RIGIDITY OF FINITE CIRCLE DOMAINS

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A plane domain $\Omega = \mathbb{C} \setminus E$ is called a circle domain if each connected component of E is closed disk or a point. We say it is a finite circle domain if E has a finite number of connected components. A circle domain is called rigid if any conformal mapping of it to another circle domain is the restriction of a Möbius transformation.

Theorem 0.1. *Finite circle domains are rigid.*

A set E is called CH-removable is every orientation preserving homeomorphism of the sphere that is conformal off E is a Möbius transformation. The usual proof of Theorem 0.1 is to use reflections (as we shall below) to show that f extends to a homeomorphism of the sphere that is conformal off a certain Cantor set associated to the domain Ω and use one of a variety of known sufficient conditions for CH-removability show that this Cantor is removable, and hence f is Möbius. An alternate strategy would be to show that the functions F_n defined below converge to 1-quasiconformal functions with the geometric definition (the modulus of any quadrilateral is let unchanged) and then prove that a 1-QC map is conformal. This would take about 10 pages from scratch, mostly developing the basic definitions of extremal length, quasiconformal maps and finally the proof that 1-quasiconformal maps are conformal. The point of this note is to give a (mostly) self-contained, fairly elementary proof. We shall use without proof the Cauchy-Pompeiu formula for smooth functions (a variation of Stokes formula), and basic distortion estimates for conformal maps (though Harnack's inequality would probably be enough). Some of the other facts I use might require a more detailed proof, depending on the definition of "fairly elementary".

Proof. Suppose f is a conformal map between finite circle domains Ω_0 and Ω_0'. The Riemann removable singularity theorem allows us to extend f conformally to a neighborhood of any isolated boundary point of Ω_0, so we may henceforth assume that $\partial\Omega_0$ has a finite number of boundary components that are all non-degenerate circles.

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By composing with Möbius transformations, we can can also assume $\infty \in \Omega_0$ and $f(\infty) = \infty \in \Omega_0'$. Then it suffices to show that f is linear. Since we already know f is 1-to-1, it suffices to show f is homomorphic on the whole plane. By Morera's theorem it suffices to show $\int_\gamma f(z)dz = 0$ whenever γ is the boundary curve of a triangle.

We shall deduce the theorem using the Cauchy-Pompeiu formula for smooth functions: if W is a bounded planar domain with smooth boundary, F is a smooth function of W, and $z = x + iy$, then

$$F(w) = \frac{1}{2\pi i}\int_{\partial W} \frac{F(z)}{z-w}dz - \frac{1}{\pi}\int_W \frac{F_{\overline{z}}}{z-w}dxdy.$$

Using Schwarz reflection we may extend f to a conformal map between the domains Ω_n and Ω_n' where $\Omega_n = \mathbb{C} \setminus E_n$ is obtained by recursively reflecting Ω_{n-1} over all it boundary components. The number of components of E_n increases rapidly with n, but we claim that the total area decreases to zero.

To prove this, note that because the components of the original set E do not touch each other, Ω_0 contains pairwise disjoint round annulus, $\{A_j\}$, that each have a boundary circle of Ω as their "inner" boundaries. If we reflect A_j across the corresponding boundary circle of Ω we get an annulus A_j' that lies in one of the component disks D of E and takes up a fixed fraction of the area of D. Moreover, if we apply any Möbius transformation that maps D to itself, the image of the annulus A_j' always takes up a fixed fraction of the the area. One way to prove this, is to observe that if the width of $B_j = A_j \cup A_j'$ is about $r = \mathrm{diam}(D)/M$ there there is "necklace" of about M disjoint disks with tangent points on the boundary circle C and whose closures cover C. Any Möbius transformation sending C to itself maps this necklace to another one with the same number of disks, and at least one of the image disks must be at least size $\simeq r$ or the image necklace could not cover C. The part of this largest image disk inside C thus has area $\gtrsim \mathrm{area}(D)/M^2$ uniformly.

Alternatively, we can note that the image of B_j never contains ∞ (after the first generation these annuli are contained in E_0, which is bounded), and hence the image of B_j belongs to a compact family of annuli that are symmetric with respect to the corresponding boundary circle C of $\partial\Omega_n$. Thus the distance between the boundary of the image annulus and the circle C is bounded below by a fixed multiple of $\mathrm{diam}(C)$. Besides proving that the area of E_{n_1} is a fixed factor less than the area of E_n, this compactness property will be used below to extend f across E with uniform bounds.

