CONFORMAL REMOVABILITY IS HARD

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Abstract. A planar compact set E is called conformally removable if every homeomorphism of the plane to itself that is conformal off E is conformal everywhere, and hence linear. Characterizing such sets is notoriously difficult and in this paper, we partially explain this by showing that the collection of conformally removable subsets of $S = [0,1]^2$ is not a Borel subset of the space of compact subsets of S with its Hausdorff metric. We give some similar results for other classes of removable sets and pose a number of open problems related to removability and conformal welding, using the language of descriptive set theory.

1. Introduction

Several well known problems in classical complex analysis have remained open for nearly a century and seem intractable. Two of these are to characterize the compact planar sets that are removable for conformal homeomorphisms, and to characterize conformal weldings homeomorphisms among all circle homeomorphisms. The purpose of this note is to partially explain the difficulty of these problems by proving that the collection of conformally removable sets is not a Borel subset of the space of all planar compact sets with the Hausdorff metric. Much of the paper is a survey of the relevant ideas from complex analysis and descriptive set theory, and a recasting of known results into new forms. However, we also present a new result regarding a special class of removable Jordan curves, and we discuss several new open problems at the interface of classical complex analysis and descriptive set theory. We start by recalling some relevant definitions.

A planar compact set E is called removable for a property P if every function with property P on $\Omega = E^c = \mathbb{C} \setminus E$ is the restriction of a function on $\mathbb{C}$ with this property. For example, if P is the property of being a bounded holomorphic function, then E is

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removable iff every bounded holomorphic function on its complement extends to be bounded and holomorphic on the whole plane (and hence is constant by Liouville's theorem). A standard result in many introductory complex variable classes is the Riemann removable singularity theorem, that says single points are removable in this sense. While there are a wide variety of properties that could be considered, most attention has been devoted to the following cases:

• H^∞-removable: $P =$ bounded and holomorphic,
• A-removable: $P = H^\infty$ and extends continuously to E,
• S-removable: $P =$ holomorphic and 1-to-1 (also known as conformal or schlicht),
• CH-removable: $P =$ conformal and extends to a homeomorphism of $\mathbb{C}$.

For any excellent survey of what is known about each of these classes, see Malik Younsi's 2015 paper [52].

The basic problem is to find "geometric" characterizations of removable sets. For example, Xavier Tolsa has given a characterization of H^∞-removable sets in terms of the types of positive measures supported on the set (see Section 2). Alhfors and Beurling [1] gave a characterization of S-removable sets as "NED sets" (negligible sets for extremal distance). On the other hand, although there are various known sufficient conditions and necessary conditions, e.g., [25], [27], [28], there is no simple characterization of A-removable or CH-removable sets. Thus it appears that characterizing these sets is "harder" than characterizing H^∞-removable or S-removable sets. The following is a precise formulation of this idea.

Theorem 1.1. *Let $S = [0,1]^2$ be the unit square in $\mathbb{C}$ and let 2^S denote the hyperspace of S, i.e., the compact metric space consisting of all compact subsets of S with the Hausdorff metric. Within this metric space, the collection of*

(1) *H^∞-removable subsets is a G_δ,*
(2) *S-removable subsets is a G_δ,*
(3) *A-removable subsets is not Borel,*
(4) *CH-removable subsets is not Borel,*

Thus, in some sense, removability for conformal homeomorphisms is distinctly more complicated than for bounded holomorphic functions. It turns out the proof of parts (1) and (2) are relatively elementary, and parts (3) and (4) follow from well known results in descriptive set theory and complex analysis.

Given a closed Jordan curve Γ with bounded complementary component Ω and unbounded component Ω^*, there are conformal maps $f : \mathbb{D} = \{|z| < 1\} \to \Omega$ and $g : \mathbb{D}^* = \{|z| > 1\} \to \Omega^*$. Both these maps extend homeomorphically to the circle $\mathbb{T} = \partial\mathbb{D} = \{|z| = 1\}$, so $h = g^{-1} \circ f$ is a homeomorphism of the circle to itself. Such a map is called a conformal welding. A single curve Γ can give rise to several weldings due to different choices of the conformal maps f and g but all such weldings are related by Möbius transformations of the circle. Similarly, two curves that are Möbius images of each other will have the same set of associated weldings. In fact, this is true for any image of a curve Γ under a homeomorphism of the sphere that is conformal off Γ. (For brevity, we call this a CH-image of Γ.) Thus for conformally removable curves, the curve (modulo Möbius transformations of the 2-sphere) is uniquely determined by its welding (modulo Möbius transformations of the circle).

It is very tempting to claim that a non-removable curve is not uniquely determined by its welding, but this is still open; it is possible that there is some non-removable curve Γ so that any CH-image of Γ is also a Möbius image. Very likely there is no such curve. Indeed, an even stronger conjecture is that any conformally non-removable curve has a CH-image of positive area. Combined with the measurable Riemann mapping theorem, this immediately implies that every non-removable curve has a CH-image that is not a Möbius image. We will say more about these problems in a later section of the paper.

It is known that not all circle homeomorphisms are weldings, e.g., examples are given in [10] and [40]. Thus the map from curves to circle homeomorphisms is not onto. However, weldings form a "large" subset in several senses. For example, conformal weldings are dense in all circle homeomorphisms. This is easy for the uniform metric, since every circle diffeomorphism is a welding, but they are also dense in a much stricter sense: for any $\epsilon > 0$, any circle homeomorphism can be altered on set of length ϵ to become a conformal welding. See Theorem 1 of [10]. Moreover, weldings generate all circle homeomorphisms, i.e., any circle homeomorphism is the composition of two conformal weldings, [46]. It follows from a result of Pugh and Wu that conformal weldings contain a residual set in the space of all circle homeomorphisms. See Section 9. However, it is not known if weldings are a Borel subset of circle homeomorphisms. It follows from general results about Borel sets (to be stated more precisely in Section 3), that if the map from curves to weldings were injective, then conformal weldings would be a Borel subset of circle homeomorphism. Thus the question of whether conformal weldings are a Borel subset is closely linked to understanding the failure of injectivity of this map, and it seems likely that injectivity fails exactly for CH-non-removable curves, creating a strong link between these problems. I expect that the collection of conformally removable Jordan curves is non-Borel, but this seems difficult to prove. As a step in this direction, we will prove the following result.

Theorem 1.2. *As above, let $S = [0,1]^2$ be the unit square in $\mathbb{C}$ and let 2^S denote the hyperspace of S, i.e., the compact metric space consisting of all compact subsets of S with the Hausdorff metric. Within this metric space, the collection of A-removable closed Jordan curves is not Borel.*

This result does not seem to be a reformulation of any known results, and perhaps it forms a model for attacking the case of conformally non-removable curves. The proof will consist of creating a family of Jordan curves, each corresponding to a tree indexed by finite strings of integers, and so that the removable curves correspond exactly to well-founded trees, i.e., those that have no infinite branches.

Given a compact set K, we define the Hausdorff distance between compact subsets K_1, K_2 as

$$d_H(K_1, K_2) = \inf\{\epsilon : K_2 \subset K_1(\epsilon), K_1 \subset K_2(\epsilon)\},$$

where $K_j(\epsilon) = \{z : \mathrm{dist}(z, K_j) < \epsilon\}$ is an ϵ-neighborhood of K_j, $j = 1, 2$. This defines a compact metric space consisting of all compact subsets of K, called the Hausdorff hyperspace of K and denoted 2^K (e.g., see Theorem A.2.2 of [11]). In this note, we mainly deal with three examples of K: the unit interval $I = [0,1] \subset \mathbb{R}$, the unit square $S = [0,1]^2 \subset \mathbb{R}^2 = \mathbb{C}$, or the Riemann sphere $\mathbb{S}$. The collection of Borel sets is the smallest σ-algebra containing the open sets (a σ-algebra is closed under countable unions, under countable intersections and under complements). An F_σ set is a countable union of closed sets; a G_δ is a countable intersection of open sets (this terminology originates with Hausdorff in 1914). Analytic sets (also known as Suslin sets) are continuous images of Borel sets, but they need not be Borel themselves (more about this later). The complement of an analytic set is called co-analytic. The sets in parts (3) and (4) of Theorem 1.1 turn out to be co-analytic complete, a condition we will define in Section 5, and that implies that they are non-Borel in a strong sense.

