CURVES OF FINITE TOTAL CURVATURE AND THE
WEIL-PETERSSON CLASS

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Abstract. This paper gives various geometric characterizations of a certain collection of rectifiable quasicircles, known as the Weil-Petersson class; these curves correspond to the closure of the smooth curves in the Weil-Petersson metric on universal Teichmüller space defined by Takhtajan and Teo [75]. This class arises naturally in a variety of settings including geometric function theory, Teichmüller theory and probability. The new characterizations all say that a curve is in the Weil-Petersson class if and only if some measure of local curvature is square integrable over all locations and scales. The local curvature of a curve $\Gamma$ is measured using various quantities such as Peter Jones’ beta-numbers, lengths of inscribed polygons, Menger curvature, the difference between arc-length and chord-length, intersections with random lines, Sobolev norms of the tangent directions, the “thickness” of the hyperbolic convex hull of $\Gamma$, and the total curvature of a minimal surface in hyperbolic 3-space with $\Gamma$ as its asymptotic boundary. We also give a connection between Weil-Peterson curves and renormalized area of the corresponding hyperbolic minimal surface.

Date: June 27, 2019.

1991 Mathematics Subject Classification. Primary: 30C62, 30F60 Secondary:

Key words and phrases. universal Teichmüller space, the Weil-Petersson class, quasicircles, chord-arc curves, beta numbers, Schwarzian derivatives, Dirichlet class, minimal surfaces, renormalized area, finite total curvature, Loewner energy.

The author is partially supported by NSF Grant DMS 1906259.
1. **The Weil-Petersson class**

Given a Jordan arc $\gamma \subset \mathbb{R}^2$ with endpoints $z$ and $w$, let $\text{diam}(\gamma)$ denote its diameter, let $\ell(\gamma)$ be its arc-length, and let $\text{crd}(\gamma) = |z - w|$ be its “chord-length”. Note that $\text{crd}(\gamma) \leq \text{diam}(\gamma) \leq \ell(\gamma)$. Two important classes of closed Jordan curves $\Gamma$ are the quasicircles (quasiconformal images of a circle) and the sub-class of chord-arc curves (bi-Lipschitz images of a circle). Each class has a simple geometric characterization: $\Gamma$ is a quasicircle if and only if

$$\text{diam}(\gamma) = O(\text{crd}(\gamma)),$$

for all subarcs $\gamma \subset \Gamma$ with $\text{diam}(\gamma) \leq \text{diam}(\Gamma)/2$, and $\Gamma$ is a chord-arc curve if and only if it is rectifiable and for all subarcs $\gamma \subset \Gamma$ with $\ell(\gamma) \leq \ell(\Gamma)/2$ we have

$$\ell(\gamma) = O(\text{crd}(\gamma)).$$

A certain sub-class of the chord-arc curves, called the Weil-Petersson class, arises in a number of different settings. The name comes from Teichmüller theory. Each quasicircle $\Gamma$ corresponds to a point of the universal Teichmüller space $T(1)$, and in [75] Takhtajan and Teo define a Hilbert manifold structure on $T(1)$ that is related to the Weil-Petersson metric on finite dimensional Teichmüller spaces. This topology has uncountably many connected components, and the one containing the origin is denoted $T_0(1)$, and corresponds to the class of Weil-Petersson curves. Alternatively, Weil-Petersson curves are the closure of the smooth curves in the Takhtajan-Teo topology. See Appendix D for the original definition, but a simpler and equivalent one is the following: $\Gamma$ is Weil-Petersson if it is a quasicircle and if $\Gamma = f(\partial \mathbb{D})$, where $f$ is a conformal of the unit disk with $\log f'$ is in the Dirichlet class, i.e.,

$$\int_{\mathbb{D}} |(\log f')'|^2 dxdy = \int_{\mathbb{D}} \left| \frac{f''}{f'} \right|^2 dxdy < \infty. \quad (1.1)$$

There are other equivalent definitions (described in the next section), in terms of function theory, but none using the intrinsic geometry of the curve $\Gamma$ (e.g., see Remark II.1.2 of [75]). The purpose of this paper is to provide several such geometric characterizations. The most fundamental one is in terms of Peter Jones’ $\beta$-numbers [47]. Given a curve $\Gamma$, define

$$\beta_\Gamma(x, t) = \inf_L \sup_{z \in D(x, t)} \frac{\text{dist}(z, L)}{t},$$
where the infimum is over all lines hitting $x$ (we take the infimum to be zero if the disk does not hit $\Gamma$). We shall prove

**Theorem 1.1.** A closed Jordan curve $\Gamma \subset \mathbb{R}^2$ is Weil-Petersson if and only if

\[
\int_{\mathbb{R}^2} \int_0^\infty \beta_1^2(x,t) \frac{dxdt}{t^3} < \infty.
\]

(1.2)

This should be compared with Peter Jones’ traveling salesman theorem. In the special case of a Jordan curve $\Gamma$, his theorem says that

\[
\ell(\Gamma) = O\left(\text{diam}(\Gamma) + \int_{\mathbb{R}^2} \int_0^\infty \beta_1^2(x,t) \frac{dxdt}{t^2}\right).
\]

(1.3)

Thus the condition in Theorem 1.1 is a strengthening of Jones’ condition (1.3) characterizing rectifiability (for bounded sets, both integrals are automatically convergent near $\infty$). Analogs of Jones’ theorem are known in $\mathbb{R}^n$ [59] and Hilbert space [70], and some metric spaces [24], [35], [52], [53].

Condition (1.2) has many equivalent formulations. The following is one of the simplest. Choose a dyadic decomposition of $\Gamma$, i.e., nested, ordered collections of points $\{z_j^n\}$, $n \in \mathbb{N}$, $j = 1, \ldots, 2^n$ that for each $n$ divide $\Gamma$ into $2^n$ disjoint subarcs of length $2^{-n}\ell(\Gamma)$. Note that this depends on the choice of some basepoint $z_1^0$. Let $\Gamma_n$ be the inscribed polygon with vertices $\{z_j^n\}$ (vertices in order around $\Gamma$). Clearly $\ell(\Gamma_n) \nearrow \ell(\Gamma)$ and the Weil-Petersson class is characterized by the rate of this convergence:

**Theorem 1.2.** With notation as above, a curve $\Gamma$ is Weil-Petersson if and only if

\[
\sum_{n=1}^\infty 2^n \left[\ell(\Gamma) - \ell(\Gamma_n)\right] < \infty
\]

(1.4)

with a bound that is independent of the choice of the dyadic decomposition of $\Gamma$.

The proof of this requires a refinement of Peter Jones’ traveling salesman theorem [47], that may be of interest in its own right:

**Theorem 1.3.** If $\Gamma \subset \mathbb{R}^2$ is a Jordan arc, then

\[
\ell(\Gamma) = \text{crd}(\Gamma) + O\left(\int_{\mathbb{R}^2} \int_0^\infty \beta_1^2(x,t) \frac{dxdt}{t^2}\right).
\]

(1.5)

The point of Theorem 1.3 is that the diam(\Gamma) term in (1.3) can be replaced by the smaller value crd(\Gamma), and that this term is only multiplied by “1” in the estimate (1.5). This will be crucial for proving the “Jones Conjecture” described in Section 3.
The integral (1.2) is clearly invariant under rescaling by Euclidean similarities and we will see later that the finiteness of this integral is invariant under Möbius transformations of \( \Gamma \) that keep \( \Gamma \) bounded, so it is natural to seek characterizations in terms of quantities that are Möbius invariant. For example, consider the upper half-space \( \mathbb{H}^3_+ = \{(x, y, t) : x, y \in \mathbb{R}, t > 0\} \) with its hyperbolic metric. We will prove:

**Theorem 1.4.** \( \Gamma \subset \mathbb{R}^2 \) is a Weil-Petersson curve if and only if it is the boundary of a smooth embedded surface \( S \subset \mathbb{H}^3_+ \) that is asymptotically flat with finite total curvature, i.e., \(|K(z)| \to 0\) as \( z \) tends to \( \mathbb{R}^2 \) though \( S \) and \( \int_S |K(z)|^2dA_\rho(z) < \infty \), where \( K \) is the second fundamental form on \( S \). If these conditions hold for any surface with boundary \( \Gamma \), then they hold for any embedded minimal surface with boundary \( \Gamma \).

This condition is clearly invariant under isometries of hyperbolic space. Michael Anderson [6] has shown that every closed Jordan curve on \( \mathbb{R}^2 \) bounds an embedded minimal surface in \( \mathbb{H}^3_+ \) that is topologically a disk. (For simplicity, “surface” in this paper will always mean an embedded surface, although our results might hold more generally.) Note that any surface in \( \mathbb{H}^3_+ \) that meets \( \mathbb{R}^2 \) in a non-degenerate curve \( \Gamma \) has infinite hyperbolic area, so “minimal” means with respect to compact perturbations. For a minimal surface, the mean curvature vanishes and hence the second fundamental form \( K \) agrees with the trace free second fundamental form \( K_0 \). Hence we can replace \( K \) by \( K_0 \) in the theorem above. The integral \( \int_S |K_0|^2dA_\rho \) is called the Willmore energy of the surface \( S \), so Theorem 1.4 can be restated by saying \( \Gamma \) is Weil-Petersson if and only if it bounds a surface in \( \mathbb{H}^3_+ \) of finite Willmore energy. (This use of the term “Willmore energy” differs from the usual definition for compact surfaces; see page 370 of [5].)

Finite total curvature is closely related to another concept: finite renormalized area. The hyperbolic area of a surface \( Y \subset \mathbb{H}^3_+ \) whose boundary is a non-trivial curve \( \Gamma \subset \mathbb{R}^2 \) is always infinite but we say that it has finite renormalized area if

\[
\mathcal{A}_R(Y) = \lim_{t \to 0} \left[ A_\rho(Y_t) - \ell_\rho(\partial Y_t) \right]
\]

exists and is finite. Here

\[
Y_t = Y \cap \{(x, y, s) \in \mathbb{H}^3_+ : s > t\}, \quad \text{and} \quad \partial Y_t = Y \cap \{(x, y, s) \in \mathbb{H}^3_+ : s = t\},
\]

and \( \ell_\rho \) denotes hyperbolic length (in this case \( \ell_\rho(\partial Y_t) = \ell(\partial Y_t)/t \)). This was shown to be well defined by Graham and Witten [42] (i.e., invariant under isometries of...
Alexakis and Mazzeo \cite{AlMa1}, \cite{AlMa2} showed that if $\Gamma$ is sufficiently smooth (e.g., $C^{3,\alpha}$), then a minimal surface $Y$ with boundary $\Gamma$ has finite renormalized area and this value can be computed in terms of the total curvature,

$$A_R(Y) = -2\pi \chi(Y) - \frac{1}{2} \int_Y |K_0|^2 d\rho,$$

where $\chi(Y)$ is the Euler characteristic of $Y$. Weil-Petersson curves are not smooth enough to apply the Alexakis-Mazzeo result, but their conclusion is still correct:

**Theorem 1.5.** If $\Gamma$ is a Weil-Petersson curve and $S \subset \mathbb{H}^3_+$ is a minimal surface with boundary $\Gamma$, then $S$ has finite renormalized area.

Theorem 1.5 is closely linked to Theorem 1.2. Define the “dyadic cylinder” for $\Gamma$ as

$$X = \bigcup_{n=1}^{\infty} \Gamma_n \times [2^{-n}, 2^{-n+1}),$$

where $\{\Gamma_n\}$ are as in Theorem 1.2. See Figure 17.1. Theorem 1.2 implies that $X$ has finite renormalized area if and only if $\Gamma$ is Weil-Petersson. Using known curvature estimates on the minimal surface $S$ and our characterizations of the Weil-Petersson class, we will show that $A_\rho(S_t) - A_\rho(X_t)$ and $\ell_\rho(S_t) - \ell_\rho(X_t)$ both have finite limits as $t \searrow 0$, if $\Gamma$ is Weil-Petersson. This implies $S$ has finite renormalized area. It seems possible that the converse is also true, but we leave this as an open problem.

Renormalized area has strong motivations arising from string theory. Maldacena \cite{Ma} proposed that the expectation value of the Wilson loop operator (a precursor of string theory) should be the area of a minimal surface with asymptotic boundary $\Gamma$. It was pointed out by Hennington and Skenderis \cite{HeSk}, and by Graham and Witten \cite{GrWi}, that area should be interpreted as renormalized area. More recently, it has been suggested that renormalized area be used to measure the entanglement entropy of regions in conformal field theory (CFT), in a way that is analogous to how the entropy of a black hole is measured by the area of its event horizon, e.g., \cite{W}. See the introduction of \cite{AlMa1} for references to these developments up to 2010.

All of our geometric conditions will state that some scale invariant measure of the curvature $\Gamma$ is square integrable: they differ in how the curvature is defined and by how we integrate it (but we want every position and every scale to get comparable weight). Indeed, it seems as if almost any reasonable definition of this form defines
the same class of curves. Thus Weil-Petersson curves are really “curves with finite $L^2$ curvature over all points and scales” or, more concisely, “curves of finite total curvature”.

We will start by giving several known characterizations of Weil-Petersson curves using ideas from function theory: the conformal mapping $f$, its Schwarzian derivative, the complex dilatation $\mu$ of a quasiconformal extension of $f$, and the conformal welding homeomorphism $\varphi : \mathbb{T} \to \mathbb{T}$ associated with $f$. After that, we list our new geometric characterizations: one section is devoted to quantities defined in the Euclidean plane and another to quantities defined in hyperbolic 3-space. Many other minor variations are possible and, no doubt, other distinct characterizations remain to be discovered. See Section 18 for some possibilities. Table 2 summarizes the definitions discussed in this paper, and Figure 3 indicates the implications between them proven in this paper. Appendix D gives even more known characterizations.

The problem of geometrically characterizing Weil-Petersson curves was suggested to me in December of 2017 by David Mumford. However, I did not start to think seriously about this problem until attending a lecture of Yilin Wang at an IPAM workshop in January of 2019 that addressed connections between SLE and the Weil-Petersson class (see Appendix D). Her lecture presented several equivalent definitions of this class that I had been unaware of, and it immediately suggested the connection to Peter Jones’ $\beta$-numbers. This connection was confirmed by the paper [37] of Gallardo-Gutiérrez, González, Pérez-González, Pommerenke and Rättyä, which states a conjecture of Peter Jones that has an obvious re-formulation in terms of $\beta$-numbers (essentially Theorem 1.2). I am very grateful to Atul Shekhar for pointing me to this interesting and important paper. I also thankful acknowledge discussions with Kari Astala, Blaine Lawson, Raanan Schul and Leon Takhhtajan. I thank Jack Burkart and María González for reading various early drafts and providing many helpful comments and corrections. I am especially appreciative to Mike Anderson and Rafe Mazzeo for very helpful discussions of curvature, minimal surfaces, renormalized area and Willmore energy.
2. Function theoretic characterizations

We now give the function theoretic definitions, starting with the one mentioned earlier. These definitions are already known to be equivalent to each other; we shall give references, but also sketch proofs of the equivalences in Appendix C.

Suppose \( \Gamma \) is a closed curve in the plane and let \( f \) be a conformal map from the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \) to \( \Omega \), the bounded complementary component of \( \Gamma \). If \( f \) is conformal on \( \mathbb{D} \), then \( f' \) is never zero, so \( \Phi = \log f' \) is a well defined holomorphic function on \( \mathbb{D} \). Recall that the Dirichlet class is the Hilbert space of holomorphic functions \( F \) on the unit disk such that

\[
|F(0)|^2 + \int_{\mathbb{D}} |F'(z)|^2 dxdy < \infty.
\]

**Definition 1.** \( \Gamma \) is a quasicircle and \( \Gamma = f(T) \), where \( f \) is conformal on \( \mathbb{D} \) and \( \log f' \) is in the Dirichlet class.

This definition immediately gives some geometric information about the curve \( \Gamma \). If \( \log f' \) is in the Dirichlet class, then it is also in VMOA (vanishing mean oscillation; a definition is given at the end of this section), and a theorem of Pommerenke [64] then implies that \( \Gamma \) is asymptotically smooth, i.e., \( \ell(\gamma)/\crd(\gamma) \to 1 \) as \( \ell(\gamma) \to 0 \). In other words, a Weil-Petersson curve has “no corners”. It is not hard to prove that asymptotically smooth quasicircles are chord-arc, and so Weil-Petersson curves must be chord-arc curves. This was observed in [37].

It is easy to see that a function \( F \) is in the Dirichlet class if and only if \( F(\mathbb{D}) \) has finite area, when counted with multiplicity. It is easy to construct unbounded simply connected domains that have finite area, and the conformal map \( \Phi \) into such a region is in the Dirichlet class. Using this, one can construct conformal maps \( f \) with \( \log f' = \Phi \in \mathcal{D} \) and so that the curves \( \Gamma = f(T) \) have infinite spirals and hence are not \( C^1 \) (even examples where the spirals are dense on the curve).

Condition (1.1) can also be written as

\[
\int_{\mathbb{D}} |(\log f'(z))'|^2 (1 - |z|^2)^2 dA_p(z) < \infty
\]

where \( dA_p \) is the hyperbolic area on \( \mathbb{D} \) and the integrand is now invariant under pre-compositions by Möbius transformations of the disk.

It is simple to see (e.g., as in Appendix C) that for any holomorphic function on \( \mathbb{D} \)

\[
|F(0)|^2 + \int_{\mathbb{D}} |F'(z)|^2 dxdy < \infty
\]
if and only if
\[ |F(0)|^2 + |F'(0)|^2 + \int_D |F''(z)|^2 (1 - |z|^2)^2 \, dx \, dy < \infty. \]

Applying this to \( F = \log f' \), we see that (1.1) could be replaced by the condition
\[ \int_D \left| \left( \frac{f''}{f'} \right)' - \left( \frac{f''}{f'} \right)^2 \right|^2 (1 - |z|^2)^2 \, dx \, dy < \infty. \]

This integrand is reminiscent of the Schwarzian derivative of \( f \) given by
\[ S(f) = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 = \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2. \]

The quantities in (2.2) and (2.3) are the same, except that a factor of 1 has been changed to 3/2. However, this represents a non-linear change, and it is difficult to compare the two quantities directly, e.g., for a Möbius transformation sending \( \mathbb{D} \) to a half-plane, the Schwarzian is constant zero, but the expression in (1.1) blows up to infinity at a boundary point. Nevertheless, for conformal maps into bounded quasidisks, the integrals of these two quantities are simultaneously finite or infinite:

**Definition 2.** \( \Gamma \) is quasicircle and \( \Gamma = f(\mathbb{T}) \), where \( f \) is conformal on \( \mathbb{D} \) and satisfies
\[ \int_D |S(f)(z)|^2 (1 - |z|^2)^2 \, dx \, dy < \infty. \]

As with (2.1), we can rewrite (2.4) as
\[ \int_D |S(f)(z)|^2 (1 - |z|^2)^4 \, dA_\rho(z) < \infty, \]
where the integrand is now conformally invariant.

Proposition 1 of Cui’s paper [22] says that Definitions 2 and 1 are equivalent to each other; the arguments closely follow a similar result of Astala and Zinsmeister for BMOA in [7]. See also Theorem II.1.12 of Takhtajan and Teo’s book [75] and Theorem 1 of [63] by Pérez-González and Răță"y.

A mapping \( h \) is called quasiconformal if it is absolutely continuous on almost all lines and its partial derivatives satisfy the Beltrami equation \( h_\bar{z} = \mu h_z \), almost everywhere, for some measurable \( \mu \) with \( \|\mu\|_\infty < 1 \). See [2] or [50]. Moreover, the measurable Riemann mapping theorem says that given such a \( \mu \), there is a quasiconformal map \( h \) of the disk to itself that satisfies this equation. A quasiconformal map \( h \) is called \( K \)-quasiconformal if its dilatation satisfies \( \|\mu\|_\infty \leq k = (K - 1)/(K + 1) \).
More geometrically, at almost every point \( h \) is differentiable and its derivative (which is a real linear map) send circles to ellipses of eccentricity at most \( K \) (the eccentricity of an ellipse is the ratio of the major to minor axis).

It is well known that if \( f \) is univalent then
\[
\sup_{z \in \mathbb{D}} |S(f)(z)|(1 - |z|^2)^2 \leq 6. \tag{2.6}
\]
See Chapter II of [51] for this and following properties of the Schwarzian. Conversely, if \( f \) is holomorphic on the disk and satisfies this inequality with 6 replaced by 2 then \( f \) is a conformal map. If 2 is replaced by a value \( t < 2 \), then \( f \) has a \( K \)-quasiconformal extension to the plane, where \( K \) depends only on \( t \). This was proven by Ahlfors and Weill [1], who even gave a formula for the dilatation of the extension:
\[
\mu(z) = -\frac{1}{2z}(1 - |z|^2)^2S(f)(z), \quad z \in \mathbb{D}^*.
\]
See, e.g., Formula (3.34) of [60]. The Alhfors-Weill extension shows that Definition 2 implies (after reversing the roles of \( \mathbb{D} \) and \( \mathbb{D}^* \)):

**Definition 3.** \( \Gamma = f(T) \) where \( f \) is a quasiconformal map of the plane that is conformal on \( \mathbb{D}^* \) and whose dilatation \( \mu \) on \( \mathbb{D} \) satisfies satisfies
\[
\int_{\mathbb{D}} \frac{|\mu(z)|^2}{(1 - |z|^2)^2}dxdy < \infty. \tag{2.8}
\]

This was shown to be equivalent to Definition 2 by Guizhen Cui; see Theorem 2 of [22]. This can also be proven using the ideas in the proof of Lemma 1 from Astala and Zinsmeister’s paper [7] (which proves the BMO version of this result, giving a new proof of a result of Stephen Semmes [71]). The integral in (2.8) is the same as
\[
\int_{\mathbb{D}} |\mu(z)|^2dA_\rho < \infty, \tag{2.9}
\]
where \( dA_\rho \) denotes integration against hyperbolic area.

Another variation on this theme is to consider the map \( R(z) = f(1/f^{-1}(z)) \). This is a quasiconformal map of the sphere to itself that fixes \( \Gamma \) pointwise, exchanges the two complementary components of \( \Gamma \) and whose dilatation satisfies
\[
\int_{\Omega, \Omega^*} |\mu(z)|^2dA_\rho(z) < \infty, \tag{2.10}
\]
where \( dA_\rho \) is hyperbolic area on each of the domains \( \Omega, \Omega^* \). This variant is sometimes easier to check, and we will use it interchangeably with Definition 3. The map
$R$ is called a quasiconformal reflection across $\Gamma$. We shall see later that the easiest path from our new geometric conditions back to the known function theoretic characterizations is by construction of a quasiconformal reflection satisfying (2.10).

Each dilatation $\mu$ on $D^*$ extends to a dilatation on $\mathbb{C}$ by the reflection formula

$$\nu(z) = \mu(1/z)z^2/z^2, \quad z \in D.$$ 

Applying the measurable Riemann mapping theorem to $\nu$ gives a quasiconformal mapping of the plane $\omega_{\nu}$ that fixes both 0 and 1, and sends the unit circle to itself. If $f$ is as in Definition 3, then $g = \omega_{\nu}^{-1} \circ f$ is a quasiconformal map of the plane taking $\mathbb{T}$ to the same curve $\Gamma = g(\mathbb{T})$, but now $g$ is conformal on $D$ and its dilatation satisfies (2.9) on $D^*$. The map $\omega_{\nu}$ restricted to $\mathbb{T}$ is a homeomorphism $\varphi = f^{-1} \circ g : \mathbb{T} \to \mathbb{T}$.

A circle homeomorphism that arises in this way is called a conformal welding homeomorphism associated to $\Gamma$ (there are many welding associated to $\Gamma$, but they all differ by compositions with Möbius transformations of $\mathbb{T}$).

A circle homeomorphism is called $M$-quasisymmetric if it maps adjacent arcs in $\mathbb{T}$ of the same length to arcs whose length differ by a factor of at most $M$; we call $\varphi$ quasisymmetric if it is $M$-quasisymmetric for some $M$. The quasisymmetric maps are exactly the circle homeomorphisms that can be continuously extended to quasiconformal self-maps of the disk. The quasisymmetric homeomorphisms corresponding to Weil-Petersson curves have been characterized by Yuliang Shen [73]:

**Definition 4.** $\Gamma$ is a quasicircle and the welding homeomorphism $\varphi$ associated to $\Gamma$ is quasisymmetric, absolutely continuous, and $\log \varphi'$ is in the Sobolev space $H^{1/2}$.

The space $H^{1/2}$ will be defined below. Gay-Balmaz and Ratiu [40] proved that the welding homeomorphism $\varphi$ is in $H^s$ for every $s < 3/2$, but Shen has proven it need not be in $H^{3/2}$, nor in the Lipschitz class $\Lambda^1$ [73]. Further characterizations of the Weil-Petersson welding maps are given in [46] using higher order analogs of the Schwarzian derivative.

To see that Definition 4 should be the correct characterization, note that Definition 1 can be rewritten as

$$(2.11) \quad \int_D |(\log f')'|^2dxdy = \int_D \left| \frac{f''}{f'} \right|^2dxdy < \infty.$$
If $\Phi = \log f'$ is given by the power series
\[ \Phi(z) = \sum_{n=0}^{\infty} a_n z^n, \]
then the Dirichlet condition above is equivalent to
\[ D(f) \equiv \int_{\mathbb{T}} |\Phi'(z)|^2 dx dy = \sum_{n=0}^{\infty} n^2 |a_n|^2 \int_0^{2\pi} \int_0^{1} r^{2n-2} r dr d\theta = \pi \sum_{n=0}^{\infty} n |a_n|^2 < \infty. \]
The last equation is the definition for the boundary values of $\Phi = \log f'$ on $\mathbb{T}$ to be in the Sobolev space $H^{1/2}$ (this is also a special case of the trace theorem for Sobolev spaces, where we restrict from the disk to its boundary). Since
\[ \log \varphi'(x) = - \log f'(\varphi(x)) + \log g'(x), \]
we might hope $\log \varphi' \in H^{1/2}$, assuming composition with $\varphi$ preserves this class. This is proven correct in [73]. $H^{1/2}$ is equivalently defined by using the seminorm
\[ D(f) = \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|\Phi(z) - \Phi(w)|^2}{|z - w|^2} |dz||dw| < \infty. \]
This called the Douglas formula, after Jesse Douglas who introduced it in his solution of the Plateau problem [25]. See also Theorem 2.5 of [3] (for a proof of the Douglas formula) and [67] (for more information about the Dirichlet space). Several of our geometric conditions will involve conditions very similar to (2.12) on various quantities associated to $\Gamma$.

The fact that Definitions 1-4 are all equivalent is analogous to other equivalences that have been studied over the last 40 years. Table 1 summarizes some of these results: the first column gives five quantities associated to $f$: $\log f'$, $S(f)(1 - |z|^2)^2$, the dilatation $\mu$ of a QC extension of $f$, the conformal welding homeomorphism $\varphi$, and $\Gamma = f(\mathbb{T})$. Boxes is each row claim that the corresponding quantity in a certain space (defined below), and each column represents a theorem that these inclusions are equivalent to each other. The contribution of this paper is to fill in the “$\sum \beta^2 < \infty$” in the lower right corner. Quasidisks $\Omega = f(\mathbb{D})$ with $\log f' \in \text{BMO}$ are called BMO domains, so it would be natural to call a quasidisk $\Omega$ a Dirichlet domain if $\log f'$ is in the Dirichlet class. Then Weil-Peterson curves would be exactly the boundaries of Dirichlet domains.
Table 1. Each row consists of different conditions that can be placed on a quantity associated to a conformal map $f : \mathbb{D} \to \Omega$ (the second row are just the definitions of the names in the first row). The conditions in each column are equivalent, e.g., $\log f' \in \text{BMO}$ holds if and only if $S(f)(z)^2(1 - |z|^2)^2 dxdy$ is a Carleson measure. The conditions grow more stringent as we move left-to-right in the table. Precise definitions of all the entries are given in the text. The purpose of this paper is to fill in the lower right corner. The proofs of the equivalences come from many sources including: [7], [10], [15], [22], [27], [33], [57], [64], [74].

