will be approximately δ^{n-1} instead of $\simeq 1$), but we have to integrate over some appropriate 2-surface, such as the minimal 2-chain or current described above, or the dyadic dome. In the latter case, we don't know that the dyadic dome is contained inside $\mathrm{CH}(\Gamma)$), so $\delta(z)$ as given above is not defined there, but it suffices to integrate $\delta^2(R(z))$ over the dome, where $R : \mathbb{H}^{n+1} \to \mathrm{CH}(\Gamma)$ is the nearest point retraction. We can also state the condition as $\sum_Q \delta^2(Q) < \infty$, where the sum is over dyadic cubes in $\mathbb{R}^n$ and $\delta(Q)$ is defined as the maximum of $\delta(z)$ over $\mathrm{CH}(\Gamma) \cap \mathrm{T(Q)}$, where $T(Q) = Q \times [\frac{1}{2}\ell(Q), \ell(Q)] \subset \mathbb{H}^{n+1}$.

7. Summary of definitions

For the reader's convenience, Table 1 gives a summary of the definitions from the preceding sections, as well as some definitions that are briefly discussed in Appendix A, but do not play a role our proofs. The graph in Figure 7 has vertices representing definitions and edges representing proofs; the edge labels say where the corresponding proof may be found. Our goal is to prove:

Theorem 7.1. *For curves in $\mathbb{R}^2$, Definitions (1)-(20) are equivalent. For curves in $\mathbb{R}^n$, $n \geq 3$, Definitions (6)-(20), properly modified, are all equivalent.*

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions
14	between disjoint disks
15	thickness of convex hull
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder
21	closure of smooth curves in $T_0(1)$
22	P_φ^- is Hilbert-Schmidt
23	double hits by random lines
24	finite Loewner energy
25	large deviations of SLE(0^+)
26	Brownian loop measure

TABLE 1. The definitions above the first double line are the previously known function theoretic definitions. The second group are the new definitions given in this paper. The third group consists of some other known characterizations of the Weil-Petersson class that are not used in this paper; these are briefly described in Appendix A.

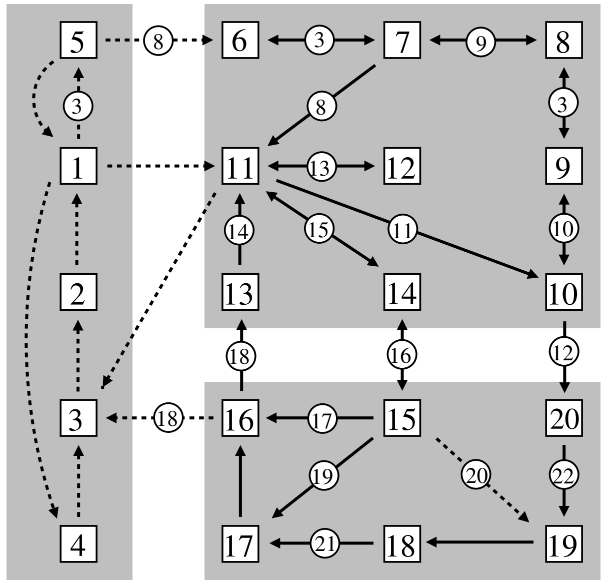

FIGURE 7. A diagram of the implications between the definitions. Each number inside a square refers to a definition given in the text and Table 1. The dashed arrows are only valid for $n = 2$, either because one of the definitions only makes sense there, or we only give the proof in that case. Numbers on an arrow indicate which section the corresponding implication is proven in; unlabled dashed edges are proven in [16] and unlabled solid edges are immediate from the definitions. The shaded blocks group definitions based on conformal maps (left), Euclidean geometry (upper right) and hyperbolic geometry (lower right).

8. (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (11): FROM FUNCTION THEORY TO β-NUMBERS

Lemma 8.1. *Definition 5 implies Definition 6.*

Proof. Suppose f is a conformal map from $\mathbb{D}$ to the bounded complementary component of Γ. Let $a : \mathbb{T} \to \Gamma$ be an orientation preserving arclength parameterization and let $\varphi = a^{-1} \circ f : \mathbb{T} \to \mathbb{T}$. We claim this circle homeomorphism is quasisymmetric. To prove this, consider to adjacent arcs I, J of the same length. Since Definition 1 is known to be equivalent to Definition 3, f has a quasiconformal extension to the whole plane, hence it is also a quasisymmetric map and this implies that $f(I)$ and $f(J)$ have comparable diameters. See [62] or Section 4 of [68]. Since we also know that Γ is chord-arc (see our remarks in Section 2), this implies that $f(I)$ and $f(J)$

have comparable lengths, hence $\varphi(I)$ and $\varphi(J)$ also have comparable lengths, since a preserves arclength. This is the definition of quasisymmetry for φ.

Note that $a' = \exp(i \arg f') \circ \varphi$. Beurling and Ahlfors proved in [12] that $H^{1/2}$ is invariant under composition with a quasisymmetric homeomorphism of $\mathbb{T}$. Thus $a' \in H^{1/2}$ iff $\exp(i \arg f) \in H^{1/2}$. Since a is Lipschitz, it is also absolutely continuous, so its weak derivative agrees with its pointwise derivative a'. Hence $a \in H^{3/2}(\mathbb{T})$. $\square$

A direct proof of the converse is given in [16]. A more roundabout proof uses the implications (6) $\Rightarrow$ (7) $\Rightarrow$ (11) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (5), which are all proven in this paper. By our remarks in Section 3, we already know that Definition 6 is equivalent to Definition 7, so we need only prove:

Lemma 8.2. *Definition 7 implies Definition 11.*

Proof. Let U be the torus $\mathbb{T} \times \mathbb{T}$ minus the diagonal. Take a Whitney decomposition of U, i.e., a covering of U by squares Q with disjoint interiors and the property that $\operatorname{diam}(Q) \simeq \operatorname{dist}(Q, \partial U)$. We will think of $\mathbb{T}$ as $[0,1]$ with its endpoints identified, and use dyadic squares in $[0,1]^2$ as elements of our decomposition. See Figure 8. Each element W_j of the decomposition can be written as $W_j = \gamma_j \times \gamma_j'$ where $\gamma_j \cup \gamma_j' = \Gamma_j \setminus \Gamma_j'$ and all these arcs have comparable lengths (in fact, γ_j and γ_j' have the same length).

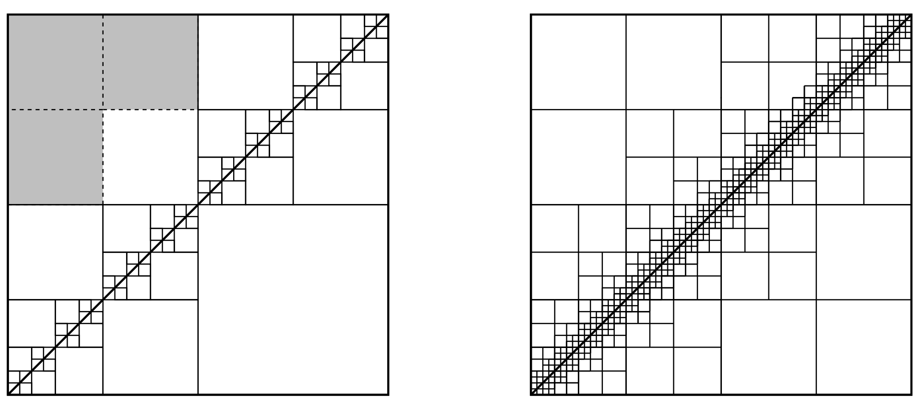

FIGURE 8. On the left is the obvious packing of $[0,1]^2$ minus the diagonal by maximal dyadic squares, but this is not a Whitney decomposition, since some squares touch the diagonal. However, if we recursively subdivide each of these squares into four sub-squares and keep the three not touching the diagonal (shaded on left), we generate the Whitney decomposition on the right.

For each Whitney piece $W_j = \gamma_j \times \gamma_j'$, choose a $w_0 \in \gamma_j'$ so that

$$\ell(\gamma_j') \int_{\gamma_j} |\tau(z) - \tau(w_0)|^2 |dz| \leq 2 \int_{\gamma_j'} \int_{\gamma_j} |\tau(z) - \tau(w)|^2 |dz||dw|.$$

(We can do this because a positive measurable function must take a value that is less than or equal to twice its average.) Let L be the line through one endpoint of γ_j' in direction $\tau(w)$. Then the maximum distance d that γ_j can attain from L satisfies

$$d \lesssim \int_{\gamma_j} |\tau(z) - \tau(w_0))||dz| \leq \left(\int_{\gamma_j} |\tau(z) - \tau(w_0)|^2 |dz| \right)^{1/2} \ell(\gamma_j)^{1/2}.$$

Therefore (using the fact that γ is chord-arc),

$$\begin{aligned} \beta^2(\gamma_j) \simeq d^2/\mathrm{diam}(\gamma_j) &\lesssim \frac{1}{\ell(\gamma_j)} \int_{\gamma_j} |\tau(z) - \tau(w_0)|^2 |dz| \\ &\leq \frac{2}{\ell(\gamma_j)^2} \int_{\gamma_j} \int_{\gamma_j'} |\tau(z) - \tau(w)|^2 |dz||dw| \\ &\lesssim \int_{\gamma_j} \int_{\gamma_j'} \left| \frac{\tau(z) - \tau(w)}{z - w} \right|^2 |dz||dw|. \end{aligned}$$

Summing over all Whitney pieces proves that the β^2-sum is finite when taken over all arcs of the form $\{\gamma_j\}$. By construction (see Figure 8), every dyadic interval in $[0,1]$ (except for $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$ and $[0,1]$) occurs as a γ_j at least once and at most three times, so this bounds the sum of $\beta^2(\gamma)$ over all dyadic subintervals of Γ for a fixed base point, with an estimate independent of the basepoint. Thus it holds for some multi-resolution family of arcs (recall the $\frac{1}{3}$-trick for making such a family from three translates of the dyadic family). Because of Lemma 4.2, this proves the lemma. $\square$

9. (7) ⇔ (8): TANGENT'S CONTROL MÖBIUS ENERGY

The following proof follows an argument in [20].

Lemma 9.1. *Definition 7 is equivalent to Definition 8.*

Proof. We want to show

$$\mathrm{M\ddot{o}b}(\Gamma) = \int_\Gamma \int_\Gamma \frac{1}{|z-w|^2} - \frac{1}{\ell(z,w)^2} |dz||dw| < \infty, \tag{9.1}$$

if and only if

$$\int_\Gamma\int_\Gamma \frac{|\tau(x)-\tau(y)|^2}{|x-y|^2}|dx||dy|<\infty. \tag{9.2}$$

First we collect a few relevant formulas about rectifiable arcs.

Suppose γ is rectifiable with endpoints z,w, and let $\tau(x)$ denote the unit tangent vector at $x\in\gamma$ (well defined almost everywhere on γ; we assume γ is oriented from w to z). Then

$$u=\frac{z-w}{|z-w|}=\frac{1}{|z-w|}\int_\gamma \tau(y)|dy|,$$

is the unit vector in direction $z-w$ and hence

$$|z-w|^2 = |z-w|\int_\gamma\langle\tau(x),u\rangle|dx| = \int_\gamma\int_\gamma\langle\tau(x),\ \tau(y)\rangle|dy||dx|. \tag{9.3}$$

Next, using $|\tau|=1$, we get

$$\begin{aligned}\int_\gamma\int_\gamma|\tau(x)-\tau(y)|^2|dx||dy| &= \int_\gamma\int_\gamma\langle\tau(x)-\tau(y),\tau(x)-\tau(y)\rangle|dx||dy|\\ &= \int_\gamma\int_\gamma\left(|\tau(x)|^2-2\langle\tau(y),\tau(x)\rangle+|\tau(y)|^2\right)|dx||dy|\\ &= 2\ell(\gamma)^2-2\int_\gamma\int_\gamma\langle\tau(x),\tau(y)\rangle|dx||dy|.\end{aligned}$$

Let $\gamma=\gamma(z,w)\subset\Gamma$ be the shorter sub-arc with end points z,w. Combining the equality above with (9.3) and the assumption that Γ is chord-arc, we get

$$\begin{aligned}\text{Möb}(\Gamma) &= \int_\Gamma\int_\Gamma\frac{\ell(z,w)^2-|z-w|^2}{\ell(z,w)^2|z-w|^2}|dz||dw|\\ &\simeq \int_\Gamma\int_\Gamma\frac{\ell(z,w)^2-\int_\gamma\int_\gamma\langle\tau(x),\tau(y)\rangle|dx||dy|}{|z-w|^4}|dz||dw|\\ &= \frac{1}{2}\int_\Gamma\int_\Gamma\frac{\int_\gamma\int_\gamma|\tau(x)-\tau(y)|^2|dx||dy|}{|z-w|^4}|dz||dw|\end{aligned}$$

Given $x,y\in\Gamma$ with $\ell(x,y)\le\frac{1}{8}\ell(\Gamma)$, set $\sigma(x,y)=\{(z,w)\in\Gamma\times\Gamma : x,y\in\gamma(z,w)\}$. If, in addition, $0<t<\ell(\Gamma)/2$, let $\sigma(x,y,t)\subset\Gamma$ be the arc of length t with one endpoint x that is disjoint from the arc $\gamma(x,y)$. Using the fact that Γ is chord-arc, it is not hard to show that if $m\ge2$, $t\in[\ell(x,y),\mathrm{diam}(\Gamma/8)]$, and $w\in\sigma(y,x,t)$, then

$$\int_{\sigma(x,y,t)}\frac{|dz|}{|z-w|^m}\simeq\frac{1}{|x-w|^{m-1}} \tag{9.4}$$

(hint: divide the integral using the annuli $\{z : 2^n|w-x| < |z-w| \le 2^{n+1}|w-x|\}$). Set $s = \ell(\Gamma)/2$ and set $t = \ell(\Gamma)/8$. Note that

$$\ell(x,y) \le t, w \in \sigma(y,x,t), z \in \sigma(x,y,t) \quad \Rightarrow \quad (z,w) \in \sigma(x,y)$$

$$\ell(x,y) \le t, (z,w) \in \sigma(x,y) \quad \Rightarrow \quad z \in \sigma(x,y,s) \text{ and } w \in \sigma(y,x,s).$$

Let $\Sigma(t) = \{(x,y) \in \Gamma \times \Gamma : \ell(x,y) \le t\}$. By the first implication and Fubini's theorem,

$$\begin{aligned}
&\iint_{\Gamma\times\Gamma} |\tau(x)-\tau(y)|^2 \iint_{(z,w)\in\sigma(x,y)} \frac{|dz||dw|}{|z-w|^4}|dx||dy| \\
&\qquad \ge \iint_{\Sigma(t)} |\tau(x)-\tau(y)|^2 \iint_{(z,w)\in\sigma(x,y)} \frac{|dz||dw|}{|z-w|^4}|dx||dy| \\
&\qquad \gtrsim \iint_{\Sigma(t)} |\tau(x)-\tau(y)|^2 \int_{z\in\sigma(x,y,t)} \int_{w\in\sigma(y,x,t)} \frac{|dz||dw|}{|z-w|^4}|dx||dy|
\end{aligned}$$

and using (9.4),

$$\begin{aligned}
&\simeq \iint_{\Sigma(t)} |\tau(x)-\tau(y)|^2 \int_{w\in\sigma(y,x,t)} \frac{|dw|}{|x-w|^3}|dx||dy| \\
&\simeq \iint_{\Sigma(t)} \frac{|\tau(x)-\tau(y)|^2}{|x-y|^2}|dx||dy|.
\end{aligned}$$

Bounding this restricted integral suffices to show Definition 8 implies Definition 7 since the integral over the remaining, well separated, points is obviously bounded.