The removable sets in the first three cases of Theorem 1.1 all form σ-ideals of compact sets, i.e., they are closed under taking compact subsets and under compact countable unions. The subset property is obvious, and the fact that a compact set that is a countable union of compact removable sets is also removable is proven in [52] for each of these three classes. The dichotomy theorem for co-analytic σ-ideals (e.g., Theorem IV.33.3 in [30]) then says these collections must be either G_δ or co-analytic complete in 2^S. Theorem 1.1 indicates which possibility occurs in each case. It is not known whether the CH-removable sets form a σ-ideal; indeed, it is not even known if the union of two overlapping CH-removable sets is CH-removable. If the sets are disjoint, then this is true, but is remains open even if both sets are Jordan arcs sharing a single endpoint.

Although it is a basic theorem of descriptive set theory that every uncountable Polish space X contains analytic and co-analytic sets that are not Borel (see Section 4), it is very interesting to obtain "natural" examples. For example, if $X = C([0,1])$ (continuous functions on $[0,1]$ with the supremum norm) the following subsets of functions are all known to be co-analytic complete, and hence non-Borel:

- everywhere differentiable [38],
- differentiable except on a finite set [47] or countable set [23],
- nowhere differentiable [37],
- everywhere convergent Fourier series [3].

For the space $C([0,1])^{\mathbb{N}}$ of sequences of continuous functions on $[0,1]$ the space CN of everywhere convergent sequences is co-analytic complete, as is the space CN_0 of sequences converging to zero everywhere. See Theorem IV.33.11 of [30] by Kechris. When X is the hyperspace of the unit circle $\mathbb{T}$, we have already mentioned the countable compact sets are co-analytic non-Borel. Other known examples of non-Borel subsets of 2^I are:

- sets of uniqueness [31],
- sets of strict multiplicity [29].

A closed set $E \subset \mathbb{T}$ is a set of uniqueness if any trigonometric series that converges to zero everywhere off E must be the all zeros series. E is a set of strict multiplicity if it supports a measure whose Fourier coefficients tend to zero; the Fourier series of such a measure shows that its support is not a set of uniqueness in a strong way. These particular examples have an intimate connection to the foundations of modern mathematics: Cantor showed that finite sets are sets of uniqueness and the problem of extending this to infinite sets led him to the creation of set theory. For more about this fascinating episode in the history of mathematics, see e.g., [14], [15], [36], [48]. For further "natural" examples of non-Borel sets from analysis and topology, see [6] by Howard Becker.

This note was prompted by email discussions with Guillaume Baverez, in which he proposed a possible characterization of CH-removable Jordan curves in terms of their conformal weldings (see Section 9). I doubted such a concise criterion could be given, and eventually I found a counterexample to his conjecture, but the interchange raised the question of quantifying the difficulty of the problem. This note was written in the hope that gathering the basic facts needed from descriptive set theory might be of interest to fellow complex analysts, and perhaps motivate some of them to attack harder variants of these problems, e.g., those discussed in Sections 8, 9 and 10.

2. H^∞-removability is "easy"

As we shall explain below, identifying removable sets isn't exactly easy in the usual sense, but in terms of descriptive set theory the collection of such sets is pretty simple:

Lemma 2.1. *The collection of H^∞-non-removable subsets of $S = [0,1]^2$ is an F_σ subset of 2^S. The H^∞-removable sets are therefore a G_δ subset.*

Proof. Suppose $E \subset [0,1]^2$ is non-removable for H^∞. Then there is a non-constant, bounded holomorphic function f defined on the complement of E. Near infinity f has a Laurent expansion
$$f(z) = c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \dots$$
and this has at least one non-zero coefficient c_k for some $k \geq 1$. If $c_1 = 0$, the function
$$f_1(z) = z(f(z) - c_0) = \frac{c_2}{z} + \frac{c_3}{z_2} + \dots$$
is also bounded, non constant and holomorphic off E. Continuing in this way, we see that we eventually obtain a bounded holomorphic function on $\Omega = \mathbb{C} \setminus E$ that has non-zero coefficient c_1 in its Laurent expansion.

Let X_n be the collection of non-removable sets in $[0,1]^2$ whose complements support a holomorphic function whose absolute value is bounded by 1 and whose Laurent coefficient satisfies $|c_1| \geq 1/n$. We claim X_n is a closed set in 2^S. Suppose $\{K_n\} \subset X_n$ are compact sets converging to K in the Hausdorff metric. Assume f_n is the holomorphic function on K_n^c attesting to its membership in X_n. Each compact disk D in the complement of K is eventually contained in the complements of the K_n for n large enough, and by a normal families argument, we may extract a subsequence that converges to a holomorphic function f_D on D. Covering K^c by a countable union of such disks and applying a diagonalization argument, we may extract a subsequence converging to a holomorphic function f bounded by 1. Applying the Cauchy integral formula to a fixed circle surrounding $[0,1]^2$ we see that the Laurent coefficients of f_n converge to the Laurent coefficients of f and hence $|c_1(f)| \geq 1/n$. Thus $K \in X_n$. Since every non-removable set is in some X_n, the collection of all non-removable sets is an F_σ in 2^S. $\square$

The proof that S-removable sets form a G_δ is very similar, except that the trick of replacing $f(z)$ by $z(f(z)-c_0)$ to get $|c_1| > 0$ might not give a 1-to-1 map. Instead, we may assume the map conformal off E has an expansion $f(z) = z + c_1/z + c_2/z^2 + \dots$ and that $c_k \neq 0$ for some k. Thus it suffices to prove each member of the countable family $K_{n,m}$ where $|c_k| \geq 1/n$ is closed. Then the proof proceeds just as above.

Of course, just because H^∞-non-removable sets are Borel in 2^S does not mean that it is an easy task to find an elegant characterization of them. Indeed, it is a deep result of Xavier Tolsa that E is non-removable for bounded holomorphic functions if and only if it supports a positive measure μ of linear growth, i.e.,

$$\mu(D(x,r)) \leq Mr, \tag{2.1}$$

(for some $M < \infty$ and all $x \in \mathbb{R}^2$ and $r > 0$) and that has finite Menger curvature in the sense that

$$c^2(\mu) = \int\int\int c^2(x,y,z)d\mu(x)d\mu(y)d\mu(z) < \infty, \tag{2.2}$$

where $c(x,y,z)$ is the reciprocal of the radius of the unique circle passing thorough (x,y,z) (linear growth implies μ^3 gives zero measure to the set were two or more of x,y,z agree).

We know the proof of Lemma 2.1 must break down for A-removable and CH-removable sets. In both cases, we can find non-removable sets contained in the strip $[0,1]\times[0,\frac{1}{n}]$ and with corresponding functions that converge to non-constant functions on $\mathbb{C}\setminus[0,1]$ as $n\to\infty$. For A-removability this is done in [8]; one simply has to take an Jordan arc that has tangents only on a set of zero linear measure. Thus any "flat enough" fractal arc will work. For CH-removability, a similar argument works by approximating a segment by "flat" flexible curves, as constructed in [9] or [10].