Here are the definitions needed to interpret Table 1. We let $\mathcal{B}_0$ denote the little Bloch class of holomorphic functions on $\mathbb{D}$ such that

$$\mathcal{B}_0 = \{ f : \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|^2) \to 0 \text{ as } |z| \to 1 \}$$

or, more concisely, $|f(z)|(1 - |z|^2) \in C_0(\mathbb{D})$ (continuous functions that tend to zero at the boundary). A Carleson measure on $\mathbb{D}$ is a non-negative measure $\mu$ so that

$$\mu(D(x,r)) \leq Cr,$$

for some fixed $C < \infty$ and all $x \in \mathbb{T}, r > 0$. We define a certain class of functions that give Carleson measures as

$$\text{CM}(\mathbb{D}) = \{ f : |f|^2(1 - |z|^2)^{-1} dxdy \text{ is a Carleson measure } \},$$

and $\text{CM}_0(\mathbb{D}) \subset \text{CM}(\mathbb{D})$ are the functions so that

$$\frac{1}{r} \int_{D \cap D(x,r)} |f|^2(1 - |z|^2)^{-1} dxdy \to 0,$$

as $r \to 0$. The space BMOA (bounded mean oscillation) on the unit circle has several equivalent definitions (see Chapter VI of Garnett’s book [38]); a convenient one to use here is that $f \in \text{BMO}$ if its harmonic extension $u$ to $\mathbb{D}$ satisfies $|\nabla u(z)|(1 - |z|^2) \in \text{VMO}$.

<table>
<thead>
<tr>
<th>log $f'$</th>
<th>$\mathcal{B}_0$</th>
<th>BMOA</th>
<th>VMOA</th>
<th>Dirichlet</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\log f')'(1 -</td>
<td>z</td>
<td>^2)$</td>
<td>$C_0(\mathbb{D})$</td>
<td>CM(\mathbb{D})</td>
</tr>
<tr>
<td>$S(z)(1 -</td>
<td>z</td>
<td>^2)^2$</td>
<td>$C_0(\mathbb{D})$</td>
<td>CM(\mathbb{D})</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$C_0(\mathbb{D})$</td>
<td>CM(\mathbb{D})</td>
<td>CM_0(\mathbb{D})</td>
<td>A_2(\mathbb{D})</td>
</tr>
<tr>
<td>$\varphi = g^{-1} \circ f$</td>
<td>symmetric</td>
<td>strongly quasisymmetric</td>
<td>log $\varphi' \in \text{VMO}$</td>
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<tr>
<td>$\Gamma = f(\mathbb{T})$</td>
<td>asymptotically conformal</td>
<td>Bishop-Jones condition</td>
<td>asymptotically smooth</td>
<td>$\sum \beta^2 &lt; \infty$</td>
</tr>
</tbody>
</table>
CM(\(D\)). BMOA is the subspace of functions such that the harmonic extension \(u\) is holomorphic. We say a function \(f \in \text{BMOA}\) is in \(\text{VMOA}\) if 
\[|f'(1 - |z|^2)| \in \text{CM}_0(D).\]

The space \(A_2(D)\) is defined as 
\[A_2(D) = \{f : \int_D |f(z)|^2 dA_\rho(z) < \infty\}.\]

For a circle homeomorphism \(\phi\), symmetric means \(\phi(I)/\phi(J) \to 1\), for adjacent intervals with \(|I| = |J| \to 0\), and \(\phi\) is strongly quasisymmetric if for there are \(\delta, \epsilon > 0\) such that 
\[|E| \leq \delta |I| \implies |\phi(E)| \leq \epsilon |I|\]
for every measurable \(E \subset I\), and arc \(I \subset \mathbb{T}\).

Given an arc \(\gamma\) with endpoints \(z, w\), let \(L\) be the line through \(z\) and \(w\) and let \(\beta(\gamma) = \max_{z \in \gamma} \text{dist}(z, L)/|z - w|\). A curve \(\Gamma\) is called asymptotically conformal if \(\beta(\gamma) \to 0\) as \(\text{diam}(\gamma) \to 0\), and is called asymptotically smooth if \(\Delta(\gamma) = (\ell(\gamma) - \text{crd}(\gamma)) = o(\text{crd}(\gamma))\) as \(\text{diam}(\gamma) \to 0\). Asymptotically smooth curves are rectifiable, but asymptotically conformal curves need not be, e.g., a variable snowflake construction in which the triangle heights tend to zero, but are not square summable. The Bishop-Jones condition [15] on \(\Gamma\) says that there is a \(M < \infty\) so that for each \(z \in \Omega = \mathbb{R}^2 \setminus \Gamma\) there is a subdomain \(U_z\) bounded by a \(M\)-chord-arc curve, so that \(z \in U_z \subset \Omega\) and such that 
\[\text{dist}(z, \partial U_z) \simeq \ell(\partial U_z) \simeq \ell(\partial U_z \cap \Gamma).\]

Although awkward to state, this is a useful condition, e.g., it shows that such curves are invariant under Lipschitz mappings of the plane. What natural family of planar homeomorphisms preserves the Weil-Petersson class? What other functions spaces have results similar to Table 1? For example, the paper [62] by Pau and Peláez describes a column corresponding to \(Q_p\) spaces.

3. Euclidean conditions

We start by recalling some standard notation. Given two quantities \(A, B\) that both depend on a parameter, we write \(A \lesssim B\) if there is a constant \(C\) so that \(A \leq CB\) holds independent of the parameter. We write \(A \gtrsim B\) if \(B \lesssim A\), and we write \(A \simeq B\) if both \(A \lesssim B\) and \(A \gtrsim B\) hold. The notation \(A \lesssim B\) means the same the as the “big-Oh” notation \(A = O(B)\).

A dyadic interval \(I\) in \(\mathbb{R}\) is one of the form \(2^{-n}j, 2^{-n}(j + 1)\]. A dyadic square in \(\mathbb{R}^2\) is of the form \(Q = (2^{-n}j, 2^{-n}(j + 1)] \times (2^{-n}k, 2^{-n}(k + 1)]\). For a positive number
\( \lambda > 0 \), we let \( \lambda Q \) denote the square concentric with \( Q \) but with diameter \( \lambda \text{diam}(Q) \), e.g., \( 2Q \) is the “double” of \( Q \). We let \( Q^\uparrow \) denote the parent of \( Q \); the unique interval (or square) containing \( Q \) and having twice the length.

A multi-resolution family in metric space \( X \) (in this paper, \( X \) will usually be \( \mathbb{R}, \mathbb{R}^2, \mathbb{T} \) or a Jordan curve \( \Gamma \)) is a collection of sets \( \{X_j\} \) in \( X \) such that

1. For each \( r > 0 \), the sets with diameter comparable to \( r \) cover \( X \),
2. each bounded subset of \( X \) is hits finitely many of the sets \( X_k \) of comparable size,
3. any bounded subset of \( X \) is contained in one of the \( X_j \) of comparable diameter.

Dyadic intervals do not form a multi-resolution family, e.g., \([-1,1] \subset \mathbb{R} \) is not contained in any dyadic interval. However, the family of triples of all dyadic intervals (or squares) do form a multi-resolution family. Similarly, if we “triple” the collection of dyadic intervals by adding all translates by \( \pm 1/3 \), this property is true: every interval is contained in an element of one of the three families that has comparable size. The analogous construction for dyadic squares in \( \mathbb{R}^2 \) is to take all nine translates by elements of \( \{ -\frac{1}{3}, 0, \frac{1}{3} \} \times \{ -\frac{i}{3}, 0, \frac{i}{3} \} \).

During the course of this paper, we will frequently deal with functions \( \alpha \) that map a collection of sets into the non-negative reals, and will wish to decide if the sum \( \sum_j \alpha(X_j) \) over some multi-resolution family converges or diverges. If \( X_j \subset X_k \) and \( \text{diam}(X_j) \simeq \text{diam}(X_k) \) implies that \( \alpha(X_j) \lesssim \alpha(X_k) \), then it is easy to check that the convergence or divergence does not depend on the particular choice of the family.

Given a planar set \( E \) and a dyadic square \( Q \), define Peter Jones’ \( \beta \)-numbers as

\[
\beta(Q) = \beta_E(Q) = \frac{1}{\text{diam}(Q)} \inf_{L} \sup \{ \text{dist}(z, L) : z \in 3Q \cap E \},
\]

where the infimum is over all lines \( L \) that hit \( 3Q \). See Figure 1. The traveling salesman theorem (TST) [47] says \( E \) is contained in a curve of finite length if and only if

\[
\text{diam}(\Gamma) + \sum_{Q} \beta(Q)^2 \text{diam}(Q) < \infty,
\]

where the sum is over all dyadic squares \( Q \) in the plane and the length of the optimal curve containing \( E \) is comparable to the left-hand side above. If \( E = \Gamma \) is a closed Jordan curve, this result implies that \( \Gamma \) has length comparable to the sum in (3.2).
Our first geometric condition is a simple strengthening of (3.1).

**Definition 5.** \( \Gamma \) is a closed Jordan curve that satisfies

\[
\sum_Q \beta_\Gamma(Q)^2 < \infty,
\]

where the sum is over all dyadic squares in the plane.

In other words, we merely drop the length factor “diam(\( Q \))” from the sum in (3.1). This is not terribly surprising. Peter Jones and I proved (Lemma 3.9 of [15], or Theorem X.6.2 of [39]) that if \( \Gamma \) is a \( M \)-quasicircle, then its length comparable to

\[
\iint |f'(z)||S(f)(z)|^2(1-|z|^2)^3dx\,dy
\]

with constants depending only on \( M \). By Koebe’s distortion theorem

\[ |f'(z)|(1-|z|^2) \simeq \text{dist}(f(z), \partial \Omega), \]

and thus the factor on the left is analogous to the diam(\( Q \)) in Jones’ \( \beta^2 \)-sum. Dropping this term from (3.3) gives exactly the integral in Definition 2. Thus Definition 5 is the direct geometric analog of Definition 2.

Define

\[
\beta_\Gamma(x, t) = \frac{1}{t} \inf_L \max \{ \text{dist}(z, L) : |x - z| \leq t \},
\]

where the infimum is over all lines passing though \( x \). See Figure 1. Then the discrete sum in Definition 5 can be replaced by the integral

\[
\int_{R^2} \int_0^\infty \beta_\Gamma^2(x, t) \frac{dx\,dt}{t^3} < \infty,
\]

as in Theorem 1.1. If \( \Gamma \) is chord-arc, this is equivalent to

\[
\int_{\Gamma} \int_0^\infty \beta_\Gamma^2(x, t) \frac{dt(x)dt}{t^2} < \infty,
\]

where we use arclength measure to integrate along \( \Gamma \).

Another way to express the \( \beta^2 \)-sum condition is to divide \( \Gamma \) into a multi-resolution family (e.g., triples of dyadic subarcs) \( \{\Gamma_j\} \) and define

\[
\beta(\Gamma_j) = \frac{1}{\text{crd}(\Gamma_j)} \sup_{z \in \Gamma_j} \text{dist}(z, L_j),
\]
where \( \text{crd}(\Gamma_j) = |z_j - w_j| \) is the distance between the endpoints of \( \Gamma_j \), and \( L_j \) is the line through these points. Then it is easy to check (and we leave it as an exercise) that Definition 5 holds if and only if \( \Gamma \) is chord-arc and

\[
\sum_j \beta^2(\Gamma_j) < \infty \tag{3.4}
\]

for some, hence every, multi-resolution family for \( \Gamma \). This formulation will be used several times later in this paper.

If \( z, w \in \Gamma \), define \( \ell(z, w) \) to be the length of the shorter subarc \( \gamma \) of \( \Gamma \) connecting \( z \) and \( w \).

**Definition 6.** \( \Gamma \) is chord-arc and satisfies

\[
\int_{\Gamma} \int_{\Gamma} \frac{\ell(z, w) - |z - w|}{|z - w|^3} |dz||dw| < \infty. \tag{3.5}
\]

In [37], Gallardo-Gutiérrez, González, Pérez-González, Pommerenke and Rättyä state that this condition follows from Definition 1, but their proof contains a small error; we will provide an alternative proof and also prove the converse, thereby verifying a conjecture of Peter Jones stated in [37] (Definition 1 \( \Leftrightarrow \) Definition 6).

Set

\[
\kappa(z, w) = \sqrt{\frac{24(\ell(z, w) - |z - w|)}{|z - w|^3}}.
\]

If \( \Gamma \) is smooth, then it is easy to check that

\[
\kappa(x) = \lim_{y \to x} \kappa(x, y),
\]

is the usual Euclidean curvature of \( \Gamma \) at \( x \). Thus (3.5) can be rewritten as

\[
\int_{\Gamma} \int_{\Gamma} \kappa^2(z, w)|dz||dw| < \infty, \tag{3.6}
\]
and this has more of a "$L^2$-curvature" flavor. As with the $\beta^2$-sum and $\beta^2$-integral, Definition 6 has an equivalent discrete counterpart:

**Definition 7.** $\Gamma$ is chord-arc and

$$\sum_j \Delta(\Gamma_j)/\ell(\Gamma_j) < \infty$$

for some multi-resolution family $\{\Gamma_j\}$ of arcs on $\Gamma$. Here $\Delta(\gamma) = \ell(\gamma) - \text{crd}(\gamma)$.

This definition is just a reformulation of (1.4) given in Section 1, since if $\{\gamma_j\}$ corresponds to a dyadic decomposition of $\Gamma$ we have

$$\sum_n 2^n[\ell(\Gamma) - \ell(\Gamma_n)] = \sum_j \Delta(\gamma_j)/\ell(\gamma_j).$$

Thus proving that Definition 7 is equivalent to being Weil-Petersson proves Theorem 1.2. If $\Gamma$ is rectifiable, then it has a well defined tangent direction $\tau(z) \in \mathbb{T}$ at almost every point $z \in \Gamma$ (traverse $\Gamma$ in the counter-clockwise direction).

**Definition 8.** $\Gamma$ is chord-arc and

$$\int_\Gamma \int_\Gamma \left| \frac{\tau(z) - \tau(w)}{z - w} \right|^2 |dz||dw| < \infty.$$

In other words, $\Gamma$ is Weil-Petersson iff its tangent directions are in $H^{1/2}(\Gamma)$. The equivalence of this to Definition 1 strengthens Theorem 6 of [37]. That result gives the same conclusion under the hypothesis that $\log f'$ is Dirichlet and $0 < c \leq |f'| \leq C < \infty$. Moreover, Theorem 5 of [37] states that the finiteness of integral above above is equivalent to being in the Weil-Petersson class if arc-length measure is replaced by harmonic measure for either side. Definition 8 also has a discrete formulation. For a subarc $\{\gamma\}$ of $\Gamma$, let $\tau(\gamma) = (z - w)/|z - w|$, where $\{z, w\}$ are the endpoints of $\gamma$ with $w$ being the initial endpoint for the counter-clockwise orientation.

**Definition 9.** $\Gamma$ is chord-arc and

$$\sum_j \int_{\Gamma_j} |\tau(z) - \tau(\Gamma_j)|^2 \frac{ds}{\ell(\Gamma_j)} < \infty,$$

where the sum is over a multi-resolution family of arcs $\{\Gamma_j\}$ for $\Gamma$.

Another measure of how much $\Gamma$ deviates from a straight line can be given in terms of how random lines hit $\Gamma$. Suppose we parameterize lines $L$ in $\mathbb{R}^2 \setminus \{0\}$ by
\( (r, \theta) \in (0, \infty) \times (0, 2\pi) \) where \( z = r \exp(i\theta) \) is the point of \( L \) closest to 0. It is well known fact from integral geometry (e.g., [68]) that the measure \( d\mu = drd\theta \) on lines is invariant under Euclidean isometries of the plane, and the measure of the set of lines hitting a non-degenerate convex set \( X \) equals the length of the boundary of \( X \) (for a line segment, it is twice the length of the segment). For a dyadic cube \( Q \) let \( S(Q, \Gamma) \) be the set of lines that hits \( 3Q \) also hits both \( \Gamma \cap \frac{5}{3}Q \) and \( \Gamma \cap (3Q \setminus 2Q) \).

**Definition 10.** Any translate of \( \Gamma \) satisfies

\[
\sum_Q \frac{\mu(S(Q, \Gamma))}{\text{diam}(Q)} < \infty.
\]

The equivalence of Definitions 5 and 10 follows immediately from Theorem 10.2.1 of [16], so need not be proven here. Note that \( 12 \text{diam}(Q)/\sqrt{2} \) is the perimeter of \( 3Q \) and hence is the \( \mu \) measure of the set of random lines hitting \( 3Q \). Dividing the sum by \( 6\sqrt{2} \), each term becomes the probability that if a line \( L \) hits \( 3Q \), then \( L \in S(Q, \Gamma) \):

\[
\sum_Q \mathbb{P}(S(Q, \Gamma)|3Q) < \infty.
\] (3.8)

Roughly speaking, this is the probability that a random line hitting \( 3Q \) will hit \( \Gamma \cap 3Q \) at two points \( \simeq \ell(Q) \) apart; can this be made precise? The particular values \( \frac{5}{3}, 2 \) and 3 in the definition of \( S(L, \Gamma) \) are probably not very important; they were chosen to make the proof in [16] work out easily.

The Menger curvature of three points \( x, y, z \in \mathbb{R}^2 \) is \( c(x, y, z) = 1/R \) where \( R \) is the radius of the circle passing through these points. Equivalently,

\[
c(x, y, z) = \frac{2\text{dist}(x, L_{yz})}{|x - y||x - z|},
\]

where \( L_{yz} \) is the line passing through \( y \) and \( z \), or

\[
c(x, y, z) = 2\frac{\sin \theta}{|x - y|},
\]

where \( \theta \) is the angle opposite \([x, y]\) in the triangle with vertices \( \{x, y, z\} \). Let

\[
\ell(x, y, x) = |x - y| + |y - z| + |z - x|
\]

be the perimeter of this triangle.
**Definition 11.** $\Gamma$ is chord-arc and satisfies

\[
\int_{\Gamma} \int_{\Gamma} \int_{\Gamma} \frac{c(x, y, z)^2}{\ell(x, y, z)}|dx||dy||dz| < \infty.
\]

(3.9)

Again, this is not unexpected. It is known that the conditions

\[
\int_{\Gamma} \int_{\Gamma} \int_{\Gamma} c(x, y, z)^2|dx||dy||dz| < \infty.
\]

(3.10)

are equivalent, and the analog of dropping the length term from (3.11), would be to divide by a term that scales like length in (3.10), which is what we have done in (3.9). Indeed, to prove that Definitions 5 and 11 are equivalent, I will simply indicate how to modify the proof of the equivalence of (3.10) and (3.11) given in Pajot’s book [61].

Definition 11 does not quite have the same form as many of our earlier conditions: a scale invariant quantity summed over a discrete set of locations/scales. However, we can easily convert it into such a form by setting

\[
c^2(Q) = \int_{\Gamma \cap 3Q} \int_{\Gamma \cap 3Q} \int_{\Gamma \cap 3Q} \frac{c(x, y, z)^2|dx||dy||dz|}{\ell(Q)}
\]

and requiring

\[
\sum_{Q} c^2(Q) < \infty.
\]

(3.12)

Note that $c(x, y, z)\ell(Q)$ is scale invariant so rewriting the condition this way puts it into the “scale invariant curvature summed over dyadic squares” format.

Finally, we give a variation of the $\beta$-numbers that uses disks instead of strips. This will provide a “stepping-stone” to the hyperbolic conditions discussed in the next section. Given a dyadic square $Q$ let $\varepsilon_\Gamma(Q)$ be the smallest $\varepsilon \in (0, 1]$ so that $3Q$ intersects two disjoint disks $D_1, D_2$ of radius $\ell(Q)/\varepsilon$ that are separated by $\Gamma$, and are within distance $\varepsilon \cdot \ell(Q)$ of each other. If no such disks exist, we set $\varepsilon_\Gamma(Q) = 1$.

**Definition 12.** $\Gamma$ is chord-arc and satisfies

\[
\sum_{Q} \varepsilon_\Gamma^2(Q) < \infty
\]

(3.13)

where the sum is over all dyadic squares $Q$ that hit $\Gamma$ and satisfy $\text{diam}(Q) \leq \text{diam}(\Gamma)$. 

Figure 2. The definition of $\varepsilon_\Gamma(Q)$.

4. Hyperbolic Conditions

Each disk in the plane is characterized by three real numbers: the coordinates of its center and its radius. Thinking of the radius as a height above the center, we see that planar disks correspond to points in the hyperbolic upper half-space, $H^3_+ = \{(x, y, t) : x, y \in \mathbb{R}, t > 0\}$. We start by recalling the basic definitions concerning the hyperbolic metric on $H^3_+$.

The hyperbolic length of a (Euclidean) rectifiable curve in the unit disk $\mathbb{D}$ or in the 3-dimensional ball $\mathbb{B}^3$ is given by integrating $\frac{ds}{1 - |z|^2}$ along the curve. In the upper half-plane $H^+_2 = \{(x, t) : t > 0\}$ or upper half-space $H^3_+ = \{(x, y, t) : t > 0\}$ we integrate $ds/2t$. Note that this definition differs by a factor of 2 from that given in some sources. The hyperbolic distance between two points is given by taking the infimum of all hyperbolic lengths of paths connecting the points. In the disk and ball, hyperbolic geodesics are either diameters or subarcs of circles perpendicular to the boundary. In $H^+_2$ and $H^3_+$ the hyperbolic geodesics are either vertical rays or semi-circles centered on the boundary.

Koebe’s estimate (e.g., Theorem I.4.3 of [39]) says that if $\varphi$ is conformal from $\mathbb{D}$ to a simply connected domain $\Omega$, then

$$\frac{1}{4} |\varphi'(z)|(1 - |z|^2) \leq \text{dist}(\varphi(z), \partial \Omega) \leq |\varphi'(z)|(1 - |z|^2).$$
The hyperbolic metric $\rho = \rho_\Omega$ on a simply connected planar domain $\Omega$ is defined by transferring the hyperbolic metric on $\mathbb{D}$ by a conformal map (the choice of the map makes no difference). The quasi-hyperbolic metric on $\Omega$ is defined by integrating

$$\tilde{\rho} = \frac{ds}{\text{dist}(z, \partial \Omega)}.$$ 

Koebe’s theorem implies these two metrics are comparable to within a factor of 4,

$$\rho_\Omega(z, w) \leq \tilde{\rho}_\Omega(z, w) \leq 4 \cdot \rho_\Omega(z, w).$$

Given $\Omega \subset \mathbb{R}^2$, consider the union of all solid hemispheres whose base disk is in $\Omega$. The boundary of this region is a surface $S \subset \mathbb{H}_3^+$ that meets $\mathbb{R}^2$ along $\partial \Omega$ and is called the dome of $\Omega$. Using hyperbolic isometries we can also define the domes of any region on the sphere. The dome of $\Omega$ is the boundary in $\mathbb{H}_3^+$ of the hyperbolic convex hull of $\Omega^c$. For a set $E \subset \mathbb{R}^2$, its hyperbolic convex hull $\text{CH}(E)$ is the convex hull in $\mathbb{H}_3^+$ of all the hyperbolic geodesics whose endpoints both lie in $E$; if $E = \Gamma$ is a closed curve, the convex hull is actually the union of these geodesics. See Lemma 18.1. For closed curves, it is known that the boundary of $\text{CH}(\Gamma)$ in $\mathbb{H}_3^+$ has two connected components, $S_1, S_2$. Each of these surfaces meets $\mathbb{R}^2$ exactly along $\Gamma$ and each is isomorphic to the hyperbolic unit disk when given its hyperbolic path metric.

Each of these surfaces is also a pleated surface. This means that it is a disjoint union of non-intersecting infinite geodesics for $\mathbb{B}^3$ (possibly uncountably many) and at most countably many regions lying on hyperbolic planes, each region bounded by disjoint hyperbolic geodesics. Roughly speaking each surface is a copy of the hyperbolic disk that has been “bent” along a collection of disjoint geodesics, and there is an associated bending measure that gives the amount of bending on each geodesic. The bending measure actually measures arcs that are transverse to the bending geodesic and in general it may have both atoms and continuous parts. However, for convex hulls of Weil-Petersson curves, no atoms will occur. For more about convex hulls and pleated surfaces, see [29] by Epstein and Marden (or the revised version [30]).

For a point $z \in \text{CH}(\Gamma)$ we define $\delta(z) = \max(\text{dist}_\rho(z, S_1), \text{dist}_\rho(z, S_2))$, i.e., $\delta(z)$ is the hyperbolic distance to farther of the two boundary components of $\text{CH}(\Gamma)$. Thus $\delta(z)$ measures the “thickness” of the convex hull of $\Gamma$ near $z$.

Although Möbius images of disks are still disks, Definition 12 is not quite Möbius invariant, since two disks of equal size might not map to disks of equal size, as required.
in the definition of \( \varepsilon_\Gamma(Q) \). If we consider the domes of two disjoint disks \( D_1, D_2 \) each of Euclidean radius \( r = 1/\varepsilon \geq 1 \), and so that \( \text{dist}(D_1, D_2) \leq \varepsilon \), then the hyperbolic distance between the corresponding domes is \( O(\varepsilon) \) (see Lemma 13.1).

**Definition 13.**

\[
\int_{\partial \text{CH}(\Gamma)} \delta^2(z) dA_\rho(z) < \infty,
\]

where \( dA_\rho \) denotes hyperbolic surface area on \( \partial \text{CH}(\Gamma) \).

We have integrated over both boundary components of \( \partial \text{CH}(\Gamma) \), but the proof will show that if the integral over one component is finite, then so is the integral over the other one. If \( \Gamma \) is a quasicircle, then each point \( z \) of one boundary component is within a uniformly bounded hyperbolic distance \( \delta(z) \) of the other boundary component, i.e., if \( \Gamma \) is a quasicircle then \( \delta(z) \in L^\infty(\partial \text{CH}(\Gamma), dA_\rho) \). Definition 13 says that the Weil-Petersson class corresponds to \( \delta(z) \in L^2(\partial \text{CH}(\Gamma), dA_\rho) \). Similarly, \( \delta(z) \in L^1(\partial \text{CH}(\Gamma), dA_\rho) \) is equivalent to \( \text{CH}(\Gamma) \) having finite hyperbolic volume. Does this correspond to an “interesting” class of planar curves?

If \( 0 \in \text{CH}(\Gamma) \), and \( \delta(0) \) is small, then the part of the convex hull that is within unit distance of 0 is trapped between two hyperbolic planes that are only distance \( O(\delta(0)) \) apart; see Lemma 13.2. Hence this part of the convex hull has hyperbolic volume bounded by \( O(\delta(0)) \). Therefore Definition 13 can be restated as:

\[
\int_\mathbb{B}^3 \text{dist}(z, \mathbb{B}^3 \setminus \text{CH}(\Gamma)) dV(z) < \infty,
\]

where \( dV \) denotes hyperbolic volume.