Similarly, to prove Möb(Γ) is finite, it suffices to evaluate the energy integral (9.1) over $\Sigma(t)$. A calculation similar to the one above gives

$$\begin{aligned}
&\iint_{\Sigma(t)} |\tau(x)-\tau(y)|^2 \iint_{(z,w)\in\sigma(x,y))} \frac{|dz||dw|}{|z-w|^4}|dx||dy| \\
&\qquad \lesssim \iint_{\Sigma(t)} |\tau(x)-\tau(y)|^2 \int_{z\in\sigma(x,y,s)} \int_{w\in\sigma(y,x,s)} \frac{|dz||dw|}{|x-w|^4}|dx||dy| \\
&\qquad \simeq \iint_{\Sigma(t)} |\tau(x)-\tau(y)|^2 \int_{w\in\sigma(y,x,s)} \frac{|dw|}{|x-w|^3}|dx||dy| \\
&\qquad \simeq \iint_{\Sigma(t)} \frac{|\tau(x)-\tau(y)|^2}{|x-y|^2}|dx||dy|. \quad \square
\end{aligned}$$

10. (9) $\Leftrightarrow$ (10): CONTINUOUS AND DISCRETE ASYMPTOTIC SMOOTHNESS

Lemma 10.1. *Definition 9 is equivalent to Definition 10.*

Proof. Without loss of generality we may rescale Γ so that is has length 1. We identify $\Gamma \times \Gamma$ with the torus $\mathbb{T}^2 = [0,1]^2$, let U be the torus minus the diagonal, and take a Whitney decomposition of U by dyadic squares $\{Q_j\}$ as in the proof of Lemma 8.2. Elements of the decomposition are denoted $\{W_j\}$, and each is a product of dyadic arcs $W_j = \gamma_j \times \gamma_j'$. For each W_j, we can write $\gamma_j \cup \gamma_j' = \Gamma_j \setminus \Gamma_j'$ for arcs Γ_j, Γ_j' so that all four arcs have comparable lengths.

Recall that $\mathrm{crd}(\gamma) = |z-w|$ where z, w are the endpoints of γ and that $\Delta(\gamma) \equiv \ell(\gamma) - \mathrm{crd}(\gamma)$. We sometimes write $\Delta(z,w)$ for $\Delta(\gamma)$ when γ has endpoints z, w, and it is clear from context which arc connecting these points we mean. We say two subarcs of Γ are adjacent if they have disjoint interiors, but share a common endpoint.

Lemma 10.2. *If $\gamma, \gamma' \subset \Gamma$ are adjacent, then $\Delta(\gamma) + \Delta(\gamma') \leq \Delta(\gamma \cup \gamma')$.*

Proof. Note that $\ell(\gamma \cup \gamma') = \ell(\gamma) + \ell(\gamma')$, and $\mathrm{crd}(\gamma \cup \gamma') \leq \mathrm{crd}(\gamma) + \mathrm{crd}(\gamma')$, so

$$\begin{aligned}\Delta(\gamma \cup \gamma') &= \ell(\gamma \cup \gamma') - \mathrm{crd}(\gamma \cup \gamma') \\ &\geq \ell(\gamma) + \ell(\gamma') - \mathrm{crd}(\gamma) - \mathrm{crd}(\gamma') = \Delta(\gamma) + \Delta(\gamma'). \quad \square\end{aligned}$$

Corollary 10.3. *If $\gamma \subset \gamma'$ then $\Delta(\gamma) \leq \Delta(\gamma')$.*

Now, fix j and consider the Whitney box $W_j = \gamma_j \times \gamma_j'$. If $\gamma \subset \Gamma_j$ is any arc with one endpoint in γ_j and the other in γ_j' then $\Gamma_j' \subset \gamma \subset \Gamma_j$, and hence $\Delta(\Gamma_j') \leq \Delta(\gamma) \leq \Delta(\Gamma_j)$. Because Γ is chord-arc, if $z \in \gamma_j'$ and $w \in \gamma_j$, then $|z-w| \gtrsim \ell(\Gamma_j') \simeq \ell(\Gamma_j)$. We can therefore write the integral from Definition 9 as

$$\begin{aligned}\int_\Gamma \int_\Gamma \frac{\ell(z,w) - |z-w|}{|z-w|^3} |dz||dw| &= \sum_j \int_{W_j} \frac{\Delta(z,w)}{|z-w|^3} |dz||dw| \\ &\lesssim \sum_j \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)^3} \ell(\Gamma_j)^2 = \sum_j \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)}.\end{aligned}$$

Reversing the argument, now assume Γ_j' is some dyadic subinterval of Γ and let γ_j, γ_j' be the equal length dyadic arcs adjacent to Γ_j'.

$$\int_{\gamma_j} \int_{\gamma_j'} \frac{\ell(z,w) - |z-w|}{|z-w|^3} |dz||dw| \quad \gtrsim \quad \frac{\Delta(\Gamma_j')}{\ell(\Gamma_j')}.$$

The squares $W_j = \gamma_j \times \gamma'_j$ arising in this way have bounded overlap, so

$$\int_\Gamma \int_\Gamma \frac{\ell(z,w) - |z-w|}{|z-w|^3} |dz||dw| \gtrsim \sum_j \frac{\Delta(\Gamma'_j)}{\ell(\Gamma'_j)},$$

where the sum is over all dyadic subintervals of Γ. This works for any dyadic decomposition $\{\Gamma_j\}$ of Γ, and hence for a multi-resolution family. This gives the equivalence of Definitions 9 and 10. □

11. (11) ⇐ (10): β^2-SUM IMPLIES ASYMPTOTIC SMOOTHNESS

The following us where we use Theorem 1.5, the strengthening of Peter Jones's traveling salesman theorem mentioned in the introduction.

Lemma 11.1. *Definition 11 implies Definition 10.*

Proof. We continue using the notation from the previous section. Let $\{\Gamma_j\}$ be a dyadic decomposition of Γ. For each j, choose a dyadic cube Q_j that hits Γ_j and has diameter between $\mathrm{diam}(\Gamma_j)$ and $2 \cdot \mathrm{diam}(\Gamma_j)$. Note that any such dyadic square can only be associated to a uniformly bounded number of arcs Γ_j in this way, because there are only a bounded number of arcs Γ_j that have the correct size and are close enough to Q_j; this uses the fact that Γ is chord-arc. Because Γ is chord-arc, $\mathrm{diam}(\Gamma_j) \simeq \ell(\Gamma_j) \simeq \mathrm{diam}(Q_j)$. Therefore, by the strengthened TST (1.5)

$$\Delta(\Gamma_j) \simeq \sum_{Q \subset 3Q_j} \beta^2_{\Gamma_j}(Q) \ell(Q).$$

Since $\beta_{\Gamma_j}(Q) \leq \beta_\Gamma(Q)$, we get

$$\begin{aligned}
\sum_j \frac{\Delta_j}{\ell(\Gamma_j)} &\simeq \sum_j \sum_{Q \subset 3Q_j} \beta^2_{\Gamma_j}(Q) \frac{\ell(Q)}{\ell(Q_j)} \\
&\lesssim \sum_j \sum_{Q \subset 3Q_j} \beta^2_\Gamma(Q) \frac{\ell(Q)}{\ell(Q_j)} \simeq \sum_Q \beta^2_\Gamma(Q) \cdot \sum_{j: Q \subset 3Q_j} \frac{\ell(Q)}{\ell(Q_j)}.
\end{aligned}$$

Note that for each Q with $\mathrm{diam}(Q) \leq \mathrm{diam}(\Gamma)$ and $Q \cap \Gamma \neq \emptyset$, there is a cube of the form Q_j from above, that has diameter comparable to $\mathrm{diam}(Q)$ and such that $Q \subset 3Q_j$. Moreover, there there can only be a uniformly bounded number of dyadic squares Q_j of a given size so that $3Q_j$ contains Q, so each Q_j can only be chosen a bounded number of times. Thus the sum over the j's in the last line above is bounded by a multiple of a geometric series and so is uniformly bounded. Thus

$$\sum_j \frac{\Delta(\Gamma_j)}{\ell(Q_j)} \lesssim \sum_Q \beta_\Gamma^2(Q). \quad \square \tag{11.1}$$

12. (10) $\Leftrightarrow$ (20): DYADIC CYLINDERS AND DOMES

Next we introduce a discrete version of a surface in $\mathbb{H}^{n+1}$ with asymptotic boundary Γ and show that Γ is Weil-Petersson iff this surface has finite renormalized area. If $\Gamma \subset \mathbb{R}^n$ is rectifiable and $Y = \Gamma \times (0,1] \subset \mathbb{H}^{n+1}$, then

$$\mathrm{A}_\rho(Y_t) = \int_t^1 \int_\Gamma \frac{dsdt}{t^2} = \ell(\Gamma)(\frac{1}{t} - 1) = \ell(\Gamma_t)(\frac{1}{t} - 1) = L_\rho(\Gamma_t) - O(1),$$

so the cylinder Y always has finite renormalized area. Roughly speaking, we expect renormalized area to measure how orthogonal the surface is to the boundary.

Define a "dyadic cylinder" associated to Γ by $X = \bigcup_{n=0}^\infty \Gamma_n \times (2^{-n-1}, 2^{-n}]$, where Γ_n is the 2_n-gon inscribed in Γ corresponding to a dyadic decomposition of Γ into subarcs of length $2^{-n}\ell(\Gamma)$. Note that each "layer" of X between heights 2^{-n} and 2^{-n+1} consists of 2^n Euclidean rectangles (or "panels") in vertical planes that meet along vertical edges (called "hinges"). See Figure 12.1. Alternate vertices of the top edge of one layer agree with the bottom vertices of the next layer up, but there are triangular horizontal "holes" between the layers.

These holes can be eliminated as followed. Suppose $[z, w]$ is an edge segment of Γ_n and let $a_1, a_2 \in \mathbb{H}^3$ be the points of height 2^{-n} and 2^{-n-1} above z. Similarly b_1, b_2 above w. Let v be the vertex of Γ_{n+1} between z and w and let c_2 be the point at height 2^{-n-1} above v. The rectangular face of the dyadic cylinder X with corners a_1, a_2, b_2, b_1 is replaced by the three Euclidean triangles with vertices (a_1, b_1, c_2), (c_2, b_2, b_1) and (a_1, c_2, b_1). Doing this for every edge of Γ_n and adding the interior of the polygon Γ_2 raised to height $1/4$ defines a closed surface Y that we will call a dyadic dome of Γ. See Figure 9 for two examples of dyadic cylinders and the corresponding domes.

Lemma 12.1. *If Γ a closed rectifiable Jordan curve, then Γ is Weil-Petersson if and only if every corresponding dyadic cylinder X has finite renormalized area.*

Proof. First we show that the Weil-Petersson condition implies finite renormalized area. A simple calculation as above shows that the part of X between heights 2^{-n}

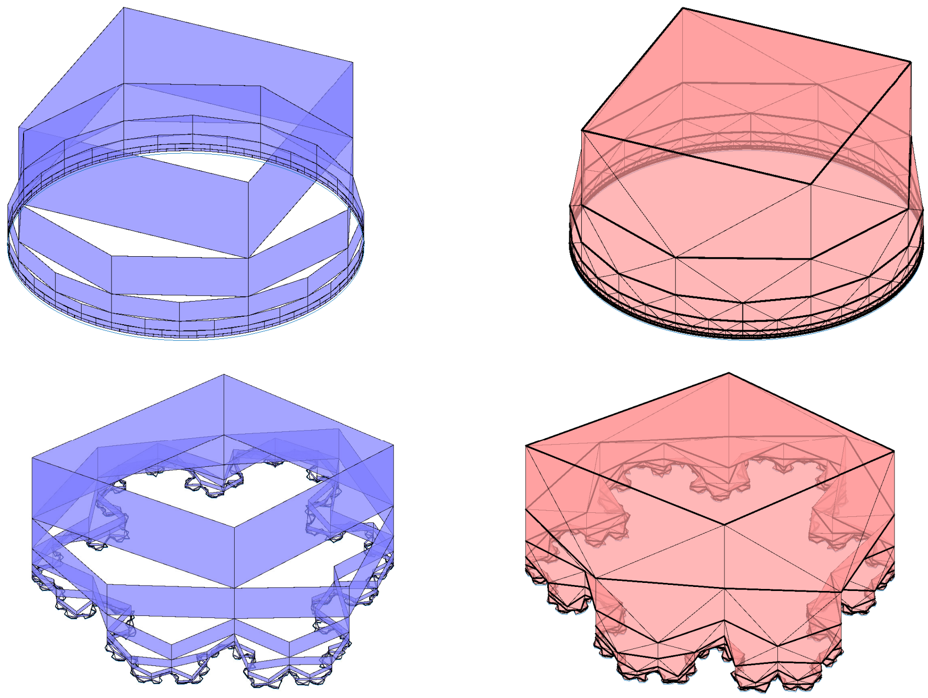

FIGURE 9. The dyadic cylinder and dome of a circle and snowflake; the first has finite renormalized area and the second does not.

and 2^{-n+1} has hyperbolic area $2^{n-1}\ell(\Gamma_n)$. Similarly, if $2^{-n-1} \le t \le 2^{-n}$, then

$$\mathrm{A}_\rho(X_t) = \sum_{k=0}^{n} 2^{k-1}\ell(\Gamma_k) + (\frac{1}{t} - 2^n)\ell(\Gamma_{n+1}).$$

and hence

$$\begin{aligned}
\mathrm{A}_\rho(X_t) - \frac{1}{t}\ell(\Gamma) &= \mathrm{A}_\rho(X_t) - (\frac{1}{t} - 2^n + 1 + \sum_{k=1}^{n} 2^{k-1})\ell(\Gamma) \\
&= -\ell(\Gamma) - \sum_{k=1}^{n} 2^k[\ell(\Gamma) - \ell(\Gamma_k)] + (\frac{1}{t} - 2^n)(\ell(\Gamma) - \ell(\Gamma_{n+1}) \\
&= -\ell(\Gamma) - \sum_{k=1}^{n} 2^k[\ell(\Gamma) - \ell(\Gamma_k)] + O(2^n[\ell(\Gamma) - \ell(\Gamma_{n+1})]) \\
&\to -\ell(\Gamma) - \sum_{k=1}^{\infty} 2^k[\ell(\Gamma) - \ell(\Gamma_k)]
\end{aligned}$$

since the infinite series is convergent when Γ is Weil-Petersson by (1.2). Finally, for $2^{-n-1} \leq t \leq 2^{-n}$, note that $\ell(\partial X_t) = \ell(\Gamma_{n+1})/t$, so

$$\frac{1}{t}[\ell(\partial X_t) - \ell(\Gamma)] \leq 2^{n+1}[\ell(\Gamma_{n+1}) - \ell(\Gamma)] \to 0,$$

since these are terms of a summable series. Thus $\mathrm{A}_\rho(X_t) - L_\rho(\partial X_t)$ has a finite limit and X has finite renormalized area.

Next we consider the converse: finite renormalized area implies Γ is Weil-Petersson. Suppose $\mathcal{RA}(X) < \infty$. First we deduce that Γ is rectifiable. If $t = 2^{-n}$, then

$$\mathrm{A}_\rho(X_t) - L_\rho(\partial X_t) = \left(\sum_{k=1}^{n} 2^{k-1}\ell(\Gamma_k)\right) - 2^n \ell(\Gamma_n) = O(1),$$

or equivalently,

$$\ell(\Gamma_n) = \frac{1}{2}\ell(\Gamma_n) + \frac{1}{4}\ell(\Gamma_{n-1}) + \cdots + 2^{-n}\ell(\Gamma_1) + O(2^{-n}),$$

and hence (since $\{\ell(\Gamma_n)\}$ is non-decreasing),

$$\begin{aligned} \ell(\Gamma_n) &= \frac{1}{2}\ell(\Gamma_{n-1}) + \frac{1}{4}\ell(\Gamma_{n-2}) + \cdots + O(2^{-n}) \\ &\leq \frac{1}{2}\ell(\Gamma_{n-1}) + \frac{1}{4}\ell(\Gamma_{n-1}) + \cdots + O(2^{-n}) \\ &\leq \ell(\Gamma_{n-1}) + O(2^{-n}) \end{aligned}$$

which clearly implies $\ell(\Gamma) < \infty$. To show that Γ is Weil-Petersson, note that

$$\begin{aligned} \mathrm{A}_\rho(X_t) - L_\rho(\partial X_t) &= \left(\sum_{k=1}^{n} 2^{k-1}\ell(\Gamma_k)\right) - 2^n \ell(\Gamma_n) \\ &= \left(\sum_{k=1}^{n} 2^{k-1}\ell(\Gamma_k)\right) - (1 + 1 + 2 + \ldots 2^{n-1})\ell(\Gamma_n) \\ &= -\frac{1}{2}\sum_{k=1}^{n} 2^k[\ell(\Gamma_n) - \ell(\Gamma_k)] - \ell(\Gamma_n). \end{aligned}$$

By the Monotone Converge Theorem (for counting measure on $\mathbb{N}$), this tends to

$$-\tfrac{1}{2}\textstyle\sum_{k=1}^{\infty} 2^k[\ell(\Gamma) - \ell(\Gamma_k)] - \ell(\Gamma).$$

Thus if $\mathrm{A}_\rho(X_t) - L_\rho(\partial X_t)$ is bounded below, then

$$\sum_{k=1}^{\infty} 2^k[\ell(\Gamma) - \ell(\Gamma_k)] < \infty,$$

with a bound independent of the choice of the dyadic decomposition. Hence finite renormalized area implies Γ is Weil-Petersson by Theorem 1.3. $\square$

It is not hard to show that the dyadic dome has finite renormalized area iff the dyadic cylinder does, by considering a horizontal projection between the surfaces that changes hyperbolic area and lengths by at most a bounded additive factor. A similar argument will be used in Section 22 to show that Weil-Petersson curves bound minimal surfaces with finite renormalized area.