3. Analytic sets

A topological space X is called Polish if it is separable (has a countable dense set) and has a compatible metric that makes it complete (Cauchy sequences converge). Standard examples include Euclidean space $\mathbb{R}^n$, the continuous functions on $[0,1]$ with the supremum norm $C([0,1])$, and the collection of compact subsets of a compact set $K\subset\mathbb{R}^n$ with the Hausdorff metric. Another important example is the Baire space $\mathbb{N}^\mathbb{N}$ of infinite sequences of positive integers equipped with the metric given by $d((a_n),(b_n))=e^{-m}$, where $m=\max\{n\geq 0\colon a_k=b_k \text{ for all } 1\leq k\leq n\}$. One can show $\mathbb{N}^\mathbb{N}$ is homeomorphic to the irrational numbers (with the usual topology) although they are different as metric spaces (one is complete and the other is not). Every Polish space is the continuous image of the Baire space (Theorem B.1.2, [11]).

If X is a Polish space, then $A\subset X$ is called analytic if there is another Polish space Y and a Borel set $E\subset X\times Y$ so that A is the projection on E onto A, i.e.,

$$A=\{x\in X: \exists\, y\in Y \text{ such that } (x,y)\in E\}.$$

In a Polish space every open set and every closed set is analytic, and the analytic sets are closed under countable unions and intersections. See [30] or Appendix B of [11]. From this it follows that every Borel set is analytic. However, it is known that any uncountable Polish space contains an analytic set that is not Borel (see Lemma 4.1), and several explicit examples were already mentioned in Section 1.

If $A\subset X$ is analytic, then $A^c=X\setminus A$ is called co-analytic. In descriptive set theory, analytic sets are denoted $\mathbf{\Sigma^1_1}$ and co-analytic sets $\mathbf{\Pi^1_1}$ (using light-faced characters refers to something else). These form the simplest elements of the projective hierarchy of sets, much as closed and open sets are the simplest sets of the Borel hierarchy. Analytic and co-analytic sets can be quite complicated, e.g., although every uncountable analytic set contains a perfect subset, Gödel [24] showed that this question for co-analytic sets is undecidable (similar to his results for the Axiom of Choice and the Continuum Hypothesis). Similarly, all analytic sets are Lebesgue measurable, but proving general projective sets are measurable requires additional axioms, e.g., the assumption that certain "large cardinals" exist. See [49].

There are several equivalent characterizations of analytic sets, including

(1) A is the projection of a closed set in $X \times \mathbb{N}^{\mathbb{N}}$,
(2) A is the continuous image of $\mathbb{N}^{\mathbb{N}}$,
(3) A is a continuous image of a Polish space,
(4) A is the continuous image of a Borel subset of a Polish space,
(5) A is the Borel image of a Borel subset of a Polish space.

In comparison, Borel subsets of a Polish space are characterized by being

(1) a continuous 1-to-1 image of $\mathbb{N}^{\mathbb{N}}$,
(2) a continuous 1-to-1 image of a Borel subset of a Polish space,
(3) a 1-to-1 projection of a closed set in $X \times \mathbb{N}^{\mathbb{N}}$,
(4) both a co-analytic and analytic set (see below).

Analytic sets are also known as Suslin sets in honor of Mikhail Yakovlevich Suslin, who proved that a set is Borel if and only it is both analytic and co-analytic. While a research student of Lusin in 1917, Suslin constructed a Borel set in the plane whose projection on the real axis is not Borel, contradicting a claim in a 1905 paper of Lebesgue, (Cooke [14] refers to this as "one of the most fruitful mistakes in all the history of analysis"). Suslin died of typhus in 1919 at the age of 24, having published just one 4-page paper while alive, and one posthumously with Sierpinski. His work was further developed by Lusin[1], Sierpinski[2] and others, and Suslin's legacy remains very active a century later.

[1] In 1936 Lusin was the victim of a political attack that included charges of taking credit for Suslin's work and publishing too much in Western journals. Lusin survived the incident and was officially rehabilitated in 2012. See [18]. However, Lusin's thesis advisor, Egorov, died in 1931 following a hunger strike in prison after similar attacks.

[2] According to [14] although Sierpinski was technically under arrest in Moscow during World War I as an Austrian citizen, he was allowed to participate in the academic life of Moscow University.

To prove that the conformally non-removable subsets of $S = [0,1]^2$ form an analytic subset of the hyperspace of S, we first record a few simple facts. A mapping is called Borel if the inverse image of every open set is a Borel set.

Lemma 3.1. *For any Borel map $f : X \to Y$ between Polish spaces, the graph of f is a Borel set in $X \times Y$.*

Proof. It suffices to prove the complement of the graph is Borel. Since Y is separable, there is a countable basis $\{B_k\}$ for the topology. Thus given any $x \in X$ and $y \in Y$ so that $y \neq f(x)$ there is a basis element B_k so that $f(x) \in B_k$ and $y \neq B_k$. In other words, (x, y) is contained in the Borel product set $f^{-1}(B_k) \times (Y \setminus B_k) \subset X \times Y$ and this set is disjoint from the whole graph of f. Thus the complement of the graph of f is a countable union of Borel sets, and hence is Borel itself. □

Lemma 3.2. *If A is an analytic subset of 2^K, then the collection of supersets of A is also analytic.*

Proof. Since A is analytic, it is the continuous image of some Polish space, say $A = f(X)$. Define a map $X \times 2^K \to 2^K$ by $(x, E) \mapsto f(x) \cup E$. It is easy to check that taking unions is a continuous map from $2^K \times 2^K \to 2^K$. Since products of Polish spaces are also Polish, we see the union of supersets is a continuous image of a Polish space, hence is analytic. □

Lemma 3.3. *Suppose X is a Polish space. Suppose $K \subset \mathbb{C}$ is compact and that each open $U \subset \mathbb{C}$ is associated to a closed set $X(U) \subset X$ so that $\cap_\alpha X(U_\alpha) = X(\cup_\alpha U_\alpha)$ for any collections of open sets $\{U_\alpha\}$. Then the map Λ from points of X to compact subsets of $K = [0,1]^2$ defined by*

$$\Lambda : x \to K_x = K \setminus \cup\{U : x \in X(U)\},$$

is Borel from X to 2^K.

Proof. Note that if $V \subset W$ are open sets, then $V \cup W = W$, and hence

$$X(V) \supset X(V) \cap X(W) = X(V \cup W) = X(W),$$

so our map has a "reverse monotone" property. For each closed set $E \subset K$ and $\epsilon > 0$ consider the open ball in 2^K

$$B(E, \epsilon) = \{F \subset K : d_H(F, E) < \epsilon\}$$

These form a basis of the topology of the hyperspace, so it suffices to show preimages of such sets are Borel. Each such set is a countable union of closed balls

$$\overline{B}(E,\delta) = \{F \subset K : d_H(F,E) \le \delta\},$$

for some sequence $\delta_n \nearrow \epsilon$. Thus it suffices to show that sets of the form $\Lambda^{-1}(\overline{B}(E,\delta))$ are Borel, i.e., $\{x \in X : d_H(K_x, E) \le \delta)\}$ is a Borel subset of X.

Let $\overline{N}(E,\epsilon) = \{y \in \mathbb{C} : \mathrm{dist}(y,E) \le \delta\}$ and similarly for $\overline{N}(K_x,\delta)$. It is easy to check that the condition $d_H(K_x,E) \le \delta$ holds for some $x \in X$ if and only if x is in the intersection of the sets $Y_1 = \{x : K_x \subset \overline{N}(E,\delta)\}$ and $Y_2 = \{x : E \subset \overline{N}(K_x), \delta)\}$. Hence so if suffices to show both Y_1 and Y_2 are Borel.