Since the boundary components of \( \partial \text{CH}(\Gamma) \) are pleated surfaces, there is always at least one direction through each point where the curvature of the surface is zero (the direction of a geodesic contained in the surface passing through that point). Thus all the “curvature” of the surface is in the orthogonal direction, transverse to the geodesic foliation. Domes of simply connected domains need not be even \( C^1 \); for a union of two overlapping disks, the dome has a definite angle along an infinite geodesic. But for Weil-Petersson curves the transverse bending measure cannot have an atom; this would violate Definition 13 by producing an infinite sequence of unit disks centered along the geodesic that all have a fixed, positive amount of bending.
However, the bending measure for such a curve can be singular to Lebesgue measure in the transverse direction (e.g., look like the Cantor singular measure). Even for analytic domains, the dome may be no better than $C^1$; see [34]. The amount of bending, $B(z)$, within unit distance of $z \in S_1$ is $O(\delta(z))$. Indeed,

\begin{equation}
\int_S B^2(z)dA_\rho < \infty
\end{equation}

gives another characterization of such surfaces. By smoothing the convex hull boundary, we can obtain a surface $S$ with finite total curvature, defined as follows. If we use the ball model of hyperbolic space, assume $0 \in S$, then we can rotate so that the $xy$-plane is the tangent plane to $S$ at 0 and $S$ looks like the graph of $\kappa_1 x^2 + \kappa_2 y^2$ (plus higher order terms), where $\kappa_1, \kappa_2$ are the principle curvatures of $S$ at 0. The mean curvature is $H = (\kappa_1 + \kappa_2)/2$, and the norm-squared of the second fundamental form is $|K|^2 = \kappa_1^2 + \kappa_2^2$, the norm of the $2 \times 2$ diagonal matrix $K$ with entries $\kappa_1, \kappa_2$. The trace free second fundamental form is $K - HI$, and its norm-squared is $|K_0|^2 = (\kappa_1 - \kappa_2)^2/2$. By “finite total curvature” we mean that $\int_S |K|^2dA_\rho < \infty$.

**Definition 14.** $\Gamma \subset S^2$ is the boundary of a smooth surface $S \subset \mathbb{H}_+^3$ such that $K(z) \to 0$ as $z$ tends to the boundary of hyperbolic space and

\begin{equation}
\int_S |K(z)|^2dA_\rho(z) < \infty,
\end{equation}

where $K$ is second fundamental form of $S$.

By a result of Charles Epstein [28], Definition 14 implies that $\Gamma$ has a quasiconformal reflection satisfying (2.10) and hence $\Gamma$ satisfies Definition 3. This will complete our circle of equivalences. An alternate way to prove that Definition 13 implies Definition 14 is to show that a minimal surface in $\mathbb{H}_+^3$ with boundary $\Gamma$ must have finite total curvature (recall that all our minimal surfaces are embedded). The existence of such a minimal surface $S$ for any closed Jordan curve in $\mathbb{R}^2$, is due to Michael Anderson [6] and we have the estimate

\begin{equation}
\max(|\kappa_1(z)|, |\kappa_2(z)|) = O(\delta(z)^2),
\end{equation}

for $z \in S \subset CH(\Gamma)$ due to Andrea Seppi (Equation (32) of [72]). Because of this we see that $\Gamma$ is Weil-Petersson if and only if it spans an embedded minimal surface that is topologically a disk and has finite total curvature. In this case, the assumption that $S$ is asymptotically flat is unnecessary; it follows from (4.5).
In order to provide an easy reference guide, Table 2 gives a brief description of each definition of the Weil-Petersson class used in this paper. The arrows in Figure 3 indicate which implications will be proved in this paper.

<table>
<thead>
<tr>
<th>Definition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \log f' ) in Dirichlet class</td>
</tr>
<tr>
<td>2</td>
<td>( \int</td>
</tr>
<tr>
<td>3</td>
<td>( \int</td>
</tr>
<tr>
<td>4</td>
<td>( \log \varphi' \in H^{1/2} )</td>
</tr>
<tr>
<td>5</td>
<td>( \sum_Q \beta_1^2(Q) &lt; \infty )</td>
</tr>
<tr>
<td>6</td>
<td>( \int \kappa^2(z, w)</td>
</tr>
<tr>
<td>7</td>
<td>( \sum \Delta_j/\ell(\Gamma_j) &lt; \infty )</td>
</tr>
<tr>
<td>8</td>
<td>( \int</td>
</tr>
<tr>
<td>9</td>
<td>( \sum_{\Gamma_j}</td>
</tr>
<tr>
<td>10</td>
<td>( \sum_Q \mu_\Gamma(Q) &lt; \infty )</td>
</tr>
<tr>
<td>11</td>
<td>( \int[c^2(x, y, z)/\ell(x, y, z)]</td>
</tr>
<tr>
<td>12</td>
<td>( \sum \varepsilon_1^2(Q) &lt; \infty )</td>
</tr>
<tr>
<td>13</td>
<td>( \int \delta_1^2(z)dA_\rho(z) &lt; \infty )</td>
</tr>
<tr>
<td>14</td>
<td>( \int S K^2(z)dA_\rho &lt; \infty )</td>
</tr>
<tr>
<td>15</td>
<td>( T_0(1) )</td>
</tr>
<tr>
<td>16</td>
<td>( P_\varphi ) is Hilbert-Schmidt</td>
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<tr>
<td>17</td>
<td>finite Loewner energy</td>
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<td>18</td>
<td>large deviations of SLE((0^+))</td>
</tr>
<tr>
<td>19</td>
<td>Brownian loop measure</td>
</tr>
</tbody>
</table>

**Table 2.** The definitions above the first double line are the previously known function theoretic definitions. The second group are new geometric definitions given in this paper. The third group consists of other known characterizations of the Weil-Petersson class that are not used in this paper; these are briefly described in Appendix D.
Figure 3. A diagram of the implications between the definitions. Each number refers to a definition given in the text (or Table 2). Definitions to the left of the vertical dashed line were previously known to be equivalent; the upward arrows indicate proofs we will sketch in an appendix for completeness. The remaining arrows are new results. Definition 6 was earlier formulated in [37]. Note that \( (5) \Rightarrow (3) \) and \( (13) \Rightarrow (3) \) are redundant; however, they are so simple that it seems worth including them in appendices.

5. \( (1) \Rightarrow (5): \) Dirichlet implies \( \sum \beta^2 < \infty \)

In this section we assume that the conformal map \( f: \mathbb{D} \to \Omega \) satisfies \( \log f' \in \mathcal{D} \) and will prove that Definition 5 holds, i.e., \( \sum_Q \beta_I^2(Q) < \infty \). More formally,

**Lemma 5.1.** Definition 1 implies Definition 5.

**Proof.** We start with some standard definitions that we will use throughout the paper. Given an arc \( I \subset \mathbb{T} = \partial \mathbb{D} \) of length \( |I| \) less than 1, we define the corresponding Carleson square \( Q_I \) as

\[
Q_I = \{ z \in \mathbb{D} : z/|z| \in I, 1 - |z| < |I| \}.
\]

We define the “top half” of \( Q_I \) as

\[
W_I = T(Q_I) = \{ z \in \mathbb{D} : z/|z| \in I, |I|/2 < 1 - |z| < |I| \}.
\]

When \( I \) ranges over the dyadic subintervals of \( \mathbb{T} \) these sets (together with a single disk around the origin) form a Whitney decomposition \( \mathcal{W} \) of \( \mathbb{D} \) (a collection of closed sets with disjoint interiors that cover \( \mathbb{D} \) and satisfy \( \text{diam}(X) \simeq \text{dist}(X, \partial \mathbb{D}) \)).
For each element $W \in \mathcal{W}$, let $\eta(W)$ be the maximum of $|(\log f')'(1-|z|^2)|$ over $W$. Standard results, e.g., Theorem VII.2.1 of [39], imply that $\eta(W) \leq 6$ for all $W \in \mathcal{W}$ and any conformal $f$. Each element $W \in \mathcal{W}$ has hyperbolic area comparable to 1, and Euclidean area comparable to $(1-|z|^2)^2$ for any $z \in W$. Hence

$$\int \left| \frac{f''}{f'} \right|^2 \, dx \, dy = \sum_{W} \int_{W} \left| \frac{f''}{f'} \right|^2 \, dx \, dy \lesssim \sum_{W} \eta(W)^2.$$  

Let $u(z) = f''(z)/f'(z)$ and let $B_z = D(z, (1-|z|)/2)$. Since $u$ is a holomorphic function, the mean value property implies

$$|u(z)|^2 \leq \left( \frac{1}{\text{area}(B_z)} \left| \int_{B_z} u(z) \, dx \, dy \right| \right)^2 \leq \frac{1}{\text{area}(B_z)} \int_{B_z} |u(z)|^2 \, dx \, dy.$$  

Thus the maximum of $|u|^2$ over $W$ is bounded above by the integral of $|u|^2$ over the union of $W$ and a bounded number of adjacent Whitney boxes (enough to cover the set $\cup_{z \in W} B_z$). Each Whitney box is only used a bounded number of times, so we see that Definition 1 is equivalent to

$$\sum_{W \in \mathcal{W}} \eta(W)^2 < \infty.$$  

(5.1)

The next step closely follows a calculation from Pommerenke’s paper [64]. Pommerenke’s result implies that if $\eta(W) \leq \epsilon$ for all Whitney boxes inside $2Q_I$, then $f(I)$ is a quasi-arc with small constant. Clearly (5.1) implies that this holds for all sufficiently small Carleson boxes, say smaller than some $r > 0$. The conformal map $f$ restricted to such a Carleson square $2Q_I$ has a $K$-quasiconformal extension to the reflection across $2I$, with $K$ close to 1. Hence a simple consequence of Mori’s theorem (Theorem III.C of [2]) gives

$$\frac{\text{diam}(f(J))}{\text{diam}(f(I))} \leq C \left( \frac{\text{diam}(J)}{\text{diam}(I)} \right)^{\alpha},$$  

(5.2)

for some uniform $C < \infty$ and $\alpha < 1$ as close to 1 as we wish. For our purposes, it will suffice to take $\alpha = 3/4$ (any value $> 1/2$ would work). Henceforth we assume $r$ has been chosen so that (5.2) holds for all arcs of length less than $r$.

Suppose $z_0 \in W = T(Q_I) \in \mathcal{W}$, and suppose $z \in 2Q_I$. Let $W = W_0, \ldots, W_N$ be the list of Whitney squares hit by the hyperbolic geodesic from $z_0$ to $z$; note that $N = N(z) \simeq 1 + \rho(z_0, z)$ and that the boxes can be ordered so that $W_0 = W$, ...
\[ k \simeq 1 + \rho(W_0, W_k), \text{ and} \]
\[ \text{diam}(W_k) \simeq \text{diam}(W_0) \exp(-\rho(W_0, W_k)). \]  
\[ \text{(5.3)} \]

Moreover, we can choose points \( z_k \in \gamma \cap W_k \) which are ordered by increasing distance from \( z_0 \). See Figure 4.

By a linear rescaling we may assume \( f'(z_0) = 1 \), without changing the values of \((\log f')'\) and hence without changing the \( \eta \)'s. It is convenient to truncate the sequence of Whitney boxes in some cases. Define \( M \leq N \) to be the first index where
\[ \sum_{k=0}^{M} \eta(W_k) \geq 1, \]  
\[ \text{(5.4)} \]

or set \( M = N \) if there is no such index. If \( M = N \), let \( z_M = z \); otherwise let \( z_M \) be some point in \( \gamma \cap W_M \). Note that by our assumptions either \( z_M = z \) or
\[ |f(z_M) - f(z)| \lesssim \text{diam}(f(W_0)) \left( \frac{\text{diam}(W_M)}{\text{diam}(W_0)} \right)^{\alpha} \]
\[ \lesssim \text{diam}(f(W_0)) \cdot \exp(-\alpha \rho(W_M, W_0)). \]

If \( z + M \neq z \) then (5.4) holds, so
\[ |f(z_M) - f(z)| \lesssim \text{diam}(f(W_0)) \left( \sum_{k=0}^{M} \eta(W_k) \right) \exp(-\alpha \rho(W_M, W_0)) \]
\[ \lesssim \text{diam}(f(W_0)) \left( \sum_{k=0}^{M} \eta(W_k) \exp(-\alpha \rho(W_k, W_0)) \right). \]
Similarly, either $z_M = z$ or

$$|z_M - z| \lesssim \text{diam}(W_M) \lesssim \text{diam}(W_0) \exp(-\rho(W_M, W_0)) \lesssim \text{diam}(W_0) \left( \sum_{k=0}^{M} \eta(W_k) \exp(-\alpha \rho(W_M, W_0)) \right) \lesssim \text{diam}(W_0) \left( \sum_{k=0}^{M} \eta(W_k) \exp(-\alpha \rho(W_k, W_0)) \right).$$

Pommerenke’s estimate (proof of (i) $\Rightarrow$ (ii) of Theorem 1 on page 201 of [64]), says that if $z_n \in \gamma \cap W_n$,

$$|\log f'(z_n)| \leq \int_{z_0}^{z_n} \left| \frac{f''(t)}{f'(t)} \right| \, |dz| \leq \sum_{k=0}^{n-1} \int_{z_k}^{z_{k+1}} \frac{\eta(W_k)}{\text{diam}(W_k)} \, |dz| \lesssim \sum_{k=0}^{n-1} \eta(W_k).$$

Thus

$$|f'(z_n) - 1| \leq \exp \left( C \sum_{k=0}^{n-1} \eta(W_k) \right) - 1.$$

Integrating, and using $e^x - 1 \lesssim x$ for $x \in [0, 1]$,

$$\left| \frac{f(z_M) - f(z_0)}{z_M - z_0} - 1 \right| \lesssim \int_{z_0}^{z_M} \left[ \exp(C \sum_{k=0}^{M} \eta(W_k)) - 1 \right] \, |dz| \lesssim \sum_{k=0}^{M-1} \int_{z_k}^{z_{k+1}} \left[ \sum_{j=0}^{k-1} \eta(W_j) \right] \, |dz| \lesssim \sum_{k=0}^{M-1} \text{diam}(W_k) \left[ \sum_{j=0}^{k-1} \eta(W_j) \right] \lesssim \sum_{j=0}^{M-1} \eta(W_j) \sum_{k=j+1}^{M-1} \text{diam}(W_k) \lesssim \sum_{j=0}^{M-1} \eta(W_j) \text{diam}(W_j).$$

Where the last inequality uses that fact that the $\{\text{diam}(W_k)\}$ decrease geometrically, and hence the sum is dominated by a multiple of its largest term. Using (5.3) and
the fact that \( \alpha < 1 \), we see that the last term above is

\[
\lesssim \text{diam}(W_0) \sum_{k=0}^{M} \eta(W_k) \exp(-\alpha \rho(W_0, W_k)).
\]

Multiplying by \((z_M - z_0)\) and using \(|z_0 - z_M| \lesssim 1\), we get

\[
(f(z_M) - f(z_0)) - (z_M - z_0) = O \left( \text{diam}(W_0) \sum_{k=0}^{M} \eta(W_k) \exp(-\alpha \rho(W_0, W_k)) \right).
\]

Thus by adding \(f(z) - f(z)\) and \(z - z\) to both sides and rearranging we get

\[
f(z) - f(z_0) = (z - z_0) + O \left( \text{diam}(W_0) \sum_{k=0}^{M} \eta(W_k) \exp(-\alpha \rho(W_0, W_k)) \right)
\]

since we have previously shown that each of the last three terms in the first line is dominated by the sum in the second line. This equation says that \(f\) restricted to \(I\) is linear with a small error, hence \(f(I)\) will be close to a line segment with small error.

More precisely, if \(\gamma = f(I)\), then

\[
\beta(\gamma) \lesssim \sup_{z \in I} \sum_{k=0}^{M(z)} \eta(W_k) \exp(-\alpha \rho(W, W_k)).
\]

In particular, for each \(W = T(Q_I) \in \mathcal{W}\) that is small enough, we can choose a boundary point \(z \in 2I\) and a chain of adjacent Whitney boxes \(C(W)\) so that the corresponding sum is within a factor of 2 of the supremum over all points in \(2I\). Choose \(s\) so that \(1 < s < 2\alpha = 3/2\). Then summing over all sufficiently small Whitney boxes in \(\mathbb{D}\)

\[
\sum_{W: \text{diam}(W) < r} \beta^2(\gamma_W) \lesssim \sum_{W} \left( \sum_{W' \in C(W)} \eta(W') \exp(-\alpha \rho(W, W')) \right)^2
\]

\[
\lesssim \sum_{W} \left( \sum_{W' \in C(W)} \eta^2(W') \exp(-s \rho(W, W')) \right) \times \left( \sum_{W' \in C(W)} \exp((s - 2\alpha) \rho(W, W')) \right).
Since \( s - 2\alpha < 0 \) and the distances between \( W \) and \( W' \) grow linearly, the sum in the second term of the product is a geometric sum, and hence converges. Thus

\[
\sum_{W : \text{diam}(W) < r} \beta^2(\gamma_W) \lesssim \sum_{W'} \eta^2(W') \sum_{W : W' \in C(W)} \exp(-s\rho(W, W'))
\]

Since \( s > 1 \), the second sum above is bounded, even when taken over every Whitney box \( W \) in \( D \), so we obtain

\[
\sum_W \beta^2(\gamma_W) \lesssim \sum_{W'} \eta^2(W').
\]

The collection \( \{\gamma_W\} \) is not itself a multi-resolution family for \( \Gamma \), but a finite family of rotations of \( D \) gives such a family and the proof above applies equally well to each family. Thus the \( \beta^2 \)-sum is finite over some multi-resolution family of arcs for \( \Gamma \), and hence Definition 1 implies Definition 5. \( \square \)

6. \((5) \Rightarrow (3)\): \( \beta^\prime \)'s give QC reflections:

Lemma 6.1. Definition 5 implies Definition 3.

Proof. Since \( \sum_Q \beta^2_{\Gamma}(Q) < \infty \), only finitely many of of the \( \beta^\prime \)'s can be larger than \( 1/1000 \). Let \( U(\epsilon) \) denote the \( \epsilon \)-neighborhood of \( \Gamma \), and choose \( \epsilon_0 \) so that \( U(\epsilon_0) \) only contains dyadic squares \( Q \) with \( \beta_{\Gamma}(Q) < 1/1000 \).

Let \( \Omega \) be the bounded complementary component of \( \Gamma \) and consider a Whitney decomposition for \( \Omega \) using dyadic squares. Form a triangulation of \( \Omega \) by connecting the center of each square to the vertices on its boundary. Note that neighboring triangles have comparable diameters and that all angles are bounded uniformly above 0 and below \( \pi \).

We will define a QC reflection across \( \Gamma \) that is defined on a neighborhood of \( \Gamma \) and is piecewise affine on the above triangles. Let \( S_k \) be the collection of squares \( Q \) in the Whitney decomposition so that \( \ell(Q) = 2^{-k} \) and let \( S = \cup_{k > k_0} S_k \) where \( k_0 \) is chosen so that the elements of \( S \) are all contained in \( U(\epsilon/100) \). Order the elements of \( \{Q_j\}_{j=1}^{\infty} = S \) so that side lengths are non-decreasing. For each \( Q_j \) choose a dyadic square \( Q'_j \) that hits \( \Gamma \) and so that so that \( 3Q'_j \) contains \( Q_j \). Note that \( Q'_j \subset U \), so \( \beta_{\Gamma}(Q'_j) \) is small. To begin, choose a line \( L_j \) that minimizes the definition of \( \beta_{\Gamma}(Q'_j) \). Reflect all four vertices of \( Q_1 \) across \( L_1 \). In general, reflect each vertex \( v \) of \( Q_j \) across
$L_j$ to a point $v^*$ in $\Omega^*$, if it was not already reflected by belonging to some $Q_k$ with $k < j$. See Figure 5.

![Figure 5](image)

**Figure 5.** Each vertex on one side of $\Gamma$ is reflected across a line approximating $\Gamma$. Since different vertices may use different lines the corresponding triangles become distorted, but since the lines come within $O(\beta)$ of each other and have angles differing by $O(\beta)$, the triangles are related by an affine map that is quasiconformal with dilatation $O(\beta)$.

The main point is that each vertex $v$ belongs a uniformly bounded number of $Q_j$'s and the different possible reflections $v^*$ of $v$ corresponding to these different squares all lie within distance $\beta_j \cdot \text{dist}(v, \Gamma)$ of each other, where $Q_j$ is any of the Whitney squares having $v$ as a corner and $\beta_j = \beta_\Gamma(Q'_j)$. This occurs because the finitely many different lines we might use all have angles directions that differ by at most $O(\beta_j)$, and they all pass within $O(\beta_j \ell(Q'_j))$ of some point in $Q'_j$. We now define affine maps on each element of our triangulation that lies inside $U(\epsilon_0/1000)$ by sending each vertex to its reflection $v^*$. Suppose $T$ is a triangle associated to $Q_j$. Then $\text{diam}(T) \simeq \text{dist}(T, \Gamma)$. The reflected vertices of $T$ form a triangle $T^*$ that is within $O(\beta_j)$ of being congruent to $T$. Thus the following simple result applies:

**Lemma 6.2.** Suppose $T = (v_1, v_2, v_3)$ is a planar triangle of diameter 1 and that the three interior angles are bounded uniformly away from 0 and $\pi$. Suppose $T'$ is another triangle whose vertices $(v'_1, v'_2, v'_3)$ are each within $\epsilon$ of the corresponding...
vertex of $T$ the affine map that extends $v_j \to v'_j$, $j = 1, 2, 3$ is quasiconformal with complex dilatation bounded by $O(\epsilon)$.

Proof. One way to check this is by an explicit computation: the affine map that sends $\{0, 1, a\}$ to $\{0, 1, b\}$ with $a, b$ in the upper half-plane is $z \to \alpha z + \beta \bar{z}$ where $\alpha = (b - \bar{a})/(\bar{a} - a)$ and $\beta = 1 - \alpha$. From this one sees that the complex dilatation is the constant $\mu = \beta/\alpha = (a - b)/(b - \bar{a}) = \tanh(\rho(a, b))$, where $\rho$ denotes the hyperbolic metric on $\mathbb{H}^+_2$.

Extending the map between vertices linearly, we get an affine map from $T$ to $T^*$ that is quasiconformal with dilatation bounded $O(\beta_j)$. Putting together the different triangle we get a homeomorphism on a neighborhood of $U$ of $\Gamma$ that fixes $\Gamma$ pointwise and is quasiconformal on $U \setminus \Gamma$. Since $\Gamma$ is a quasicircle, it is removable for quasiconformal homeomorphisms and hence our map is quasiconformal on all of $U$, i.e., we have defined a quasiconformal reflection across $\Gamma$ on a neighborhood $U$ of $\Gamma$. Each triangle $T$ has hyperbolic area $\simeq 1$, so

$$\int_T |\mu(z)|^2 dA_\rho(z) = O(\beta_j^2(Q)),$$

for some dyadic square that has diameter and distance from $T$ that are both comparable to $\text{diam}(T)$. Each such $Q$ can therefore only occur for boundedly many $T$ and hence

$$\int_U |\mu(z)|^2 dA_\rho(z) = O\left(\sum_Q \beta_j^2(Q)\right).$$

We now extend this map quasiconformally to the rest of $\Omega$ anyway that we want and we obtain a map satisfying Definition 3. \qed

This completes the proof that $(1) \Rightarrow (5) \Rightarrow (3)$ and hence the proof of Theorem 1.1 since $(3) \Rightarrow (1)$ is already known. (For completeness we will sketch a proof that $(3) \Rightarrow (2) \Rightarrow (1)$ in Appendix C.) The construction in this section gives a quasiconformal reflection for a quasicircle $\Gamma$ whenever the $\beta$’s are sufficiently small (say $\beta \leq 1/100$) for all small enough squares hitting $\Gamma$, not just Weil-Petersson curves.
7. An improved traveling salesman theorem

Peter Jones’ traveling salesman theorem (TST) [47] says that for a planar curve $\Gamma$,
\[ \ell(\Gamma) \simeq \text{diam}(\Gamma) + \sum_Q \beta^2_{\Gamma}(Q) \text{diam}(Q), \]
where $\ell(\Gamma)$ denotes the length (Hausdorff 1-measure) of $\Gamma$ and $\ell(Q)$ denotes the side length of $Q$. Actually, Jones’ paper [47] states
\[ \ell(\Gamma) \leq (1 + \delta) \text{diam}(\Gamma) + C(\delta) \sum_Q \beta^2_{\Gamma}(Q) \text{diam}(Q), \]
for any $\delta > 0$. In this paper we will need to use a slightly stronger version of this:

**Theorem 7.1.** With notation as above,
\[ (7.1) \quad \ell(\Gamma) - \text{diam}(\Gamma) \simeq \sum_Q \beta^2_{\Gamma}(Q) \text{diam}(Q) \]
holds for any Jordan arc $\Gamma \subset \mathbb{R}^2$.

Recall that $\text{crd}(\gamma) = |z - w|$ where $\{z, w\}$ are the endpoints of $\gamma$. For an arc $\gamma$, we always have $\text{crd}(\Gamma) \leq \text{diam}(\gamma)$, so Theorem 7.1 implies
\[ (7.2) \quad \Delta(\Gamma) \equiv \ell(\Gamma) - \text{crd}(\Gamma) \simeq \sum_Q \beta^2_{\Gamma}(Q) \text{diam}(Q). \]
We will need the opposite directions as well:

**Theorem 7.2.** With notation as above,
\[ (7.3) \quad \Delta(\Gamma) = \ell(\Gamma) - \text{crd}(\Gamma) \simeq \sum_Q \beta^2_{\Gamma}(Q) \text{diam}(Q). \]
holds for any Jordan arc $\Gamma \subset \mathbb{R}^2$.

**Proof of Theorem 7.1.** Although Theorem 7.1 is formally stronger than the usual statement of Jones’s theorem (e.g., [47], [39], [61]), this formulation follows from existing proofs. To verify this, we review the proof of the TST given in [16].

First consider the direction
\[ (7.4) \quad \ell(\Gamma) - \text{diam}(\Gamma) \lesssim \sum_Q \beta^2_{\Gamma}(Q) \text{diam}(Q). \]

The proof of Theorem 10.5.1 in [16] involves an inductive construction that covers $\Gamma$ by a nested sequence of closed sets $\{\Gamma_n\}$ that shrink down to $\Gamma$. There are two cases
that determine how $\Gamma_{n+1}$ is constructed from $\Gamma_n$, but when $\Gamma$ is connected, only the first case ever occurs. Thus in our situation, $\Gamma_n$ is a union of convex hulls of subsets of $\Gamma$, and $\Gamma_{n+1}$ is formed by replacing each of these convex sets $R$ by two disjoint convex subsets, $R_1$ and $R_2$, each the convex hull of two subsets of $\Gamma \cap R$. Moreover, Lemma 10.5.2 of [16] says that the sum of the diameters of $R_1$ and $R_2$ is less than

$$\text{diam}(R) + O\left(\beta^2(R)\text{diam}(R)\right),$$

where

$$\beta(R) = \sup_L \sup_{z \in R} \frac{\text{dist}(z, L)}{\text{diam}(R)},$$

and the first supremum is over all lines $L$ determined by diameters of $R$ (diameters are segments connecting pairs of points in $\partial R$ that are diam$(R)$ apart). Since the first convex set $R$ used in the construction is the convex hull of $\Gamma$, and since a set and its convex hull have the same diameter, it follows that

$$(7.5) \quad \ell(\Gamma) \leq \text{diam}(\Gamma) + O\left(\sum_R \beta^2(R)\text{diam}(R)\right),$$

where the sum is over all convex sets that occur at all generations. Lemma 10.5.4 of [16] shows that $\sum_R \beta^2(R)\ell(R)$ is bounded by a multiple of the usual $\beta^2$-sum over dyadic squares, and hence (7.4) follows.