When we reflect over a boundary component C of Ω_n to obtain Ω_{n+1}, there is a an annulus with outer boundary C that is a Möbius image of one of the A_j and that misses all the boundary components of Ω_n. Thus the area of all the complementary components of Ω_n is less, by a fixed positive factor, than the area of the complement of Ω_{n-1}. This proves the claim that the area of E_n tends to zero. The same proof applies to show that area of E'_n also tends to zero. This also implies that the diameters of the components of E_n and E'_n tend to zero uniformly with n. Thus any homeomorphic extension of f from Ω_n to $\mathbb{C}$ will converge uniformly to f as $n \nearrow \infty$.

Now fix n and let $\{D_j\}_1^N$ be an enumeration of the disks in E_n, and $\{D'_j\}$ the corresponding enumeration of components of E'_n. Define a diffeomorphism F_n by setting $F = f$ in Ω_n and extending F_n to map D_j to D'_j so that $|F_{\overline{z}}|$ is bounded by $C\text{diam}(D_j)/\text{diam}(D_j)$ for some fixed $C < \infty$ independent of n and j. We can accomplish this condition because f is conformal on an annulus of fixed modulus containing each boundary component of $\partial\Omega_n$, and so has derivatives bounded in terms of the given ratio of diameters because of the Koebe distortion theorem. As noted above, F_n tends to f uniformly on the whole plane.

As above, let γ be the boundary of a triangle T and let W be some smooth domain compactly containing T (say a disk of 10 times the diameter of T centered inside T). By the Cauchy-Pompeiu formula

$$F_n(w) = \frac{1}{2\pi i}\int_{\partial W} \frac{F_n(z)}{z-w}dz - \frac{1}{\pi}\int_{\Omega} \frac{(F_n)_{\overline{z}}}{z-w}dxdy$$

and the first term on the right is holomorphic in W, so its integral around γ is zero. Thus

$$\int_\gamma F_n(w)dw = \int_\gamma \left[\frac{1}{\pi}\int_\Omega \frac{(F_n)_{\overline{z}}}{z-w}dxdy\right]dw = \frac{1}{\pi}\int_\Omega (F_n)_{\overline{z}}\left[\int_\gamma \frac{1}{z-w}dw\right]dxdy.$$

The interchange of integrals is allowed by Fubini's theorem, because $(F_n)_{\overline{z}}$ is bounded (recall F_n is a diffeomorphism) and $1/|z-w|$ is in $L^1(W, dxdy)$ with a bound independent of $w \in \gamma$ (thus $1/|w-z| \in L^1((\Omega, dxdy) \times (\gamma, |dw|))$).

The integral of $1/(z-w)$ around γ is zero of z is outside T and is $2\pi i$ if z is inside T (we don't need to worry about what happens when $z \in \partial T = \gamma$ since this has zero area). Thus

$$\begin{aligned} \left|\int_\gamma F_n(w)dw\right| &= 2\left|\int_T \int_\gamma (F_n)_{\overline{z}}dxdy\right| \\ &= 2\sum_j \int_{T\cap D_j} |(F_n)_{\overline{z}}|dxdy \\ &\lesssim \sum_j \text{area}(D_j \cap T)\frac{\text{diam}(D'_j)}{\text{diam}(D_j)}. \end{aligned}$$

Now apply the Cauchy-Schwarz inequality and $\mathrm{diam}^2(D_j) \simeq \mathrm{area}(D_j)$ to get

$$\begin{aligned} |\int_\gamma F_n(w)dw| &\lesssim \sum_j \mathrm{diam}(D_j')\mathrm{diam}(D_j) \\ &\lesssim \left(\sum_j \mathrm{area}(D_j')\right)^{1/2}\left(\sum_j \mathrm{area}(D_j)\right)^{1/2} \\ &\lesssim \sqrt{\mathrm{area}(E_n)\mathrm{area}(E_n')} \end{aligned}$$

which we proved above tends to zero. Since F converges to f uniformly on compact sets,

$$\int_\gamma f(w)dw = \lim_n \int_\gamma F_n(w)dw = 0.$$

Since this holds for the boundary of every triangle, Morera's theorem implies f is holomorphic, and hence linear, on $\mathbb{C}$. □

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