First consider Y_1. We claim that $x \in Y_1$ if and only if $x \in X(U)$ where $U = \{z : \mathrm{dist}(z,E) > \delta\}$. Suppose $x \in X(U)$. Then K_x is in the complement of U, and hence every point of K_x is within distance δ of E, i.e., $K_x \subset \overline{N}(E,\delta)$. Hence $x \in Y_1$. Conversely, suppose $x \in Y_1$. Then any point $y \in U$ is strictly more than distance δ from E and so y cannot be in K_x. Therefore y is in one of the open sets (call it U_y) that was subtracted from K in the definition of K_x, and hence $x \in X(U_y)$. Thus $K_x \subset \cap_{y\in U} X(U_y) = X(\cup_{y\in U} U_y)$. Since every point of U is in this union, we have $U \subset \cup_{y\in U} U_y$, so $\cap_{y\in U} X(U_y) \subset X(U)$ by the reverse monotone property. By assumption, $X(U)$ is a closed subset of X, so Y_1 is closed, and hence it is Borel.

Next we consider Y_2. The complement $X \setminus Y_2$ consists of points x so that E contains some point y that is strictly more than distance δ from K_x, i.e., K_x misses some closed disk $D' = \{z : |z \in \mathbb{C} : |z-y| \le \delta\}$ with $r > \delta$. Thus K_x also misses some closed disk $D \subset D'$ that is centered at a rational point of the plane and that has rational radius $> \delta$. For each point $z \in D$, $z \notin K_x$ implies $x \in X(U_z)$ for some open set U_z containing z, hence $x \in \cap_{z\in D} X(U_z) = X(\cup_{z\in D} U_z) = X(V_D)$ where V_D is some open set containing D but disjoint from K_x. For each rational closed disk chosen in this way, the corresponding set $X(V_D)$ is closed. If $x \in X \setminus Y_2$, then it is in one of these closed sets and hence $X \setminus Y_2$ is contained in the union of these countably many closed sets. Conversely, if x is in some $X(V_D)$, then K_x omits D and hence every point of K_x is strictly more than distance δ from some point of E. Thus $X \setminus Y_2 = \cup_D X(V_D)$ is F_σ, and hence Y_2 is also Borel, as desired. □

Next we want to specialize to the case when X is the space of homeomorphisms of the 2-sphere to itself that are holomorphic off $K = [0,1]^2$ and normalized to be

$h(z) = z + O(1/|z|))$ at infinity. The space of homeomorphisms of a compact Polish space (like the 2-sphere) is always a Polish space itself, but in this case we can be more explicit and take the metric $d(f, g) = \sup |f - g| + \sup |f^{-1} - g^{-1}|$, where distances are measures in the spherical metric. It is not completely trivial to find a countable dense subset, but we leave this as an exercise for the reader, with the following hint: subdivide $K = [0, 1]^2$ into a $n \times n$ square grid Γ, and consider homeomorphisms that map each edge of the grid to a polygonal curve with rational vertices. Then "fill in" the squares, using homeomorphism of squares to polygons that preserve arclength on each side up to a multiplicative factor (so definitions on adjacent squares agree).

Lemma 3.4. *The CH-non-removable subsets of $[0, 1]^2$ form an analytic subset of the hyperspace of $[0, 1]^2$. Thus the removable sets are co-analytic.*

Proof. Let X be the space of homeomorphisms of the 2-sphere to itself that are holomorphic off $K = [0, 1]^2$ and normalized to be $h(z) = z + O(1/|z|))$ at infinity. For each open set $U \subset \mathbb{C}$ let $X(U)$ be the elements of X that are holomorphic on U. Since uniform limits of holomorphic functions are holomorphic, this is a closed subset of X. Moreover, if h is holomorphic on each set in a collection $\{U_\alpha\}$ it is holomorphic on the union so $X(\cup_\alpha U_\alpha) = \cap_\alpha X(U_\alpha)$. (All functions in this set may be homomorphic on a strictly larger set, e.g., if the union has removable complement, but this equality still holds, and simply gives an example where $X(V) = X(W)$ even if V is strictly contained in W.)

For each $h \in X$, and let $U_h = \mathbb{C}\backslash K_h$ be the largest open set so that h is holomorphic on some neighborhood of every $z \in U_h$. Lemma 3.3 says that $h \mapsto K_h$ from X to Y is a Borel map, Lemma 3.1 says its graph $\{(h, K_h)\}$ is a Borel set in $X \times Y$, and the projection onto the second coordinate gives an analytic set $A = \cup_{h\in X} K_h$ (projections of Borel sets are analytic). By definition, a compact subset of K is conformally non-removable if and only if it contains a non-empty set in A. Removing a point from an analytic set gives another analytic set, so by Lemma 3.2 the super-sets of non-empty elements of A form another analytic set. Thus conformally non-removable sets are analytic in 2^K. $\square$

Corollary 3.5. *The A-removable subsets of $[0, 1]^2$ are co-analytic in 2^S.*

Proof. This is exactly the same as the proof of Lemma 3.4, except that now we work in the Polish space of all continuous functions on the Riemann sphere that are holomorphic off $[0,1]^2$. (This space is complete with the usual supremum metric, and a countable dense set is not hard to construct.) As before, the map sending each such function to the complement of the set where it is holomorphic is a Borel mapping of this Polish space into 2^S, and the projection of its graph onto the second coordinate gives an analytic subset of 2^S. Taking all supersets of all non-empty projections gives all A-non-removable sets, and shows this collection is analytic. □

4. Analytic non-Borel sets exist

This is another standard result, but we include the simple proof for completeness. We follow the argument in Section 11.5 of [12].

Lemma 4.1. $\mathbb{N}^{\mathbb{N}}$ *contains an analytic set that is not Borel. Thus the complement of this set is co-analytic and not Borel.*

Proof. This is a diagonalization argument. We claim it that suffices to show there is an analytic subset $X \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ so that every analytic subset $A \subset \mathbb{N}^{\mathbb{N}}$ occurs as a slice $A = X_y = \{x \in \mathbb{N}^{\mathbb{N}} : (x,y) \in X\}$ for some y. Given such a set X, then

$$B = \{x \in \mathbb{N}^{\mathbb{N}} : (x,x) \in X\}$$

is the projection of the intersection of X with the (closed) diagonal of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ and hence is the continuous image of an analytic set, and therefore is itself analytic. The complementary set $B^c = \{x \in \mathbb{N}^{\mathbb{N}} : (x,x) \notin X\}$ is automatically co-analytic, and if B^c were also analytic, then it would be equal to a slice X_y of X for some y. In this case,

$$X_y = \{x : (x,y) \in X\} = \{x : (x,x) \notin X\} = B^c.$$

However, assuming either $y \in B$ or $y \in B^c$ both lead to contradictions. Thus B^c can't be analytic, and hence neither B nor B^c is Borel (since Borel sets are closed under complements, and all Borel sets are analytic). Thus we have reduced proving the existence of a non-Borel analytic set to finding an analytic set $X \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ which has every analytic subset of $\mathbb{N}^{\mathbb{N}}$ as a slice.

First we show this is possible for closed slices. If Y is a Polish with a countable basis $\{B_k\}$ for the topology, and if $y \in Y$, then let $S(y) \subset \mathbb{N}$ be the set of all k's with $y \notin B_k$ and let $T(y) \subset \mathbb{N}^{\mathbb{N}}$ be all the sequences with elements in $S(y)$. Then $\{(y, T(y)) \in Y \times \mathbb{N}^{\mathbb{N}}\}$ is a closed set: if $y_n \to y$ and $y_n \notin B_k$ for large n, then $y \notin B_k$, since B_k^c is closed. The second coordinates also converge, since the topology on $\mathbb{N}^{\mathbb{N}}$ agrees with the product topology. If we fix a sequence $(a_k) \in N^{\mathbb{N}}$ as the second coordinate, the first coordinate ranges over $Y \setminus \cup_k B_{a_k}$, so any closed subset of Y can occur as a slice.