Next we consider the opposite direction:

$$(7.6) \quad \ell(\Gamma) \geq \text{diam}(\Gamma) + O\left(\sum_Q \beta^2(Q)\text{diam}(Q)\right).$$

I claim that proof of Theorem 10.2.3 in [16] actually shows

$$(7.7) \quad \ell(\Gamma) \geq \frac{1}{2}\text{prm}(\Gamma) + O\left(\sum_Q \beta^2(Q)\text{diam}(Q)\right),$$

where $\text{prm}(\Gamma) = \ell(\partial(\text{ch}(\Gamma)))$ denotes the perimeter of $\Gamma$, i.e., the length of the boundary of its planar convex hull (twice the length of $\Gamma$ if it is a line segment). By noting that the orthogonal projection of a closed curve onto a diameter segment is 1-Lipschitz and at least 2-to-1, we see that the perimeter of $\Gamma$ is at least twice its diameter. Hence (7.7) implies (7.6). (We have used the notation $\text{ch}(\Gamma)$ to denote the planar convex hull of $\Gamma$ for the Euclidean metric; later we will use $\text{CH}(\Gamma)$ to denote the hyperbolic convex hull of $\Gamma \subset \mathbb{R}^2$ in hyperbolic 3-space.)
Estimate (7.7) is proven using ideas from integral geometry. For the following facts, see Section 10.1 of [16] (for a brief summary) or [68] (for a more encyclopedic treatment). Each line $L$ in $\mathbb{R}^2 \setminus \{0\}$ is identified with the point $z = re^{i\theta} \in L$ that is closest to the origin. The measure $d\mu = drd\theta$ on $(0, \infty) \times [0, 2\pi)$ becomes a measure on lines and is invariant under Euclidean isometries of the plane. It is known that if $\Gamma$ is connected then the $\mu$ measure of the set of lines that hit $\Gamma$ is exactly its perimeter (length of its convex hull boundary). If $n(L, \Gamma)$ is the number of times the line $L$ intersects $\Gamma$ (possibly infinite), then Crofton’s formula says

$$
\ell(\Gamma) = \frac{1}{2} \int n(L, \Gamma) d\mu(L).
$$

Let $\mathcal{D}$ denote the collection of dyadic squares in the plane and let $\mathcal{D}^*$ be the union of translates of $\mathcal{D}$ by the nine elements of $\{-\frac{1}{3}, 0, \frac{1}{3}\} \times \{-\frac{i}{3}, 0, \frac{i}{3}\}$. This creates a multi-resolution family from the family of dyadic squares. For $Q \in \mathcal{D}^*$ let $S(Q, \Gamma)$ be the set of lines that intersect both $\frac{5}{3}Q \cap \Gamma$ and $(3Q \setminus 2Q) \cap \Gamma$. Let $N(L, \Gamma)$ be the number of squares $Q \in \mathcal{D}^*$ such that $L \in S(Q, \Gamma)$.

I claim that

$$
n(L, \Gamma) = 1 + O(N(L, \Gamma)),
$$

whenever $n(L, \Gamma) > 0$. Actually, only the weaker statement

$$
n(L, \Gamma) \simeq 1 + N(L, \Gamma),
$$

is stated in [16], but the proof of Theorem 10.2.2, explicitly shows that

$$
n(L, \Gamma) \leq 1 + 2N(L, \Gamma).
$$

Moreover, if $n(L, \Gamma) > 0$, then the proof shows

$$
n(L, \Gamma) \geq \frac{1 + N(L, \Gamma)}{M},
$$

for some uniform $M$. Since $n$ is integer valued, these imply (7.8).
Following the proof of Theorem 10.2.3 in [16], if we integrate over the set of lines that hit $\Gamma$, we get

$$
\ell(\Gamma) = \frac{1}{2} \int n(L, \Gamma) d\mu
= \frac{1}{2} \int 1 + O(N(L, \Gamma)) d\mu
= \frac{1}{2} \brm(\Gamma) + O \left( \int N(L, \Gamma) d\mu \right),
$$

and the rest of the proof given in [16] shows that

$$
\int N(L, \Gamma) d\mu \simeq \sum_{Q^*} \mu(S(Q^*, \Gamma) \simeq \sum_{Q} \beta_2^2(Q) \text{diam}(Q),
$$

where the second sum is the usual $\beta^2$-sum over all dyadic squares $Q$ and the first is summed over all squares in $D^*$. This completes the proof of (7.6), and hence of (7.1).

\textit{Proof of Theorem 7.2.} As noted earlier, the $\geq$ direction is immediate from Theorem 7.1 since $\text{crd}(\Gamma) \leq \text{diam}(\Gamma)$.

To prove the other direction, we may assume the $\beta^2$-sum in (7.2) is finite, for otherwise there is nothing to prove. Thus we may assume $\gamma$ is rectifiable. Let $Q_0$ be a dyadic square hitting $\Gamma$ with $\text{diam}(\Gamma) \leq \text{diam}(Q_0) \leq 2\text{diam}(\Gamma)$, hence $\Gamma \subset 3Q$. Suppose $\beta_0$ is a small positive number (chosen to satisfy various conditions described below). If $\beta_1(Q_0) > \beta_0$, then the result is trivially true since then

$$
\text{crd}(\Gamma) \leq \text{diam}(\Gamma) \leq \frac{1}{\beta_0^2} \beta_1^2(Q_0) \text{diam}(Q_0) \lesssim \beta_1^2(Q_0) \text{diam}(Q_0),
$$

and hence the $\text{crd}(\Gamma)$ term in (7.2) can be absorbed into the $\beta^2$-sum term.

Therefore we may assume $\beta_1(Q_0) \leq \beta_0$. Let $S = [x, y]$ be a diameter segment of $\Gamma$ and let $\gamma_0$ be the open subarc of $\Gamma$ connecting $x$ and $y$. Then $\Gamma \setminus \gamma_0$ consists of two arcs, $\gamma_1$ connecting $x$ to an endpoint $z$ (possibly $z = x$) and $\gamma_2$ connecting $y$ to the other endpoint $w$ (possibly $w = y$). By rotating and rescaling, we may assume that $x = 1, y = -1$. See Figure 6. It suffices to show that $\ell(\gamma_1)$ and $\ell(\gamma_2)$ are both bounded by a uniform multiple of the $\beta^2$-sum on the right side of (7.2). Because of the traveling salesman theorem, it is enough to bound the diameters of these arcs by the $\beta^2$-sum; then the diameters can be absorbed into the $\beta^2$-sum by making the
comparability constant larger. The arguments for both arcs are the same, so we only discuss $\gamma_1$.

![Figure 6. Definitions for the proof of Theorem 7.2.](image)

Let $\epsilon = \text{diam}(\gamma_1)$. Assume $\epsilon > 0$ (otherwise there is nothing to do). Let $Q_1, \ldots, Q_k$ be nested dyadic squares containing $x$ with diameters going from $\text{diam}(Q_0)$ to $\epsilon$. Note that $k \approx \log \text{diam}(Q_0)/\epsilon$. If any one of these squares satisfies $\beta_T(Q_j) \geq \beta_0$, then

$$\text{diam}(\gamma_1) \leq \frac{\beta^2_T(Q_j)}{\beta_0^2} \text{diam}(\gamma_1) \lesssim \beta^2_T(Q_j) \text{diam}(Q_j),$$

and hence $\text{diam}(\gamma_1)$ is dominated by the $\beta^2$-sum, as desired. For the remainder of the proof we may therefore assume that $\beta_T(Q_j) \leq \beta_0$ for all $j \in \{0, 1, \ldots, k\}$. Let $L_j$ be a best line in the definition of $\beta_T(Q_j)$.

**Case 1:** Assume that for some $j \in \{1, \ldots, k\}$, the line $L_j$ makes an angle larger than $10\beta_0$ with the horizontal. Since the angle between $L_0$ and $L_j$ is bounded by $O\left(\sum_{i=0}^{j} \beta(Q_i)\right)$, and we have normalized so that the best line for $Q_0$ is within $\beta_0$ of horizontal, we must have $\sum_{j=1}^{k} \beta_T(Q_j) \gtrsim \beta_0 \gtrsim 1$. The Cauchy-Schwarz inequality then implies

$$1 \lesssim \left(\sum_{j=1}^{k} \beta_T(Q_j)\right)^2 \leq \left(\sum_{j=1}^{k} \beta^2_T(Q_j)2^{-j}\right) \cdot \left(\sum_{j=1}^{k} 2^j\right) \approx 2^k \sum_{j=1}^{k} \beta^2_T(Q_j)2^{-j}$$

so

$$\sum_{j=1}^{k} \beta^2_T(Q_j) [\beta_T(Q_0)2^{-j}] \gtrsim \beta_T(Q_0)2^{-k} \gtrsim \frac{\beta_T(Q_0)}{\beta_T(Q_0)/\epsilon},$$

and hence

$$\epsilon = \text{diam}(\gamma_1) \lesssim \sum_{j=1}^{k} \beta^2_T(Q_j)\text{diam}(Q_j),$$

as desired.

**Case 2:** Next we assume that all the lines $L_j, j = 0, \ldots, k$ make angle $\leq 10\beta_0$ with the horizontal. Consider a subarc $\gamma'_1 \subset \gamma_1$ that is contained in, and connects the
boundary components of the annulus

\[ \{ p \in \mathbb{R}^2 : \frac{1}{10} \text{diam}(\gamma_1) \leq |p - x| \leq \frac{1}{5} \text{diam}(\gamma_1) \}. \]

Since \( \gamma_1 \) and \( \gamma'_1 \) have comparable diameters, it is enough to bound \( \text{diam}(\gamma'_1) \).

For each \( p \in \gamma'_1 \) a dichotomy holds: either every dyadic square \( Q \) containing \( p \) with \( \text{diam}(Q) \leq \text{diam}(\gamma_1)/10 \) satisfies \( \beta_T(Q) \leq \beta_0 \) or there is a square \( Q_p \) of this form such that \( \beta_T(Q_p) > \beta_0 \).

Let \( E \subset \gamma'_1 \) be the set of points \( p \) where such a \( Q_p \) exists. Suppose \( \ell(E) \approx \ell(\gamma'_1) \). Since the squares are dyadic, we can find a collection \( \{ Q^j_p \} \) of such squares with disjoint interiors that covers \( E \). Hence

\[
\ell(\gamma'_1) \approx \ell(E) \leq \sum_j \ell(Q^j_p \cap E) \lesssim \sum_j \left[ \text{diam}(Q^j_p) + \sum_{Q < Q^j_p} \beta^2_E(Q) \text{diam}(Q) \right]
\]

where we have applied the traveling salesman theorem to each set \( Q^j_p \cap E \). Note that usual formulation of the TST is to sum over all dyadic squares in the plane, but if \( E \subset Q \), then it suffices to sum over all squares contained in \( Q \) (including \( Q \) itself) since the \( \beta^2 \)-sum over all larger squares that hit \( E \) form a geometric series whose sum is \( O(\beta^2(Q) \text{diam}(Q)) \). Now we use the fact that \( \beta_E \leq \beta_T \), to show

\[
\ell(\gamma'_1) \approx \sum_j \sum_{Q < Q^j_p} \beta^2_T(Q) \text{diam}(Q)
\]

where we have also used \( \beta_T(Q^j_p) \approx 1 \) to absorb the \( \text{diam}(Q^j_p) \) terms into the \( \beta^2 \)-sums. Since this is a \( \beta^2 \)-sum over disjoint collections of dyadic squares, it is dominated by the full \( \beta^2 \)-sum, and we get the desired bound.

Since we can assume \( \gamma_1 \) is rectifiable, almost every point of \( \gamma_1 \) is a tangent point.

**Claim 1:** if \( p \in \gamma'_1 \setminus E \) and \( p \) is a tangent point of \( \gamma \), then \( p \) has the following “crossing property”: if \( Q \) is a dyadic square containing \( p \) with \( \text{diam}(Q) \leq \text{diam}(\gamma_1)/10 \) then \( \gamma_1 \) must “cross” \( Q \) in the sense that the orthogonal projection of \( \gamma_1 \cap 3Q \) onto \( L_Q \) covers \( L_Q \cap Q \), where \( L_Q \) is a best approximating line for the definition of \( \beta_T(Q) \).

To prove this claim, note that because the \( \beta \)-numbers for all smaller squares containing \( p \) are small, we can construct a Jordan curve \( \sigma \) connecting distinct components of \( \partial(3Q) \setminus \gamma_1 \), passing through \( p \), hitting no other points of \( \gamma_1 \), and so that \( \sigma \) is nearly orthogonal to \( L_{Q'} \) for each sub-square \( Q' \subset Q \) containing \( p \). To do this, choose antipodal points on each circle of radius \( t = \text{diam}(Q)2^{-n} \) that are as far as possible from
the optimal line passing through \( p \) for the definition of \( \beta(p,t) \) and then connecting these points in the obvious way. See Figure 7. If \( p \) is a tangent point of \( \gamma_1 \), then \( \sigma \) also has a tangent at \( p \) and the two tangent directions are perpendicular to each other. From this we see that \( \gamma_1 \) crosses \( \sigma \), i.e., it hits both components of \( 3Q \setminus \sigma \). Since \( \gamma_1 \) only hits \( \sigma \) once, it must leave \( 3Q \) through a different component of \( \partial(3Q) \cap S \) than it entered through. This implies Claim 1.

![Figure 7](image-url)

**Figure 7.** If all the \( \beta \)'s are small at \( p \), and \( p \) is a tangent point of \( \gamma \), then \( \gamma \) must cross \( Q \) in the sense that the orthogonal projection of \( \gamma \cap 3Q \) on the line \( L \) must cover \( L \cap Q \), i.e., \( \gamma \) can’t “double back” and leave using the same end of \( S \) that it entered.

**Claim 2:** \( \ell(E) = \ell(\gamma'_1) \). If not, then we can choose a non-empty subset \( F \subset \gamma'_1 \setminus E \) that consists entirely of tangent points of \( \gamma \). Suppose \( p \in F \) and define \( d \) to be the distance from \( p \) to \( \gamma_0 \). By the assumption that every \( L_j \) is close to horizontal, we know \( d = O(\beta_0 \text{diam}(\gamma_1)) < \text{diam}(\gamma_1)/10 \). Also note that \( d \) is positive since \( p \) is not on \( \gamma_0 \). Let \( Q_p \) be the dyadic square containing \( p \) with diameter \( 2d < \text{diam}(Q_p) \leq 4d \). Because \( \text{diam}(Q_p) \leq \text{diam}(\gamma_1)/10 \), the argument in the previous paragraph applies, and \( \gamma_1 \) must cross \( Q_p \) inside a strip \( S \) of width \( \beta_0 \text{diam}(Q_p) \). Moreover, since \( \text{diam}(Q_p) > 2d \), the curve \( \gamma_0 \) also hits \( 3Q \) and hence contains a point \( q \) in the same strip \( S \), and hence \( \gamma_0 \) is at most distance \( \beta_0 \text{diam}(Q_p) \) from \( \gamma_1 \). For \( \beta_0 \) small, this value is much smaller than \( d \), giving a contradiction. Thus no such \( p \) exists, and hence \( \ell(E) = \ell(\gamma'_1) \), so Claim 2 holds. This completes the proof of Theorem 7.2. \( \square \)
In the special case when \( \Gamma \) is a closed arc, we can take its endpoints to be \( z = w \) and \( \text{crd}(\Gamma) = 0 \), so a special case of the previous theorem is

**Corollary 7.3.** If \( \Gamma \) is a closed Jordan curve, then

\[
\ell(\Gamma) \simeq \sum_Q \beta^2_\Gamma(Q) \text{diam}(Q),
\]

where the sum is over all dyadic squares in the plane.

As above, this is a slight improvement of the usual version of the traveling salesman theorem which includes a \( O(\text{diam}(\Gamma)) \) term on the right-hand side.

8. \( (6) \Leftrightarrow (7) \): CONTINUOUS AND DISCRETE ASYMPTOTIC SMOOTHNESS

In Section 3 we formulated both continuous and discrete versions of the quantitative asymptotic smoothness condition given in [64]. In this section we show these two versions are equivalent to each other and in the next section we show the discrete variant is equivalent to Definition 5.

**Lemma 8.1.** Definition 6 is equivalent to Definition 7.

**Proof.** Without loss of generality we may rescale \( \Gamma \) so that it has length 1. Using the arclength parameterization, we may identify \( \Gamma \) with the unit interval \([0, 1]\) with its endpoints identified, and identify \( \Gamma \times \Gamma \) with the torus \( \mathbb{T}^2 = [0, 1]^2 \), with its top/bottom and left/right sides identified in the usual way. The dyadic subintervals of \([0, 1]\) correspond to intervals on \( \Gamma \) of the same length, and we will refer to this collection as the dyadic subarcs of \( \Gamma \).

Recall that if \( \gamma \) is an arc, then \( \ell(\gamma) \) denotes its arclength. Consider the open subset \( U \) of the torus \( \mathbb{T}^2 = [0, 1]^2 \) formed by removing the diagonal \( \{ x = y \} \). Take a Whitney decomposition of \( U \) by dyadic squares \( \{ Q_j \} \). Each such square consists of points \( (x, y) \) so that \( |x - y| \simeq \text{diam}(Q_j) \). The corresponding decomposition of \( \Gamma \times \Gamma \) minus its diagonal is a union of sets \( \{ W_j \} \), each of which are products of dyadic arcs: we will write \( W_j = \gamma'_j \times \gamma''_j \). For each \( W_j \), there is a minimal length subarc \( \Gamma_j \) of \( \Gamma \) (not necessarily dyadic) that contains \( \gamma'_j \cup \gamma''_j \). Note that \( \ell(\Gamma_j) \simeq \ell(\gamma'_j) \). Moreover, because of the Whitney condition, \( \gamma'_j \) and \( \gamma''_j \) are separated by an arc \( \Gamma'_j \subset \Gamma_j \) that also has length comparable to \( \ell(\Gamma) \).
Recall that \( \text{crd}(\gamma) = |z - w| \) where \( z, w \) are the endpoints of \( \gamma \). Recall that \( \Delta(\gamma) \equiv \ell(\gamma) - \text{crd}(\gamma) \). We say two subarcs of \( \Gamma \) are adjacent if they have disjoint interiors, but share a common endpoint.

**Lemma 8.2.** If \( \gamma, \gamma' \subset \Gamma \) are adjacent, then \( \Delta(\gamma) + \Delta(\gamma') \leq \Delta(\gamma \cup \gamma') \).

**Proof.** Note that \( \ell(\gamma \cup \gamma') = \ell(\gamma) + \ell(\gamma') \), and \( \text{crd}(\gamma \cup \gamma') \leq \text{crd}(\gamma) + \text{crd}(\gamma') \), so
\[
\Delta(\gamma \cup \gamma') = \ell(\gamma \cup \gamma') - \text{crd}(\gamma \cup \gamma') \\
\geq \ell(\gamma) + \ell(\gamma') - \text{crd}(\gamma) - \text{crd}(\gamma') = \Delta(\gamma) + \Delta(\gamma').
\]
\( \square \)

The following is now immediate:

**Corollary 8.3.** If \( \gamma \subset \gamma' \) then \( \Delta(\gamma) \leq \Delta(\gamma') \).

Now fix \( j \) and consider the Whitney box \( W_j = \gamma'_j \times \gamma''_j \). If \( \gamma \subset \Gamma_j \) is any arc with one endpoint in \( \gamma'_j \) and the other in \( \gamma''_j \) then \( \Gamma'_j \subset \gamma \subset \Gamma_j \), and hence \( \Delta(\Gamma'_j) \leq \Delta(\gamma) \leq \Delta(\Gamma_j) \). Because \( \Gamma \) is chord-arc, if \( z, \gamma_j \) and \( w \in \gamma'_j \), then \( |z - w| \gtrsim \ell(\Gamma'_j) \approx \ell(\Gamma_j) \). We can therefore write the integral from Definition 6 as
\[
\int_\Gamma \int_\Gamma \frac{\ell(z, w) - |z - w|}{|z - w|^3} |dz||dw| = \sum_j \int_{W_j} \frac{\Delta(z, w)}{|z - w|^3} |dz||dw| \\
\lesssim \sum_j \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)^3} \ell(\Gamma_j)^2 \\
= \sum_j \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)}.
\]

Reversing the argument, now assume \( \Gamma'_j \) is some dyadic subinterval of \( \Gamma \) and let \( \gamma'_j, \gamma''_j \) be the equal length dyadic arcs adjacent to \( \Gamma'_j \).
\[
\int_{\gamma'_j} \int_{\gamma''_j} \frac{\ell(z, w) - |z - w|}{|z - w|^3} |dz||dw| \gtrsim \frac{\Delta(\Gamma'_j)}{\ell(\Gamma'_j)}.
\]
Moreover, the squares \( W_j = \gamma'_j \times \gamma''_j \) that arise in this way only have bounded overlap, and hence
\[
\int_\Gamma \int_\Gamma \frac{\ell(z, w) - |z - w|}{|z - w|^3} |dz||dw| \gtrsim \sum_j \frac{\Delta(\Gamma'_j)}{\ell(\Gamma'_j)},
\]
where the sum is over all dyadic subintervals of \( \Gamma \). This works for any dyadic decomposition \( \{\Gamma_j\} \) of \( \Gamma \), and hence for a multi-resolution family. This gives the equivalence of Definitions 6 and 7. \( \square \)
Lemma 8.2 and Corollary 8.3 hold for any Jordan arc $\Gamma$, but we used the chord-arc assumption to prove Lemma 8.1. Can it be removed, i.e., are the finiteness of (3.5) and (3.7) equivalent for general curves?

9. $(7) \iff (5)$: the Jones conjecture

**Lemma 9.1.** Definition 5 implies Definition 7.

**Proof.** We continue using the notation from the previous section. Let $\{\Gamma_j\}$ be a dyadic decomposition of $\Gamma$. For each $j$ choose a dyadic square $Q_j$ in the plane that hits $\Gamma_j$ and has diameter between $\text{diam}(\Gamma_j)$ and $2 \cdot \text{diam}(\Gamma_j)$. Note that any such dyadic square can only be associated to a uniformly bounded number of arcs $\Gamma_j$ in this way, because there are only a bounded number of arcs $\Gamma_j$ that have the correct size and are close enough to $Q_j$; this uses the fact that $\Gamma$ is chord-arc. Also because $\Gamma$ is chord-arc, the diameter of $\Gamma_j$ is comparable to $\ell(\Gamma_j)$ and is also comparable to $\text{diam}(Q_j)$. Therefore, by (7.3)

$$\Delta(\Gamma_j) \approx \sum_{Q \subset 3Q_j} \beta^2_{\Gamma_j}(Q) \ell(Q).$$

Since $\beta_{\Gamma_j}(Q) \leq \beta_{\Gamma}(Q)$, we get

$$\sum_j \frac{\Delta_j}{\ell(\Gamma_j)} \approx \sum_j \sum_{Q \subset 3Q_j} \beta^2_{\Gamma_j}(Q) \frac{\ell(Q)}{\ell(Q_j)} \sum_{Q \subset 3Q_j} \beta^2_{\Gamma_j}(Q) \frac{\ell(Q)}{\ell(Q_j)} \sum_{j, Q \subset 3Q_j} \frac{\ell(Q)}{\ell(Q_j)}.$$

(9.1)

Note that for each $Q$ with $\text{diam}(Q) \leq \text{diam}(\Gamma)$ and $Q \cap \Gamma \neq \emptyset$, there is a square of the form $Q_j$ from above, that has diameter comparable to $\text{diam}(Q)$ and such that $Q \subset 3Q_j$. Moreover, there can only be a uniformly bounded number of dyadic squares $Q_j$ of a given size so that $3Q_j$ contains $Q$, so each $Q_j$ can only be chosen a bounded number of times. Thus the sum over the $j$'s in the last line above is bounded by a multiple of a geometric series and so is uniformly bounded. Thus

$$\sum_j \frac{\Delta_j}{\ell(Q_j)} \lesssim \sum_Q \beta^2_{\Gamma}(Q).$$

(9.2)
This proves that Definition 5 implies Definition 7. □

Lemma 9.2. Definition 7 implies Definition 5.

Proof. The only step in the argument above where \( \simeq \) became \( \lesssim \) was when we replaced \( \beta_{\Gamma_j}(Q) \) by \( \beta_{\Gamma}(Q) \) in (9.1). This is a real issue, because it is possible to have \( \beta_{\Gamma_j}(Q) \ll \beta_{\Gamma}(Q) \) in some situations. However, it is clear that \( \beta_{\Gamma_j}(Q) = \beta_{\Gamma}(Q) \) if \( \Gamma \cap 3Q \subset \Gamma_j \). Therefore, the proof of the previous lemma would also give a lower bound, if it were true that for every dyadic square \( Q \) such that \( 3Q \cap \Gamma \neq \emptyset \), there is an arc \( \Gamma_j \) of comparable size to \( Q \) so that \( 3Q \cap \Gamma \subset \Gamma_j \). This is true (by definition) if we use a multi-resolution family \( \{\Gamma_j\} \) or arcs instead of the dyadic family (e.g., use the family of triples of dyadic arcs). In that case, we have

\[
\sum_{Q:3Q\cap\Gamma \neq \emptyset} \beta^2_{\Gamma}(Q) \lesssim \sum_{Q:3Q\cap\Gamma_j \neq \emptyset} \beta^2_{\Gamma_j}(Q)
\]

Then our earlier argument shows

\[
\sum_j \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)} \simeq \sum_j \left[ \sum_{Q:3Q\cap\Gamma_j \neq \emptyset} \beta^2_{\Gamma_j}(Q) \frac{\ell(Q)}{\ell(\Gamma_j)} \right]
\]

\[
\simeq \sum_{Q:3Q\cap\Gamma \neq \emptyset} \sum_{j:3Q\cap\Gamma_j \neq \emptyset} \beta^2_{\Gamma_j}(Q) \frac{\ell(Q)}{\ell(\Gamma_j)}
\]

\[
\geq \sum_{Q:3Q\cap\Gamma \neq \emptyset} \sum_{j:3Q\cap\Gamma_j \neq \emptyset} \beta^2_{\Gamma_j}(Q) \frac{\ell(Q)}{\ell(\Gamma_j)}
\]

\[
\geq \sum_{Q:3Q\cap\Gamma_j \neq \emptyset} \beta^2_{\Gamma_j}(Q) \sum_{j:3Q\cap\Gamma_j \neq \emptyset} \frac{\ell(Q)}{\ell(\Gamma_j)}
\]

\[
\geq \sum_{Q:3Q\cap\Gamma \neq \emptyset} \beta^2_{\Gamma}(Q).
\]

This proves that Definition 7 implies Definition 5. □

Combined with our earlier results and known equivalences, this proves the Jones conjecture stated in [37]: Definition 1 is equivalent to Definition 6.

10. (5) \( \Leftrightarrow \) (11): \( \beta \)'s and Menger curvature are equivalent

In this section we prove that Definitions 5 and 11 are equivalent. The necessary estimates are contained in Pajot’ book [61]. Here we will just indicate where to find them in his book.
We start with bounding Menger curvature by the $\beta$'s. This is contained in the proof of Theorem 31 of [61]. In this proof, we will take $\mu$ to be arclength measure on $\Gamma$; this satisfies the linear growth condition of Theorem 31 because $\Gamma$ is chord-arc.

Pajot defines

$$c^2(\mu) = \int_G \int_G \int_G c^2(x, y, z) d\mu(x) d\mu(y) d\mu(z),$$

and on the bottom of page 37 notes that $c^2(\mu) \leq 3c^2(\mu)$ where

$$c^2(\mu) = \int_A c^2(x, y, z) d\mu(x) d\mu(y) d\mu(z),$$

$$A = \{(x, y, z) \in \Gamma \times \Gamma \times \Gamma : |x - z| \leq |x - y|, |y - z| \leq |x - y|\}.$$

He states that

$$c^2(\mu) \leq \sum_Q \int_{(x,z) \in 3Q} \left( \sum_{R \subset Q} \int_{x,y \in \tilde{R}} c^2(x, y, z) d\mu(y) \right) d\mu(x) d\mu(z).$$

where the inner sum is over dyadic sub-squares $R \subset Q$ and

$$\tilde{R} = \{(x, y) \in 3R : |x - y| \geq \text{diam}(R)/3\}.$$

Recall that $\ell(x, y, z) = |x - y| + |y - z| + |z - y|$ is defined as the perimeter of the triangle with vertices $(x, y, z)$, and it is comparable to the longest of the three sides. Note that for $(x, y, z) \in A$ and $(x, y) \in \tilde{R}$, we have $\ell(x, y, z) \simeq |x - y| \simeq \text{diam}(R)$.