13. (11) $\Leftrightarrow$ (12): β's and Menger curvature are equivalent

In this section we prove that Definitions 11 and 12 are equivalent. The necessary estimates are contained in Pajot' book [101]; we will just indicate where to find them.

We start with bounding Menger curvature by the β's. This is contained in the proof of Theorem 31 of [101]. In this proof, we will take μ to be arclength measure on Γ; this satisfies the linear growth condition of Theorem 31 because Γ is chord-arc. Pajot defines

$$c^2(\mu) = \int_\Gamma \int_\Gamma \int_\Gamma c^2(x,y,z) d\mu(x) d\mu(y) d\mu(z),$$

and on the bottom of page 37 notes that

$$c^2(\mu) \le 3\overline{c}^2(\mu) \tag{13.1}$$

where

$$\overline{c}^2(\mu) = \int_A c^2(x,y,z) d\mu(x) d\mu(y) d\mu(z),$$

$$A = \{(x,y,z) \in \Gamma \times \Gamma \times \Gamma : |x-z| \le |x-y|, |y-z| \le |x-y|\}.$$

He states that

$$\overline{c}^2(\mu) \le \sum_Q \int_{(x,z)\in 3Q} \left(\sum_{R \subset Q} \int_{x,y \in \widetilde{R}} c^2(x,y,z) d\mu(y) \right) d\mu(x) d\mu(z).$$

where the inner sum is over dyadic sub-cubes $R \subset Q$ and

$$\widetilde{R} = \{(x,y) \in 3R : |x-y| \ge \mathrm{diam}(R)/3\}.$$

Recall that $\ell(x,y,z) = |x-y| + |y-z| + |z-y|$ is defined as the perimeter of the triangle with vertices (x,y,z), and it is comparable to the longest of the three sides.

Note that for $(x,y,z) \in A$ and $(x,y) \in \widetilde{R}$, we have $\ell(x,y,z) \simeq |x-y| \simeq \operatorname{diam}(R)$. Thus we can replace (13.1) by

$$\begin{aligned}\int_\Gamma \int_\Gamma \int_\Gamma & \frac{c^2(x,y,z)}{\ell(x,y,z)} d\mu(x) d\mu(y) d\mu(z) \\ &\lesssim \sum_Q \int_{x,z\in 3Q} \left(\sum_{R\subset Q} \int_{x,y\in \widetilde{R}} \frac{c^2(x,y,z)}{\ell(x,y,z)} d\mu(y)\right) d\mu(x) d\mu(z) \\ &\simeq \sum_Q \int_{x,z\in 3Q} \left(\sum_{R\subset Q} \int_{x,y\in \widetilde{R}} \frac{c^2(x,y,z)}{\operatorname{diam}(R)} d\mu(y)\right) d\mu(x) d\mu(z).\end{aligned}$$

We now follow the rest of the proof on page 38, replacing the factor $\operatorname{diam}(R)^{-2}$ that occurs throughout by $\operatorname{diam}(R)^{-3}$. At the end we obtain

$$\int_\Gamma \int_\Gamma \int_\Gamma \frac{c^2(x,y,z)}{\ell(x,y,z)} d\mu(x) d\mu(y) d\mu(z) \quad \lesssim \quad \sum_Q \beta^2(Q).$$

Thus Definition 11 implies Definition 12, as desired.

Next we deal with the opposite inequality: bounding $\sum \beta^2$ in terms of the Menger curvature. The relevant estimates are given in the proof of Theorem 38 of [101]. On the bottom of page 43 Pajot gives the inequality

$$\beta_\Gamma^2(Q)\operatorname{diam}(Q) \lesssim \sum_{P\subset Q} \int_{P^*}\int c^2(x,y,z) d\mu(x) d\mu(y) d\mu(z) \left(\frac{\operatorname{diam}(P)}{\operatorname{diam}(Q)}\right)^{1/2},$$

where

$$P^* = \{(x,y,z) \in (3P)^3 : |x-y| \simeq |x-z| \simeq |y-z| \simeq \operatorname{diam}(P)\}.$$

Divide both sides by $\operatorname{diam}(Q)$ and note that for $(x,y,z) \in P^*$ we have $\ell(x,y,z) \simeq \operatorname{diam}(P)$. This gives

$$\begin{aligned}\beta_\Gamma^2(Q) \quad &\lesssim \quad \sum_{P\subset Q} \int_{3P}\int \frac{c^2(x,y,z)}{\operatorname{diam}(Q)} d\mu(x) d\mu(y) d\mu(z) \left(\frac{\operatorname{diam}(P)}{\operatorname{diam}(Q)}\right)^{1/2} \\ &\lesssim \quad \sum_{P\subset Q} \int_{3P}\int \frac{c^2(x,y,z)}{\ell(x,y,z)} d\mu(x) d\mu(y) d\mu(z) \left(\frac{\operatorname{diam}(P)}{\operatorname{diam}(Q)}\right)^{1/2}\end{aligned}$$

On the top of the next page, this modified expression leads to

$$\sum_{S\subset Q} \beta_\Gamma^2(S) \lesssim \int_Q \int_Q \int_Q \frac{c^2(x,y,z)}{\ell(x,y,z)} d\mu(x) d\mu(y) d\mu(z).$$

Since $d\mu$ is arclength measure, this shows Definition 12 implies Definition 11.

14. (13) $\Rightarrow$ (11): REFLECTIONS CONTROL β'S

Lemma 14.1. *A map $R : U \to U'$ satisfying Definition 13 is biLipschitz on U.*

Proof. Suppose $z, w \in U$, and and $|z - w| \leq 3 \max(\mathrm{dist}(z, \Gamma), \mathrm{dist}(w, \Gamma))$. Without loss of generality we may assume $\mathrm{dist}(z, \Gamma) \geq \mathrm{dist}(w, \Gamma)$. Let S be the segment between z and w. Then $|R(z) - R(w)| \leq \ell(R(S))$. The segment S may hit Γ, but R is the identity at such points, and $S \setminus \Gamma$ consists of at most countably many open subsegments, each covered by its intersection with Whitney cubes Q for $\mathbb{R}^n \setminus \Gamma$. The length of each such intersection is increased by at most a factor of $\rho(Q)$. Therefore,

$$|R(z) - R(w)| - |z - w| \lesssim \sum_{Q \cap S \neq \emptyset} \rho(Q) \mathrm{diam}(Q)),$$

where the sum is over all Whitney cubes that hit S. By the Cauchy-Schwarz inequality, the right side is less than

$$\begin{aligned}
&\lesssim \left(\sum_{Q \cap S \neq \emptyset} \rho^2(Q) \mathrm{diam}(Q) \right)^{1/2} \left(\sum_{Q \cap S \neq \emptyset} \mathrm{diam}(Q) \right)^{1/2} \\
&\lesssim \left(\ell(S) \sum_{Q \cap S \neq \emptyset} \rho^2(Q) \mathrm{diam}(Q) \right)^{1/2} \\
&\lesssim \left(\ell(S) \sum_{Q \subset 3Q'} \rho^2(Q) \mathrm{diam}(Q) \right)^{1/2}.
\end{aligned}$$

Let Q' be the minimal dyadic cube containing w with $\ell(Q') \geq 6\mathrm{dist}(z, \Gamma)$ and define

$$P(Q') = \left(\frac{1}{\mathrm{diam}(Q')} \sum_{Q \subset 3Q'} \rho^2(Q) \mathrm{diam}(Q) \right)^{1/2},$$

where we sum over Whitney cubes inside $3Q'$. This gives

$$|R(z) - R(w)| - |z - w| \lesssim P(Q') \mathrm{diam}(Q'),$$

and since $P(Q') \leq (\sum_Q \rho^2(Q))^{1/2} < \infty$, we get $|R(z) - R(w)| = O(|z - w|)$ for all $z, w \in U$ with $|z - w| \leq 3\mathrm{dist}(z, \Gamma)$. Reversing the roles of z and w gives the same estimate when $|z - w| \leq 3\mathrm{dist}(w, \Gamma)$. When $|z - w| \geq 3 \max(\mathrm{dist}(z, \Gamma), \mathrm{dist}(w, \Gamma)$ we can choose $z', w'\Gamma$, with $|z - z'| = \operatorname{dist}(z, \Gamma)$ and similarly for w, w' and since z', w' are fixed by R we have

$$|R(z) - R(w)| \leq |R(z) - z'| + |z' - w'| + |w' - R(w)| \lesssim |z - w|.$$

Thus R is Lipschitz. Since $R = R^{-1}$ is an involution, it is automatically biLipschitz. $\square$

Lemma 14.2. *Definition 13 implies Definition 11.*

Proof. First note that

$$\begin{aligned} \sum_{Q'} P^2(Q') &= \sum_{Q'} \sum_{Q \subset 3Q'} \rho^2(Q) \frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')} \\ &= \sum_{Q} \rho^2(Q) \sum_{Q': Q \subset 3Q'} \frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')} \lesssim \sum_{Q} \rho^2(Q), \end{aligned}$$

since the sum over Q' only involves $O(1)$ cubes of each size. Thus it suffices to show that $\beta(Q') = O(P(Q'))$. Normalize so $\ell(Q') = 1$. Choose two points $p, q \in \Gamma \cap 3Q'$ with $|p-q| \simeq 1$ and let L be the line through p and q. Choose $w \in \Gamma \cap 3Q'$ Now choose w to maximize the distance on $\Gamma \cap 3Q'$ from L. Let $\beta = \operatorname{dist}(w, L0$. It suffices to show that $\beta = O(P(Q'))$. We may fix a large $M < \infty$ and assume that $P(Q') \leq 1/M^2$ and $MP(Q') \leq \beta \leq 1/M$, for otherwise there is nothing to do. We will show this gives a contradiction if M is large enough.

Let w' be the closest point on L to w and let z be the point on the ray from w' through w so that $\operatorname{dist}(z, L) = \frac{1}{2}\ell(Q')$. Let Q be the Whitney square for $\mathbb{R}^n \setminus \Gamma$ containing z and let $z' = R(z)$. Note that the p, q, w, w', z, z' all lie in a three dimensional sub-space, so, without loss of generality, we may assume L is the z-axis in $\mathbb{R}^3$, $w' = 0$, $w = (\beta, 0, 0)$, and $z = (1, 0, 0)$. The points p, q satisfy $|p| \simeq |q| \simeq |p - q| \simeq 1$. Since z and z' are the same distance from each of these points, up to a factor of $O(P(Q'))$, we deduce z' lies inside a $O(P(Q'))$ neighborhood of the circle $x^2 + y^2 = 1$ in the xy-plane. See Figure 10.

Similarly, since z and z' are equidistant from w, up to a factor of $O(P(Q'))$, the points z' lies within a $O(P(Q'))$ neighborhood of the sphere of radius $1 - \beta$ around z. However, since $P(Q') \ll \beta \ll 1$, these two regions only intersect in the half-space $\{x > 0\}$ and thus z' also lies in this half-space. Thus $q = (z + z')/2$ has x-coordinate $\geq 1/2$ and, by the definition of ρ is within $\rho(Q)$ of a point $q' \in \Gamma$. But

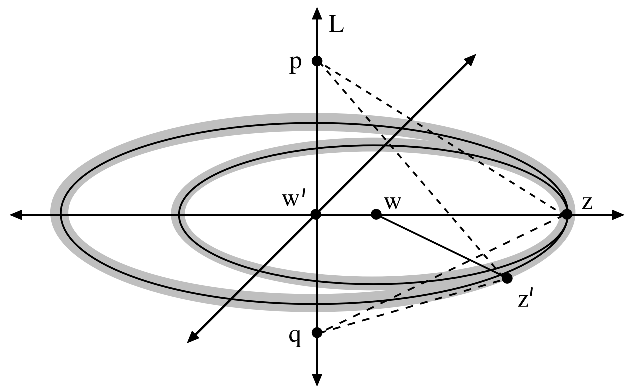

FIGURE 10. Proof that $\beta = O(P)$. The two points z, z' cannot be almost equidistant from p, p' and w without their average being far from from L, contradicting how these points were all chosen.

$\rho(Q) \lesssim P(Q') \ll 1$ (since it is one of the cubes in the sum defining $P(Q')$). This implies there is a point q' of Γ that is about unit distance from L, contradicting the assumption that the maximum distance was $\beta \leq 1/M \ll 1$. Thus $\beta(Q') \leq M \cdot P(Q')$, as desired, and we have proven that Definition 13 implies Definition 11. □

15. (11) $\Leftrightarrow$ (14): β_Γ IS EQUIVALENT TO ε_Γ

Recall that for a dyadic cube Q, $\varepsilon_\Gamma(Q)$ is the infimum of $\epsilon \in (0, 1]$ so that $3Q$ hits a line L, a ball B of radius $\mathrm{diam}(Q)/\epsilon$, that B attains its minimum distance $\leq \epsilon$ from L at a point $z \in Q$, and so that every rotation of B around L is disjoint from Γ.

Lemma 15.1. *Definition 11 is equivalent to Definition 14.*

Proof. It is easy to see that $\beta_\Gamma(Q) \lesssim \varepsilon_\Gamma(Q)$, but the reverse direction can certainly fail for a single square Q. However, we shall prove that the sum of $\varepsilon_\Gamma^2(Q)$ over all dyadic squares is bounded iff the sum of $\beta_\Gamma^2(Q)$ is.

Fix $x \in \Gamma$ and a dyadic cube Q_0 containing x with $\mathrm{diam}(Q_0) \leq \mathrm{diam}(\Gamma)$, for some $N \geq 10$. Renormalize so $\mathrm{diam}(Q_0) = 1$. For $k \geq 1$, let Q_k be the dyadic cube containing Q_0 and with diameter $\mathrm{diam}(Q_k) = 2^k \mathrm{diam}(Q_0)$. Let

$$\epsilon = 2A \sum_{k=1}^{\infty} 2^{-k} \beta_\Gamma(Q_k) = 2A \sum_{Q': Q \subset Q'} \beta_\Gamma(Q_k) \frac{\mathrm{diam}(Q)}{\mathrm{diam}(Q')},$$

where the constant $0 < A < \infty$ will be chosen later. I claim that $\varepsilon_\Gamma(Q) \lesssim \epsilon$.