Next, to obtain every analytic subset of $\mathbb{N}^{\mathbb{N}}$ as a slice, we apply the previous argument to $Y = \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ to get a closed set $X \subset (\mathbb{N}^{\mathbb{N}})^3 = \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^n$ so that every closed subset of $(\mathbb{N}^{\mathbb{N}})^2$ occurs as a slice of X. Hence every analytic subset of $\mathbb{N}^{\mathbb{N}}$ occurs when we project X onto the first coordinate. Since projections of analytic sets are analytic, projecting X onto the first and third coordinates gives an analytic subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ where the first coordinate ranges over all analytic subsets of $\mathbb{N}^{\mathbb{N}}$, as desired. $\square$

Note that this implies the cardinality of the analytic subsets of a Polish space is at most the cardinality of $\mathbb{N}^{\mathbb{N}}$, i.e., the same as $\mathbb{R}$, the continuum c. Since single points are analytic sets, the analytic subsets of $\mathbb{R}$ have cardinality exactly c. In particular, the collection of all Borel subsets of $\mathbb{R}$ also has cardinality c.

5. Co-analytic complete sets

A co-analytic subset $A \subset X$ of Polish space is called co-analytic complete if for any co-analytic set B of $\mathbb{N}^{\mathbb{N}}$ there is a Borel map $f : \mathbb{N}^{\mathbb{N}} \to X$ so that $f(y) \in A$ iff $y \in B$. Thus membership in any such B can be reduced to checking membership in A. Since Borel pre-images of Borel sets are Borel, and we know that $\mathbb{N}^{\mathbb{N}}$ contains a non-Borel co-analytic set, we can deduce that any complete co-analytic set A must be non-Borel. Thus a simple strategy for proving a co-analytic set $A \subset X$ is non-Borel is to find a Borel map $f : \mathbb{N}^{\mathbb{N}} \to X$ so that its inverse image $f^{-1}(A)$ is a known non-Borel set. A common set to use for $f^{-1}(A)$ is the collection of well-founded trees (which we define next).

Let $\mathbb{N}^*$ be the set of finite sequences of natural numbers (including the empty sequence). A tree T is a subset of $\mathbb{N}^*$ that is closed under removing the final element, i.e., if a finite sequence is in T, so is every initial segment, including the empty one (this labels the root vertex of T). An infinite branch of T is an element of $\mathbb{N}^{\mathbb{N}}$, all of whose finite initial segments belong to T. The set of all infinite branches of T is denoted $[T]$.

A tree is well-founded if it has no infinite branches. Finite trees are obviously well-founded, and the infinite set of finite sequences $(n, n-1, n-2, \dots, 1)$ with $n \in \mathbb{N}$, together with all initial segments of these sequences, form an infinite well-founded tree. See Figure 1.

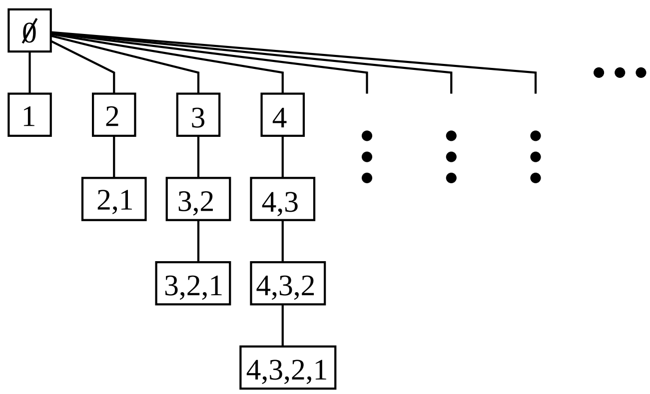

FIGURE 1. An example of a well-founded tree. It is an infinite tree, but has no infinite branches.

Since $\mathbb{N}^*$ is countable and a subset can be identified with its indicator function, any tree can be identified with a point of $2^{\mathbb{N}}$, i.e., the Cantor set of infinite binary sequence. In fact, the set of all trees corresponds to a closed subset of $2^{\mathbb{N}}$ with the usual metric, making it a Cantor set itself. However, the collection of well-founded trees is co-analytic complete, and hence non-Borel, in this space. To prove this, we use the following.

Lemma 5.1. *Every closed set in $\mathbb{N}^{\mathbb{N}}$ is of the form $[T]$ for some tree T. For every analytic set $A \subset \mathbb{N}^{\mathbb{N}}$ there is a tree T so that $a = (a_1, a_2, \dots) \in A$ if and only if there is some $b = (b_1, b_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ so that*

$$W(a, b) = (a_1, b_1, a_2, b_2, \dots) \in [T].$$

Proof. The first part is straightforward. Suppose $K \subset \mathbb{N}^{\mathbb{N}}$ is closed, and let T be the tree of all finite initial segments of all elements in K. Using the definitions and the fact that $\mathbb{N}^{\mathbb{N}}$ is complete, we see that K is closed if and only if the limit $x = (x_1, x_2, \dots)$ of any Cauchy sequence $\{x^n\} = \{(x_1^n, x_2^n, \dots)\}_{n=1}^{\infty}$ in K is also in K. This occurs if and only if for every n, the sequence of initial n-segments $(x_1^m, \dots, x_n^m)_{m=1}^\infty$ equals $(x_1, \dots, x_n)$ for all large enough m. This holds if and only if every initial segment of x is in T, which is equivalent to $x = (x_1, x_2, \dots)$ being an infinite branch of T.

To prove the second part of the lemma, note that $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ is homeomorphic to $\mathbb{N}^\mathbb{N}$ by the 1-1, continuous map that interweaves sequences:

$$W : (a_1, a_2, \dots) \times (b_1, b_2, \dots) \mapsto (a_1, b_1, a_2, b_2, \dots).$$

Thus, if A has the form given in the lemma, then it is the projection onto the first coordinate of the closed set $W^{-1}([T]) \subset \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$, and hence A is analytic. Conversely, if A is analytic, then it is a continuous image $A = f(\mathbb{N}^\mathbb{N})$ and hence A is the projection of the closed set $(f(x), x) \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ (recall that graphs of continuous functions are closed sets). Taking the W-image of this closed graph gives an analytic set in $\mathbb{N}^\mathbb{N}$ whose corresponding tree T satisfies the interweaving condition in the lemma. □

Lemma 5.2. *The well-founded trees are co-analytic complete in* $2^\mathbb{N}$.

Proof. Suppose A is analytic in $\mathbb{N}^\mathbb{N}$. Then there is a tree T so that $a = (a_1, a_2, \dots) \in A$ iff $W(a, b) \in [T]$ for some $b = (b_1, b_2, \dots) \in \mathbb{N}^\mathbb{N}$. If we fix a, then the map $\mathbb{N}^\mathbb{N}$ to itself given by $b \mapsto W(a, b)$ is continuous. Since the inverse image of a closed set is closed, we see that $T(a) = \{b \in \mathbb{N}^\mathbb{N} : W(a, b) \in T\}$ is a closed set in $\mathbb{N}^\mathbb{N}$, and this corresponds to a tree by the previous lemma. We claim the mapping $\mathbb{N}^\mathbb{N} \to 2^{N^*}$ given by $a \mapsto T(a)$ is Borel. If this is true, then it suffices to show that the image of A^c under this map is the set of well-founded trees (since A^c can be any co-analytic set in $\mathbb{N}^\mathbb{N}$). Note that a sequence $a \in A^c$ if and only if $W(a, b) \notin [T]$ for all $b \in \mathbb{N}^\mathbb{N}$. Thus $a \in A^c$ if and only if $T(a)$ is a well-founded tree.