Thus we can replace the above estimate by

$$\int_G \int_G \int_G c^2(x, y, z) \ell(x, y, z)^{-2} d\mu(x) d\mu(y) d\mu(z) \lesssim \sum_Q \int_{x,z \in 3Q} \left( \sum_{R \subset Q} \int_{x,y \in \tilde{R}} \frac{c^2(x, y, z)}{\ell(x, y, z)} d\mu(y) \right) d\mu(x) d\mu(z),$$

$$\simeq \sum_Q \int_{x,z \in 3Q} \left( \sum_{R \subset Q} \int_{x,y \in \tilde{R}} \frac{c^2(x, y, z)}{\text{diam}(R)} d\mu(y) \right) d\mu(x) d\mu(z).$$

We now follow the rest of the proof on page 38, replacing the factor $\text{diam}(R)^{-2}$ that occurs throughout by $\text{diam}(R)^{-3}$. At the end we obtain

$$\int_G \int_G \int_G c^2(x, y, z) \ell(x, y, z) d\mu(x) d\mu(y) d\mu(z) \lesssim \sum_Q \beta^2(Q).$$

Thus Definition 5 implies Definition 11, as desired.
Next we deal with the opposite inequality: bounding $\sum \beta^2$ in terms of the Menger curvature. The relevant estimates are given in the proof of Theorem 38 of [61]. On the bottom of page 43 Pajot gives the inequality

$$\beta^2(Q) \operatorname{diam}(Q) \lesssim \sum_{P \subset Q} \int_{P^*} \int c^2(x, y, z) d\mu(x) d\mu(y) d\mu(z) \left( \frac{\operatorname{diam}(P)}{\operatorname{diam}(Q)} \right)^{1/2},$$

where

$$P^* = \{(x, y, z) \in (3P)^3 : |x - y| \simeq |x - z| \simeq |y - z| \simeq \operatorname{diam}(P)\}.$$  

Divide both sides by $\operatorname{diam}(Q)$ and note that for $(x, y, z) \in P^*$ we have $\ell(x, y, z) \simeq \operatorname{diam}(P)$. This gives

$$\beta^2(Q) \lesssim \sum_{P \subset Q} \int_{P^*} \int \frac{c^2(x, y, z)}{\ell(x, y, z)} d\mu(x) d\mu(y) d\mu(z) \left( \frac{\operatorname{diam}(P)}{\operatorname{diam}(Q)} \right)^{1/2}.$$  

On the top of the next page, this modified expressions leads to

$$\sum_{S \subset Q} \beta^2(F) \lesssim \int_Q \int_Q \int_Q c^2(x, y, z) \frac{d\mu(x) d\mu(y) d\mu(z)}{\ell(x, y, z)}.$$  

Since $d\mu$ is arclength measure, this shows Definition 11 implies Definition 5.

11. $(5) \iff (8) \iff (9)$: $\beta$’s control tangents (and conversely).

We will prove this in the order $(9) \Rightarrow (8) \Rightarrow (5) \Rightarrow (9)$.

**Lemma 11.1.** Definition 9 implies Definition 8.

*Proof.* Take a Whitney decomposition of $U$ (the torus minus its diagonal) as we did in Section 8. Recall that each element can be written as $W_j = \gamma_j' \times \gamma_j''$ where $\gamma_j' \cup \gamma_j'' = \Gamma_j \setminus \Gamma_j'$ and all these arcs have comparable lengths. For any such $W_j$ we
have
\[
\int_{\gamma_j'} \int_{\gamma_j''} \left| \frac{\tau(z) - \tau(w)}{z - w} \right|^2 |dz||dw| \leq \int_{\gamma_j'} \int_{\gamma_j''} \left| \tau(z) - \tau(w) \right|^2 \frac{|dz||dw|}{\ell(\Gamma_j)^2}
\]
\[
\lesssim \int_{\gamma_j'} \int_{\gamma_j''} \left( |\tau(z) - \theta(\Gamma_j)|^2 + |\tau(w) - \theta(\Gamma_j)|^2 \right) \frac{|dz|}{\ell(\Gamma_j)^2}
\]
\[
+ \ell(\gamma_j') \int_{\gamma_j''} \left| \tau(w) - \theta(\Gamma_j) \right|^2 \frac{|dw|}{\ell(\Gamma_j)^2}
\]
\[
\lesssim \int_{\Gamma_j} |\tau(z) - \theta(\Gamma_j)|^2 \frac{|dz|}{\ell(\Gamma_j)}
\]
which proves that the discrete version implies the continuous version. \[\square\]

**Lemma 11.2.** Definition 8 implies Definition 5.

**Proof.** Decompose \( \Gamma \times \Gamma \) as above. For each Whitney piece \( W_j = \gamma_j' \times \gamma_j'' \), choose a \( w \in \gamma_j'' \) so that
\[
\ell(\gamma_j'') \int_{\gamma_j'} |\tau(z) - \tau(w)|^2 |dz| \leq 2 \int_{\gamma_j'} \int_{\gamma_j''} |\tau(z) - \tau(w)|^2 |dz||dw|.
\]
(We can do this because a positive measurable function must take a value that is less than or equal to twice its average.) Let \( L \) be the line through one endpoint of \( \gamma_j' \) in direction \( \tau(w) \). Then the maximum distance \( D \) that \( \gamma_j' \) can attain from \( L \) satisfies
\[
d \lesssim \int_{\gamma_j'} |\tau(z) - \tau(w)||dz| \leq \left( \int_{\gamma_j'} |\tau(z) - \tau(w)|^2 |dz| \right)^{1/2} \ell(\gamma_j')^{1/2}.
\]
Therefore
\[
\beta^2(\gamma_j') \simeq d^2 / \text{diam}(\gamma_j') \lesssim \frac{1}{\ell(\gamma_j')} \int_{\gamma_j'} |\tau(z) - \tau(w)|^2 |dz|
\]
\[
\lesssim \frac{1}{\ell(\gamma_j')^2} \int_{\gamma_j'} \int_{\gamma_j''} |\tau(z) - \tau(w)|^2 |dz||dw|
\]
\[
\lesssim \int_{\gamma_j'} \int_{\gamma_j''} \left| \frac{\tau(z) - \tau(w)}{z - w} \right|^2 |dz||dw|.
\]
Summing over all Whitney pieces proves that the \( \beta^2 \)-sum is finite when taken over all arcs of the form \( \{\gamma_j'\} \). However, every dyadic interval is contained in the union
of a uniformly bounded number of such arcs, so the sum over all dyadic arcs is also finite. By moving the base point, the same conclusion holds for any translate of the dyadic intervals, hence for some multi-resolution family of arcs. Finally, because we assume $\Gamma$ is chord-arc, the intersection of $\Gamma$ with $3Q$ for any dyadic square $Q$, is contained in an arc $\gamma \subset \Gamma$ of the multi-resolution family of comparable diameter, and so $\beta_\Gamma(Q) \lesssim \beta(\gamma)$. This proves the lemma. □

Now we come to the most difficult implication: $(5) \Rightarrow (9)$. Before giving the details, we recall a few basic facts about martingales. We will only need to consider the case when we have a closed curve $\Gamma$ and collections of subarcs $\mathcal{I} = \{I_n\} = \{I^n_j\} \subset \Gamma$, for $n \in \mathbb{N}$ and $j = 1, \ldots, 2^n$. We let $I^n_0 = \Gamma$ and for each generation $n \geq 0$ the interval $I^n_j$ is split into two “children” intervals in generation $n + 1$. The typical example is a dyadic decomposition of $\Gamma$, but we will only require that each interval breaks into two intervals of comparable (rather than equal) length. This implies that for a fixed $n, j$ we have

$$
\sum_{m,k: I^n_m \subset I^n_k} \frac{\ell(I^n_j)}{\ell(I^n_m)},
$$

is dominated by a geometric series, and hence converges. Given a collection of intervals as above, we define $L^p(\mathcal{I})$ as

$$
L^p(\mathcal{I}) = \left\{ (a_{n,j}) : \sum_{n,j} |a_{n,j}|^p \ell(I^n_j) \right\} < \infty.
$$

A martingale on $\mathcal{I}$ is a sequence of functions $\{f_n\}$ defined on $\Gamma$ so that $f_n$ is constant on each interval of the form $I^n_j$ for $j = 1, \ldots, 2^n$ and

$$
\int_{I^n_j} f_{n+1} \, dx = \int_{I^n_j} f_n \, dx.
$$

Since $f_n$ is constant on $I^n_j$, we let $f_n(I^n_j)$ denote its value there. The functions $\Delta_n f = f_n - f_{n+1}$ are orthogonal in $L^2$, but additionally, the martingale property implies that if

$$
\sup_n \|f_n\|_2^2 = \sum_{n=0}^{\infty} \|\Delta_n f\|_2^2 < \infty,
$$

then $\{f_n\}$ converges pointwise almost everywhere and in $L^2$ to a limiting function $f$ (this is not true assuming orthogonality alone). See Chapter 4 (and particularly Theorem 4.4) of Durrett’s book [26] for details.
Recall that \( \chi_E \) is the characteristic function of the set \( E \), i.e., the function that is 1 on \( E \) and is 0 elsewhere. We define \( \Delta f_{n,j} = \Delta f_n \cdot \chi_{I^n_j} \); this denotes the function that equals \( f_{n+1} - f_n \) on \( I^n_j \) (where it takes two values with opposite signs) and is zero off \( I^n_j \).

We would like to apply martingale convergence to sequences that are “almost” martingales. The following lemma makes this idea precise.

**Lemma 11.3.** With notation as above, suppose that for \( n \in \mathbb{N} \), \( g_n = f_n + h_n \) where \( \{f_n\} \) is a martingale with respect to \( \{I^n_j\} \) and each \( h_n \) is constant on every interval of the form \( I^n_j \), for all \( n \). Assume that

1. \( \|\Delta h_{n,j}\|_\infty \leq a_{n,j} \)
   
   \[
   \{A_{n,j}\} = \left\{ \sum_{I^n_k \subset I^n_j} a_{m,k} \sqrt{\ell(I^m_k)} \right\} \in L^2(\mathcal{I}),
   \]

2. \( \|\Delta f_{n,j}\|_\infty = b_{n,j} \)
   
   \[
   \{B_{n,j}\} = \left\{ \sum_{I^n_k \subset I^n_j} b_{m,k}^2 \ell(I^m_k) \right\} \in L^1(\mathcal{I}),
   \]

Then \( g_n \) converges pointwise almost everywhere to a function \( g \in H^{1/2}(\Gamma) \), i.e.,

\[
\int_\Gamma \int_\Gamma \left| g(x) - g(y) \right|^2 |x - y|^2 d\ell(x) d\ell(y) < \infty.
\]

**Proof.** By the triangle inequality for \( L^2 \),

\[
\|h\|_2 \leq \sum_{n,j} \|\Delta h_{n,j}\|_2 \leq \sum_{n,j} \left( \int_{I^n_j} a^2(I^n_j) dx \right)^{1/2} \leq \sum_{n,j} a(I^n_j) \sqrt{\ell(I^n_j)} = A_{0,1}
\]

is finite by assumption. More generally,

\[
\|(h - h(I^n_j)) \cdot \chi_{I^n_j}\|_2 \leq \sum_{I^n_k \subset I^n_j} a(I^n_k) \sqrt{\ell(I^n_k)} = A_{n,j}.
\]

Since \( \ell(\Gamma) < \infty \),

\[
\sum_n \|\Delta h_n\|_1 \leq \sum_{n,j} \|\Delta h_{n,j}\|_1 = \sum_{n,j} \int_{I^n_j} a(I^n_j) dx = \sum_{n,j} a(I^n_j) \ell(I^n_j)
\]

\[
\leq \sum_{n,j} a(I^n_j) \sqrt{\ell(I^n_j)} = A_{0,1} < \infty,
\]

and hence \( h_n \) converges pointwise almost everywhere to some \( L^1 \) function \( h \).
By the orthogonality of martingale increments,
\[ \sum_n \| \Delta f_n \|_2^2 \leq \sum_{n,j} \int_{I_j^n} b_{n,j}^2 |dz| \leq \sum_{n,j} b_{n,j}^2 \ell(I_j^n) = B_{0,1} < \infty, \]
so by the martingale convergence theorem, \( f_n \) converges pointwise almost everywhere and in \( L^2 \) to a function \( f \in L^2 \). Moreover,
\[ \| (f - f(I_j^n)) \cdot \chi_{I_j^n} \|_2^2 \leq \sum_{m,k: I_k^m \subset I_j^n} \int_{I_k^m} b_{m,k}^2 |dz| \leq \sum_{m,k: I_k^m \subset I_j^n} b_{m,k}^2 \ell(I_k^m) = B_{n,j}. \]

We can now deduce that \( g_n \) converges pointwise a.e. to a function \( g = h + f \), that and for each arc \( I_j^n \),
\[ \int_{I_j^n} |g - g_n(I_j^n)|^2 |dz| \lesssim \int_{I_j^n} |h - h_n(I_j^n)|^2 |dz| + \int_{I_j^n} |f - f_n(I_j^n)|^2 |dz| \lesssim A_{n,j}^2 + B_{n,j}. \]
Thus
\[ \sum_{n,j} \int_{I_j^n} |g - g_n(I_j^n)|^2 \frac{|dz|}{\ell(I_j^n)} \lesssim \sum_{n,j} \frac{A_{n,j}^2}{\ell(I_j^n)} + \sum_{n,j} \frac{B_{n,j}}{\ell(I_j^n)} \lesssim \|A_{n,j}\|_{L^2(I)}^2 + \|B_{n,j}\|_{L^1(I)}. \]
This implies \( g \in H^{1/2}(\Gamma) \) exactly as in the proof of Lemma 11.1. \( \square \)

**Lemma 11.4.** Definition 5 implies Definition 9.

**Proof.** Let \( \Gamma_0 \) be a subarc of \( \Gamma \), let \( \{ \Gamma_j \} \) denote a dyadic decomposition of \( \Gamma_0 \), and let \( \{ z_j, w_j \} \) denote the endpoints of \( \Gamma_j \). Assume Definition 5 holds and suppose \( \Gamma_0 \) has been chosen so small that \( \beta_\Gamma(Q) < 1/100 \) for every square that hits \( \Gamma_0 \) and satisfies \( \text{diam}(Q) \leq 4 \cdot \text{diam}(\Gamma) \). Since we already know that Definition 5 implies Definition 7, we may also assume that \( \sum_j \Delta(\Gamma_j)/\ell(\Gamma_j) < \infty \).

Define \( \tau(\Gamma_j) \) to be \( (z_j - w_j)/|z_j - w_j| \). Define \( \theta(\Gamma_0) \) to some choice of \( \arg(z_0 - w_0) \). If \( \Gamma_j \) is the parent arc of \( \Gamma_j' \), define \( \theta(\Gamma_j') \) to be the value of \( \arg(z_j' - w_j') \) that is closest to \( \theta(\Gamma_j) \). Then \( \tau(\Gamma_j) = \exp(i\theta(\Gamma_j)) \), but the \( \theta \)'s might be unbounded if \( \Gamma \) contains infinite spirals.

Suppose \( \Gamma_j'' \) is the sibling of \( \Gamma_j' \), i.e., \( \ell(\Gamma_j') = \ell(\Gamma_j'') \) and they have the same parent, \( \Gamma_j \). Considering all the dyadic arcs of length \( 2^{-m}\ell(\Gamma_0) \), \( \theta(\Gamma_k) \) defines a function \( \theta_m \) on \( \Gamma_0 \) that is piecewise constant on these arcs. We would like to say that \( \{ \theta_m \} \)
forms a martingale, but this is not quite true; we need to add a small (and harmless) correction term $h_m$ and then apply Lemma 11.3. The following bound will control this term.

We claim that

$$\left| \theta(\Gamma_j) - \frac{1}{2} [\theta(\Gamma_j') + \theta(\Gamma_j'')] \right| \lesssim \frac{\beta(\Gamma_j) \Delta(\Gamma_j)}{\ell(\Gamma_j)}. \tag{11.3}$$

To prove (11.3), normalize so that the parent interval, $\Gamma_j$ has its endpoints at $-1 \in \Gamma_j'$ and $1 \in \Gamma_j''$ and let $z = x + iy$ denote the common endpoint of these two children arcs. By the definition of $\beta(\Gamma_j)$, $|y| \lesssim \beta(\Gamma_j)$. By the definition of $\Delta(\gamma)$,

$$\frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)} \leq x \leq \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)}.$$

Thus by elementary trigonometry (see Figure 8),

$$|\tan(\theta(\Gamma_j) - \theta(\Gamma_j'))| = \frac{y}{1 + x},$$

$$|\tan(\theta(\Gamma_j) - \theta(\Gamma_j''))| = \frac{y}{1 - x},$$

and the angles on the left hand side have opposite signs, so for small $x$ and $y$ we get

$$|\theta(\Gamma_j') - \theta(\Gamma_j'')| = \left| \frac{y}{1 - x} - \frac{y}{1 + x} \right| = \frac{|y(1 + x) - y(1 - x)|}{1 - x^2} = \frac{|y||(1 + x) - (1 - x)|}{1 - x^2} \leq 4|xy| \lesssim \beta(\Gamma_j) \Delta(\Gamma_j).$$

Above we normalized the endpoints of $\Gamma_j$ to be $\pm 1$. In general we get

$$|\theta(\Gamma_j') - \theta(\Gamma_j'')| \lesssim \frac{\beta(\Gamma_j) \Delta(\Gamma_j)}{\ell(\Gamma_j)}.$$

With notation as above, inductively define a sequence of functions $\{h_m\}$ (the harmless corrections) on $\Gamma_0$ as follows. Each $h_m$ will be constant of each dyadic interval of length $2^{-m} \ell(\Gamma_0)$. First, set $h_0 = 0$ on $\Gamma_0$. If we have defined $h_{m-1}$, and $\Gamma_j$ is a dyadic interval of length $2^{-m+1} \ell(\Gamma_0)$, we define $h_m$ on each of its two children $\Gamma_j', \Gamma_j''$. 


The point \( z \) denotes the center of the arc \( \Gamma_j \) with respect to arclength. It must be close to the average of the endpoints: within \( \beta(\Gamma_j) \) of the line between the endpoints and within \( \Delta(\Gamma_j) \) of the perpendicular bisector of the segment connecting the endpoints. Thus the angles formed by \([x, z]\) and \([y, z]\) with \([x, y]\) are almost equal, with an error of size \( O(\beta(\Gamma_j)\Delta(\Gamma_j)\ell(\Gamma_j)) \).

by \( h_m = h_{m-1} \) on \( \Gamma_j' \) and set \( h_m = h_j + (\theta(\Gamma_j') - \theta(\Gamma_j'')) \) on \( \Gamma_j' \). Note that by the estimates above

\[ \|\Delta h_{n,j}\|_\infty = a_{n,j} = O \left( \beta(I^n_j)\Delta(I^n_j)/\ell(I^n_j) \right). \]

Define \( f_m = \theta_m + h_m \); the definition of \( \{h_m\} \), implies that \( \{f_m\} \) is a martingale and

\[ \|\Delta f_{n,j}\|_\infty = b_{n,j} = O(\beta(I^n_j)). \]

We now only have to verify the hypotheses of Lemma 11.3. We start with (11.1). First note that using the Cauchy-Schwarz inequality and Corollary 8.3

\[
A_{n,j}^2 = \left| \sum_{I^m_k \subset I^n_j} a_{m,k} \sqrt{\ell(I^m_k)} \right|^2 \lesssim \sum_{I^m_k \subset I^n_j} \frac{\beta(I^m_k)\Delta(I^m_k)}{\sqrt{\ell(I^m_k)}} \\
\lesssim \left( \sum_{I^m_k \subset I^n_j} \beta^2(I^m_k) \right) \cdot \left( \sum_{I^m_k \subset I^n_j} \frac{\Delta^2(I^m_k)}{\ell(I^m_k)} \right) \\
\lesssim \left( \sum_{I^m_k \subset I^n_j} \beta^2(I^m_k) \right) \cdot \Delta(I^n_j) \cdot \left( \sum_{I^m_k \subset I^n_j} \frac{\Delta(I^m_k)}{\ell(I^m_k)} \right) \\
\lesssim \Delta(I^n_j),
\]
since the two terms in the parentheses are bounded by Definitions 5 and 7. Therefore
\[
\sum_{n,j} \frac{A_{n,j}^2}{\ell(I^n_j)} = \sum_{n,j} \frac{\Delta(I^n_j)}{\ell(I^n_j)} < \infty,
\]
as required.

Next we verify (11.2). Note that
\[
B_{n,j} \lesssim \sum_{I^n_k \subset I^n_j} \beta^2(I^n_k) \ell(I^n_k)
\]
and hence
\[
\sum_{n,j} \frac{B_{n,j}}{\ell(I^n_j)} \lesssim \sum_{n,j} \sum_{I^n_k \subset I^n_j} \frac{\beta^2(I^n_k)}{\ell(I^n_j)} \ell(I^n_k)
\]
\[
\lesssim \sum_{m,k} \beta^2(I^n_k) \sum_{I^n_j \supset I^n_k} \frac{\ell(I^n_j)}{\ell(I^n_k)}
\]
\[
\lesssim \sum_{m,k} \beta^2(I^n_k) < \infty
\]
because the inner sum in the next to last line is a geometric series. Thus $B_{n,j}$ is in $L^1(\mathcal{I})$, as desired, completing the proof that Definition 5 implies Definition 9.

12. (5) $\Leftrightarrow$ (12): $\beta$’s control $\varepsilon$’s (and conversely)

Recall that for a dyadic square $Q$, $\varepsilon_\Gamma(Q)$ denotes the smallest $\varepsilon \geq 0$ so that $3Q$ intersects two disjoint disks $D_1, D_2$ of radius $\ell(Q)/\varepsilon$ that are separated by $\Gamma$, but are within distance $\varepsilon \cdot \ell(Q)$ of each other.

It is easy to see that $\beta_\Gamma(Q) \lesssim \varepsilon_\Gamma(Q)$, but the reverse direction can certainly fail for a single square $Q$. However, we shall prove that the sum of $\varepsilon^2_\Gamma(Q)$ over all dyadic squares is bounded iff the sum of $\beta^2_\Gamma(Q)$ is.

**Lemma 12.1.** Definition 5 is equivalent to Definition 12.

**Proof.** As noted above, one direction is obvious (12 $\Rightarrow$ 5), so we only prove the other direction. Fix $x \in \Gamma$ and a dyadic square $Q_0$ containing $x$ with $\text{diam}(Q_0) \simeq 2^{-N}\text{diam}(\Gamma)$, for some $N \geq 10$. Renormalize so $\text{diam}(Q_0) = 1$. For $1 \leq k \leq N$, let $Q_n$ be the dyadic square containing $Q_0$ and with diameter $\text{diam}(Q_k) = 2^k\text{diam}(Q_0)$. 

Let
\[ \epsilon = 2A \sum_{k=1}^{\infty} 2^{-k} \beta_{\Gamma}(Q_k) = 2A \sum_{Q' : Q \subset Q'} \beta_{\Gamma}(Q_k) \frac{\text{diam}(Q)}{\text{diam}(Q')}, \]
where the constant \(0 < A < \infty\) will be chosen later. I claim that each complementary component of \(\Gamma\) contains disks of radius \(1/\epsilon\) that are only distance \(\epsilon\) apart. If this holds, then \(\epsilon_{\Gamma}(Q) \lesssim \epsilon\).

To prove the claim, let \(L\) be a line through \(x\) that minimizes in the definition of \(\beta_{\Gamma}(Q_0)\). Let \(L^\perp\) be the perpendicular line through \(x\) and let \(z \in L^\perp\) be distance \(1/\epsilon\) from \(x\). Let \(D = D(z, r)\) be a disk \(x\) and with radius \(r = (1/\epsilon) - \epsilon/2\).

Note that the distance between \(D\) and \(L\) is \(\epsilon/2\). More generally, for \(0 \leq n \leq N\), simple trigonometry shows that the distance between \(D \setminus 3Q_n\) and \(L\) is \(\geq C_1 \epsilon 2^{2n}\). On the other hand, the distance between \(\Gamma \cap 3Q_n\) and \(L\) is \(\leq C_2 \sum_{k=0}^{n} \beta_{\Gamma}(Q_k) 2^k\).

Therefore \(D\) and \(\Gamma \cap 2Q_{2N}\) are disjoint if for every \(0 \leq n \leq 2N\), we have
\[ \sum_{k=0}^{n} \beta_{\Gamma}(Q_k) 2^k < (C_1/C_2) \epsilon 2^{2n}, \]
or equivalently, if
\[ \epsilon > (C_2/C_1) 2^{-2n} \sum_{k=0}^{n} \beta_{\Gamma}(Q_k) 2^k, \quad n = 0, \ldots, N. \]
If we take $A = 4C_2/C_1$, then this is true by the definition of $\epsilon$, because

$$A \max_{0 \leq n \leq 2N} 2^{-2n} \sum_{k=0}^{n} \beta_T(Q_k)2^k \leq A \sum_{n=0}^{2N} 2^{-2n} A \sum_{k=0}^{n} \beta_T(Q_k)2^k$$

$$\leq A \sum_{k=0}^{2N} \beta_T(Q_k)2^k \sum_{n=k}^{n} 2^{-2n}$$

$$\leq 2A \sum_{k=0}^{2N} \beta_T(Q_k)2^{-k}$$

$$= 2A \epsilon.$$

The same argument shows the reflection of $D$ across $L$ also misses $\Gamma$, and hence

$$\epsilon(\Gamma) \leq \epsilon.$$

Using the bound and summing over all dyadic squares gives

$$\int_Q \epsilon^2(T) \lesssim \sum_Q \left[ \sum_{Q' \subset Q} \beta_T(Q') \frac{\text{diam}(Q)}{\text{diam}(Q')} \right]^2$$

$$\lesssim \sum_Q \left[ \sum_{Q' \subset Q} \beta_T(Q') \left( \frac{\text{diam}(Q)}{\text{diam}(Q')} \right)^{3/4} \left( \frac{\text{diam}(Q)}{\text{diam}(Q')} \right)^{1/4} \right]^2$$

$$\lesssim \sum_Q \left[ \sum_{Q' \subset Q} \beta_T^2(Q') \left( \frac{\text{diam}(Q)}{\text{diam}(Q')} \right)^{3/2} \right] \cdot \left[ \sum_{Q' \subset Q} \left( \frac{\text{diam}(Q)}{\text{diam}(Q')} \right)^{1/2} \right].$$

The second term in the last line is bounded because it is a geometric series. Thus

$$\int_Q \epsilon^2(T) \lesssim \sum_{Q'} \beta_T^2(Q') \sum_{Q' \subset Q} \frac{\text{diam}(Q)}{\text{diam}(Q')}^{3/2}.$$

Since $\Gamma$ is a chord-arc curve, the number of dyadic squares $Q$ of size $\text{diam}(Q')2^{-k}$ that are inside $Q'$ and also hit $\Gamma$ is at most $O(2^k)$. Thus the last line above is bounded by

$$\lesssim \sum_{Q'} \beta_T^2(Q') \sum_{k=0}^{\infty} O(2^k)2^{-3k/2} \lesssim \sum_{Q'} \beta_T^2(Q') \sum_{k=0}^{\infty} 2^{-k/2} \lesssim \sum_{Q'} \beta_T^2(Q').$$

Thus the $\epsilon^2$-sum is finite if the $\beta^2$-sum is finite, as desired. \hfill \Box

Later it will be convenient to assume that the disjoint disks in the definition of $\epsilon$ are small compared to $\text{diam}(\Gamma)$. This is easy to obtain if we replace $\epsilon_T(Q)$ by

$$\bar{\epsilon}_T(Q) = \max(\epsilon_T, (\text{diam}(Q)/\text{diam}(\Gamma))^\alpha)$$
for some $1/2 < \alpha < 1$. Because $\alpha < 1$, the corresponding disjoint disks have diameter

$$\frac{\text{diam}(D)}{\text{diam}(\Gamma)} = \frac{\text{diam}(Q)/\tilde{\varepsilon}_\Gamma(Q)}{\text{diam}(\Gamma)} \leq \left(\frac{\text{diam}(Q)}{\text{diam}(\Gamma)}\right)^{1-\alpha} \to 0,$$

and because $\alpha > 1/2$,

$$\sum_{Q : Q \cap \Gamma \neq \emptyset} \varepsilon^2_\Gamma(Q) \lesssim \sum_{Q} \varepsilon^2_\Gamma(Q) + \sum_{Q} \left(\frac{\text{diam}(Q)}{\text{diam}(\Gamma)}\right)^{2\alpha}.$$

The second sum is finite because $\Gamma$ is chord-arc and so the number of dyadic squares of size $\simeq 2^{-n}$ hitting $\Gamma$ is $O(2^n)$.