To prove this, we construct a large ball B whose rotations are disjoint from Γ. Let L be a line through x that minimizes in the definition of $\beta_\Gamma(Q_0)$. Let $L^\perp$ be the perpendicular hyperplane through x and let $z \in L^\perp$ be distance $1/\epsilon$ from x. Let $B = B(z,r)$ where $r = (1/\epsilon) - \epsilon$. Then $\mathrm{dist}(B, L) = \epsilon$ and for $0 \le n \le N = \lfloor \log_2 \frac{1}{\epsilon} \rfloor$, simple trigonometry shows that $\mathrm{dist}(B \setminus 3Q_n, L) \ge C_1 \epsilon 2^{2n}$ (we can do the calculation in the plane generated by L and z; see Figure 11 and recall that $\mathrm{diam}(Q_0) = 1$). On the other hand, the distance between $\Gamma \cap 3Q_n$ and L is $\le C_2 \sum_{k=0}^n \beta_\Gamma(Q_k)2^k$, because the angle between the best approximating lines for Q_k and Q_{k+1} is $O(\beta_\Gamma(Q_{k+1}))$. Therefore B and $\Gamma \cap 2Q_N$ will be disjoint, if for every $0 \le n \le N$ we have

$$\sum_{k=0}^{n} \beta_\Gamma(Q_k) 2^k < (C_1/C_2)\epsilon 2^{2n}.$$

Note that

$$\begin{aligned}
\max_{0\le n\le N} 2^{-2n} \sum_{k=0}^{n} \beta_\Gamma(Q_k)2^k &\le \sum_{n=0}^{N} 2^{-2n} \sum_{k=0}^{n} \beta_\Gamma(Q_k)2^k \\
&\le \sum_{k=0}^{N} \beta_\Gamma(Q_k)2^k \sum_{n=k}^{N} 2^{-2n} \le \sum_{k=0}^{N} \beta_\Gamma(Q_k)2^{-k} = \epsilon/(2A) = (C_1/C_2)\epsilon,
\end{aligned}$$

if we take $A = \frac{1}{2}C_2/C_1$. This holds for every choice of z in $L^\perp$ that is distance $1/\epsilon$ from L, so we have proven that $\varepsilon_\Gamma(Q) \lesssim \epsilon$, as claimed. Summing over all dyadic

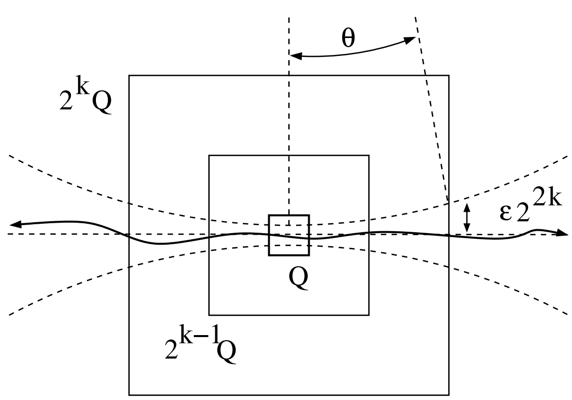

FIGURE 11. The part of the ball of radius $\mathrm{diam}(Q)/\varepsilon(Q)$ that lies in $2^kQ \setminus 2^{k+1}Q$ makes angle $\theta \simeq \varepsilon 2^k$ with the perpendicular ray from L to z and hence (since we are assuming $\mathrm{diam}(Q) = 1$) is distance approximately $\varepsilon^{-1}(1 - \cos(\theta)) \simeq \varepsilon\theta^2 = \varepsilon 2^{2k}$ from the line L.

cubes gives

$$\sum_Q \varepsilon_\Gamma^2(Q) \lesssim \sum_Q \left[\sum_{Q':Q\subset Q'} \beta_\Gamma(Q')\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right]^2$$
$$\lesssim \sum_Q \left[\sum_{Q':Q\subset Q'} \beta_\Gamma(Q')\left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^{3/4}\left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^{1/4}\right]^2$$

and by Cauchy-Schwarz we get

$$\lesssim \sum_Q \left[\sum_{Q':Q\subset Q'} \beta_\Gamma^2(Q')\left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^{3/2}\right]\cdot\left[\sum_{Q':Q\subset Q'}\left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^{1/2}\right].$$

The second term is dominated by a geometric series, hence bounded. Thus

$$\sum_Q \varepsilon_\Gamma^2(Q) \lesssim \sum_{Q'} \beta_\Gamma^2(Q') \sum_{Q:Q\subset Q'} \frac{\operatorname{diam}(Q)^{3/2}}{\operatorname{diam}(Q')^{3/2}}.$$

Since Definition 11 implies Γ is chord-arc, the number of dyadic cubes inside Q' of size $\operatorname{diam}(Q')2^{-k}$ and hitting Γ is at most $O(2^k)$. Thus the right side is bounded by

$$\lesssim \sum_{Q'} \beta_\Gamma^2(Q')\sum_{k=0}^\infty O(2^k)2^{-3k/2} \lesssim \sum_{Q'} \beta_\Gamma^2(Q')\sum_{k=0}^\infty 2^{-k/2} \lesssim \sum_{Q'} \beta_\Gamma^2(Q')$$

and so the ε^2-sum is finite if the β^2-sum is finite, as desired. □

It is sometimes convenient to assume that the balls in the definition of ε_Γ are small compared to $\operatorname{diam}(\Gamma)$. This is easy to obtain if we replace $\varepsilon_\Gamma(Q)$ by

$$\widetilde{\varepsilon}_\Gamma(Q) = \max(\varepsilon_\Gamma, (\operatorname{diam}(Q)/\operatorname{diam}(\Gamma))^\alpha)$$

for some $1/2 < \alpha < 1$. This corresponds to never taking a ball larger than diameter $(\operatorname{diam}(\Gamma)/\operatorname{diam}(Q))^\alpha \cdot \operatorname{diam}(Q)$ in the definition of $\epsilon(Q)$. Thus only balls of diameter $\lesssim \operatorname{diam}(Q)^{1-\alpha}$ are used to bound $\widetilde{\varepsilon}_\Gamma(Q)$. Clearly $\epsilon_\Gamma(Q) \le \widetilde{\epsilon}_\Gamma(Q)$, and

$$\sum_{Q:Q\cap\Gamma\neq\emptyset} \widetilde{\varepsilon}_\Gamma^2(Q) \lesssim \sum_Q \varepsilon_\Gamma^2(Q) + \sum_Q \left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(\Gamma)}\right)^{2\alpha}$$

where the second sum is finite for chord-arc curves because $\alpha > 1/2$ the number of dyadic squares of size $\simeq 2^{-n}$ hitting Γ is $O(2^n)$. Thus bounding $\sum \widetilde{\epsilon}_\Gamma^2$ is equivalent to bounding $\sum \epsilon_\Gamma^2$ for chord-arc curves. This is helpful if one wants to control $\epsilon_\Gamma(Q)$

in terms of the local behavior of Γ, and it also gives local control of the hyperbolic convex hull Γ, as we shall see in the following section.

16. (14) $\Leftrightarrow$ (15): ε_Γ IS EQUIVALENT TO δ

Recall that each ball $B \subset \mathbb{R}^n$ is the boundary of a hyperbolic half-space H in $\mathbb{H}^{n+1}$, and two balls are disjoint iff the corresponding half-spaces are disjoint.

Lemma 16.1. *Suppose $B_1, B_2 \subset \mathbb{R}^n$ are disjoint balls of radius r that are distance ϵ apart. Then the hyperbolic distance between the corresponding half-spaces is $\simeq \sqrt{\epsilon/r}$.*

Proof. The nearest points on the half-spaces will occur over the line connecting the centers of B_1 and B_2, so it suffices to do this calculation in the copy of the hyperbolic upper half-plane lying above this line; this is a simple calculus exercise. We can normalize so the balls both have have radius 1 and the distance between them is $\eta = \epsilon/r$. The intersection of the hemispheres with this plane are two half-circles. At height t above $\mathbb{R}^2$, these circles are Euclidean distance $\eta + O(t^2)$ apart, hence hyperbolic distance $\simeq t + \eta/t$ apart. This is minimized when $t = \sqrt{\eta} = \sqrt{\epsilon/r}$. $\square$

Lemma 16.2. *If $z \in \mathrm{CH}(\Gamma)$, then $\delta(w) = O(\delta(z))$ for all $w \in \mathrm{CH}(\Gamma) \cap \mathrm{B}_\rho(\mathrm{z}, 1)$.*

Proof. The point is that if H_1, H_2 are two disjoint hyperbolic half-planes that both within distance δ of a point z, then their boundaries remain within distance $O(\delta)$ of each other inside $B_\rho(z, 1)$ (imagine $z = 0$ in the ball model). $\square$

Lemma 16.3. *Definition 14 implies Definition 15.*

Proof. Lemmas 16.1 and 16.2 imply that if $\varepsilon_\Gamma(Q)$ is small (say less than 1/100), then $\delta(z) \lesssim \varepsilon_\Gamma(Q)$ for every point $z \in T(Q) = Q \times [\ell(Q)/2, \ell(Q)] \subset \mathbb{H}^{n+1}$. Thus $\sum_Q \delta^2(Q)$ is bounded by a uniform multiple of $\varepsilon_\Gamma^2(Q)$. $\square$

Alternatively, we could integrate δ^2 over any surface with $\mathrm{A}_\rho(S \cap T(Q)) = O(1)$, e.g., the dyadic dome of Γ, or a smoothed version of the dyadic dome, or a minimal surface with asymptotic boundary Γ, or (in the case $n = 2$) a boundary component of the hyperbolic convex hull of Γ. The latter has been extensively studied, e.g., [14], [15], [25], [26], [27], [47], [50].

Lemma 16.4. *Definition 15 implies Definition 14.*

Proof. If $\Gamma \subset \mathbb{R}^2$ is a circle, then $\delta(z)$ vanishes everywhere but ε_Γ does not. Thus $\varepsilon_\Gamma(Q)$ cannot be bounded using δ alone; there must also be some dependence on size of Q. Without loss of generality, we assume $\operatorname{diam}(\Gamma) = 1$, $\operatorname{diam}(Q) \leq 1/100$, and $\delta(z) < 1/100$ for z in $\mathrm{CH}(\Gamma) \cap \mathrm{T(Q)}$ ($T(Q) \subset \mathbb{H}^{n+1}$ are the points that project vertically into Q have have height between $\ell(Q)/2$ and $\ell(Q)$). For $z \in T(Q)$ and each normal direction at z that is perpendicular to an optimal plane in the definition of $\delta(z)$, there are a pair of disjoint hyperbolic half-spaces H and H^* connected by a geodesic segment of hyperbolic length at most $O(\delta)$ running through z and perpendicular to each half-space. These half-spaces intersect $\mathbb{R}^n$ in disjoint regions B, B^* that do not hit Γ and are bounded by spheres.

If both regions are bounded balls then they are separated a $(n-1)$-plane, which extends to a vertical n-plane in $\mathbb{H}^{n+1}$ which separates the hyperbolic half-spaces and thus comes within $O(\delta)$ of the point z. This implies B, B^* each have radius $\gtrsim \delta(z) \cdot \operatorname{diam}(Q)$. Otherwise one region, say B, is a bounded ball and the other region B^* is the exterior of a ball. Since B^* doesn't hit Γ its boundary sphere must have diameter ≥ 1, and therefore it makes angle of at most $O(\operatorname{diam}(Q))$ with the vertical near z. Since the other half-space H is also within $O(\delta)$ of z, it makes and angle of at most $\theta = O(\delta(z)) + O(\operatorname{diam}(Q))$ with the vertical, and hence B has radius $\gtrsim \operatorname{diam}(Q)/(\delta(z) + \operatorname{diam}(Q))$.

In either case we have $\varepsilon_\Gamma^2(Q) = O(\delta^2(z)) + O(\operatorname{diam}^2(Q))$. The δ^2-sum is bounded by assumption. This assumption also implies that given $\delta_0 = 2^{-m} > 0$, all but finitely many terms of the δ sum are less than δ_0. Assume we are at a scale below which all cubes satisfy this. Given such a cube Q, $\Gamma \cap 3Q$ can hit only $O(1/\delta_0)$ sub-dyadic-cubes of $3Q$ of size $\delta_0 \operatorname{diam}(Q)$. Iterating, we see that Γ hits at most $O(C^k \delta^k) = O(2^{(m+\log_2 C)k})$ dyadic cubes of size $\gtrsim 2^{-mk}$. Thus

$$\sum_{k=0}^{\infty} \sum_{Q: 2^{-m(k+1)} < \ell(Q) \leq 2^{-mk}, Q \cap \Gamma \neq \emptyset} \operatorname{diam}^2(Q) \leq \sum_k 2^{(m+\log_2 C - 2m)k} < \infty$$

if $m > \log_2 C$, which occurs if δ_0 is small enough. This proves the lemma. $\square$

17. (15) $\Rightarrow$ (16): δ CONTROLS SURFACE CURVATURE

Lemma 17.1. *Definition 15 implies Definition 16.*

Proof. In both $n = 2$ and higher dimensions we create a triangulated surface where adjacent triangles are very close to parallel, and smooth this surface to obtain a surface with small principle curvatures. In dimensions ≥ 2, the discrete surface can be the dyadic dome, introduced in Section 12, and the principle curvatures are controlled by the β-numbers. In the special case $n = 2$, we can also use a discretization of the usual hyperbolic dome of one side of Γ. Since we this case is of particular interest, we describe it first, despite the redundancy with the more general argument given later.

Suppose S is one component of $\partial \mathrm{CH}(\Gamma)$. It is known that S, with its hyperbolic path metric, is isomorphic to the hyperbolic disk (e.g., [46], [85], [84]). The hyperbolic unit disk can be triangulated by geodesic triangles with hyperbolic diameters $\simeq 1$ and angles bounded strictly between 0 and π, e.g., take the tesselation corresponding to a Fuchsian triangle group, or obtain a triangulation by connecting the center of each Whitney box for $\mathbb{D}$ to the box's vertices.

Fix such a triangulation of $\mathbb{D}$ and map the vertices to S via the isometry. Each triple of image vertices corresponding to a triangle on $\mathbb{D}$ lies on a hyperbolic plane and determines a triangle on this plane. Create a new surface S_1 by gluing these triangles together along their edges. Because the vertices lie in $\mathrm{CH}(\Gamma)$, convexity implies each triangle, and hence all of S_1, also lie in $\mathrm{CH}(\Gamma)$.

Consider two triangles T_1, T_2 in S_1 that meet along a common edge e. Normalize so that one endpoint of e is the origin in the ball model of hyperbolic 3-space, e lies along the x axis and T_1 lies in the xy-plane. Then T_2 lies in Euclidean plane that makes some angle θ with the xy-plane, and by our assumptions, it contains a point p (e.g., the vertex of T_2 not on e) that is hyperbolic distance $\simeq 1$ from 0 and Euclidean distance $\simeq 1$ from the x-axis. Then p is Euclidean distance $\simeq \theta$ from the xy-plane. Because both triangles lie inside $\mathrm{CH}(\Gamma)$ and $\mathrm{CH}(\Gamma)$ is trapped between two hyperbolic half-planes that each come within hyperbolic distance $\delta(0)$ of the origin, we must have $\theta \lesssim \delta(0)$ (we are using Lemma 16.2).

If T is component triangle of S_1, let $\theta(T)$ be the maximum angle T makes with any of its neighboring triangles, and think of $\theta(z)$ as a function on S_1 that is constant on triangles. Since $\theta(z)$ can be bounded by a uniform multiple of $\delta(w)$ for a point w

that is a uniform hyperbolic distance away, we get

$$\int_{S_1} \theta^2(z) d\mathrm{A}_\rho(z) \lesssim \int_{S_1} \delta^2(z) d\mathrm{A}_\rho(z) < \infty.$$

The principle curvatures of S_1 are zero inside each triangle and a measure along the edges. However, by smoothing S_2 we can obtain a surface S_2 so that the principle curvatures tend to zero as we approach infinity and are bounded by $O(\max_{T^*} \theta(z))$, where T^* denotes the union of all component triangles that touch T (including those that only touch at a vertex). Then

$$\int_{S_2} |K|^2(z) d\mathrm{A}_\rho(z) \lesssim \int_{S_1} \delta^2(z) d\mathrm{A}_\rho(z) < \infty.$$

For $n \geq 2$ essentially the same proof works if we take the dyadic dome for our triangulated surface with asymptotic boundary Γ. The angles between adjacent faces are easily bounded by the β-numbers of the corresponding arcs of Γ, which, after smoothing, proves that Definition 11 implies Definition 16. $\square$

18. (16) $\Rightarrow$ (3): Surface curvature bounds QC reflections.

Lemma 18.1. *For $n = 2$, Definition 16 implies Definition 3.*

Proof. For $n = 2$, this implication is due to Charles Epstein [44]. He proves that for a surface $S \subset \mathbb{H}^3$ whose principle curvatures $|\kappa_1(p)|, |\kappa_2(p)|$ are bounded strictly below 1, the Gauss map from the surface to the plane at infinity is quasiconformal. Recall that the Gauss map sends a point p on S to the endpoint on $\mathbb{R}^2$ of the hyperbolic geodesic ray starting at p that is normal to S. There are actually two Gauss maps from S to $\mathbb{R}^2$ depending on which "side" of S the geodesic ray is in. In the case when the surface has asymptotic limit Γ, a curve on $\mathbb{R}^2$, the composition of one of these maps with the inverse of the other defines a quasiconformal reflection across Γ. By Proposition 5.1 of [44], the dilatation of the composed Gauss maps is

$$\begin{aligned} D(z) &= \max\left(\left| \frac{1+\kappa_1(p)}{1-\kappa_1(p)} \cdot \frac{1-\kappa_2(p)}{1+\kappa_2(p)} \right|^{1/2}, \left| \frac{1-\kappa_1(p)}{1+\kappa_1(p)} \cdot \frac{1+\kappa_2(p)}{1-\kappa_2(p)} \right|^{1/2} \right) \\ &= 1 + O(|\kappa_1(p)| + |\kappa_2(p)|), \end{aligned}$$

where $p \in S$ is the point corresponding to $z \in \mathbb{R}^2$. Therefore the dilatation satisfies

$$|\mu(z)| = O(|\kappa_1(p)| + |\kappa_2(p)|).$$

Moreover, on page 121 of [44], Epstein shows that the Jacobian J of this map satisfies

$$C_1|(1 \mp \kappa_1)(1 \mp \kappa_2)| \leq J \leq C_2|(1 \pm \kappa_1)(1 \pm \kappa_2)|.$$

In particular, $J \simeq 1$ if $|\kappa_1|, |\kappa_2|$ are both uniformly bounded below 1.