To check that the map $a \mapsto T(a)$ is Borel, we recall $T(a)$ is a closed set of sequences in $\mathbb{N}^\mathbb{N}$ and that a neighborhood of such a set is a countable union of basis elements where we specify a finite initial segment and allow the remaining elements to be free. The inverse image of one such basis element is the collection of all sequences a, so that (1) interweaving the initial elements of a with the specified elements of the basis gives a finite string in T and (2) there is some continuation of the specified elements to an infinite sequence so that interweaving is a branch of T. Thus a is simply the sequence of odd coordinates of all branches of T that pass through the specified vertex, and this is a closed set. Thus inverse images of open sets are countable unions of closed sets, so the mapping is Borel, as desired. □

Theorem 5.3 (Hurewicz, [26])**.** *The compact countable subsets of $I = [0,1]$ are co-analytic complete in 2^I.*

Proof. We have to construct a continuous map from the space of trees into 2^I, so that the image of T is countable if and only if T is well-founded. For each $n = 1, 2, \dots,$ let $A_n = \{x \in [0,1] : \frac{1}{2n+1} \leq |x - \frac{1}{2}| \leq \frac{1}{2n}\}$. Then the A_n are all disjoint and each consist of two compact intervals. For any $S \subset \mathbb{N}$ define

$$A_S = \{\frac{1}{2}\} \bigcup \cup_{n \in S} A_n$$

This is a compact subset of $[0,1]$, and equals $\{1/2\}$ if and only if S is empty.

Suppose we are given a tree T. The root vertex (labeled by the empty string) is associated to $E_0 = I_\emptyset = [0,1]$. In general, suppose E_n is a compact subset of $[0,1]$ whose connected components are a countable number of points labeled by strings of length $< n$, and a countable number of non-trivial closed intervals I_s labeled by strings of length n. All strings that occur as labels of intervals in E_n correspond to labels of vertices in level n of T, and for each such label, 2^n intervals in E_n will have that label. To construct E_{n+1} from E_n, we keep every point component from E_n (and leave the label the same) and replace each interval component J_s labeled by a string s of length n by $L_S(A_S)$, where S is the set of integers that can be appended to S to give a length $n+1$ string in T (i.e., these correspond to the edges leading out of vertex s), where A_S is as above, and where L_S is a linear map from J to J_s. Since each A_n consists of two intervals, each nth generation interval with a given label gives rise to two intervals in the next generation with identical labels. Let $E_T = \cap E_n$. Since the E_n are nested compact sets, this is a non-empty compact subset of $[0,1]$.

If T as an infinite branch, then following this branch through the construction gives a Cantor subset of E, hence E is uncountable. Conversely, if E is uncountable, then $E \cap J_1$ must be uncountable for one of the countably many connected components of E_1. Then $E \cap J_2$ must be uncountable for one of the countably many components of E_2 contained in J_1. Continuing in this way, we obtain nested, non-degenerate components $J_1 \supset J_2 \supset J_3 \supset \dots$ whose labels form an infinite branch of T, so T is not well-founded. □

The endpoints of all the components of E_n in the previous proof are rational numbers. Thus the sets E that arise from well-founded trees are subsets of $\mathbb{Q}$, and we could reformulate the result to say that compact subsets of $\mathbb{Q} \cap I$ are co-analytic complete in 2^I. Theorem 5.3 also gives a rather concrete example of a non-Borel set in $[0, 1]$. Let $\{r_n\}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$ and for $K \in 2^I$ define

$$f(K) = \sum_{r_n \notin K} 3^{-n}.$$

Clearly f is 1-to-1 (since distinct sums of powers of 3 are distinct). The sets $\{K : f(K) > \alpha\}$ are easily checked be open in 2^I, so f is Borel. Thus

$$X = \{f(K) : K \subset \mathbb{Q} \cap [0, 1] \text{ and is compact } \} \subset [0, 1],$$

cannot be Borel.

6. A-removable sets are co-analytic complete

We start with a well known fact from complex analysis.

Lemma 6.1. *If $E \subset [0, 1]$ has positive length, then it is H^∞-non-removable.*

Proof. If E is an interval, then we simply apply the Riemann mapping theorem to conformally map the complement of I (on the sphere) to the unit disk. This gives a non-constant bounded holomorphic function on the complement.

The general case was proven by Ahlfors and Beurling in [1] (or see Section I.6 of Garnett's book [19]). Note that

$$F(z) = \int_E \frac{dz}{z - w} = \int_E \frac{t - x}{(t - x)^2 + y^2} + i \int_E \frac{y}{(t - x)^2 + y^2}$$

is holomorphic on $\Omega = E^c$, has imaginary part in $[-\pi, \pi]$, and Laurent expansion $c_1/z + c_2/z^2 + \dots$ near infinity satisfies $c_1 = \ell(E)$. Thus $G = \exp(F/2)$ takes values in the right half-plane, and $(G - 1)/(G + 1)$ maps Ω holomorphically into the disk and one can compute its leading Laurent coefficient as$c_1 = \ell(E)/4$. $\square$

Extending this result from subsets of $\mathbb{R}$ to subsets of graphs of real Lipschitz functions was a major breakthrough by Alberto Calderon which led to many important developments in in harmonic analysis and geometric measure theory over the last fifty years, including Tolsa's result, discussed in Section 2. For some of the related history, see [17], [41], [50], [51].

The following is stated and proved on page 117 of Carleson's 1951 paper [13]:

Theorem 6.2. *If $E_1, E_2 \subset [0,1]$ are compact and if E_2 has positive Lebesgue measure, then $E = E_1 \times E_2$ is A-removable iff E_1 is countable.*

Proof. For completeness, we recreate Carleson's proof. Let $Q_0 = [0,1]^2$. Suppose f is continuous on the sphere and holomorphic off $E = E_1 \times E_2$. Then f is uniformly continuous on the whole sphere; thus its modulus of continuity

$$\omega(f,\delta) = \max_{|z-w|\le\delta} \max_{z,w\in Q_0} |f(z) - f(w)|$$

tends to zero uniformly with the diameter of Q. Near infinity $f(z) = c_0 + c_1/z + c_2/z^2 + \dots$.

Now fix $\epsilon > 0$. If E_1 is countable, enumerate it as $E_1 = \{x_n\}$ and for each n choose an open interval I_n that contains x_n and so that $\omega(f, 2|I_n|) < \epsilon 2^{-n}$. We can do this since f is continuous at x. Since E_1 is compact, a finite number of these intervals cover E_1. For each I_n used in this finite cover, $I_n \times [0,1]$ can be covered by at most $O(1/|I_n|)$ closed squares of side length $|I_n|$ and disjoint interiors, so and so that each square projects vertically onto I_n. Doing this for each I_n in the covering of E_1 gives a covering of E by squares $\{Q_k\}$.