13. $(12) \Rightarrow (13)$: $\varepsilon$’s control $\delta$’s.

If $\Omega \subset \mathbb{R}^2$ is open, and $D \subset \Omega$ is an open disk, consider the hemisphere $H$ in the upper half-space $\mathbb{H}^3_+$ that has $D$ as its base. The boundary of $H$ is a geodesic plane for the hyperbolic metric and the regions above and below this surface are both hyperbolically convex. If we take the union of these hemispheres over all disks in $\Omega$, we get an open region $W$ in $\mathbb{H}^3_+$ whose complement is a closed, hyperbolically convex set (it is an intersection of closed half-spaces). The boundary surface $S = \partial W \cap \mathbb{H}^3_+$ is called the “dome” of $\Omega$. It is well known that this surface, with its hyperbolic path metric, is isomorphic to the hyperbolic unit disk. For an excellent review of its properties and importance, see Al Marden’s survey article [56].

There is a natural map $N : \Omega \to S$ defined as follows. Given $z \in \Omega \subset \mathbb{R}^2$, consider a Euclidean closed ball in the upper half-space that is tangent to $\mathbb{R}^2$ at $z$. If the radius is small enough (say half dist$(z, \partial \Omega)$), then this ball lies entirely underneath $S$. Now expand the ball, keeping the tangent point at $z$, until it first hits $S$ at a point $w$. This point is uniquely determined, because the hyperbolic tangent plane to the ball at $w$ only touches the ball at $w$ and the ball is below this plane; $S$ is contained in the closed hyperbolic half-space above this plane. The map $N$ is called the “nearest point projection” since it is the continuous extension to $\Omega$ of the usual nearest point map onto $S$ defined on $W$. This map is known to be 2-Lipschitz [31] from the hyperbolic metric on $\Omega$ and the hyperbolic path metric on $S$, but need not be a homeomorphism (if $\Omega$ is the union of two overlapping disks, there are arcs that collapse to a single point under $N$). However, $N$ is a quasi-isometry between the two
metric spaces, i.e., [13],
\[
\frac{1}{A} \rho_\Omega(z, w) - B \leq \rho_S(N(z), N(w)) \leq 2 \rho_\Omega(z, w).
\]
Explicit estimates for $A$ and $B$ are given in [18]. See also: [12], [17], [18], [19], [32], [34]. In particular, there is a $r < \infty$, so that any hyperbolic ball of radius $r$ in $\Omega$ projects to a set that contains, and is contained in, balls of radius comparable to $r$. Thus any sufficiently large hyperbolic ball in $\Omega$ is sent to a set of comparable hyperbolic area on $S$.

**Lemma 13.1.** Suppose $D_1, D_2 \subset \mathbb{R}^2$ are disjoint disks of radius $r$ that are distance $\epsilon$ apart. Then the hyperbolic distance between the corresponding domes is $\simeq \sqrt{\epsilon/r}$.

**Proof.** By a linear rescaling we can assume $D_1 = D(0, 1)$ and $D_2 = (2 + \frac{\epsilon}{r}, 1)$. Let $z_1 = 1$, $z_2 = -1$, $z_3 = 1 + \epsilon/r$ and $z_4 = 3 + \epsilon/r$. Let $\tau$ be the Möbius transformation that fixes each of $z_1 = 1$ and $z_2 = -1$ and sends $z_4 = 3 + \frac{\epsilon}{r}$ to $z'_4 = -1 - \sqrt{\epsilon/r}$. Let
\[
z_3 = 1 + \epsilon/r.
\]
By the invariance of cross products, we know
\[
\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} = \frac{(z_1 - z'_3)(z_2 - z'_4)}{(z_2 - z'_3)(z_1 - z'_4)}.
\]
Moreover, by our choices
\[
\frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \simeq \frac{\epsilon}{r}
\]
and
\[
\frac{(z_1 - z'_3)(z_2 - z'_4)}{(z_2 - z'_3)(z_1 - z'_4)} \simeq \sqrt{\epsilon/r} \cdot |z_1 - z_3|,
\]
and hence $|z_1 - z_3| \simeq \sqrt{\epsilon/r}$. Therefore $\tau(\partial D_2)$ is contained in the annulus $1 < |z| < 1 + O(\sqrt{\epsilon/r})$, and this implies the domes of $D_1$ and $D_2$ are at most hyperbolic distance $O(\sqrt{\epsilon/r})$ apart. 

For a closed curve $\Gamma \subset S^2$, the boundary of $\text{CH}(\Gamma)$ consists of two surfaces in $\mathbb{B}^3$ that both meet $S^2$ along $\Gamma$. If $z \in \text{CH}(\Gamma)$, let $\delta(z)$ denote the hyperbolic distance to the farther boundary component of $\text{CH}(\Gamma)$. We want to show that if $\delta$ is small at one point, it is also small at all nearby points.

**Lemma 13.2.** If $z \in \text{CH}(\Gamma)$ and $\delta(z) \leq \epsilon < 1$ for some $z \in S_1$, then $\delta(w) = O(\epsilon)$ for all $w \in S_1 \cap B_\rho(z, 1)$. 

Proof. Assume $z = 0$ and $S_1, S_2$ are both above $\mathbb{R}^2$ in the ball model. Then the convex hull is trapped between $\mathbb{R}^2$ and a Euclidean sphere $S$ of radius $\simeq 1/\epsilon$ that is contained in the upper half-space and contains a point within Euclidean distance $O(\epsilon)$ of the origin. Then $S \cap \mathbb{B}^3$ must stay within Euclidean distance $O(\epsilon)$ of $\mathbb{R}^2$, which means the hyperbolic distance between any point of $S_1 \cap B_\rho(0, 1)$ and $S_2$ is also at most $O(\epsilon)$ (near the origin Euclidean and hyperbolic distances are comparable). \qed

Lemma 13.3. Definition 12 implies Definition 13.

Proof. Lemma 13.1 implies that when $\varepsilon_\Gamma(Q)$ is small, say less than $1/100$, we have $\delta(z) \lesssim \varepsilon_\Gamma(Q)$ some point $z \in \mathbb{H}^3_+$ that lies above $Q$ at Euclidean height $\simeq \ell(Q)$. Lemma 13.2 then implies $\delta$ is comparably small on a unit neighborhood of $z$. Thus the integral of $\delta^2(z)$ with respect to hyperbolic area measure over such a neighborhood is bounded by a uniform multiple of $\varepsilon_\Gamma^2(Q)$. This proves that Definition 12 implies Definition 13. \qed

14. $(13) \Rightarrow (14)$: $\delta$ controls surface curvature


Proof. Suppose $S$ is one component of $\partial \text{CH}(\Gamma)$. It is known that $S$, with its hyperbolic path metric, is isomorphic to the hyperbolic disk. See Al Marden’s book [55] for the precise definitions and basic properties of the convex hull boundaries. The hyperbolic unit disk can be triangulated by geodesic triangles with hyperbolic diameters $\simeq 1$ and angles bounded strictly between 0 and $\pi$, e.g., take the tessellation corresponding to a triangle Fuchsian group, or obtain a triangulation by connecting the center of each Whitney Fuchsian group, or obtain a triangulation by connecting the center of each Whitney box to the box’s vertices.

Fix such a triangulation of $\mathbb{D}$ and map the vertices to $S$ via the isometry. Each triple of image vertices corresponding to a triangle on $\mathbb{D}$ lies on a hyperbolic plane and determines a triangle on this plane. Create a new surface $S_1$ by gluing these triangles together along their edges. Because the vertices lie in $\text{CH}(\Gamma)$, convexity implies each triangle, and hence all of $S_1$, also lie in $\text{CH}(\Gamma)$.

Consider two triangles $T_1, T_2$ in $S_1$ that meet along a common edge $e$. Normalize so that one endpoint of $e$ is the origin in the ball model of hyperbolic 3-space, $e$ lies along the $x$ axis and $T_1$ lies in the $xy$-plane. Then $T_2$ lies in Euclidean plane...
that makes some angle $\theta$ with the $xy$-plane, and by our assumptions, it contains a point $p$ (e.g., the vertex of $T_2$ not on $e$) that is hyperbolic distance $\simeq 1$ from 0 and Euclidean distance $\simeq 1$ from the $x$-axis. Then $p$ is Euclidean distance $\simeq \theta$ from the $xy$-plane. Because both triangles lie inside CH($\Gamma$) and CH($\Gamma$) is trapped between two hyperbolic half-planes that each come within hyperbolic distance $\delta(0)$ of the origin, we must have $\theta \lesssim \delta(0)$ (we are using Lemma 13.2).

If $T$ is component triangle of $S_1$, let $\theta(T)$ be the maximum angle $T$ makes with any of its neighboring triangles, and think of $\theta(z)$ as a function on $S_1$ that is constant on triangles. Since $\theta(z)$ can be bounded by a uniform multiple of $\delta(w)$ for a point $w$ that is a uniform hyperbolic distance away, we get

$$\int_{S_1} \theta^2(z) dA_{\rho}(z) \lesssim \int_{S_1} \delta^2(z) dA_{\rho}(z) < \infty.$$ 

The mean curvature of $S_1$ is zero inside each triangle and is a $\delta$-mass measure along the edges. However, by smoothing $S_2$ we can obtain a surface $S_2$ so that the principle curvatures tend to zero as we approach infinity and are bounded by $O(\max_T \theta(z))$, where $T^*$ denotes the union of all component triangles that touch $T$ (including those that only touch at a vertex). Then

$$\int_{S_2} |K|^2(z) dA_{\rho}(z) \lesssim \int_{S_1} \delta^2(z) dA_{\rho}(z) < \infty.$$

\[ \square \]

15. $(14) \Rightarrow (3)$: Surface curvature bounds QC reflections.

**Lemma 15.1.** Definition 14 implies Definition 3.

**Proof.** This is due to Charles Epstein [28]. He proves that if a surface $S \subset \mathbb{H}^3_+$ whose principle curvatures $|\kappa_1(p)|, |\kappa_2(p)|$ are bounded strictly below 1, then the Gauss map from the surface to the sphere at infinity is quasiconformal. Recall that the Gauss map sends a point $p$ on $S$ to the endpoint on $S^2$ of the hyperbolic geodesic ray starting at $p$ that is normal to $S$. There are actually two Gauss maps from $S$ to $S^2$ depending which “side” of $S$ the geodesic ray is in. In the case when the surface has asymptotic limit $\Gamma$, a curve on $S^2$, the composition of one of these maps with the inverse of the other defines a quasiconformal reflection across $\Gamma$. 

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15. $(14) \Rightarrow (3)$: Surface curvature bounds QC reflections. 

**Lemma 15.1.** Definition 14 implies Definition 3.
In particular, Proposition 5.1 of [28] computes the dilatation of the composed Gauss maps as

\[
D(z) = \max \left( \left| \frac{1 + \kappa_1(p)}{1 - \kappa_1(p)} \cdot \frac{1 - \kappa_2(p)}{1 + \kappa_2(p)} \right|^{1/2}, \left| \frac{1 - \kappa_1(p)}{1 + \kappa_1(p)} \cdot \frac{1 + \kappa_2(p)}{1 - \kappa_2(p)} \right|^{1/2} \right) = 1 + O(|\kappa_1(p)| + |\kappa_2(p)|),
\]

where \( p \in S \) is the point corresponding to \( z \in S^2 \). Therefore the complex dilatation satisfies

\[
\mu(z) = O(|\kappa_1(p)| + |\kappa_2(p)|).
\]

Moreover, on page 121 of [28], Epstein shows that the Jacobian \( J \) of this map satisfies

\[
C_1 |(1 + \kappa_1)(1 - \kappa_2)| \leq J \leq C_2 |(1 - \kappa_1)(1 + \kappa_2)|.
\]

In particular, \( J \approx 1 \) if \( |\kappa_1|, |\kappa_2| \) are both uniformly bounded below 1.

Definition 14 implies that \( \kappa_1, \kappa_2 \) are both small outside some compact ball \( B \) around the origin. Thus the Gauss map for \( S \) defines a quasiconformal reflection in some neighborhood \( U \) of \( \Gamma \) and inside this neighborhood

\[
\int_U |\mu(z)|^2 dA_\rho(z) \leq \int_{S \setminus B} |K(z)|^2 dA_\rho(z),
\]

where \( dA_\rho \) is the hyperbolic area measure on \( S^2 \setminus \Gamma \) and \( S \) respectively and \( K_0 \) is the trace-free second fundamental form of \( S \). Extend this reflection to the rest of \( S^2 \) by some diffeomorphism of one component of \( S^2 \setminus U \) to the other that agrees with the reflection given by the Gauss map on \( \partial U \). This gives a global quasiconformal reflection across \( \Gamma \) that satisfies (2.10), as desired. \( \square \)

This completes the circle of implications that proves a curve is Weil-Petersson if and only if it is the boundary of a surface in \( \mathbb{H}_+^3 \) that has finite Willmore energy. In fact, we have now proven all the implications given in Figure 3, assuming we make use of the previously known equivalence of (1)-(4). In Appendix C we shall sketch proofs of 1 \( \Rightarrow \) 2 \( \Rightarrow \) 3 \( \Rightarrow \) 4, and also sketch proofs of the “redundant” arrows in Figure 3, so that the proof of every arrow in that figure is discussed (at least briefly) in this paper.
16. Theorem 1.4: Minimal surfaces

We have already proven the first claim of Theorem 1.4: $\Gamma$ is Weil-Petersson if and only if it is the boundary of some surface in $\mathbb{H}^3_+$ that is asymptotically flat and has finite total curvature. In this section we prove the final claim of the theorem that any embedded minimal surface with boundary $\Gamma$ also has this property. The key fact is the following

**Lemma 16.1.** Suppose $S$ is an embedded minimal surface in $\mathbb{H}^3$ that is topologically a disk and has boundary curve $\Gamma \subset \mathbb{S}^2$. Suppose $0 \in S$ and that $S$ lies between two disjoint hyperbolic planes that both come within distance $\epsilon$ of 0, one on either side of the $xy$-plane. Then the tangent plane of $S$ at 0 makes angle at most $O(\epsilon)$ with the $xy$-plane and the absolute values of the principle curvatures of $S$ at 0 are both bounded by $O(\epsilon)$.

This is proven by Andrea Seppi in [72]; it is essentially Propositions 4.14 and 4.15 of that paper; see Equation (32) in particular. Given a minimal surface $S$ that is trapped between two hyperbolic planes $P_-, P_+$, Seppi considers the function $u(z) = \sinh(\text{dist}(z, P_-))$ for $z \in S$ and uses the fact that this satisfies the equation $\Delta_S u - 2u = 0$, where $\Delta_S$ is the Laplace-Beltrami operator for the surface $S$. The Schauder estimates for this equation imply that

$$\|u\|_{C^2(B(x,r/2))} \leq C\|u\|_{C^0(B(x,r))}.$$ 

In order to get a uniform bound for $C$, we must bound the curvature of $S$, and Seppi gives an argument for this assuming the boundary of $S$ is a quasicircle (this covers our application, since Weil-Petersson curves are quasicircles). Finally, the sup norm of $u$ is bounded in terms of the distance between $P_-$ and $P_+$ near $z$, and which we have shown is $O(\delta(z))$, e.g. Lemma 13.2. One small technical point is that Seppi requires the point $z$ to be on a geodesic segment that meets both $P_-$ and $P_+$ orthogonally. However, it is very simple to see that if $z$ is between two disjoint hyperbolic planes that each come within $\epsilon$ of $z$, then there are two disjoint planes that come within $O(\epsilon)$ and satisfy the orthogonality condition for $z$.

The lemma implies that near the boundary of hyperbolic space we have

$$\int_S |K|^2 dA_\rho \lesssim \int_{\partial CH(\Gamma)} \delta^2(z) dA_\rho < \infty,$$
when $\Gamma$ is a Weil-Petersson curve. This verifies the final claim in Theorem 1.4.

17. Renormalized area

As we discussed in Section 1, a surface $Y \subset \mathbb{H}_3^+ \cap \mathbb{R}^2$ with boundary curve $\Gamma \subset \mathbb{R}^2$ is said to have finite renormalized area if

$$A_R(Y) = \lim_{t \to 0} \left[ A_\rho(Y_t) - \ell_\rho(\partial Y_t) \right] = \lim_{t \to 0} \left[ A_\rho(Y_t) - \ell(\partial Y_t)/t \right],$$

exists and is finite, where

$$Y_t = \{(x, y, s) \in Y : s \geq t\},$$

$$\partial Y_t = \{(x, y, s) \in Y : s = t\}.$$ 

Here $\ell_\rho$ denotes hyperbolic arclength and $\ell$ Euclidean arclength. For any rectifiable $\Gamma$, if we define the cylinder $Y = \Gamma \times (0, 1] \subset \mathbb{H}_3^+$, then

$$A_\rho(Y_t) = \int_0^1 \int_{\Gamma} \frac{dsdt}{t^2} = \ell(\Gamma)(\frac{1}{t} - 1) = \ell(\Gamma_t)(\frac{1}{t} - 1),$$

so the cylinder $Y$ always has finite renormalized area. Thus we expect that more general surface having finite renormalized area is roughly a measure of how orthogonal the surface is to the boundary.

Next consider the “dyadic cylinder”

$$X = \bigcup_{n=0}^\infty \Gamma_n \times (2^{-n-1}, 2^{-n}],$$

where $\Gamma_n$ is the $2^n$-gon inscribed in $\Gamma$ corresponding to a dyadic decomposition of $\Gamma$ into subarcs of length $2^{-n}\ell(\Gamma)$. Note that each “layer” of $X$ between heights $2^{-n}$ and $2^{-n+1}$ consists of $2^n$ Euclidean rectangles in vertical planes that meet along vertical edges that we will call “hinges”. See Figure 17.1. Alternate vertices of the top edge of one layer agree with the bottom vertices of the next layer up, but there are triangular horizontal “holes” between the layers. Thus $X$ is not really a surface, but we shall see below that it is a good approximation to the minimal surface $S$ near the boundary of $\mathbb{H}_3^+$. 

**Lemma 17.1.** If $\Gamma$ a closed rectifiable Jordan curve, then it is Weil-Petersson if and only if every corresponding dyadic cylinder $X$ has finite renormalized area.
Proof. First we show that the Weil-Petersson condition implies finite renormalized area. A simple calculation as above shows that the part of $X$ between heights $2^{-n}$ and $2^{-n+1}$ has hyperbolic area $2^{n-1}\ell(\Gamma_n)$. Similarly, if $2^{-n-1} \leq t \leq 2^{-n}$, then

$$A_\rho(X_t) = \sum_{k=0}^{n} 2^{k-1}\ell(\Gamma_k) + \left(\frac{1}{t} - 2^n\right)\ell(\Gamma_{n+1}).$$

so

$$A_\rho(X_t) - \frac{1}{t}\ell(\Gamma) = A_\rho(X_t) - \left(\frac{1}{t} - 2^n\right) + \sum_{k=1}^{n} 2^{k-1}\ell(\Gamma)$$

$$= -\ell(\Gamma) - \sum_{k=1}^{n} 2^k[\ell(\Gamma) - \ell(\Gamma_k)] + \left(\frac{1}{t} - 2^n\right)\ell(\Gamma) - \ell(\Gamma_{n+1})$$

$$= -\ell(\Gamma) - \sum_{k=1}^{n} 2^k[\ell(\Gamma) - \ell(\Gamma_k)] + O(2^n[\ell(\Gamma) - \ell(\Gamma_{n+1})])$$

$$\to -\ell(\Gamma) - \sum_{k=1}^{\infty} 2^k[\ell(\Gamma) - \ell(\Gamma_k)]$$

since the series is convergent when $\Gamma$ is Weil-Petersson by (1.4). Finally, for $2^{-n-1} \leq t \leq 2^{-n}$, note that $\ell(\partial X_t) = \ell(\Gamma_{n+1})/t$, so

$$\frac{1}{t}[\ell(\partial X_t) - \ell(\Gamma)] \leq 2^{n+1}[\ell(\Gamma_{n+1}) - \ell(\Gamma)] \to 0,$$

since these are terms of a summable series. Thus $A_\rho(X_t) - \ell(\partial X_t)$ has a finite limit and $X$ has finite renormalized area.

Next we consider the converse: finite renormalized area implies $\Gamma$ is Weil-Petersson. First we suppose that $A_R(X) < \infty$ and deduce that $\Gamma$ must be rectifiable. Consider
$t = 2^{-n}$. Then
\[
A_\rho(X_t) - \ell_\rho(\partial X_t) = \left( \sum_{k=1}^{n} 2^{k-1} \ell(\Gamma_k) \right) - 2^n \ell(\Gamma_n) = O(1),
\]
or equivalently,
\[
\ell(\Gamma_n) = \frac{1}{2} \ell(\Gamma_n) + \frac{1}{4} \ell(\Gamma_{n-1}) + \cdots + 2^{-n} \ell(\Gamma_1) + O(2^{-n}),
\]
and hence (since $\{\ell(\Gamma_n)\}$ is non-decreasing),
\[
\ell(\Gamma_n) \leq \frac{1}{2} \ell(\Gamma_{n-1}) + \frac{1}{4} \ell(\Gamma_{n-2}) + \cdots + \ell(\Gamma_{n-1}) + O(2^{-n})
\]
which clearly implies $\ell(\Gamma) < \infty$.

To show that $\Gamma$ is Weil-Petersson, note that
\[
\ell(\Gamma) = \sum_{k=1}^{\infty} 2^{k-1} [\ell(\Gamma) - \ell(\Gamma_k)] - \ell(\Gamma).
\]
By the Monotone Convergence Theorem (for counting measure on $\mathbb{N}$), this tends to
\[
\sum_{k=1}^{\infty} 2^{k}[\ell(\Gamma) - \ell(\Gamma_k)] < \infty,
\]
with a bound independent of the choice of the dyadic decomposition. Thus $\Gamma$ is Weil-Petersson by Theorem 1.2. $\square$

Next we will show that if $A_R(X)$ is finite and $\Gamma$ is Weil-Petersson, then $A_R(S)$ is also finite, where $S$ is any minimal surface with boundary $\Gamma$. The idea is that when we are close enough to the boundary of $\mathbb{H}_+^3$, there is a obvious normal projection from $X$ to $S$ that preserves hyperbolic areas and lengths up to a factor of $1 + O(\delta^2(z))$, \[\delta^2(z)\]
and this is integrable over $X$ because of Definition 13. We handle the area and length estimates as two lemmas.

**Lemma 17.2.** Suppose notation is as above and $S$ is a minimal surface with boundary curve $\Gamma$. If $\Gamma$ is Weil-Petersson, then $A_\rho(S_t) - A_\rho(X_t)$ has a finite limit as $t \searrow 0$.

**Proof.** Renormalize $\Gamma \subset \mathbb{R}^2$ by a conformal linear map, so that the highest point in the hyperbolic convex hull of $\Gamma$ is $z_0 = (0, 0, 1) \in \mathbb{H}_+^3$. Then $\Gamma$ is contained in the unit disk. Consider the geodesic connecting $z_0$ to a point $p \in \Gamma$. At height $t$ above $\mathbb{R}^2$, this geodesic makes angle $O(t)$ with the vertical direction and is at most hyperbolic distance $O(t)$ from the vertical ray with endpoint $p$. Since this geodesic is inside the convex hull of $\Gamma$ this means the both boundary components of the convex hull are within distance $O(t + \delta)$ of the same vertical ray at height $t$. In the following paragraphs, $\delta$ will refer to the maximum of $\delta(z)$ over a unit neighborhood of the face of $X$ that we are considering. Our assumption that $\Gamma$ is Weil-Petersson implies that the sum of $\delta^2$ over all faces of $X$ is finite.

Assume $t > 0$ is small enough that that $\delta(z) \leq 1/100$ for all points $z = (x, y, s)$ on $\partial CH(\Gamma)$ below height $t$. Suppose such a point $z = (x, y, s)$ is on one of the rectangular faces of of the dyadic cylinder $X$. Then $z$ is within hyperbolic distance $O(s)$ of some point $w \in CH(\Gamma)$ and hence is within $O(s) + O(w)$ of the minimal surface $S$. Map $z$ to a point on $S$ by following the line orthogonal to $X$ at $z$ until it hits $S$. The tangent space of $S$ at this point is within $O(s + \delta)$ of orthogonal to this ray, and hence the projection from $X$ to $S$ preserves area up to a factor of $1 + O(s^2) + O(\delta^2)$. For Weil-Petersson curves, the $O$-terms are integrable for hyperbolic area over $X$. Because of the discontinuities of the normal to $X$ along the hinges, some parts of $S$ may correspond to points on two different faces of $X$ or be missed completely. However, the angles between faces at each hinge are bounded by $O(\delta)$ and the distance from the hinge to $S$ is also bounded by $O(\delta)$, so the total error we make for each hinge is $O(\delta^2)$, and the sum over all hinges of a given level, controlled by the integral of $\delta^2$ over the part of convex hull boundary at the same height as the hinge. Thus

$$A_\rho(S_s) - A_\rho(X_s) = O(1)$$

and the difference has a limit as $s \searrow 0$. \qed
Lemma 17.3. Suppose notation is as above and $S$ is a minimal surface with boundary curve $\Gamma$. If $\Gamma$ is Weil-Petersson, then $\ell_\rho(S_t) - \ell_\rho(X_t)$ had a finite limit as $t \searrow 0$.

Proof. This is analogous to the previous proof comparing areas. Again there is a horizontal projection from each horizontal segment crossing a face of $X$ to an arc of $S$, and the Seppi’s estimates show that the image arc is within $O(t + \delta)$ of parallel to the segment and hence has the same length up to a factor of $O(t^2 + t\delta + \delta^2)$, which is finite when we integrate over $\partial X_t$. Similarly, some pieces of the $\partial S_t$ may be counted twice or missed completely, but each such occurrence is controlled by $O(\delta^2)$ and these terms are summable. \hfill \Box

Clearly Lemmas 17.1, 17.2 and 17.3 imply that if $\Gamma$ is Weil-Petersson, then the corresponding minimal surface $S$ has finite renormalized area. This proves Theorem 1.5. A similar, but simpler, argument also shows that the two boundary components of the hyperbolic convex hull have finite renormalized area. In this case, we do not need Seppi’s estimate; we merely use the characterization of Weil-Petersson curves in terms of the $L^2$ bending of the convex core boundaries to show that the projection from the dyadic cylinder to each boundary component preserves hyperbolic area and length up to a bounded additive factor.

A second approach to proving Theorem 1.5 uses a calculation of Alexakis and Mazzeo. We sketch the idea. In their proof of Proposition 3.1 in [4] they show that

$$A_\rho(S_t) - \frac{1}{t} \ell(\partial S_t) = -2\pi \chi(S_t) - \frac{1}{2} \int_{S_t} |K|^2 dA_\rho - \int_{\partial S_t} (\kappa_t - 1) d\ell_\rho, \quad (17.1)$$

where $\kappa_t$ is the geodesic curvature of $\partial S_t$. The estimate of Seppi discussed in Section 16 shows that

$$\int_{S_t} |K|^2 dA_\rho = O(\int_{S_t} \delta^2 dA_\rho).$$

Hence this integral converges to a finite limit as $t \searrow 0$.