Definition 16 implies that κ_1, κ_2 are both small outside some compact ball B around the origin. Thus the Gauss map for S defines a quasiconformal reflection in some neighborhood U of Γ and inside this neighborhood

$$\int_U |\mu(z)|^2 d\mathrm{A}_\rho(z) \lesssim \int_{S\setminus B} |\mathcal{K}_0(z)|^2 d\mathrm{A}_\rho(z),$$

where $d\mathrm{A}_\rho$ is the hyperbolic area measure on $\mathbb{R}^2 \setminus \Gamma$ and S respectively and $\mathcal{K}_0$ is the trace-free second fundamental form of S. Extend this reflection to the rest of $\mathbb{R}^2$ by some diffeomorphism of one component of $\mathbb{R}^2 \setminus U$ to the other that agrees with the reflection given by the Gauss map on ∂U. This gives a global quasiconformal reflection across Γ that satisfies (2.10), as desired. □

Next we give an argument for $n \geq 3$ that gives a biLipschitz involution.

Lemma 18.2. *For $n \geq 2$, Definition 16 implies Definition 13.*

Proof. We consider only points z on S that are at height $\leq t_0$ above $\mathbb{R}^n$ where t_0 is chosen so small that that if $z = (x,t) \in \mathbb{R} \times (0, t_0)$, then the principle curvatures at z are all very small, say $\leq 1/100$. There is an $(n-1)$-sphere of directions in the tangent space of $\mathbb{H}^{n+1}$ at z that are perpendicular to S. These directions define a tangent $(n-1)$-dimensional hyperbolic hyper-plane H_z that passes through z, and the boundary of H_z on $\mathbb{R}^n$ is a Euclidean $(n-2)$-sphere S_z that whose center is within $O(t \cdot \sup_w \max_j |\kappa_j(w)|)$. We define R on this sphere by taking the antipodal map.

We claim that such circles foliate a neighborhood U of Γ and that R is Lipschitz. If so, then R is a biLipschitz involution that fixes Γ. Let $K_r = K_r(z)$ be an upper bound for $\max |\kappa_j\|$ in a hyperbolic r-ball around z. Given $z, w \in S$ that are $t < r$ apart in the hyperbolic metric, let γ be the geodesic segment in $\mathbb{H}^{n+1}$ connecting them. The perpendicular hyperplanes H_z, H_w are both within $O(K_r)$ of orthogonal to γ and hence the corresponding spheres S_z, S_w are within $O(K_r \cdot t)$ of each other, but are also at least distance $\gtrsim K \cdot t$ apart (this is easiest to see in the ball model of hyperbolic space, setting $z = 0 \in \mathbb{B}^{n+1}$). Thus the antipodal maps preserve distance between points on the same sphere and increase the distance between points on different spheres by at most $O(K \cdot t)$. Thus R is Lipschitz, as desired. Moreover, if two such spheres intersect the same Whitney cube Q of $\mathbb{R}^n \setminus \Gamma$, then then both have radii $\simeq \ell(Q)$ and centers that are within $O(\ell(Q))$ of each other. Thus the corresponding points on S are within hyperbolic distance $O(1)$ of each other. Therefore the argument above implies that $\rho(Q) = O(K_r(z))$ for some point $z \in S$ and $\sum_Q \rho^2(Q)$ is finite if $\leq \int_S |K_r(z)|^2$ is. Hence Definition 16 implies Definition 13. $\square$

19. (15) $\Rightarrow$ (17): Minimal surfaces with finite total curvature

We already know that Γ is Weil-Petersson if and only if it is the boundary of some surface in $\mathbb{H}^{n+1}$ that is asymptotically flat and has finite total curvature. Next we prove this surface can be taken to be minimal if $n = 2$. Later we show that for $n \geq 3$, Γ bounds some minimal 2-chain that agrees with a surface of finite total curvature near the boundary of $\mathbb{H}^{n+1}$. First, we need to know that a minimal surface that is trapped between parallel planes in necessary close to flat. This is obvious for minimal surfaces in $\mathbb{R}^n$ because the coordinates give harmonic functions, but in hyperbolic 3-space, the corresponding estimate is due to Andrea Seppi [117]:

Lemma 19.1. *Suppose S is an embedded minimal disk in $\mathbb{B}^3$ that has an asymptotic bounding quasicircle $\Gamma \subset \mathbb{S}^2$. Suppose $0 \in S$ and that S lies between two disjoint hyperbolic planes that both at most distance ϵ from 0, one on either side of the xy-plane. Then the tangent plane of S at 0 makes angle at most $O(\epsilon)$ with the xy-plane and the absolute values of the principle curvatures of S at 0 are both bounded by $O(\epsilon)$.*

This is essentially Propositions 4.14 and 4.15 of [117]; see Equation (32) in particular. Given a minimal surface S that is trapped between two hyperbolic planes P_-, P_+, Seppi considers the function $u(z) = \sinh(\mathrm{dist}(z, P_-))$ for $z \in S$ and uses the fact that this satisfies the equation $\Delta_S u - 2u = 0$, where Δ_S is the Laplace-Beltrami operator for the surface S. The Schauder estimates for this equation imply that

$$\|u\|_{C^2(B(x,r/2))} \leq C\|u\|_{C^0(B(x,r))}.$$

In order to get a uniform bound for C, we must bound the curvature of S, and Seppi gives an argument for this assuming the boundary of S is a quasicircle (this covers our application, since Weil-Petersson curves are quasicircles). Finally, the sup norm of u is bounded in terms of the distance between P_- and P_+ near z, and that we have shown is $O(\delta(z))$, e.g. Lemma 16.2. One small technical point is that Seppi requires the point z to be on a geodesic segment that meets both P_- and P_+ orthogonally. However, it is very simple to see that if z is between two disjoint hyperbolic planes that each come within ϵ of z, then there are also two disjoint planes that come within $O(\epsilon)$ and satisfy the orthogonality condition for z.

The lemma implies that near the boundary of hyperbolic space we have

$$\int_S |\mathcal{K}|^2 d\mathrm{A}_\rho \lesssim \int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z) d\mathrm{A}_\rho < \infty,$$

when Γ is Weil-Petersson. Thus, for $n = 2$ Definition 15 implies Definition 17.

20. (15) $\Rightarrow$ (19): RENORMALIZED AREA

As we discussed in Section 1, a 2-surface $S \subset \mathbb{H}^{n+1}$ with boundary curve $\Gamma \subset \mathbb{R}^n$ is said to have finite renormalized area if

$$\mathcal{RA}(S) = \lim_{t \searrow 0} [\mathrm{A}_\rho(S_t) - L_\rho(\partial S_t)]$$

exists and is finite, where

$$S_t = \{(x, y, s) \in S : s \geq t\}, \quad \partial S_t = \{(x, y, s) \in S : s = t\}.$$

Lemma 20.1. *For $n = 2$, Definition 15 implies Definition 19.*

Proof. Using the Gauss-Bonnet theorem

$$\begin{aligned}
\mathrm{A}_\rho(S_t) - L_\rho(\partial S_t) &= \int_{S_t} 1 d\mathrm{A}_\rho - \int_{\partial S_t} 1 dL_\rho \\
&= \int_{S_t} (1 + \kappa^2) d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 dL_\rho \\
&= -\int_{S_t} K d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 dL_\rho \\
&= -2\pi\chi(S_t) + \int_{\partial S_t} \kappa_g dL_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 dL_\rho \\
&= -2\pi\chi(S_t) - \int_{S_t} \kappa^2 d\mathrm{A}_\rho + \int_{\partial S_t} (\kappa_g - 1) dL_\rho
\end{aligned}$$

where κ_t is the geodesic curvature of ∂S_t in S_t. Since we are assuming Definition 15 holds, we know from earlier results that the β's tend to zero and this implies that near the boundary, any minimal surface is nearly vertical (trapped between nearly touching hyperbolic planes) and therefore it has finite Euler characteristic.

The estimate of Seppi discussed in Section 19 shows that

$$\int_{S_t} \kappa^2 d\mathrm{A}_\rho = O(\int_{S_t} \delta^2 d\mathrm{A}_\rho).$$

Since Γ is Weil-Petersson, our earlier results imply this integral converges to a finite limit as $t \searrow 0$. Seppi's paper [117] is written for minimal 2-surfaces in $\mathbb{H}^3$; extending his bound on principle curvatures to surfaces in $\mathbb{H}^{n+1}$, would extend the lemma to this case, since the rest of this argument is valid in higher dimensions.

The geodesic curvature κ_g of the boundary curve comes from two components. There is a vertical component of size 1 due to the curve lying on the horizontal plane. There is a horizontal component due to the curvature of ∂S_t in this plane. This component has size bounded by the principle curvatures of the surface, that by Seppi's estimate are bounded by $O(\delta(z))$. The geodesic curvature κ_g is given by projecting this vector onto the tangent space of S_t, that our previous estimates show makes an angle at most $O(\delta)$ with the vertical. Thus $|\kappa_g| = 1 + O(\delta^2)$. Hence

$$\int_{\partial S_t} (\kappa_g - 1) ds = O\left(\int_{\partial S_t} \delta^2(z) ds\right) \tag{20.1}$$

Note that since δ^2 has finite integral over the whole surface its integral over the annulus $A_t = S_t \setminus S_{t+1}$ tends to zero with t. Moreover, Lemma 16.2 implies the integral of $\delta^2(z)$ over ∂S_t is dominated by a multiple of the area integral over A_t and hence the boundary integral in (20.1) must tend to zero. This proves the lemma (and also shows that the formula (1.6) holds.) $\square$

The estimate $|\kappa_g| = 1 + O(\delta^2)$ also follows from from Equation (2.4) of [34]:

$$\kappa_g = \frac{1}{\nabla r}(\coth r + \langle \mathcal{K}(e,e), \nabla \perp r\rangle),$$

where r is the hyperbolic distance to some fixed point (say the origin in the ball model), Dr is the gradient of r in $\mathbb{H}^{n+1}$, ∇r is the projection of Dr onto the tangent space of S, $\nabla^\perp r$ is the projection of Dr onto the normal space of S, and $\mathcal{K}$ is the second fundamental form of S.

Proof of Corollary 1.8. The inequality

$$\mathcal{RA}(S) \leq \sup\{\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega)\}$$

is obvious since the truncated surfaces in the definition of $\mathcal{RA}(S)$ are among the domains used in the supremum on the right.

To prove the other direction note that if $D(z,R) \subset \Omega$, then $\chi(S) = \chi(\Omega) = \chi(\Omega_t)$ for all $0 \le t \le T/2$ if R is large enough. Then by Lemma 21.1

$$\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega) \le -(2\pi \mp \epsilon)\chi(D(z,R/2)) - (1-\epsilon)\int_{D(z,R/2)} \kappa^2 d\mathrm{A}_\rho.$$

Taking $R \nearrow \infty$, and applying the Monotone Convergence Theorem, we get

$$\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega) \le -(2\pi \mp \epsilon)\chi(\Omega) - (1-\epsilon)\int_S \kappa^2 d\mathrm{A}_\rho.$$

Then taking $\epsilon \searrow 0$ gives

$$\limsup_{R\nearrow\infty} \sup_{\Omega:\Omega\supset D(z,R)} \mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega) \le -2\pi\chi(\Omega) - \int_S \kappa^2 d\mathrm{A}_\rho. \quad \square$$

The following lemma allows us to estimate renormalized areas in some special cases, e.g., the examples in Section 23.

Lemma 20.2. *Suppose that $\Gamma \subset \mathbb{R}^2$ is a C^1 closed Jordan curve and $r(z)$ is a positive Lipschitz function on Γ so that for every $z \in \Gamma$, there are tangent disks of radius $r(z)$ at z contained in each component of $\mathbb{R}^2 \setminus \Gamma$. Then for any minimal surface S with asymptotic boundary Γ, $\mathcal{RA}(S) \ge -4\pi\chi(S) - O\left(\int_\Gamma (1/r(z))dz\right)$. Equivalently, $\mathcal{RA}(S) \ge -4\pi\chi(S) - O(N)$, where N is the number of disks $\{D_j\}_1^N$ of the form $D(z_j, r(z_j)/2))$ needed to cover Γ. If, in addition, there is an $\epsilon > 0$ so that $\Gamma \cap D_j$ differs by more than $\epsilon \cdot diam(D_j)$ from every circular arc through z for more than M disjoint such disks, then $\mathcal{RA}(S) \le -4\pi\chi(S) - O(M)$.*

Proof. For $w \in \Gamma \cap D(z, r(z)/2)$, we have $r(z)/2 \le r(w) \le 2r(z)$, and $r(z) \le \ell(\Gamma \cap D(z, r(z)/2)) \le 2r(z)$. Thus any covering satisfies $\int_\Gamma (1/r(z))dz \le 4N$, and extracting a subcover that covers each point at most twice (always possible for arcs) we get $N \le 2\int_\Gamma (1/r(z))dz$. Thus $N \simeq \int_\Gamma (1/r(z))dz$.

The surface S is contained in the convex hull of Γ, so it is also trapped between the domes of Ω_1 and Ω_2, the union of tangent disks on each side of Γ. Thus there is curve $\Gamma' \subset S$ at height approximately $r(z)/2$ above z that has hyperbolic length $\simeq N$. It bounds a compact region $\Omega \subset S$ and by the isoperimetric inequality (21.3),

$$\mathrm{A}_\rho(\Omega) \le L_\rho(\Gamma') - 4\pi\chi(\Omega) = O(N) - 4\pi\chi(\Omega).$$

As before, let S_t be the part of S above height t and Γ_t its boundary. The region between Γ' and Γ_t is an annulus, so $\chi(\Omega) = \chi(S_t) = \chi(S)$. The region $S \setminus \Omega$ can be cut into N regions $\{W_j\}$ with bounded overlap, each with asymptotic boundary $\{\Gamma \cap D_j\}$. Since Γ is trapped between the domes of Ω_1 and Ω_2, $\delta_\Gamma(w) \lesssim s/r(z)$ if w is height s above $D(z, r(z)/2$. By Lemma 5.4 the same bound holds for the principle curvature, and this proves that the integral of κ^2 over each W_j is $O(1)$. Thus $\mathrm{A}_\rho(S_t) \leq -4\pi\chi(S) + L_\rho(\Gamma_t) - O(N)$, which is the desired estimate.