Let Ω be the union of the squares $\{Q_k\}$. This is finite union of rectangles. Since f is holomorphic on the complement of Ω, the Cauchy integral formula implies

$$\int_{\partial\Omega} f(z)dz = \int_{|z|=R} f(z)dz$$

for large R. The right hand side is $2\pi i c_1$, since near infinity $f(z) = c_0 + c_1/z + c_2/z^2 + \dots$ and the series converges uniformly on large circles. Thus

$$|c_1| = \frac{1}{2\pi}\int_{\partial\Omega} f(z)dz = \frac{1}{2\pi}\left|\sum_k \int_{\partial Q_k} f(z)dz\right| = \frac{1}{2\pi}\left|\sum_k \int_{\partial Q_k} f(z) - f(z_Q)dz\right|$$

where we have used the fact that integrals over common sides of the squares cancel, and constants have dz integral zero around a square. Therefore,

$$\begin{aligned} |c_1| &\leq \frac{1}{2\pi}\sum_Q \int_{\partial Q} |f(z) - f(z_Q)||dz| \\ &= O\left(\sum_Q \mathrm{diam}(Q)\omega(f, \mathrm{diam}(Q))\right) \\ &= O\left(\sum_{n>0} \omega(f, 2\mathrm{diam}(I_n))\right) \\ &= O\left(\epsilon \sum_{n>0} 2^{-n}\right) = O(\epsilon). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we deduce c_1=0. But the same argument applies to $f_1 = z(f(z) - c_0) = c_1 + c_2/z + \dots$ to show $c_2 = 0$. Continuing in this way, we see f is constant, and hence E is A-removable.

Conversely, if E_2 has positive length, then there is a non-constant bounded analytic function f on the complement of iE_2 by Lemma 6.1. If E_1 is uncountable, then it supports a non-atomic, positive, finite measure μ. Then $F(z) = \int f(z+x)d\mu(x)$ is continuous on the sphere and holomorphic off $E = E_1 \times E_2$. We may assume $c_1 \neq 0$ (otherwise recursively replace f by $z(f(z) - c_0)$ until this happens). Then the fact that

$$\frac{1}{z-x} = \frac{1}{z} + (\frac{1}{z-x} - \frac{1}{z}) = \frac{1}{z} + \frac{x}{z(z-x)},$$

implies F also has non-zero Laurent coefficient and hence is non-constant. Therefore E is A-non-removable. □

Corollary 6.3. *The A-removable compact subsets of $S = [0,1]^2$ are co-analytic complete in 2^S, hence not Borel.*

Proof. We already know this set is co-analytic by Corollary 3.5. To prove co-analytic completeness, note that the mapping $E \mapsto E \times [0,1]$ is continuous between the respective Hausdorff metrics and hence reduces the set of countable compact subsets of $[0,1]$ to the set of A-removable sets. Since the former is co-analytic complete, so is the latter. □

7. A-removable Jordan curves are co-analytic complete

A case of particular interest among compact planar sets are the closed Jordan curves. Let $\mathrm{Homeo}(X,Y) \subset C(X,Y)$ denote the 1-to-1 continuous maps of X into Y. It is easy to see that this subset is neither open nor closed in $C(X,Y)$. However, a map $f : \mathbb{T} \to \mathbb{C}$ is 1-to-1 if and only if any two disjoint closed dyadic intervals have disjoint images (an open condition) and hence $\mathrm{Homeo}(\mathbb{T}, \mathbb{C})$ is a G_δ set in $C(\mathbb{T}, \mathbb{C})$.

We can think of closed Jordan curves elements of $\mathrm{Homeo}(\mathbb{T}, \mathbb{C})/\mathrm{Homeo}(\mathbb{T}, \mathbb{T})$, i.e., modulo re-parameterizations. Thus $f, g \in \mathrm{Homeo}(\mathbb{T}, \mathbb{C})$ are equivalent if $f = g \circ \rho$ for some $\rho \in \mathrm{Homeo}(\mathbb{T}, \mathbb{T})$. We can define a metric between equivalence classes as

$$d([f],[g]) = \inf\{\|f - g \circ \rho\|_\infty : \rho \in \mathrm{Homeo}(\mathbb{T}, \mathbb{T})\},$$

although Jordan curves are not complete in this metric. A complete metric on Jordan curves separating 0 and ∞ is described by Pugh and Wu in [43], where they attribute the idea to Thurston (one takes conformal maps of $\mathbb{S} \setminus \mathbb{T}$ to $\mathbb{S} \setminus \Gamma$ normalized to fix 0 and ∞ respectively and that have positive derivative at these points, and then use the supremum metrics between conformal maps).

Theorem 7.1. *The collection of A-removable Jordan curves contained in $S = [0,1]^2$ is co-analytic complete in 2^S.*

Proof. We will construct a continuous map from trees into Jordan curves. As in earlier arguments, it suffices to show that the preimage of the removable curves is precisely the set of well-founded trees.

To simplify some formulas, we work in $[-1,1]^2$ instead of $[0,1]^2$. We start with a map from trees to compact subsets of $[-1,1]$ that maps well-founded trees into countable sets, using a slightly different map than we did in the proof of Theorem 5.3. For $n \subset \mathbb{N}$, we define

$$A_n = \{x : \frac{1}{4} + \frac{1}{2n+1} \le |x| \le \frac{1}{4} + \frac{1}{2n}\},$$

and for $S \subset \mathbb{N}$

$$A_S = \{\pm\frac{1}{4}\} \bigcup \bigcup_{n \in S} A_n \subset [-1,1].$$

This is similar to what we did before, except that now the pairs of intervals A_n converge to two different points $\pm 1/4$, instead of a single point. However, the rest of the construction is the same, and associates to each tree T a compact set E_T that is countable if and only if T is well-founded. Recall that each string s of length n is associated to 2^n intervals which we label I_s^j, $j = 1, \dots 2^n$. We assume these are numbered left to right.

Next we construct a Cantor set $K = \cap_n K_n \subset [-ii]$ of positive Lebesgue measure where each K_n is a union of 2^n equal length closed intervals which we denote $\{K_n^k\}$, $k = 1, \dots, 2^n$. We assume the components are numbered left to right.

Our Jordan curves will be constructed using a template closed set Γ_0, consisting of countable union of polygonal arcs, rectangles and copies of K. The rectangles are all of the form $I \times J$ where I is a component interval of one of the sets E_n defined above, and J is a component of one of the sets K_n (but possibly for different indices n). The template is illustrated in Figure 2.

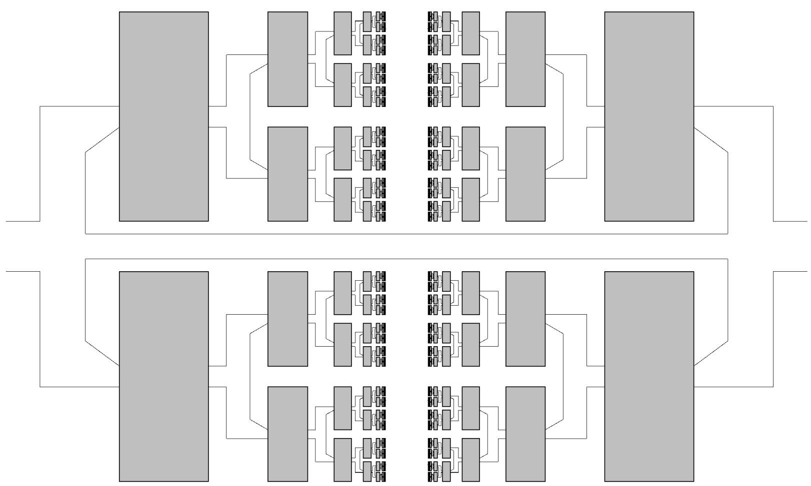

FIGURE 2. The basic template for the construction. Each column of rectangles corresponds to a positive integer.

We attempt to describe Figure 2 in words. Γ_0 has two copies of the Cantor set K, positioned in the vertcal lines $\{x = \pm 1/4\}$ near the center of the picture. There are a countably many rectangles, arranged in vertical columns which accumulate on the two Cantor sets from the left and right respectively. Each positive integer k corresponds to 2^{k+1} rectangles arranged in two columns. The integer 1 corresponds to the two leftmost and two rightmost rectangles in Figure 2. The integer 2 corresponds to the eight rectangles in the two columns adjacent to the first two, and so on. More precisely, the 2^{k+1} rectangles associated to the integer k are the components of $A_k \times K_n$. The set A_k has two components and K_k has 2^k components, giving the correct number of rectangles in the product. Each rectangle is then connected to three other rectangles in the two adjacent columns, and to one other rectangle in the same column, all as shown in Figure 2. (Slightly different arcs are used to connect the outermost rectangles to each other, as shown in Figure 2.)