By a formula on page 632 of [4], $\kappa_t = t(\bar{\kappa}_t + \partial_n \log s)$, where $\bar{\kappa}_t$ is the geodesic curvature with respect to the Euclidean metric and $\partial_n$ is the Euclidean normal derivative to $\partial Y_t$ in $Y_t$. In the integral above, subtracting the 1 cancels the normal derivative factor, up to an error of $O(\theta^2)$, where $\theta = O(t + \delta(w))$ is the angle the tangent plane to $Y$ makes with the vertical direction. The Euclidean curvature in the horizontal plane $\{(x, t, s) : s = t\}$ is bounded by the principle curvatures of the surface, which by Seppi’s estimate are bounded by $\delta(w)/t$. This is then projected into the tangent plane.
of $Y_t$, which is within $O(t + \delta)$ of orthogonal to horizontal. Thus $\kappa_t - 1 = O(t^2 + t\delta + \delta^2)$.

Hence using Cauchy-Schwarz shows that $\int_{\partial Y_t} (\kappa_t - 1) d\ell_\rho$ also converges to a finite limit bounded by $O(t + \sqrt{tD(t)} + D(t))$, where $D(t) = \int_{\partial S_t} \delta^2 d\ell_\rho \to 0$. Therefore, again, $S$ has finite renormalized area.

18. Remarks and questions

We proved that minimal surfaces bounded by Weil-Petersson curves have finite renormalized area. Is the converse true? If a non-Weil-Petersson minimal surface were to have finite renormalized area, then the left side of (17.1) has a finite limit, so the integrals on the right side of (17.1) would have to diverge in such as way as to cancel each other. Clearly the area integral is positive, so if we could prove the line integral was also positive, or at least smaller (by a fixed factor) in absolute value than the area integral, then finite renormalized area would characterize the Weil-Petersson class. The line integral looks very much like a derivative of the area integral, and this suggests that if they were of comparable size, then both integrals would have to grow exponentially. Thus even if finite renormalized area does not characterize Weil-Petersson curves, we might guess that any counterexample is “far from” Weil-Petersson. A first step might be to prove such a curve must have infinite length.

There are several other geometric quantities that might give characterizations of the Weil-Petersson class of curves. I mention a few here in the hopes of inspiring further investigation. First we record a simple lemma.

**Lemma 18.1.** Suppose $\Gamma$ is a closed Jordan curve on $S^2$. Then CH($\Gamma$) is the union of all hyperbolic geodesics that have both endpoints in $\Gamma$.

**Proof.** By convexity, CH($\Gamma$) contains all these geodesics. To prove the other direction, we consider the ball model of hyperbolic 3-space. If $z \in$ CH($\Gamma$) move $z$ to the origin by a hyperbolic isometry. We now want to prove that there is a diameter of $B^3$ with both endpoints in $\Gamma$.

Let $\Omega_1, \Omega_2$ be the components of $S^2 \setminus \Gamma$ and $\Omega_1^*$ and $\Omega_2^*$ their reflections through the origin. These reflections have the same areas as the originals, so if $\Omega_1^* \subset \Omega_1$, they must be equal. In this case $\Gamma$ reflects to itself and we are done (there are infinitely many diameters passing through the origin).
Otherwise there is a point of $\Omega_1$ that reflects to a point of $\Omega_2$; call these the north and south poles, and their bisecting great circle the equator. Since 0 is in the convex hull of $\Gamma$ and $\Gamma$ is not the equator it must enter both the northern and southern hemispheres.

If $x$ is a point of $\Gamma$ closest to the south pole (and this point must be in the southern hemisphere), then I claim that $x^*$ must be in the component containing the north pole; otherwise there would be point $y \in \Gamma$ on the geodesic arc from $x^*$ to the pole and then $y^*$ would be closer to the south pole than $x$, a contradiction. Taking the closest point to the north pole gives a point on $\Gamma$ whose reflection is in the other component. So moving $x$ continuously around $\Gamma$ must give a point of $\Gamma$ that reflects into $\Gamma$, as desired. \qed

Thus if $z \in \text{CH}(\Gamma)$ and $\sigma_z$ is any hyperbolic isometry of $\mathbb{B}^3$ that maps $z$ to the origin, then there is a diameter of $\mathbb{B}^3$ that connects two points of $\Gamma_z = \sigma(\Gamma)$. The two arcs of $\Gamma_z$ that connect these two points each have spherical length at least $\pi$ and so $\Gamma_z$ has length at least $2\pi$, the length of a great circle on $S^2$. Note that any two hyperbolic isometries that map $z$ to the origin differ by a Euclidean rotation, and hence the spherical length, $\ell_{S^2}(\Gamma_z)$, of $\Gamma_z$ depends only on $z$ and not on the particular choice of $\sigma_z$. Let $\text{ex}(\Gamma_z) = \ell_{S^2}(\Gamma_z) - 2\pi$; we refer to this as the “excess length” of $\Gamma_z$. Is there a characterization of the Weil-Petersson class in terms of some integral of $\text{ex}(\Gamma_z)$, perhaps over the convex hull of $\Gamma$ or its boundary?

Let $G$ be a quasi-Fuchsian group, $M$ its hyperbolic quotient 3-manifold, $R_1, R_2$ the two Riemann surfaces comprising the boundary at $\infty$ of $M$, and $\Gamma$ its limit set. There are a variety of papers that relate the volume $\text{CH}(\Gamma)$, the renormalized volume of $M$, and the Weil-Petersson distance between $R_1$ and $R_2$. For example, see [20], [21], [48], [69]. The ideas in these papers seem very similar to our results characterizing Weil-Petersson curves $\Gamma$ in terms of the “thickness” of the hyperbolic convex hull of $\Gamma$ and the renormalized area of a surface with boundary $\Gamma$. Is there a precise connection between the results of this paper and the papers mentioned above? In [75], Takhtajan and Teo show that the usual Weil-Petersson metric for compact surfaces can be recovered from their Weil-Petersson metric on the universal Teichmüller space. Is this helpful in making the connection suggested above?
The Hilbert manifold topology of Takhtajan and Teo divides the universal Teichmüller space into uncountable many connected components. Can we geometrically characterize when two curves belong to the same component? The current paper has done this for the component containing the unit circle. Perhaps some condition can be given saying that the convex hulls are quasi-isometric with constants that tend to 1 in a square integrable sense near the boundary of hyperbolic space. A closely related problem is to construct a natural section for universal Teichmüller space, i.e., a natural choice of one quasicircle from each connected component. A natural starting point might be Rohde’s paper [65] which gives such a choice for quasicircles modulo biLipschitz images. If two quasicircles are in the same connected component of $T(1)$, are they necessarily biLipschitz images of each other, i.e., are bi-Lipschitz equivalence classes unions of Takhtajan-Teo connected components? This is true for the Weil-Petersson class, since all such curves are chord-arc, and hence bi-Lipschitz images of circles.

In [5] Alexakis and Mazzeo impose a condition similar to (4.4), but integrate against a weight that blows up logarithmically near the boundary, in order to obtain $C^1$ control of the boundary curve $\Gamma$. They note that (4.4) by itself is not enough to allow such control: the current paper shows precisely what boundary curves are possible under this weaker hypothesis. In terms of $\beta$-numbers, their stronger condition roughly says that

$$\sum_Q \beta^2(Q)|\log \text{diam}(Q)|^{2p} < \infty,$$

for some $p > 1$. It would be interesting to explore the consequences of this assumption in terms of the conditions given in this paper, and to explore the possible analogs of this paper’s results in the higher dimensional settings considered in [4].

Although it is not clear whether the function theoretic definitions of the Weil-Petersson class make sense in higher dimensions, many of our geometric conditions, such as the $\sum \beta^2(Q) < \infty$ and $\int \delta^2(z) dA_\rho < \infty$ extend to higher dimensions. Are our geometric conditions still equivalent in settings where they makes sense? Does the convergence of $\sum \beta^2(Q)$ define a interesting class of curves or surfaces in higher dimensions? Because Definition 5 implies Definition 4, we know that $\sum \beta^2 < \infty$ in the plane implies a very strong relation between the harmonic measures for the two sides of $\Gamma$. Does an analogous result hold for surfaces in $\mathbb{R}^n$? A starting point might
be the bi-Lipschitz parameterization of Reifenberg flat surfaces due to Guy David and Tatiana Toro in [23].

Similarly, can the results relating the $\beta$-numbers to the behavior of a spanning surface in hyperbolic space be extended to higher dimensions? Again, it seems likely this is possible.

Does our refined version of Jones’ Traveling Salesman Theorem hold for arcs in higher dimensions? Is there a version for Hilbert space or other metric spaces? What can we say about a curve if e.g.,
\[ \sum_Q \beta^2(Q) \text{diam}(Q)^\alpha < \infty, \]
a condition that interpolates between rectifiability ($\alpha = 1$) and the Weil-Petersson class ($\alpha = 0$). Related sums have been considered in [8] and [9] in connection to Hölder parameterizations of curves. In [41], Silvia Ghinassi considers curves for which
\[ \int_0^1 \beta^2(x,t) t^{-2\alpha} dt < M < \infty, \]
and shows they have parameterizations that are $C^{1,\alpha}$, i.e., $f'$ is $\alpha$-Hölder. Definition 5 implies the Weil-Petersson class forms a subset of the $\alpha = 1/2$ case, and we have already noted that such curves have a parameterization with $\log f' \in H^{1/2}$ (the Sobolev space, not the Hölder space).

Our refined version of the traveling salesman theorem should have an analog for the Schwarzian derivative. In Lemma 3.9 of [15], Peter Jones and I proved that for a conformal map $f : \mathbb{D} \to \Omega$, where $\Gamma = \partial \Omega$ is a quasicircle, we have
\[ \ell(\Gamma) \lesssim \text{diam}(\Gamma) + \iint_{\mathbb{D}} |f'(z)||S(f)(z)|^2 (1 - |z|^2)^3 dx dy. \]
Recalling that $S(f) \equiv 0$ implies $\Gamma$ is a circle, can we improve this to
\[ \ell(\Gamma) - \pi \cdot \text{diam}(\Gamma) \simeq \iint_{\mathbb{D}} |f'(z)||S(f)(z)|^2 (1 - |z|^2)^3 dx dy? \]
Is there a similar estimate for estimating $\Delta(\gamma) = \ell(\gamma) - \text{crd}(\gamma)$ for sub-arcs? A first guess for a function theoretic analog to Formula (1.5) might be
\[ \Delta(\gamma) \simeq \iint_Q |f'(z)||S(f)(z)|^2 (1 - |z|^2)^3 dx dy, \]
where $\gamma = f(I)$ and $Q$ is the Carleson square with base $I$, but this fails for any Möbius transformation taking $\mathbb{D}$ to a disk (the right side is zero and the left is not),
so some modification is needed. Finding the correct version of this, or its analogs in terms of \( \log f' \) or \( \mu \), should give new arrows in Figure 3.

**Appendix A.** (13) \( \Rightarrow \) (2): \( \delta \) **controls Schwarzian.**

In the remaining appendices we give proofs of the “redundant” arrows in Figure 3 and sketch proofs of some of the previously known implications. This gives a self-contained proof of all the equivalences and provides a more geometric explanation for some of them.

We start by showing how \( \delta(z) \) directly controls the Schwarzian derivative. Suppose \( \Gamma \) is closed curve with complementary components \( \Omega_1, \Omega_2 \). Suppose \( \partial \text{CH}(\Gamma) = S_1 \cup S_2 \), the domes of \( \Omega_1 \) and \( \Omega_2 \). Using the ball model of hyperbolic 3-space, if \( 0 \in S_1 \) then there is a hyperbolic support plane for \( \text{CH}(\Gamma) \) passing through \( 0 \). This hyperbolic plane is also a Euclidean plane (since it passes through the origin), and \( \text{CH}(\Gamma) \) lies on one side of it. In the ball model we can think of the plane as \( \mathbb{R}^2 \subset \mathbb{R}^3 \) and the convex hull lying above this plane. If \( \text{dist}(0, S_2) \leq \delta \), then \( \text{CH}(\Gamma) \) lies below a hyperbolic plane that passes within hyperbolic distance \( \delta \) of \( 0 \). This hyperbolic plane lies on a Euclidean sphere of radius \( \simeq 1/\delta \) that passes within Euclidean distance \( O(\delta) \) of the orgin. Thus the boundary of the hyperbolic plane is a Euclidean circle on \( S^2 \) that lies with Euclidean distance \( O(\delta) \) of the equator \( S^2 \cap \mathbb{R}^2 \).

Next we recall Lemma 3.3 from [14]

**Lemma A.1.** Suppose \( \Omega \) is simply connected and

\[
\{ |z| < 1 - \delta \} \subset \Omega \subset \mathbb{D}.
\]

If \( F : \mathbb{D} \to \Omega \) is conformal map such that \( F(0) = 0 \) and \( F'(0) > 0 \), then

\[
|F(z) - z| = O(\delta),
\]

for all \( |z| \leq 1/2 \) and

\[
\max(|F'(0) - 1|, |F''(0)|, |F'''(0)|) = O(\delta).
\]

In particular, \( |S(F)(0)| = O(\delta) \).

Combined with our earlier discussion, and the fact that the Schwarzian is invariant under post-composition by Möbius transformations, this gives
Lemma A.2. Suppose $\Gamma$ is closed curve on the sphere with complementary domains $\Omega_1, \Omega_2$, whose domes are denoted $S_1, S_2$. If $f : \mathbb{D} \to \Omega_1$ is conformal, then
\[
|S(f)(z)|(1 - |z|^2)^2 = O(\delta(N(f(z))),
\]
where and $N$ is the nearest projection map from $\Omega_1$ to $S_1$.

Because of the quasi-area-preserving property of the projection map $N : \Omega \to S$ and Lemma 13.2 we see that Definition 13 implies Definition 2.

Appendix B. $(5) \Rightarrow (4)$: $\beta$'s control conformal welding.

In this section we discuss another “redundant” implication: that Definition 5 ($\sum \beta^2 < \infty$) implies Definition 4 ($\log \varphi' \in H^{1/2}$). Our alternate proof is not short, but it does give a direct geometric proof of this fact in the plane, and this approach may generalize to higher dimensions.

Fix $\epsilon > 0$ and let $U$ be the union of circular arcs joining $+i$ and $-i$ that are inside the left halfplane and are within distance $\epsilon$ of the unit circle. Note that $U$ forms a crescent with vertices at $\pm i$ and angles $\theta \simeq \epsilon$ at these points, and contains $L$, the arc of the unit circle that lies in the left halfplane. Let $R$ denote the arc of the unit circle that lies in the right halfplane.

Lemma B.1. If $W = \mathbb{D} \cup U$ and $z \in U$, then $\omega(z, R, W) \lesssim \epsilon$.

Proof. This is an explicit computation. Use a Möbius transformation to map $\mathbb{D}$ to the upper halfplane with $-i$ mapping to 0 and $i$ mapping to $\infty$. The domain $W$ maps to $W' = \{z : 0 < \arg(z) < \pi + \theta\}$ and $z$ maps into $W' \setminus \mathbb{H}^+_2$. It is easy to see that $\omega(z, (0, \infty), W') = 1 - \arg(z)/(\pi + \theta)$, and the lemma follows immediately. \qed
The monotonicity principle for harmonic measure states that if \( \Omega \subset W \) and \( E \subset \partial \Omega \cap \partial W \), then \( \omega(z, E, \Omega) \leq \omega(z, E, W) \) for all \( z \in \Omega \). This is a well known consequence of the maximum principle for harmonic functions, and applying it to the previous lemma immediately gives:

**Lemma B.2.** Suppose \( \Omega \) is a Jordan domain such that \( \mathbb{D} \setminus U \subset \Omega \subset \mathbb{D} \cup U \) (so one arc of \( \partial \Omega \) equals \( R \) and the other is inside \( U \)). If \( z \in \Omega \cap U \), then \( \omega(z, R, \Omega) \lesssim \epsilon \).

We now come to the main estimate of this section. We want to show that if \( \Gamma \) is a Weil-Petersson curve and \( \gamma_0 \) is a subarc of \( \Gamma \), and if we cut \( \gamma_0 \) into two pieces of equal length \( \gamma_0^+, \gamma_0^- \), then these pieces have have equal harmonic measures, with an error that is controlled by the \( \beta \)-numbers of \( \Gamma \). More precisely, let

\[
\zeta(\gamma_0) = \left| \frac{\omega(\gamma_0^+) - \omega(\gamma_0^-)}{\omega(\gamma_0)} \right|
\]

where \( \omega \) is harmonic measure for one side \( \Omega \) of \( \Gamma \) with respect to some fixed point \( p \in \Omega \). If we swap the roles of “+” and “−”, the quantity inside the absolute value only changes sign, so \( \zeta \) is well defined regardless of the choice we make.

**Lemma B.3.** With notation as above, there is an \( 1/2 < \alpha < 1 \) so that

\[
\zeta(\gamma_0) \lesssim \varepsilon(\gamma_0) + \sum_{I \subset \gamma_0} \beta(I)\ell(I)^\alpha
\]

for any arc \( \gamma_0 \subset \Gamma \). Here \( Q \) is a some dyadic square that hits \( \gamma_0 \) and has comparable diameter and the sum on the right is over a dyadic decomposition of \( \gamma_0 \cup \gamma_1 \cup \gamma_2 \), where \( \gamma_1, \gamma_2 \) are the arcs adjacent to \( \gamma_0 \) of the same length.

**Proof.** Suppose Definition 5 holds for \( \Gamma \). Then Definitions 7 and 12 also hold by results in this paper. Let \( \Omega \) be a complementary component of \( \Gamma \) and suppose \( p \in \Omega \) is the base point from which we define harmonic measure. Suppose \( \gamma_0 \subset \Gamma \) is a subarc of length at most \( r \), and \( r \) is chosen small enough that \( \beta(Q) \leq 1/100 \) and \( \varepsilon(Q) < 1/100 \) for any dyadic square that hits \( \gamma_0 \) and has diameter \( \text{diam}(Q) \leq 100\ell(\gamma_0) \). Moreover, we choose \( r \) so small that the disks of radius \( \text{diam}(Q)/\varepsilon(Q) \) in the definition of \( \varepsilon(Q) \) omit \( p \) (see the remark at the end of Section 12 that shows this is possible).

Let \( Q \) be a dyadic square hitting \( \gamma_0 \) with \( \text{diam}(\gamma_0) \leq \text{diam}(Q) \leq 2\text{diam}(\gamma_0) \). Let \( x \) and \( y \) be the endpoints of \( \gamma_0 \), chosen so that \( \Omega \) is on the left as we traverse \( \gamma_0 \) from \( x \) to \( y \). Let \( z \) be the midpoint of \( \gamma_0 \), determined by arclength. Note that the circle
determined by these three points has radius \( \geq \frac{\text{diam}(Q)}{\varepsilon} \), where \( \varepsilon = \varepsilon(Q) \). Let \( \gamma_0 \) be the arc of this circle that connects \( x \) to \( y \) and contains \( z \). Let \( \gamma_0^+ \) and \( \gamma_0^- \) be the disjoint subarcs that connect \( x \) to \( z \) and \( z \) to \( y \), respectively. See Figure 12.

Define a new closed curve \( \Gamma' \), that approximates \( \Gamma \) as follows. Let \( \gamma_1, \gamma_2 \) be arcs of \( \Gamma \) with length equal to \( \ell(\gamma_0) \), with interiors disjoint from \( \gamma_0 \), but so that \( \gamma_1 \cap \gamma_0 = x \) and \( \gamma_2 \cap \gamma_0 = y \). Let \( x_1 \) be the other endpoint of \( \gamma_1 \) and choose points \( \{z_n\}_{0}^\infty \) on \( \gamma_1 \), so that \( \ell(x_1, z_n) = 2^{-n}\ell(\gamma_1) \). Form the (infinite) polygonal path \( \tilde{\gamma}_1 \) by joining \( x \) to \( x_1 \).

We need to use the infinite path, since joining \( x \) to \( x_1 \) by a segment might not give a Jordan curve, e.g., if \( x_1 \) was the center of an infinite spiral in \( \Gamma \), but the fact that the \( \beta \)'s are all small at these scales imply \( \tilde{\gamma}_1 \) is a Jordan arc, indeed, a quasiarc with small constant.

Form a similar path \( \tilde{\gamma}_2 \) joining \( y \) to \( y_1 \) (the other endpoint of \( \gamma_2 \)). Then \( \tilde{\Gamma} \) is the union of \( \Gamma \setminus (\gamma \cup \gamma_1 \cup \gamma_2) \) with \( \tilde{\gamma}_0 \cup \tilde{\gamma}_1 \cup \tilde{\gamma}_2 \). Our assumptions imply that \( \tilde{\Gamma} \) is a closed Jordan curve. Let \( \tilde{\Omega} \) be the bounded complementary component of \( \tilde{\Gamma} \).

Let \( \tau \) be the Möbius transformation that sends \( \{x, z, y, \infty\} \) to \( \{-i, 1, i, -1\} \). Then our assumption imply \( (1 - \delta)\mathbb{D} \subset \tau(\tilde{\Omega}) \subset (1 + \delta)\mathbb{D} \), where \( \delta \simeq \varepsilon \) and \( R \subset \partial \tau(\tilde{\Omega}) \).

Also, \( \tau(p) \in D(-1, \delta) \). Let \( q = \tau(p) \) and \( V = \tau(\tilde{\Omega}) \).

Next, let \( \phi \) be the conformal map from \( V \) to \( \mathbb{D} \) that sends \( 0 \rightarrow 0 \) and maps \( q \) to a point on the negative real axis. Let \( R^+, R^- \) denote the parts of \( R \) that lie above and
below the real axis respectively. We claim that
\[ \omega(0, R^+, V) = \frac{1}{4} + O(\varepsilon), \]
and similarly for \( R^1 \). If \( V = \mathbb{D} \), then the harmonic measure would be exactly \( 1/4 \). Since these two domains are not the same, the harmonic measures of \( R^+ \) for \( \mathbb{D} \) and \( V \) differ by two probabilities: the probability that a Brownian motion in \( \mathbb{D} \) started at the origin and ending in \( R^+ \) hits \( \partial V \cap \mathbb{D} \), and the probability that a Brownian path in \( V \) started at the origin and ending in \( R^+ \) hits \( L \). However, Lemma B.2 implies that both these probabilities are \( \lesssim \varepsilon \). Therefore, the map \( \phi \) sends both \( \pm i \) to points of \( \mathbb{T} \) that are within \( O(\varepsilon) \) of themselves. The explicit form of the Poisson kernel now shows that
\[ \omega(q, R^+, \mathbb{D}) - \omega(q, R^-, \mathbb{D}) = O(\varepsilon). \]

Hence by the conformal invariance of harmonic measure,
\[ \frac{\omega(p, \gamma_0^+, \tilde{\Omega}) - \omega(p, \gamma_0^-, \tilde{\Omega})}{\omega(p, \gamma_0, \tilde{\Omega})} = O(\varepsilon). \]

We now want to show that a similar estimate still holds for the corresponding subarc of \( \Gamma \). Note that we can obtain \( \Gamma \) from \( \tilde{\Gamma} \) by a countable number of refinements: given a circular arc \( I \) (possibly a line segment) in \( \tilde{\Gamma} \) whose endpoints \( v, w \) are in \( \Gamma \) and define a subarc \( J \) of \( \Gamma \), replace \( I \) by the circular arc (possibly a line segment), that contains \( v, u, w \), where \( u \) is the midpoint of \( J \). See Figure 13. Consider the sequence of curves \( \{\Gamma_n\} \) generated in this way, that starts with \( \tilde{\Gamma} \) and converges to \( \Gamma \). Let \( \Omega_n \) be the bounded complementary component of this curve and note that each curve contains the points \( x, y, z \in \Gamma \cap \tilde{\Gamma} \). Thus it makes sense to talk about the harmonic measure in \( \Omega_n \) of the boundary arc connecting \( x \) to \( z \) and \( z \) to \( y \).

By definition, the old and new circular arcs lie within a crescent with angle measure \( O(\beta(I)) \) and so the probability that a Brownian motion starting within this crescent will leave the disk with diameter segment \([v, w]\) without hitting \( I \) (or \( J \)) is at most \( O(\beta(I)) \). Therefore, when we make a such a replacement on \( I \), the harmonic measures of the subarcs of \( \Gamma_n \) that connect \( x \) to \( z \) and \( z \) to \( y \) can change by at most \( O(\beta(I)\omega(p, I, \Omega_n)) \). Therefore the probability that it will effect the harmonic measure of an arc disjoint from \( I \) is bounded by this quantity. Moreover, since all these curves are quasicircles with a uniformly bounded constant, the harmonic measures
Figure 13. The solid curve is $\Gamma$ and the dotted and dashed curve represent two approximations using circular arcs passing through triples of points on $\Gamma$. On each arc where we replace a circular arc by two circular arcs, both the old and new curves remain in thin crescent with angle $O(\beta)$; the probability of a Brownian motion started in such a crescent leaving the shaded disk is $O(\beta)$. Combined with a standard Hölder estimate for harmonic measure on quasicircles, this implies these replacements do not greatly alter the relative harmonic measures of $\gamma_0^+$ and $\gamma_0^-$, as quantified in (B.1).

satisfy a Hölder bound

$$\omega(p, I, \Omega_n) \lesssim \omega(p, [x, y], \Omega_n) \left( \frac{\ell(I)}{|x - y|} \right)^\alpha.$$  

By assuming we are close enough to the boundary (and hence the $\beta$'s are as small as we wish) we can assume $\alpha$ is as close to 1 as we wish; we will only need $\alpha > 1/2$. Therefore we get

$$\frac{\omega(p, \gamma_0^+, \Omega) - \omega(p, \gamma_0^-, \Omega)}{\omega(p, \gamma_0, \Omega)} = O(\varepsilon) + O \left( \sum_{I \subset \gamma_0} \beta(I) \ell(I)^\alpha \right) \quad \text{(B.1)}$$

where the sum is over dyadic subarcs of $\gamma_0 \cup \gamma_1 \cup \gamma_2$.

\[\square\]

Lemma B.4. Definition 5 implies Definition 4.

Proof. Let $\Omega, \Omega^*$ be the bounded and unbounded complementary components of $\Gamma$ and $f, g$ the conformal maps from $\mathbb{D}, \mathbb{D}^*$ to $\Omega, \Omega^*$ respectively. Then $\varphi = g^{-1} \circ f$ is the welding map. For $\gamma \subset \Gamma$, let $\omega(\gamma) = |f^{-1}(\gamma)|$ be harmonic measure on $\Omega$ with respect to the point $f(0)$, and similarly for $\omega^*$ on $\Omega^*$. We want to show that $\log \varphi' \in H^{1/2}$, i.e., that

$$\int_T \int_T \left| \log \varphi'(z) - \log \varphi'(w) \right| \frac{dz}{|z - w|} \quad \text{for all } T.$$  

\[\text{(B.2)}\]
Consider a dyadic decomposition \( \{ \gamma_j^n \} \) with respect to arc-length on \( \Gamma \) and pull this back, via the conformal map \( f \), to a family of arcs \( \{ I_j^n \} \) in \( \mathbb{T} \). Define

\[
\varphi'(I_j^n) = \frac{\omega^*(\gamma_j^n)}{\omega(\gamma_j^n)} = \frac{|\varphi(I_j)|}{|I_j|} = \frac{1}{|I_j|} \int_{I_j} |\varphi'(z)||dz|.
\]

Note that \( \varphi'(I_j) \to \varphi'(x) \) almost everywhere as \( I_j \) shrinks down to \( x \).

Let \( \zeta, \zeta^* \) be the quantities defined above for the measures \( \omega, \omega^* \), respectively. These depend on the interval \( I_n^j \) we are considering, but since this is always clear from context, we will not write this explicitly.