In the other direction, fix $0 < t < r(z_j)$. If $\int_{W_j} \kappa^2 d\mathrm{A}_\rho < \delta$ for $\delta > 0$ sufficiently small (depending on t), then $W_j \cap S_t$ must be close to lying on hyperbolic half plane. Thus $\gamma_j = W_j \cap \Gamma_t$ must be close to a circular arc. But since γ_j is trapped between the domes of Ω_1 and Ω_2, its vertical projection is equally close to a circular arc in the plane that stays within Euclidean distance $O((t/r(z_j)^2 r(z_j)) = O(t^2/r(z_j))$ of Γ. This contradicts our assumption on M of the disks if t and δ are small enough, depending on ϵ. Thus

$$\int_S \kappa^2 d\mathrm{A}_\rho \gtrsim \sum_j \int_{W_j} \kappa^2 d\mathrm{A}_\rho \gtrsim \delta \cdot M \gtrsim M$$

with a constant depending only on ϵ. $\square$

21. (18) $\Rightarrow$ (17): ISOPERIMETRIC INEQUALITIES

Suppose that $S \subset \mathbb{H}^{n+1}$ is a minimal surface with asymptotic boundary curve in $\mathbb{R}^n$ (in this section, we will not denote this curve by Γ; that symbol will be used for a different curve, defined below, that approximates the asymptotic boundary). As before, for $t > 0$ let $S_t = S \cap \{(x, s) \in \mathbb{R}^n \times (t, \infty)\}$ be the part of S above height t and let $S_t^* = S \setminus S_t$ be the part below height t. We assume that for t small enough, S_t^* is real analytic and a topological annulus. Suppose $\Omega \subset S_t^*$ is a compact sub-annulus with one boundary component equal to $\Gamma_t = S \cap \mathbb{R}^n \times \{t\}$, and the other boundary component a smooth curve Γ. Let $T = T(\Omega)$ be the distance in S between Γ and Γ_t. For $0 \leq s \leq T$, let

$$\Omega(s) = \{z \in \Omega : d_S(z, \Gamma) > s\}, \quad \Gamma(s) = \{z \in \Omega : d_S(z, \Gamma) = s\}.$$

Here d_S refers to distance on the surface S. Note that $\Gamma(0) = \partial\Omega$ and $\Omega(0) = \Omega$. Also note that $\chi(\Omega) = 0$ (it is an annulus) and $\chi(\Omega(s)) \geq 0$ since $\Omega(s)$ is the union of a topological annulus and possibly some disks. Let $A(s)$ be the hyperbolic area of $\Omega(s)$

and $L(s)$ the hyperbolic length of $\Gamma(s) = \partial\Omega(s) \setminus \Gamma_t$. In particular, $A(0) = \mathrm{A}_\rho(\Omega)$ and $L(0) = L_\rho(\Gamma)$. The Gauss-Bonnet theorem says that

$$\int_{\Omega(s)} K d\mathrm{A}_\rho + \int_{\partial\Omega(s)} \kappa_g dL_\rho = 2\pi\chi(\Omega(s))$$

where κ_g is the geodesic curvature of $\partial\Omega$ in Ω. For points in $\Gamma_t \subset \partial\Omega$, this is the negative of κ_g^S, the geodesic curvature of Γ_t in S_t. Since $\partial\Omega(s) = \Gamma_t \cup \Gamma(s)$ and $\chi(\Omega(s)) \geq 0$, we get

$$-\int_{\Gamma(s)} \kappa_g dL_\rho = \int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega(s)} K d\mathrm{A}_\rho - 2\pi\chi(\Omega(s)) \leq \int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega(s)} K d\mathrm{A}_\rho$$

Lemma 21.1. *Suppose $\frac{2}{T} < \epsilon \leq 1$. With notation as above,*

$$L_\rho(\partial\Omega) - \mathrm{A}_\rho(\Omega) \geq -C(S,t) + (1-\epsilon)\int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho,$$

where

$$C(S,t) = \max\left(\int_{\Gamma_t} \kappa_g^S dL_\rho, L_\rho(\Gamma_t)\right).$$

Proof. This follows from known facts about the isoperimetric inequality on negatively curved surfaces. Our presentation follows that of Chavel and Feldman [33], although they attribute the basic facts to Faila [54].

As shown in [54], the function $A(s)$ is continuously differentiable and decreasing on $[0,T]$, and $A'(s) = -L(s)$ (Theorem 5 of [54]). Similarly, by Theorem 3 of [54], $L(s)$ is continuous on $[0,T]$, analytic except for finitely many points, and (except for these points)

$$L'(s) \leq -\int_{\Gamma(s)} \kappa_g dL_\rho.$$

Using the remarks about Gauss-Bonnet before the lemma, we get

$$L'(s) \leq \int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega(s)} K d\mathrm{A}_\rho. \tag{21.1}$$

Thus

$$\begin{aligned} L'(s) - A'(s) &\leq \int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega(s)} K d\mathrm{A}_\rho + L(s) \\ &= \int_{\Gamma_t} \kappa_g dL_\rho - \int_{\Omega(s)} (1+\kappa^2) d\mathrm{A}_\rho + L(s), \end{aligned}$$

which implies

$$L'(s) - A'(s) \quad \leq \quad L(s) - A(s) + \int_{\Gamma_t} \kappa_g dL_\rho - \int_{\Omega_s} \kappa^2 d\mathrm{A}_\rho. \tag{21.2}$$

By the isoperimetric inequality for surfaces with $K \leq -1$ (e.g., Equation (4.30) of [100]), we have

$$L_\rho(\partial\Omega(s))^2 = (L(s) + L_\rho(\Gamma_t))^2 \geq 4\pi\chi(\Omega_s)A(s) + A(s)^2,$$

and this implies

$$L(s) - A(s) \geq \frac{4\pi\chi(\Omega_s)A(s)}{L(s) + L_\rho(\Gamma_t) + A(s)} - L_\rho(\Gamma_t) \geq -L_\rho(\Gamma_t) \tag{21.3}$$

since $\chi(\Omega_s) \geq 0$. Assume for the moment that

$$L(0) - A(0) \leq -L_\rho(\Gamma_t) + \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho. \tag{21.4}$$

Then we claim there must be a $s \in [0, 1/\epsilon]$ so that

$$L'(s) - A'(s) \geq -\epsilon \int_{\Omega_{1/\epsilon}} \kappa^2 d\mathrm{A}_\rho. \tag{21.5}$$

If not, then by integrating and using (21.4) we get

$$\begin{aligned} L(\frac{1}{\epsilon}) - A(\frac{1}{\epsilon}) \quad &= \quad L(0) - A(0) + \int_0^{1/\epsilon} L'(x) - A'(x) dx \\ &< \quad -L_\rho(\Gamma_t) + \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho + \frac{1}{\epsilon}\left[-\epsilon \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho\right] \\ &= \quad -L_\rho(\Gamma_t) \end{aligned}$$

which contradicts (21.3) for $s = 1/\epsilon$, proving there is at least one such point s.

Let a be the infimum of values s where (21.5) holds. Since we have assumed that κ is not constant zero, this bound is negative if ϵ is small enough (which forces T to be large). Thus $L(s) - A(s)$ has a negative derivative except for finitely many points in $[0, a]$ and therefore $L(a) - A(a) \leq L(0) - A(0)$. Using (21.5) and (21.2) with $s = a$,

$$\begin{aligned} -\epsilon \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho \quad &\leq \quad L'(a) - A'(a) \\ &\leq \quad L(a) - A(a) + \int_{\Gamma_t} \kappa_g dL_\rho - \int_{\Omega(a)} \kappa^2 d\mathrm{A}_\rho \\ &\leq \quad L(0) - A(0) + \int_{\Gamma_t} \kappa_g dL_\rho - \int_{\Omega(a)} \kappa^2 d\mathrm{A}_\rho \end{aligned}$$

This implies

$$L(0) - A(0) \geq -\int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega_t} \kappa^2 d\mathrm{A}_\rho - \epsilon \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho.$$

Now since $0 \leq a \leq 1/\epsilon$, we have $\Omega(1/\epsilon) \subset \Omega(a)$, so

$$\int_{\Omega(a)} \kappa^2 d\mathrm{A}_\rho - \epsilon \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho \quad \geq \quad (1-\epsilon)\int_{\Omega(a)} \kappa^2 d\mathrm{A}_\rho \geq (1-\epsilon)\int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho$$

and hence

$$L(0) - A(0) \geq -\int_{\Gamma_t} \kappa_g dL_\rho + (1-\epsilon)\int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho. \tag{21.6}$$

Thus either (21.4) fails or (21.6) holds. In either case we have proven the lemma. $\square$

Lemma 21.2. *Definition 18 implies 17.*

Proof. Fix a point $z \in S$ and a large disk $D = D(z, R)$ around z. For n large enough, Ω_n contains $D(z, 2R)$ and so $\Omega_n(R)$ contains $D(z, R)$. So if R is large enough, κ is as small as we wish in $\Omega_n^*(R) = \Omega_n \setminus \Omega_n(R)$. Lemma 21.1 with $\epsilon = 1/2$ then implies

$$\int_{D(z,R)} \kappa^2 d\mathrm{A}_\rho \leq 2C(S,t) + 2[L_\rho(\partial\Omega_n) - \mathrm{A}_\rho(\Omega_n)].$$

The first term on the right is independent of n, and Definition 18 says the second term is bounded independent of n. Therefore

$$\int_{D(z,R)} \kappa^2 d\mathrm{A}_\rho = O(1),$$

with a bound independent of R. Taking $R \nearrow \infty$ and applying the Monotone Convergence Theorem shows $\int_{S_t^*} \kappa^2 d\mathrm{A}_\rho < \infty$, as desired. $\square$

22. (20) $\Rightarrow$ (19): FROM DYADIC DOMES TO RENORMALIZED AREA

In Section 20 we showed that Definition 15 ($\delta \in L^2$) implies Definition 19 ($\mathcal{RA} < \infty$) for planar curves by using a result of Seppi [117] that bounds the principle curvatures at a point z of a minimal surface in terms of $\delta(z)$, the local thickness of the hyperbolic convex hull of Γ. His proof is written for curves in $\mathbb{R}^2$ and surfaces in $\mathbb{H}^3$, but it seems very likely that his estimate remains valid for curves in $\mathbb{R}^n$ and minimal currents or chains in $\mathbb{H}^{n+1}$. However, since I lack an explicit reference for this extension, I provide an alternate approach for the higher dimensional case. We will show that Definition 19 follows from Definition 20 using a result from Seppi's paper [117], that does easily extend to higher dimensions.

We recall from the discussion of minimal currents and 2-chains in Section 6 that if Γ satisfies Definition 11, then it is the asymptotic boundary of a minimal 2-chain whose restriction to $\mathbb{H}^{n+1}_t = \{(x,s) \in \mathbb{H}^{n+1} : s < t\}$ agrees with a minimal surface S that is a topological annulus, has one boundary component on $\mathbb{R}^n \times \{t\}$, and has asymptotic boundary Γ. Lemma 1.4 in Lin's paper [82] shows that on a unit hyperbolic neighborhood of any point $z \in S$, S is a Lipschitz graph with respect to a vertical 2-plane with Lipschitz constant $o(1)$, i.e., it tends to 0 as $t \searrow 0$. In particular, the path metric on S is comparable to the ambient metric with constant tending to 1 as $t \searrow 0$. We want to show that this Lipschitz constant near $z = (x,t) \in S$ bounded by $O(\varepsilon_\Gamma(Q))$, where ε_Γ is as in Definition 14 and $Q \subset \mathbb{R}^n$ is the dyadic cube containing x and with $t < \ell(Q) \leq 2t$.

Suppose P is a n-dimensional geodesic plane in $\mathbb{H}^{n+1}$ and let $u(z) = \sinh d_{\mathbb{H}}(z,P)$ where $d_{\mathbb{H}}$ denotes the signed distance in $\mathbb{H}^{n+1}$ from z to P. Then Proposition 2.4 in [117] proves for $n=2$ that

$$\Delta_S u = 2u, \tag{22.1}$$

where $\Delta_S = \text{trace}(\nabla^S_v u)$ is the Laplace-Beltrami operator on S. The same proof works in higher codimension, except that certain terms that give projections onto the normal vector to S are replaced by the projection into the (multi-dimensional) space of normal vectors.

Definition 14 says that if $z \in S$ and Q are as above, then S is trapped between disjoint half-spaces that are at most $O(\varepsilon_\Gamma(Q))$ apart (in the hyperbolic metric) and that these half-spaces are separated by a vertical n-plane. Thus, as explained in [117], the Schauder estimates for elliptic PDE imply that $|\nabla u(z)| = O(\varepsilon_\Gamma(Q))$. The same estimate holds for $(n-1)$ mutually orthogonal choices of hyperplanes P passing through z and that are also orthogonal to the vertical direction and direction locally parallel to Γ. Since the distance function to each of these on S is Lipschitz with constant $O(\varepsilon_\Gamma(Q))$, we see that S can be parameterized by a Lipschitz function normal to a vertical 2-plane. The Schauder estimates also require that we have uniform bounds on the curvature of S, but this is standard and explained in [117].

An alternative approach that avoids using the Schauder estimates is to consider conformal map φ from the unit disk into a neighborhood of the point z on S. By standard potential theory on the disk, Equation (22.1) implies that $u\circ\varphi$ can be written as the sum of a harmonic function U bounded by $O(\varepsilon_\Gamma(Q))$ and the convolution V of $\log 1/|z|$ against a function bounded by $O(\delta)$. On a strictly smaller disk, $|\nabla U|$ is bounded by $O(\varepsilon_\Gamma(Q))$ by Harnack's inequality, and $|\nabla V|$ satisfies the same estimate because the gradient is given by convolution of $1/z$, which is in $L^1(dxdy)$, with a function bounded by $O(\varepsilon_\Gamma(Q))$. Combined with Lin's estimate showing the intrinsic path metric and ambient metrics are comparable, this gives an alternate proof that u restricted to S is Lipschitz with constant $O(\varepsilon_\Gamma(Q))$.

Lemma 22.1. *Let X denote the dyadic cylinder associated to Γ. If X has finite renormalized area, then $\mathrm{A}_\rho(S_t) - \mathrm{A}_\rho(X_t)$ has a finite limit as $t \searrow 0$*

Proof. If Definition 20 holds, so does Definition 14. For each vertical rectangle R making up a side (or a "panel") of X, we have a Lipschitz map from this panel to a portion of S that changes area by at most an additive factor of $O(\varepsilon_\Gamma^2(Q))$, where Q is the dyadic cube associated to the center of R. Due the vertical "hinges" between adjacent panels, some points of S might be hit twice or not at all by the Lipschitz maps associated to those panels. However, the angles between these panels are bounded by $O(\varepsilon_\Gamma(Q))$ and hyperbolic distance between S and X is also bounded by $O(\varepsilon_\Gamma(Q))$. Thus the total error is at most $O(\varepsilon_\Gamma^2(Q))$, which is summable over all the panels of X. Thus the difference between the hyperbolic areas of S and X above height t has a finite limit $t \searrow 0$. □

Lemma 22.2. *With X as above, $L_\rho(S_t) - L_\rho(X_t)$ had a finite limit as $t \searrow 0$*

Proof. The same argument as in the previous lemma works again: the Lipschitz map from each panel of X to S, preserves length up to an additive factor of $O(\varepsilon_\Gamma^2(Q))$ and the errors caused by the corners are bounded by the same magnitude. □

If Definition 20 holds, then $\lim_{t\searrow 0} \mathrm{A}_\rho(X_t) - L_\rho(\partial X_t)$ exists and is finite. The preceding lemmas imply the same for S, so it also has finite renormalized area.

23. Some examples

In this section we give some examples promised in the introduction that compare Loewner energy, Möbius energy and renormalized area, to illustrate how they can differ. For the sake of brevity, many details are omitted.

Example 1: We take Γ to be the nth generation approximation to the von Koch snowflake, with the corners rounded at scale 3^{-n}, as shown in Figure 12. Then $|\mathcal{RA}(\Gamma)| \simeq \ell(\Gamma) = 3 \cdot 4^n$ by Lemma 20.2. We can estimate the Möbius energy by cutting $\Gamma \times \Gamma$ minus the diagonal into Whitney pieces naturally adapted to the construction of Γ as in Section 8; we easily get $\text{Möb}(\Gamma) \simeq 4^n$. Finally, standard estimates on conformal maps and Bloch functions show that the integral of $\iint_Q |f''/f'|^2 dxdy \simeq 1$ on every Whitney square in $\mathbb{D}$ that maps to a region of Euclidean diameter larger than 3^{-n}. Therefore the integral over the whole disk is dominated by the layer of $\simeq 4^n$ squares that map to diameter $\simeq 3^{-n}$. The same argument applies to the map g from $\mathbb{D}^*$ to the outside of Γ, so $\mathcal{LE}(\Gamma) \simeq 4^n$.

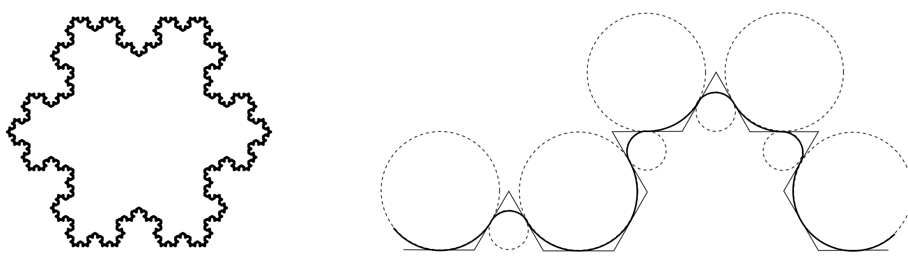

Figure 12. The smoothed nth generation approximation to the snowflake has $\mathcal{LE}(\Gamma) \simeq \text{Möb}(\Gamma) \simeq |\mathcal{RA}(\Gamma)| \simeq 4^n$.

Example 2: Consider the "snake" curve Γ on the right side of Figure 13; it has length $\simeq N^2$, comes within distance 1 of every point in a $N \times N$ box, and at each point on Γ has a tangent unit disk both inside and outside Γ. For this curve, Lemma 20.2 again implies $|\mathcal{RA}(\Gamma)| \simeq \ell(\Gamma) \simeq N^2$ and similar estimates on the conformal maps shows that $|f''/f'|$ has integral $\simeq 1$ on $\simeq N^2$ Whitney squares that map on the "axis" of the snake region and decays exponentially fast away from this axis. A similar argument works for the outside, except that now the "large" Whitney squares are located along a "comb" with N arms and $\simeq N$ squares per arm. This implies

$\mathcal{LE}(\Gamma) \simeq \ell(\Gamma) \simeq N^2$. For each $x \in \Gamma$,

$$\int_\Gamma \frac{dy}{|x-y|^2} \simeq \sum_{k=1}^N \int_{y\in\Gamma: 2^n<|x-y|2^{n+1}} \frac{dy}{|x-y|^2} \simeq \sum_{k=1}^N \frac{1}{k} \simeq \log N,$$

and then integrating over $x \in \Gamma$ gives $\text{Möb}(\Gamma) \simeq \ell(\Gamma)\log N \simeq N^2 \log N$.

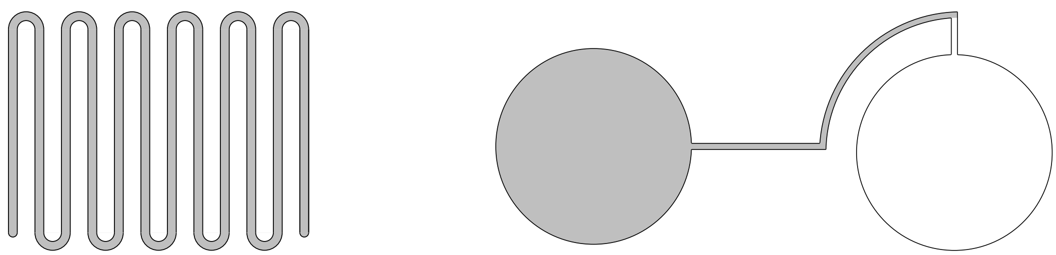

FIGURE 13. On the left is Example 2: the "snake" region that has $\text{Möb}(\Gamma) \simeq N^2 \log N$ and $|\mathcal{RA}(S)| \simeq \mathcal{LE}(\Gamma) \simeq N^2$. On the right is Example 3: the "bent barbell" has a subdomain that is a "disk-with-tube" (shaded). The renormalized areas for these two domains differ by only $O(1)$, but their Loewner energies differ by $\simeq N$.

Example 3: consider the "bent barbell" curve Γ illustrated on the right in Figure 13:3 two disks of radius $N \gg 1$ connected by a tube of width 1 and length $\simeq N$. The shaded subregion is bounded by a second curve Γ' which is a disk with just the tube attached. It is important that the distance from the second disk to the tube (except for the final vertical portion) is larger than 1, say distance 100. This ensures that a simply connected minimal surface for Γ approximates the corresponding surface for Γ' with an additional approximate hemisphere over the second disk and a connecting "tunnel" over the vertical part of the tube. Assume the corners of both curves have been rounded at scale 1. Then the arguments above show that for both curves the $1 \times n$ tube contributes $\simeq N$ to the renormalized area and the rest contributes only $O(1)$. Thus $|\mathcal{RA}(\Gamma)| \simeq |\mathcal{RA}(\Gamma')| \simeq N$ but $|\mathcal{RA}(\Gamma) - \mathcal{RA}(\Gamma')| \simeq 1$. A straightforward calculation shows that $\text{Möb}(\Gamma) \simeq N$ as well.

On the other hand, adding the extra disk increases $\int_{\mathbb{D}} |f''/f'|^2$ by about $\log N$ and, even more importantly, $\int \mathbb{D}^* |g''/g'|^2$ increases by $\simeq N$, due to the new "tube" of width 100 and length $N \gg 100$ that is formed. The values of $f'(0)$ (assume 0 is mapped to the center of the left disk) and $g'(\infty)$ are both changed by at most $O(1)$. Thus the Loewner energy for the two curves differs by $\simeq N$.

Example 4: This one comes from Michael Anderson's paper [7]. Consider the curve shown on the top left in Figure 14: it has two concentric circular arcs of radii N and $N+1$ connected by two parallel arcs distance $1/M$ apart, and the corners are rounded at scale $1/M$. In this case the area minimizer S_1 is topologically a punctured torus $(\chi(S_1) = -1)$. The general shapes of S_1 and the simply connected minimal surface S_0 $(\chi(S_0) = 1)$ are illustrated on top row of Figure 14, They are somewhat easier to visualize in the ball model, i.e., the lower row in Figure 14. These are only sketches of the general shape, not computations of the actual surfaces (drawn with [103]).

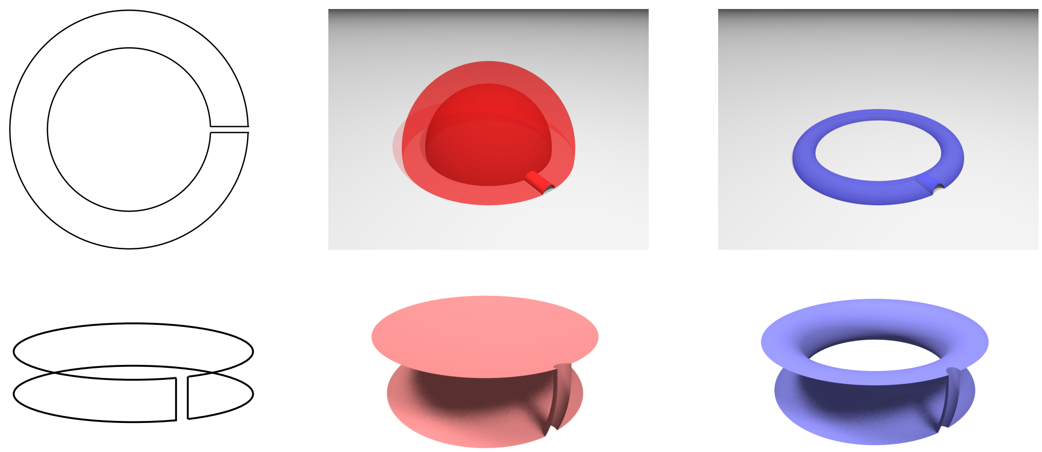

FIGURE 14. The example from Anderson's paper [7] where the absolute area minimizer is not simply connected; it can have very different renormalized area from the simply connected minimal surface. Top row shows the upper half-space model and the lower row is in the ball model.

One can show that $\mathcal{RA}(S_1) \simeq -N - M$, whereas $\mathcal{RA}(S_0) \simeq -M$. By choosing $N \gg M$ we see that the renormalized areas can quite different for the same Γ. In this case both $\mathcal{LE}(\Gamma)$ and Möb(Γ) are $\simeq N + M$, so we should take the absolute area minimizer S_1 if we want the three quantities to all be comparable in size.

24. Remarks and questions

• **Comparing different quantities:** The work of Takhtajan and Teo [125], Rohde and Wang [110] and Viklund and Wang [126] includes many explicit formulas relating the Dirichlet norm of $\log f'$ to the Kahler potential of the Weil-Petersson metric on universal Teichmuller space and the Loewner energy of the curve Γ. Are there similar formulas that relate these quantities to quantities discussed in this paper, e.g., Möbius energy, β^2-sums, Menger integrals, the curvature integral of a minimal surface associated to Γ or the renormalized area of this curve? If there is more than one such minimal surface, which surface?

The examples in Section 23 indicate there may not be any simple relation between these different quantities, but the estimates in this paper should prove that $\mathcal{LE}(\Gamma)$, $\text{Möb}(\Gamma)$, $|\mathcal{RA}(\Gamma)|$ and $\sum_Q \beta_\Gamma^2(Q)$ should all be comparable for quasicircles with small constant. Example 2 in Section 23 shows that $\text{Möb}(\Gamma)$ need not be comparable in size to the other three in general, but are the other three always comparable to each other (at least when the values are large)?

• **Other knot energies:** There are a variety of other knot energies besides Möbius energies. For example,

$$E^{j,p}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|x-y|^j} - \frac{1}{\ell(x,y)^j} \right)^p dxdy,$$

blows up for self-intersections if $jp \geq 2$ and is finite for smooth curves if $jp \leq 2p+1$. Sobolev smoothness properties for curves with finite $E^{j,p}$ energy are studied by Blatt in [20] (but there is a typo in Theorem 1.1, s should be $s = (jp-1)/(2p)$).

Another class of knot energies considered in [123] are the Menger energies

$$\mathcal{M}_p(\Gamma) = \int_\Gamma \int_\Gamma \int_\Gamma c^p(x,y,z)|dx||dy||dz|,$$

with $\mathcal{M}_2(\Gamma)$ being the usual condition that is equivalent to rectifiability (see Section 4). They show that for $p \geq 3$, finite energy curves are Jordan curves and for $p > 3$ then are even $C^{1,\alpha}$ and establish bounds on the β-numbers. The endpoint case $p = 3$ seems the most interesting, as this is the only scale-invariant Menger energy. Since $c(x,y,z) \lesssim 1/\ell(x,y,z)$, $\mathcal{M}_3(\Gamma)$ is less restrictive than the Weil-Petersson condition. What are the corresponding geometric characterizations of these curves?

• **Gradient flow on energy:** The idea of knot energies is that if we start with a knot in $\mathbb{R}^3$, and flow "downward", e.g., follow the gradient of energy, then the curve never crosses itself and we end at a "nice" representative of the same knot. This flow has been studied by various investigators; see [67], [108]. For Weil-Petersson curves in the plane does this flow always lead to a circle? Is it related to the Weil-Petersson metric or geodesics for this metric?

• **Wildness of fixed points of involutions:** Fixed point sets of C^1 involutions are locally flat by a result of Bochner [21], but this can fail for involutions in the Sobolev space $W^{1,p}$, $1 \leq p < 2$, see [98]. Can the fixed point sets of $W^{1,2}$, quasiconformal and biLipschitz involutions be wild curves?

• **Length convergence on minimal surfaces:** The estimates in this paper prove that if Γ is Weil-Petersson, S is a minimal surface with asymptotic boundary Γ and Γ_t is the curve on S at height t above the boundary, then

$$\int_0^1 |\ell(\Gamma_t) - \ell(\Gamma)| \frac{dt}{t^2} < \infty.$$

Does the converse hold? The direction stated above follows by writing

$$|\ell(\Gamma_t) - \ell(\Gamma)| \leq |\ell(\Gamma_t) - \ell(\Gamma_n)| + |\ell(\Gamma_n) - \ell(\Gamma)|,$$

where Γ_n is the usual dyadically inscribed polygon with $2^{-n-1} < t \leq 2^{-n}$. The second term on the right is integrable by Theorem 1.3, and is controlled using the β-numbers at scales smaller than t. The first term is controlled by Seppi's estimate and the ε-numbers at scale t; these, in turn, are controlled by sums of β-numbers over scales larger than t. Thus the question is whether $\ell(\Gamma_t)$ can be a much better approximation to $\ell(\Gamma)$ than $\ell(\Gamma_n)$ for some non-Weil-Petersson curves?

• **Harmonic measure on curves in $\mathbb{R}^3$:** Harmonic measure on a plane curve can be defined as the hitting distribution of a Brownian motion. In higher dimensions, a Brownian motion almost surely never hits a fixed smooth curve, so this does not make sense. An alternative possibility is to think harmonic measure as the equilibrium probability measure that minimizes energy with respect to the Newton kernel ($\log \frac{1}{z}$ in the plane, $|x|^{n-2}$ in $\mathbb{R}^n$). For curves in $\mathbb{R}^n$ no measure supported on a smooth curve gives finite energy, but we there are finite energy probability measures on an ϵ-thickening of the curve. In particular there is an energy minimizing one (the equilibrium distribution of a unit charge constrained to stay on the ϵ-tube). What measure do these converge to as $\epsilon \nearrow 0$? Is it absolutely continuous with respect to arc-length? Are $H^{3/2}$ curves in $\mathbb{R}^4$ characterized by some property of this measure, e.g., being comparable to arclength measure? Perhaps there some connection to recent work of David, Engelstein, Feneuil, Mayboroda defining harmonic measure on subsets of $\mathbb{R}^n$ with codimension greater than 1. See [38], [37].

• **Möbius energy and SLE:** As we discuss briefly in Appendix A, Weil-Petersson curves are related to the large deviations theory of Schramm-Loewner evolutions (SLE) as the parameter κ tends to zero. It is intriguing that they are also characterized in terms of the rate of blow-up of a self-repulsive energy that prevents self-intersections. Is there some more direct connection between these two ideas? A SLE(κ) curve has Hausdorff dimension $1+\kappa/8$ for $0<\kappa\le 8$ and we expect the ϵ-truncation of the energy integral for an α-dimensional measure and kernel $|x|^{2-d}$ to grow like $\epsilon^{2-d+\alpha}$. Do SLE paths have energy that grows like $\epsilon^{-1+\kappa/8}$, or are they, in some sense, optimal among such curves?

Is there something interesting to say regarding hyperbolic convex hulls and minimal surfaces of an SLE path when $\kappa > 0$, e.g. can we compute an "expected curvature" for the corresponding minimal surface? When $\kappa \ge 8$ the paths become plane filling, but do the corresponding minimal surfaces still make sense and if so, can we characterize their properties (e.g., growth rate of renormalized area) in terms of κ?

• **Brylinski's beta function:** Given a rectifiable curve $\gamma \subset \mathbb{R}^3$, in [31] Jean-Luc Brylinski defines a function $B_\gamma(s) = \int_\gamma \int_\gamma |z-w|^s dzdw$, where dz, dw denote arclength measure. This integral converges and defines a holomorphic function for $\mathbb{R}(s) > 0$. For example, if γ is a circle, then he shows

$$B_\gamma(s) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{s+1}{2})}{2\Gamma(\frac{s}{2}+1)},$$

where Γ is the usual Gamma function, the analytic extension of $\int_0^\infty x^{s-1}e^{-x}dx$. More generally, he shows that if γ is smooth, then $B_\gamma(s)$ extends to be meromorphic on the whole plane with poles only possible at the negative odd integers. He also shows that for smooth curves $B_\gamma(-2) = \text{Möb}(\gamma) - 4$ and

$$\text{Möb}(\gamma, s) = \int_\gamma \int_\gamma (\ell(z,w)^s - |z-w|^s) dzdw,$$

extends to be holomorphic in $\mathbb{R}(s) > -3$. Is this extension (or something like it) true for all Weil-Petersson curves, not just the smooth ones?

• **Renormalized volume of hyperbolic 3-manifolds:** Let G be a quasi-Fuchsian group, M its hyperbolic quotient 3-manifold, R_1, R_2 the two Riemann surfaces comprising the boundary at ∞ of M, and Γ its limit set. There are a variety of papers that relate the volume $\text{CH}(\Gamma)$, the renormalized volume of M, and the Weil-Petersson distance between R_1 and R_2. For example, see [28], [29], [75], [115]. The ideas in these papers seem very similar to our results characterizing Weil-Petersson curves Γ in terms of the "thickness" of the hyperbolic convex hull of Γ and the renormalized area of a surface with boundary Γ. Is there a precise connection between the results of this paper and the papers mentioned above? In [125], Takhtajan and Teo show that the usual Weil-Petersson metric for compact surfaces can be recovered from their Weil-Petersson metric on the universal Teichmüller space. Is this helpful in making the connection suggested above?

• **Computing minimal surfaces:** Given a planar closed curve Γ can we efficiently compute a minimal surface $S \subset \mathbb{H}^3$ that has Γ as its asymptotic boundary? Can we compute all such surfaces? Efficiently compute the number of minimal surfaces with boundary Γ?

• **Detecting Weil-Petersson components of $T(1)$:** The Hilbert manifold topology of Takhtajan and Teo divides the universal Teichmüller space into uncountable many connected components. Can we geometrically characterize when two curves belong to the same component? The current paper has done this for the component containing the unit circle. Perhaps some condition can be given saying that the convex hulls are quasi-isometric with constants that tend to 1 in a square integrable sense near the boundary of hyperbolic space. Are Γ_1, Γ_2 in the same component iff $\Gamma_2 = f(\Gamma_1)$ for some planar QC map f whose dilatation is in L^2 for hyperbolic area on the complement of Γ_1? This may be known.

A closely related problem is to construct a natural section for universal Teichmüller space, i.e., a natural choice of one quasicircle from each connected component. A good starting point might be Rohde's paper [109] that gives such a choice for quasicircles modulo biLipschitz images.