Define \( g_n = \log \varphi'(I_j^n) \) on \( I_j^n \). If \( I_j = I_m^j, J = I_m^k \) are siblings (i.e., children of the same parent interval \( K = I_l^{n-1} \)) then

\[
|\log \varphi'(I) - \log \varphi'(J)| = \left| \log \frac{\omega^*(I)\omega(J)}{\omega(I)\omega^*(J)} \right| \\
= \left| \log \frac{\left(\frac{1}{2}\omega^*(K)(1 - \zeta^*)\right)(\frac{1}{2}\omega(K)(1 + \zeta))}{\left(\frac{1}{2}\omega(K)(1 - \zeta)\right)(\frac{1}{2}\omega^*(K)(1 + \zeta^*))} \right| \\
= \left| \log \frac{(1 - \zeta)(1 - \zeta^*)}{(1 + \zeta)(1 + \zeta^*)} \right| \\
= O(|\zeta| + |\zeta^*|)
\]

Thus \( \Delta g_n \) is bounded by these quantities. Similarly,

\[
\log \varphi'(I) + \log \varphi'(J) = \log \frac{\omega^*(I)\omega^*(J)}{\omega(I)\omega(J)} \\
= \log \frac{\left(\frac{1}{2}\omega^*(K)(1 - \zeta^*)\right)(\frac{1}{2}\omega^*(K)(1 + \zeta^*))}{\left(\frac{1}{2}\omega(K)(1 - \zeta)\right)(\frac{1}{2}\omega(K)(1 + \zeta))} \\
= \log \frac{(1 - \zeta^2)}{(1 - \zeta^2)} + 2\log \frac{\omega^*(K)}{\omega(K)} \\
= O(\zeta^2 + \zeta^2) + 2\log \varphi'(K)
\]

Thus \( g_n \) differs from a martingale by a error that is bounded by \( O(\zeta^2 + \zeta^2) \). For simplicity, we will write \( \zeta \) for the sum \( |\zeta| + |\zeta^*| \) in the remainder of the proof.

Therefore we can write \( g_n \) as a sum of a martingale \( \{ f_n \} \) and piecewise constant functions \( \{ h_n \} \) as in Lemma 11.3 with (using notation from that lemma)

\[
a_{n,j} = \zeta^2(I_j^n), \quad b_{n,j} = \zeta(I_j^n),
\]

where \( \zeta(I_j^n) \) is the sum of the bounds from (B.1) for the two measures \( \omega \) and \( \omega^* \). Lemma B.4 follows once we verify that \( a_{n,j} \) and \( b_{n,j} \) satisfy the hypotheses (11.2) and
(11.1) of Lemma 11.3. We start with (11.2). Note that

\[
\| (B_{n,j})\|_{L^1(I)} = \sum_{n,j} \frac{B_{n,j}}{\ell(I_j^n)}
\]

\[
\lesssim \sum_{n,j} \sum_{m,k} \varepsilon^2(I_k^m) \frac{\ell(I_k^m)}{\ell(I_j^n)}
\]

\[
+ \sum_{n,j} \sum_{m,k} I_m^k \subset I_j^n \sum_{p,i} I_k^p \subset 3I_k^m \beta(I_i^p) \left( \frac{\ell(I_i^p)}{\ell(I_k^m)} \right)^\alpha \left( \frac{\ell(I_j^n)}{\ell(I_k^m)} \right)^2
\]

\[
= I + II.
\]

Here $3I_k^m$ means the union of $I_k^m$ with the two adjacent arcs of the same length. Rearranging the first term on the right above gives

\[
I \lesssim \sum_{m,k} \varepsilon^2(I_k^m) \sum_{n,j} \frac{\ell(I_k^m)}{\ell(I_j^n)} \lesssim \sum_k \varepsilon^2(I_k^m) < \infty
\]

since the sum of length ratios is a geometric series. Choose $s \in (0, 2\alpha - 1)$ and use Cauchy-Schwarz to rewrite the second term above as

\[
II \lesssim \sum_{n,j} \sum_{m,k} \varepsilon^2(I_k^m) \sum_{p,i} I_k^p \subset 3I_k^m \beta(I_i^p) \left( \frac{\ell(I_i^p)}{\ell(I_k^m)} \right)^\alpha \left( \frac{\ell(I_j^n)}{\ell(I_k^m)} \right)^2
\]

\[
\lesssim \sum_{n,j} \sum_{m,k} \varepsilon^2(I_k^m) \sum_{p,i} I_k^p \subset 3I_k^m \beta^2(I_i^p) \left( \frac{\ell(I_i^p)}{\ell(I_k^m)} \right)^s \left( \frac{\ell(I_j^n)}{\ell(I_k^m)} \right)^{2\alpha - s} \left( \frac{\ell(I_k^m)}{\ell(I_j^n)} \right)^\alpha
\]

Now use the fact that $2\alpha - s > 1$ to deduce

\[
\sum_{p,i} I_k^p \subset 3I_k^m \left( \frac{\ell(I_i^p)}{\ell(I_k^m)} \right)^{2\alpha - s} \leq C(s, \alpha) < \infty
\]
and hence by further rearranging

\[ II \lesssim \sum_{n,j} \sum_{m,k: I^m_k \subset I^n_j} \left[ \sum_{p,i: I^p_i \supseteq 3I^m_k} \beta^2(I^p_i) \left( \frac{\ell(I^p_i)}{\ell(I^m_k)} \right)^s \right] \frac{\ell(I^m_k)}{\ell(I^n_j)} \]

\[ \lesssim \sum_{p,i} \beta^2(I^p_i) \sum_{m,k: I^m_k \supseteq 3I^p_i} \left( \frac{\ell(I^m_k)}{\ell(I^p_i)} \right)^s \sum_{n,j: I^n_j \supseteq I^m_k} \frac{\ell(I^m_k)}{\ell(I^n_j)} \]

\[ \lesssim \sum_{p,i} \beta^2(I^p_i) < \infty \]

where the last two lines use that fact that the eliminated sums are geometric series.

Next we consider condition (11.1) in Lemma 11.3. Using Cauchy-Schwarz and (twice) the fact that \( \sum_{m,k} \zeta^2(I^m_k) < \infty \), we get

\[ \| (A_{n,j}) \|_{L^2(I^n_j)} = \sum_{n,j} A_{n,j}^2 \frac{1}{\ell(I^n_j)} \lesssim \sum_{n,j} \frac{1}{\ell(I^n_j)} \left( \sum_{m,k: I^m_k \subset I^n_j} \zeta^2(I^m_k) \sqrt{\ell(I^m_k)} \right)^2 \]

\[ \lesssim \sum_{n,j} \frac{1}{\ell(I^n_j)} \left( \sum_{m,k: I^m_k \subset I^n_j} \zeta^2(I^m_k) \right) \cdot \left( \sum_{m,k: I^m_k \subset I^n_j} \zeta^2(I^m_k) \ell(I^m_k) \right) \]

\[ \lesssim \sum_{n,j} \sum_{m,k: I^m_k \subset I^n_j} \zeta^2(I^m_k) \frac{\ell(I^m_k)}{\ell(I^n_j)} \]

\[ \lesssim \sum_{m,k} \zeta^2(I^m_k) \sum_{n,j: I^n_j \supseteq I^m_k} \frac{\ell(I^m_k)}{\ell(I^n_j)} \]

\[ \lesssim \sum_{m,k} \zeta^2(I^m_k) < \infty. \]

We have also used the observation that the sum of length ratios is a geometric series. This completes the proof of Lemma B.4.

The argument above works whenever we have two measures \( \omega, \omega^* \) that each satisfy estimates of the from (B.1); we never used the fact that they are harmonic measures of opposite sides of the same curve.
Corollary B.5. If $\Gamma_1, \Gamma_2$ are both Weil-Petersson curves, then the curve $\Gamma_3$ obtained by isometrically conformal welding these curves by any length preserving map is another Weil-Petersson curve.

More precisely, if $f : \mathbb{D} \to \Omega_1$ and $g : \mathbb{D}^* \to \Omega_2^*$ are conformal maps to the inside and outside of two Weil-Petersson curves, and $\iota : \Gamma_1 \to \Gamma_2$ is an orientation preserving, length preserving homeomorphism, then $\varphi = g^{-1} \circ \iota \circ f$ is the conformal welding homeomorphism of some Weil-Petersson curve $\Gamma_3$. In general, isometric weldings of rectifiable curves need not exist, need not be unique up Möbius transformations and need not give rectifiable curves when they do exist, e.g., [11], [45].

A stronger version of Corollary B.5 has been obtained independently by Viklund and Wang [76]: they show that the Loewner energy (see Section D) is sub-additive under isometric welding of Weil-Petersson curves.

APPENDIX C. COMPLETING THE CIRCLE

For the convenience of the reader, we sketch the implications $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$. These were all previously known to be equivalent characterizations of the Weil-Petersson class, but including sketches of these implications allows us to give a self-contained treatment of all the equivalences discussed in this paper. Together with $(1) \Rightarrow (5) \Rightarrow (4)$, these give an alternate proof of the equivalence of $(1)$-$(4)$.

• Definition 4 $\Rightarrow$ Definition 3: For each dyadic interval $I \subset \mathbb{T}$, let $z_I$ be the peak of the “tent” with base $I$, namely, choose $z_I$ so that its radial projection is the center of $I$ and the $1 - |z_I| = |I|$. Then triangulate the disk by connecting these points as indicated in Figure 14. When we apply the circle homeomorphism $\varphi$, each dyadic interval $I$ maps to an image interval $J = \varphi(I)$ with a corresponding point $z_J$. This defines a piecewise linear map of the triangulation, which is a quasiconformal homeomorphism near the boundary. Indeed the dilatation on each triangle $T$ is controlled by a constant multiple of

$$\frac{|J_1| - |J_2|}{|J|},$$

where $J_1, J_2$ are the children of $J$, and where $J$ ranges over the dyadic intervals corresponding to the vertices of $T$. These differences are square summable by Definition 4, and hence this definition implies Definition 3 (since we only need to check the integrability of $|\mu|^2$ near the boundary).
Figure 14. Dyadic intervals on the circle correspond to points in the unit disk, which form the vertices of a triangulation. The boundary map induces a map of these vertices, which in turn gives a map that is piecewise linear of the triangles. If the boundary map is quasisymmetric with small constant this map is quasiconformal with small dilatation, and if Definition 4 holds for the boundary map, then Definition 3 holds for the extension (at least near the boundary).

- **Definition 3 ⇒ Definition 2:** Assume $\Gamma = f(\mathbb{T})$ where $f$ is quasiconformal on the plane, conformal on $\mathbb{D}^*$ and has dilatation $\mu$ supported in $\mathbb{D}$.

  In [7], Astala and Zinsmeister use the identity
  \[
  \int_{\mathbb{D}} f_\pi(z)dx\,dy = \frac{\pi}{6} \lim_{w \to \infty} w^4 S(f)(w),
  \]
  a change of variables, and the Cauchy-Schwarz inequality to deduce
  \[
  |S(f)(z)|^2(|z|^2 - 1)^2 \lesssim \int_{\mathbb{D}} |w - z|^{-4} |\mu(w)|^2 \,dudv.
  \]
  Integrating this with respect $z$, and reversing the order of integration on the right side gives
  \[
  \int_{\mathbb{D}} |S(f)(z)|^2(1 - |z|^2)^2 dx\,dy \lesssim \int_{\mathbb{D}} |\mu(w)|^2(1 - |w|^2)^{-2} dudv.
  \]

- **Definition 2 ⇒ Definition 1:** The following is Lemma 3.4 of [15].
Lemma C.1. Given $\epsilon > 0$, $n \in \mathbb{N}$, there exists $C = C(\epsilon) > 0$ and $\delta = \delta(\epsilon, n) > 0$ so that the following holds. Suppose $Q$ is a Carleson square and
\[
|S(f)(z)| \leq \frac{\delta}{(1 - |z|^2)^2},
\]
for all $z \in T(Q)$. Then there is a hyperbolic geodesics $\gamma$ so that every point $z \in Q$ that satisfies both $1 - |z| \geq 2^{-n} \ell(Q)$ and $|F'(z)| \geq \epsilon/(1 - |z|^2)$ lies within hyperbolic distance $C$ of $\gamma$. Moreover, for every $\eta > 0$ there is a $\delta > 0$ so that if $\gamma_0$ is a sub-arc of $\gamma$ connecting $z_0 \in T(Q)$ to $z_1 \in Q \cap \{z : 1 - |z| = 2^{-n} \ell(Q)\}$, then $|F'(z_1)| \geq 2^{2n(1-\eta)}|F'(z_0)|$. In other words, $F'$ grows almost as fast as $(1 - |z|)^{-2}$ along $\gamma$.

If Definition 2 holds, we must have
\[
|S(f)(z)|(1 - |z|^2)^2 \to 0,
\]
as $|z| \not\to 1$. Therefore, given any $\delta > 0$, the hypothesis of Lemma C.1 holds for all sufficiently small Carleson squares. Suppose $z_0, z_1$ are as in the lemma. The Koebe $\frac{1}{4}$-theorem implies that there is an arc $J \subset \mathbb{T}$ that contains the base $I$ of $Q$ and whose chord length is comparable to $|f'(z_0)|(1 - |z_0|)$ (we can choose one endpoint of $J$ from each component of $3I \setminus I$). Similarly, there is a subarc $\gamma_1 \subset \gamma_0$ whose chord length (and hence its diameter) is comparable to
\[
|f'(z_1)|(1 - |z_1|) \geq |f'(z_0)|\left(\frac{1 - |z_0|}{1 - |z_1|}\right)^{2(1-\eta)}(1 - |z_1|).
\]
\[
= |f'(z_0)|(1 - |z_0|)\left(\frac{1 - |z_0|}{1 - |z_1|}\right)^{1-2\eta}
\]
\[
= |f'(z_0)|(1 - |z_0|)2^{n(1-2\eta)}.
\]
If $\eta < 1/2$ then this tends to infinity with $n$, and this implies that $\Gamma$ is not a quasicircle, contrary to assumption. Therefore, if $Q$ is small enough we may assume $|F'(z)| \leq \epsilon/(1 - |z|^2)$ in $Q$.

From this point we follow the proof of Lemma 1 in the paper of Astala and Zinsmeister [7]. Recall that $F = \log f'$. We want to show that $\iint_{\mathbb{D}} |F'(z)|^2 dx dy < \infty$. We claim this is equivalent to proving
\[
I = \iint_{\mathbb{D}} |F''(z)|^2(1 - |z|^2)^2 dx dy < \infty.
\]
To prove this claim, assume $F$ has the power series expansion $F(z) = \sum_{n=0}^{\infty} b_n z^n$, and then a simple computation in polar coordinates leads to

$$\int_{D} |F'(z)|^2 dx dy = 2\pi \sum_{n=1}^{\infty} n^2 |b_n|^2 \int_{0}^{1} r^{2n-1} dr = \sum_{n=1}^{\infty} (\pi n) |b_n|^2,$$

and

$$I = \int_{D} |F''(z)|^2 (1-|z|^2)^2 dx dy$$
$$= 2\pi \sum_{n=1}^{\infty} n^2 (n-1)^2 |b_n|^2 \int_{0}^{1} r^{2n-4} (1-2r^2 + r^4) dr$$
$$= 2\pi \sum_{n=1}^{\infty} n^2 (n-1)^2 |b_n|^2 \left( \frac{1}{2n-2} - \frac{2}{2n} + \frac{1}{2n+2} \right)$$
$$= \sum_{n=1}^{\infty} 2\pi n (n-1) |b_n|^2 \simeq \sum_{n=2}^{\infty} \pi n |b_n|^2$$

Thus both infinite series (and hence both integrals) diverge or converge together.

Moreover, note that $\sum_{n=N}^{\infty} \pi n |b_n|^2 \leq I$ for $N$ large enough, and $\sum_{n=0}^{2} \pi c_n |b_n|^2 \leq M$ is uniformly bounded because $F$ is in the Bloch class with norm at most 6 and hence the Cauchy estimates give bounds on $b_n$ that only depend on $n$.

Next we must show $I$ is finite. Using (2.2) and (2.3) we see that

$$F''(z) = S(f)(z) - \frac{1}{2} \left( \frac{f''}{f'} \right)^2,$$

and hence

$$I = \int_{D} |F''(z)|^2 (1-|z|^2)^2 dx dy \leq \int_{D} |S(f)(z)|^2 (1-|z|^2)^2 dx dy$$
$$+ \frac{1}{4} \int_{D} |F'(z)|^4 (1-|z|^2)^2 dx dy$$
$$+ \int_{D} |S(f)(z)|^2 |F'(z)|^2 (1-|z|^2)^2 dx dy$$
$$= I_1 + \frac{1}{4} I_2 + I_3.$$

Note that since $|xy| \leq (x^2 + y^2)/2$, we get

$$|S(f)(z)||F'(z)|^2 \leq \frac{1}{2} |k F'(z)|^4 + \frac{1}{2} |S(f)(z)|^2$$
and hence $I_3 \leq \frac{1}{2}I_1 + \frac{1}{2}I_2$. Therefore

$$ I \leq \frac{3}{2}I_1 + \frac{3}{4}I_2. \tag{C.1} $$

Now we use our assumption that $|F'(z)|(1 - |z|^2) \leq \epsilon$ is small on the annulus $A = \{z : r < |z| < 1\}$. This implies

$$ I_2 = \int_A |F'(z)|^4(1 - |z|^2)^2dxdy \leq \epsilon^2 \int_A |F'(z)|^2dxdy \leq \epsilon^2MI. $$

If $\epsilon$ is small enough, then $I_2 \leq I$, and then (C.1) becomes

$$ I \leq \frac{3}{2}I_1 + \frac{3}{4}I, $$

which implies $I \leq 6I_1 < \infty$. This completes the proof that $F = \log f'$ is in the Dirichlet class.

**APPENDIX D. OTHER CHARACTERIZATIONS OF WEIL-PETERSSON CURVES**

This appendix lists some further equivalent definitions of the Weil-Petersson class; these definitions were never used in our proofs, but I include them to illustrate the variety of problems in which the Weil-Peterson class naturally occurs. We start with the Takhtajan and Teo’s definition in terms of the universal Teichmüller space, which gives the Weil-Petersson class its name.

- **Teichmüller theory:** Recall that $\mathbb{D} = \{|z| < 1\}$ and $\mathbb{D}^* = \{|z| > 1\}$. Let $L^\infty(\mathbb{D}^*)_1$ denote the unit ball of $L^\infty(\mathbb{D}^*)$. By the measurable Riemann mapping theorem, each $\mu \in L^\infty(\mathbb{D}^*)$ determines a quasiconformal map $w^\mu$ of the plane that is conformal inside $\mathbb{D}$, and satisfies $f(0) = f''(0) = 0$, $f'(0) = 1$. We say $\mu$ and $\nu$ are equivalent if $w^\mu = w^\nu$ on $\mathbb{T}$ and we define $T(1)$ be $L^\infty(\mathbb{D}^*)_1$ quotiented by this equivalence relation. This is the universal Teichmüller space.

In [75], Takhtajan and Teo define a Weil-Petersson metric on universal Teichmüller space $T(1)$. With this metric $T(1)$ has uncountably many connected components, and $T_0(1)$ denotes the connected component containing the identity. More concretely, consider the set $U$ of holomorphic function $\phi$ on $\mathbb{D}$ that satisfy

$$ \int_{\mathbb{D}^*} |\phi(z)|^2(1 - |z|^2)^2dxdy < \sqrt{\pi/3}, $$

and for each $\phi \in U$ define a dilatation $\mu$ on $\mathbb{D}^*$ by

$$ \mu(z) = -\frac{1}{2}(1 - |z|^2)^2\phi(1/\bar{z})z^{-4}. $$
Given a fixed dilatation $\nu$ on $\mathbb{D}^*$ consider the set of all dilations of the form $\lambda = \nu \ast \mu^{-1}$ given by
\[
\lambda = \left( \frac{\nu - \mu}{1 - \bar{\nu} \mu} \right) \cdot \frac{(w_\mu)z}{(w_\mu)\bar{z}} \circ w_\mu.
\]
(This product on dilatations just corresponds to composing the corresponding quasiconformal mappings.) This defines a set $V_\nu \subset L^\infty(\mathbb{D}^*)_1$ that contains $\nu$. Projecting these sets into $T(1)$ defines a neighborhood of each point $[\nu] \in T(1)$ and $T_0(1)$ is the connected component of the identity in this topology.

**Definition 15.** $\Gamma = f(T)$, where $f$ is a quasiconformal map of the plane, conformal inside $\mathbb{D}$ and whose dilatation on $\mathbb{D}^*$ represents a point of $T_0(1)$.

In Theorem II.1.12 of [75], Takhtajan and Teo prove that this is equivalent to Definition 2: $T_0(1)$ is the inverse image of $A_2(\mathbb{D})$ under the Bers embedding $\beta : T(1) \to A_\infty(\mathbb{D})$ where
\[
A_\infty(\mathbb{D}) = \{ \phi \text{ holomorphic on } \mathbb{D} : \sup_{\mathbb{D}} |\phi(z)|(1 - |z|^2)^2 dx dy < \infty \}.
\]

The Bers embedding is defined by starting with a dilatation $\mu \in L^\infty(\mathbb{D}^*)$ representing a point of $T(1)$, and observing that $\beta(\mu) \equiv S(\mu|_{\mathbb{D}}) \in A_\infty(\mathbb{D})$ by (2.6).

- **Operator theory:** Given a circle homeomorphism $\varphi$ we can define an operator on harmonic functions on the unit disk by taking pre-composing the boundary values of $u$ with $\varphi$, taking the harmonic extension back to the disk, and subtracting the value at the origin (so the resulting harmonic function $P_\varphi u$ is zero at the origin. Given an holomorphic function in the Dirichlet class, we can apply this operator and follow it by orthogonal projection onto the anti-holomorphic Dirichlet functions and finally apply $f(z) \to f(\bar{z})$ to make is holomorphic. Nag and Sullivan [57] proved this operator $P_\varphi^-$ is bounded from the Dirichlet class to itself if and only if $\varphi$ is quasisymmetric, and Hu and Shen [44] prove it is Hilbert-Schmidt if and only if $\varphi$ is Weil-Petersson (an operator $T$ on a Hilbert space is Hilbert-Schmidt if $\sum_j \|Te_j\|^2 < \infty$ for any orthonormal basis $\{e_j\}$; equivalently $TT^*$ is trace class).

**Definition 16.** $\varphi$ is quasisymmetric and $P_\varphi^-$ is Hilbert-Schmidt from the Dirichlet space to itself.

Another operator theoretic characterization of the Weil-Petersson class is given by Takhtajan and Teo (Corollary II.2.9, [75]) in terms of Grunsky operators on $\ell^2$. 
• **Loewner energy:** Suppose $\Omega = \mathbb{C} \setminus [0, \infty)$ and suppose that $\gamma \subset \Omega$ is a curve that connects 0 to $\infty$. Suppose also that this curve corresponds to driving function $W$ via Loewner’s equation. Then the chordal Loewner energy of $\gamma$ is defined by Friz and Shekhar [36] and Wang [79] to be

$$I(\gamma) = \frac{1}{2} \int_0^\infty \dot{W}(t)^2 dt.$$ 

This was generalized by simple loops $\Gamma$ on the Riemann sphere by Steffen Rohde and Yilin Wang [66] by choosing two points $z, w \in \Gamma$ and conformally mapping the complement of the arc $\Gamma_{z,w}$ from $z$ to $w$ to $\Omega$ with $z, w$ mapping to 0, $\infty$ respectively. The image of $\Gamma \setminus \Gamma_{z,w}$ is now an arc from 0 to $\infty$ in $\Omega$, so its energy is defined as above. The energy of the loop $\Gamma$ rooted at $z$ is defined to be the limit of these energies as $w$ approaches $z$; Rohde and Wang show that this is independent of the choice of $z$.

**Definition 17.** $\Gamma$ has finite Loewner energy.

The equivalence of this to the earlier definitions was proven by Yilin Wang [79]. She showed that the Loewner energy is equal to $S_1(\varphi)/\pi$ where $S_1(\varphi)$ is the universal Liouville action defined by Takhtajan and Teo by

$$S_1(\varphi) = \iint_D \left| \frac{f''(z)}{f'(z)} \right|^2 dx dy + \iint_{D^*} \left| \frac{g''(z)}{g'(z)} \right|^2 dx dy + 4\pi \log \frac{|f'(0)|}{|g'(\infty)|}$$

where $f : \mathbb{D} \to \Omega$, $g : \mathbb{D}^* \to \Omega^*$ are the conformal maps from the two sides of the unit circle to the two sides of $\Gamma$.

• **Large deviations of SLE:** In [77], Yilin Wang interprets finite energy curves $\gamma$ from 0 to $\infty$ in $\mathbb{H}_2^+$ in terms of large deviations of SLE($\kappa$) as $\kappa \searrow 0$. Roughly speaking, the Loewner energy of $\gamma$ is equal to

$$\lim_{\epsilon \to 0} \left[ \lim_{\kappa \searrow 0} \log \mathbb{P}[\text{SLE}(\kappa) \in B(\gamma, \epsilon)] \right].$$

In words, this is the probability that SLE stays in an $\epsilon$-neighborhood of $\gamma$ decreases exponentially with decay factor equal to the energy of $\gamma$. In fact, Hausdorff neighborhoods are not suitable for this large deviation problem, so Wang’s result is actually stated in terms of sets of curves that pass to the left or right of a specified finite set of points. A little more precisely, suppose we are given a finite set $Z$ of points $\{z_n\}$ in the upper half-plane and each point is labeled with $\pm 1$. A curve $\gamma$ from 0 to $\infty$ cuts the upper half-plane into simply connected regions, which we call the “left side”
and “right side”. A curve $\gamma$ is called admissible for $Z$ (written $\gamma \in \mathcal{A}(Z)$) if every point labeled +1 is on the right side of $\gamma$ and every point labeled −1 is on the left side. If $\gamma$ is admissible for $Z$, then we say that $Z$ is consistent with $\gamma$ and we write $Z \in \mathcal{Z}(\gamma)$. Wang shows that given a set $Z$, 

$$-\lim_{\kappa \to 0} \kappa \log \mathbb{P}[\text{SLE}(\kappa) \in \mathcal{A}(Z)] = \inf \{I(\gamma) : \gamma \in \mathcal{A}(Z)\}.$$ 

Thus the Weil-Petersson class can be defined using the condition

Definition 18. $\sup_{Z \in \mathcal{Z}(\gamma)} \lim_{\kappa \to 0}(-\kappa) \log \mathbb{P}[\text{SLE}(\kappa) \in \mathcal{A}(Z)] < \infty.$

Roughly speaking, a curve in $\mathbb{H}^+_2$ from 0 to $\infty$ is a (sub-arc of a spherical) Weil-Petersson curve iff for any finite set of labeled points $Z$ consistent with $\gamma$, the probability that SLE$(\kappa)$ is also consistent with $Z$ decays at most exponentially quickly as $\kappa \to 0$. See [77] for precise statements and further details.

• Brownian loop soup: The Brownian loop measure, introduced by Greg Lawler and Wendelin Werner [49] is a measure on closed loops in a domain $\Omega$. It is conformally invariant and if $\Omega' \subset \Omega$, then the loop measure on $\Omega'$ is the just the restriction of the loop measure for $\Omega$ to loops that are contained in $\Omega'$. Given disjoint compact subsets of $\Omega$ we define $\mathcal{W}(A, B; \Omega)$ to be the loop measure of closed curves $\gamma$ in $\Omega$ so that the outer boundary of $\gamma$ hits both $A$ and $B$. Suppose $\Gamma^r$ is the image of the circle $\{|z| = r\}$ under a conformal map from $\mathbb{D}$ to the interior of $\Gamma$. Yilin Wang proves in [78] that the Loewner energy of $\Gamma$ is 12 times

$$\lim_{r \to 1} \left[\mathcal{W}(S^1, r \cdot S^1, \mathbb{C}) - \mathcal{W}(\Gamma, \Gamma^r, \mathbb{C})\right].$$

Thus being a Weil-Petersson curve is equivalent to:

Definition 19. $\Gamma$ satisfies $\lim_{r \to 1} \left[\mathcal{W}(S^1, r \cdot S^1, \mathbb{C}) - \mathcal{W}(\Gamma, \Gamma^r, \mathbb{C})\right] < \infty.$

This difference of divergent quantities is reminiscent of the definition of renormalized area. Is there any direct connection between these two concepts?
References


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