• **Characterizing subsets of Weil-Petersson curves:** Peter Jones's traveling salesman theorem characterizes the subsets of the plane that lie on some rectifiable curve by $\sum_Q \beta_E^2(Q)\text{diam}(Q) < \infty$. Does the analogous sum $\sum_W \beta_E^2(Q) < \infty$ characterize subsets of Weil-Petersson curves? This condition is obviously necessary since the β^2-sum for any such curve would dominate the sum for the set. More generally, if we define a collection of sets by the convergence of some series involving β-numbers, is every set in that collection always contained in a curve from that collection?

• **Curves with smoothness between Weil-Petersson and rectifiable:** What can we say about a curve if e.g.,

$$\sum_Q \beta_\Gamma^2(Q)\mathrm{diam}(Q)^s < \infty,$$

a condition interpolating between rectifiability ($s = 1$) and the Weil-Petersson class ($s = 0$)? Are these $H^{(3-s)/2}$-curves? See Corollary 2 of [51], but β means something different there and is not directly comparable to our β-numbers. Similar sums occur in [9] and [10] related to Hölder parameterizations of curves. In [64], Silvia Ghinassi considers curves for which

$$\int_0^1 \beta_\Gamma^2(x,t)t^{-2\alpha}dt < M < \infty,$$

and shows they have parameterizations that are $C^{1,\alpha}$, i.e., f' is α-Hölder. Definition 11 implies the Weil-Petersson class forms a subset of the $\alpha = 1/2$ case.

• **Angles of inscribed dyadic polygons:** Suppose $\{z_j^n\}$ are a choice of dyadic points in Γ, as in Theorem 1.3, and

$$\theta(n,k) = \arg\left(\frac{z_{j+1}^n - z_j^n}{z_j^n - z_{j-1}^n}\right),$$

be the angles between adjacent nth generation segments. Using Theorems 1.3 and 1.5, it is not hard to show that if Γ is Weil-Petersson, then

$$\sum_{n=1}^{\infty}\sum_{k=1}^{2^n}\theta^2(n,k) < \infty,$$

with a uniform bound independent of the choice of dyadic base point. Is the converse true? What if we also assume Γ is chord-arc? In general, θ can be zero at a point, even if β is large, e.g., at the center of a spiral. This is reminiscent of the longstanding ϵ^2-conjecture of Carleson, recently proved by Jaye, Tolsa and Villa [72].

• **The medial axis:** The medial axis $\mathrm{MA}(\Omega)$ of a domain Ω is the set of centers of disks $D(x,r) \subset \Omega$ so that $\mathrm{dist}(x,y) = r$ for at least two points $y \in \partial\Omega$. See [55] for its basic properties (it is called the skeleton of Ω there). David Mumford has asked if Weil-Petersson curves can be characterized in terms of the medial axis of their complementary domains. This means we know both the set and the distance function to the boundary, (a line segment, with different distance functions, can be the medial axis of both WP and non-WP curves). The cleanest statement I am aware of is the following. The region $\Omega \setminus \mathrm{MA}(\Omega)$ is foliated by directed line segments that connect each point to its unique nearest point on $\partial\Omega$. For each hyperbolic unit ball $B_\rho(w,1)$ in Ω we assign the supremum of the difference between directions for the segments hitting B. Then Γ is Weil-Petersson iff Γ is chord-arc and this function is in $L^2(\Omega, d\mathrm{A}_\rho)$. See Figure 15. This says Γ is Weil-Petersson iff the nearest point foliation is orthogonal to the boundary with an L^2 error. Is there a "nice" characterization in terms of the medial axis itself?

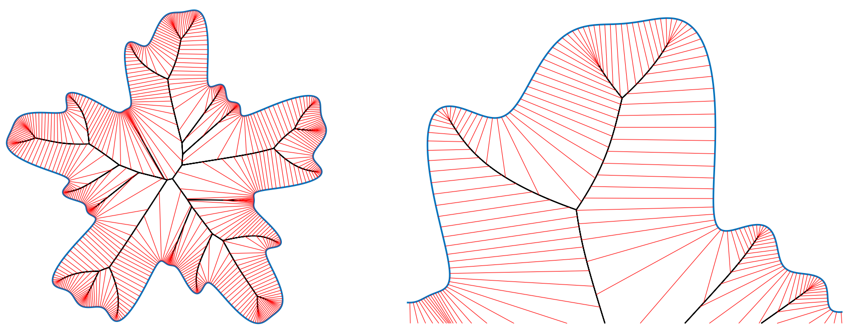

FIGURE 15. A medial axis, nearest point foliation and enlargement.

• **New characterizations of old curve families:** The function theoretic characterizations of the Weil-Petersson class are exactly analogous to known characterizations of other classes, e.g., when $\log f'$ is in VMO [104] or BMO [8], [18]. Do these classes have other characterizations analogous to the ones discussed in this paper? For example, Michel Zinsmeister has asked if anything interesting can be said about the domes and minimal surfaces associated to boundaries of BMO domains.

APPENDIX A. OTHER CHARACTERIZATIONS OF WEIL-PETERSSON CURVES

This appendix lists some further equivalent definitions of the Weil-Petersson class; these definitions were never used in our proofs, but I include them to illustrate the variety of problems in which the Weil-Petersson class naturally occurs. We start with the Takhtajan and Teo's definition that gives the class its name.

• **Teichmüller theory:** Recall that $\mathbb{D} = \{|z| < 1\}$ and $\mathbb{D}^* = \{|z| > 1\}$. Let $L^\infty(\mathbb{D}^*)_1$ denote the unit ball of $L^\infty(\mathbb{D}^*)$. By the measurable Riemann mapping theorem, each $\mu \in L^\infty(\mathbb{D}^*)$ determines a quasiconformal map w^μ of the plane that is conformal inside $\mathbb{D}$, and satisfies $f(0) = f''(0) = 0$, $f'(0) = 1$. We say μ and ν are equivalent if $w^\mu = w^\nu$ on $\mathbb{T}$ and we define $T(1)$ be $L^\infty(\mathbb{D}^*)_1$ quotiented by this equivalence relation. This is the universal Teichmüller space, $T(1)$. In [125], Takhtajan and Teo define a Weil-Petersson metric on universal Teichmüller space $T(1)$, for which $T(1)$ has uncountably many connected components, and $T_0(1)$ denotes the connected component containing the identity. More concretely, let U be the set of holomorphic ϕ on $\mathbb{D}$ so that

$$\int_{\mathbb{D}^*} |\phi(z)|^2(1-|z|^2)^2 dxdy < \sqrt{\pi/3},$$

and for each $\phi \in U$ define a dilatation μ on $\mathbb{D}^*$ by

$$\mu(z) = -\frac{1}{2}(1-|z|^2)^2\phi(1/\overline{z})z^{-4}.$$

Given a fixed dilatation ν on $\mathbb{D}^*$ consider the set of all dilations of the form

$$\lambda = \nu * \mu^{-1}\left(\frac{\nu-\mu}{1-\overline{\mu}\nu}\right)\cdot\frac{(w_\mu)_z}{(w^\mu)_{\overline{z}}}\circ w^\mu.$$

(This just corresponds to composing the corresponding quasiconformal mappings.) This defines a set $V_\nu \subset L^\infty(\mathbb{D}^*)_1$ that contains ν. Projecting these sets into $T(1)$ defines a neighborhood of each point $[\nu] \in T(1)$ and $T_0(1)$ is the connected component of the identity in this topology.

Definition 21. $\Gamma = f(\mathbb{T})$, *where f is a quasiconformal map of the plane, conformal inside $\mathbb{D}$ and whose dilatation on $\mathbb{D}^*$ represents a point of $T_0(1)$.*

In Theorem II.1.12 of [125], Takhtajan and Teo prove that this is equivalent to Definition 2: $T_0(1)$ is the inverse image of $A_2(\mathbb{D})$ under the Bers embedding $\beta : T(1) \to A_\infty(\mathbb{D})$ where

$$A_\infty(\mathbb{D}) = \{\phi \text{ holomorphic on } \mathbb{D} : \sup_{\mathbb{D}} |\phi(z)|(1-|z|^2)^2 dxdy < \infty\}.$$

The Bers embedding is defined by starting with a dilatation $\mu \in L^\infty(\mathbb{D}^*)$ representing a point of $T(1)$, and observing that $S(w^\mu|_{\mathbb{D}}) \in A_\infty(\mathbb{D})$ by (2.5).

• **Operator theory:** Given a circle homeomorphism φ we can define an operator on harmonic functions on the unit disk by pre-composing the boundary values of u with φ, taking the harmonic extension back to the disk, and subtracting the value at the origin (so the resulting harmonic function $P_\varphi u$ is zero at the origin. Given an holomorphic function in the Dirichlet class, we can apply this operator and follow it by orthogonal projection onto the anti-holomorphic Dirichlet functions and finally apply $f(z) \to f(\bar{z})$ to make it holomorphic. Nag and Sullivan [92] proved this operator P_φ^- is bounded from the Dirichlet class to itself if and only φ is quasisymmetric, and Hu and Shen [70] prove it is Hilbert-Schmidt if and only if φ is Weil-Petersson (an operator T on a Hilbert space is Hilbert-Schmidt if $\sum_j \|Te_j\|^2 < \infty$ for any orthonormal basis $\{e_j\}$; equivalently TT^* is trace class).

Definition 22. *P_φ^- is Hilbert-Schmidt on the Dirichlet space.*

Another operator theoretic characterization of the Weil-Petersson class is given by Takhtajan and Teo (Corollary II.2.9, [125]) in terms of Grunsky operators on ℓ^2.

• **Integral geometry:** Another measure of how much Γ deviates from a straight line can be given in terms of how random lines hit Γ. Suppose we parameterize lines L in $\mathbb{R}^2 \setminus \{0\}$ by $(r, \theta) \in (0, \infty) \times (0, 2\pi]$ where $z = r\exp(i\theta)$ is the point of L closest to 0. It is well known fact from integral geometry (e.g., [114]) that the measure $d\mu = dr d\theta$ on lines is invariant under Euclidean isometries of the plane, and the measure of the set of lines hitting a non-degenerate convex set X equals the length of the boundary of X (for a line segment, it is twice the length of the segment). For a dyadic cube Q let $S(Q, \Gamma)$ be the set of lines that hits $3Q$ also hits both $\Gamma \cap \frac{5}{3}Q$ and $\Gamma \cap (3Q \setminus 2Q)$.

Definition 23. *Any translate of Γ satisfies*

$$\sum_Q \frac{\mu(S(Q, \Gamma))}{\operatorname{diam}(Q)} < \infty.$$

The equivalence of Definitions 11 and 23 follows immediately from Theorem 10.2.1 of [19]. Note that $12 \operatorname{diam}(Q)/\sqrt{2}$ is the perimeter of $3Q$ and hence is the μ measure of the set of random lines hitting $3Q$. Dividing the sum by $6\sqrt{2}$, each term becomes the probability that if a line L hits $3Q$, then $L \in S(Q, \Gamma)$:

$$\sum_Q \mathbb{P}(S(Q, \Gamma)|3Q) < \infty. \tag{A.1}$$

Roughly speaking, this is the probability that a random line hitting $3Q$ will hit $\Gamma \cap 3Q$ at two points $\simeq \ell(Q)$ apart. The particular values $\frac{5}{3}$, 2 and 3 in the definition of $S(L, \Gamma)$ are probably not important; just convenient for the proof in [19].

• **Loewner energy:** Suppose $\Omega = \mathbb{C} \setminus [0, \infty)$ and suppose that $\gamma \subset \Omega$ is a curve that connects 0 to ∞. Suppose also that this curve corresponds to driving function

W via Loewner's equation. Then the chordal Loewner energy of γ is defined by Friz and Shekhar [56] and Wang [129] to be

$$I(\gamma) = \frac{1}{2}\int_0^\infty \dot{W}(t)^2 dt.$$

This was generalized to simple closed curves Γ on the Riemann sphere by Steffen Rohde and Yilin Wang [110] by choosing two points $z, w \in \Gamma$ and conformally mapping the complement of the arc $\Gamma_{z,w}$ from z to w to Ω with z, w mapping to $0, \infty$ respectively. The image of $\Gamma \setminus \Gamma_{z,w}$ is now an arc from 0 to ∞ in Ω, so its energy is defined as above. The energy of the loop Γ rooted at z is defined as the limit of these energies as $w \to z$; Rohde and Wang showed this is independent of the choice of z.

Definition 24. Γ *has finite Loewner energy.*

The equivalence with the earlier definitions was proven by Yilin Wang [129]. She showed that the Loewner energy equals $\mathbf{S_1}(\varphi)/\pi$ where $\mathbf{S_1}(\varphi)$ is the universal Liouville action defined by Takhtajan and Teo by

$$\mathbf{S_1}(\varphi) = \iint_{\mathbb{D}} \left|\frac{f''(z)}{f'(z)}\right|^2 dxdy + \iint_{\mathbb{D}^*} \left|\frac{g''(z)}{g'(z)}\right|^2 dxdy + 4\pi \log \frac{|f'(0)|}{|g'(\infty)|}$$

given in the introduction.

• **Large deviations of SLE:** In [128], Yilin Wang interprets finite energy curves γ from 0 to ∞ in $\mathbb{H}^2$ in terms of large deviations of SLE(κ) as $\kappa \searrow 0$. Roughly speaking, the Loewner energy of γ is equal to

$$\lim_{\epsilon \to 0} \left[\lim_{\kappa \searrow 0} \log \mathbb{P}[\mathrm{SLE}(\kappa) \in B(\gamma, \epsilon)] \right].$$

In words, this is the probability that SLE stays in an ϵ-neighborhood of γ decreases exponentially with decay factor equal to the energy of γ. In fact, Wang's result is not stated using Hausdorff neighborhoods, but in terms of sets of curves that pass to the left or right of a specified finite set of points. A little more precisely, suppose we are given a finite set Z of points $\{z_n\}$ in the upper half-plane and each point is labeled with ± 1. A curve γ from 0 to ∞ cuts the upper half-plane into simply connected regions, that we call the "left side" and "right side". A curve γ is called admissible for Z (written $\gamma \in \mathcal{A}(Z)$) if every point labeled $+1$ is on the right side of γ and every point labeled -1 is on the left side. If γ is admissible for Z, then we say that Z is consistent with γ and we write $Z \in \mathcal{Z}(\gamma)$. Wang shows that given a set Z,

$$-\lim_{\kappa \to 0} \kappa \log \mathbb{P}[\mathrm{SLE}(\kappa) \in \mathcal{A}(Z)] = \inf\{I(\gamma) : \gamma \in \mathcal{A}(Z)\}.$$

Thus the Weil-Petersson class can be defined using the condition

Definition 25. $\sup_{Z \in \mathcal{Z}(\gamma)} \lim_{\kappa \to 0} (-\kappa) \log \mathbb{P}\left[\mathrm{SLE}(\kappa) \in \mathcal{A}(Z)\right] < \infty$.

Roughly speaking, a curve in $\mathbb{H}^2$ from 0 to ∞ is a (sub-arc of a spherical) Weil-Petersson curve iff for any finite set of labeled points Z consistent with γ, the probability that $\mathrm{SLE}(\kappa)$ is also consistent with Z decays at most exponentially quickly as $\kappa \to 0$. See [128] for precise statements and further details.

• **Brownian loop soup:** The Brownian loop measure, introduced by Greg Lawler and Wendelin Werner [77] is a measure on closed loops in a domain Ω. It is conformally invariant and if $\Omega' \subset \Omega$, then the loop measure on Ω' is the just the restriction of the loop measure for Ω to loops that are contained in Ω'. Given disjoint compact subsets of Ω we define $\mathcal{W}(A, B; \Omega)$ to be the loop measure of closed curves γ in Ω so that the outer boundary of γ hits both A and B. Suppose Γ^r is the image of the circle $\{|z| = r\}$ under a conformal map from $\mathbb{D}$ to the interior of Γ. Yilin Wang proves in [127] that the Loewner energy of Γ is 12 times

$$\lim_{r \to 1} \left[\mathcal{W}(S^1, r \cdot S^1, \mathbb{C}) - \mathcal{W}(\Gamma, \Gamma^r, \mathbb{C})\right].$$

Thus being a Weil-Petersson curve is equivalent to:

Definition 26. Γ *satisfies* $\lim_{r \to 1} \left[\mathcal{W}(S^1, r \cdot S^1, \mathbb{C}) - \mathcal{W}(\Gamma, \Gamma^r, \mathbb{C})\right] < \infty$.

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