Given this template, we construct a Jordan curve Γ as an intersection $\Gamma = \cap_n \Gamma_n$ of compact connected sets each consisting of a countable union of rectangles, polygonal arcs and copies of the Cantor set K. The steps of the construction are controlled by the choice of a subtree T of $\mathbb{N}^{\mathbb{N}}$, and is designed so that Γ will be A-removable if and only if T is well-founded.

So suppose T is fixed. The construction always starts with a copy of Γ_0 that has two short polygonal arcs added at the far left and far right, to join the upper and lower halves of the template set, making it connected. These are shown in Figure 2, but are given in several of the following figures.

We will induct over levels of the tree, starting at the root vertex (labeled by the empty string) and at each stage of the construction, we will have a set Γ_n consisting of a countable collection of rectangles joined by polygonal arcs and accumulating on translates of the set K. At the nth stage, each rectangle R is labeled by a n-long string of positive integers that is a label of some vertex v of the tree T. To go from Γ_n to Γ_{n+1},we replace each rectangle R in Γ_n by a rescaled copy of the template (rescaled affinely to exactly fit into R). If vertex v is a leaf of T (i.e., it has no children), then every rectangle R' in the rescaled copy of the template is replaced by a pair of horizontal line segments that connect the vertical sides of R' exactly at the points where arcs of the template connect R' to other rectangles in the template. If v is not a vertex then there is a set of positive integers that when appended to the label of v give labels of its children. For the template rectangles corresponding to these integers we leave the rectangle alone. For the other integers (those that do not correspond to children of v, we replace the corresponding rectangles with horizontal line segments, as above. Doing this for every rectangle in Γ_n gives a closed connected subset $\Gamma_{n+1} \subset \Gamma_n$.

The simplest case is when the tree T has only one vertex (labeled by the empty string). Then every rectangle of the template Γ_0 is replaced by pair of horizontal segments. The result is illustrated in Figure 3. Here, Γ_1 is a closed Jordan curve that is a countable union of polygonal arcs and two copies of the Cantor set K, and is clearly an A-removable set.

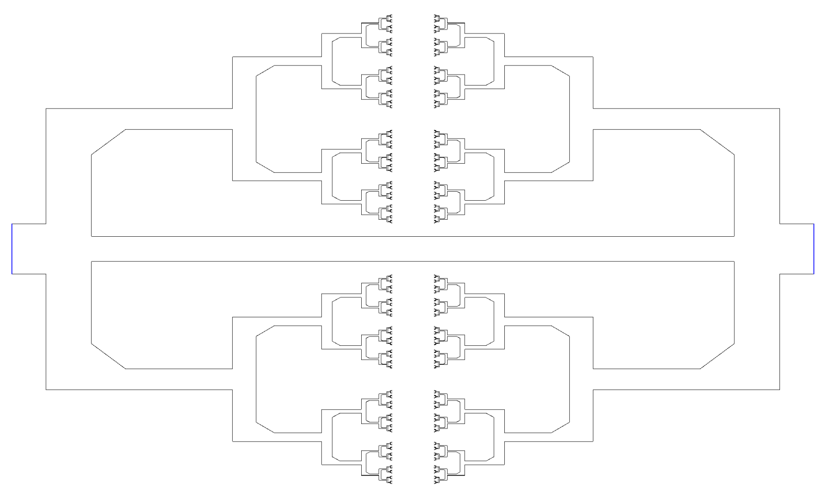

FIGURE 3. The curve corresponding to the 1-vertex tree, labeled by the empty string. This is a countable union of line segments and two linear Cantor sets and hence is A-removable. It is the "simplest" curve in our collection.

The next easiest case is when we have a rooted 2-vertex tree, say with root labeled by the empty string and the single leaf labeled by "1". If we replace the four rectangles in Γ_0 that correspond to the integer "1" with rescaled copies of Γ_0, the result is shown in Figure 4. Any curve that corresponding to a tree that contains the edge connecting the root to vertex "1", will be the a subset of the illustrated set. When T consists only of this one edge, then every rectangle in Figure 4 is replaced by a pair of horizontal edges, giving the closed Jordan curve shown in Figure 5. If the second vertex were labeled "k" instead, the replacements would occur in corresponding columns of the template.

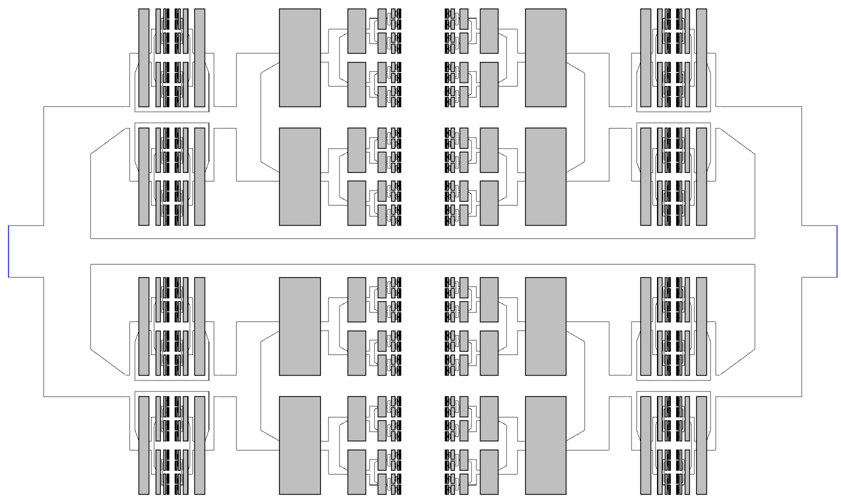

FIGURE 4. The four rectangles corresponding to “1” in the template have been replaced by rescaled copies of the template. Any curve containing the vertices $\{\emptyset, 1\}$ will contain these arcs.

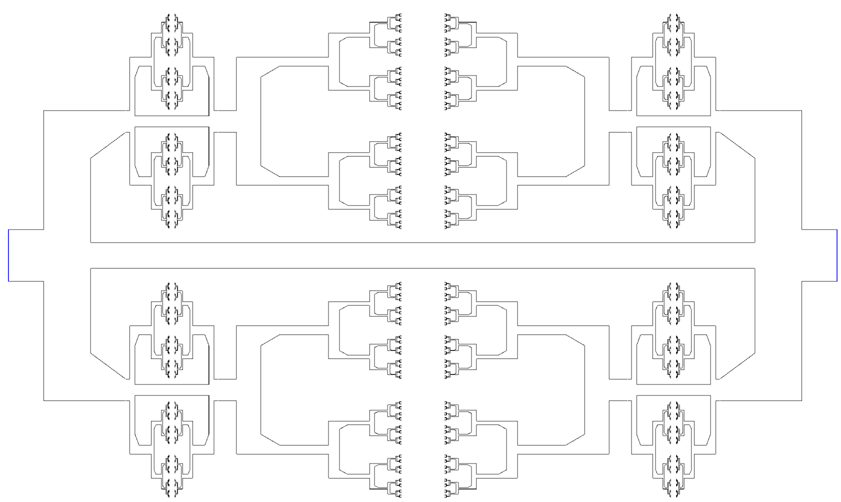

FIGURE 5. The curve corresponding to the tree with vertices $\{\emptyset, 1\}$.

Finally, we have to observe that the resulting curve is A-removable if and only if the associated tree T is well-founded. If T is well-founded, then the final curve is

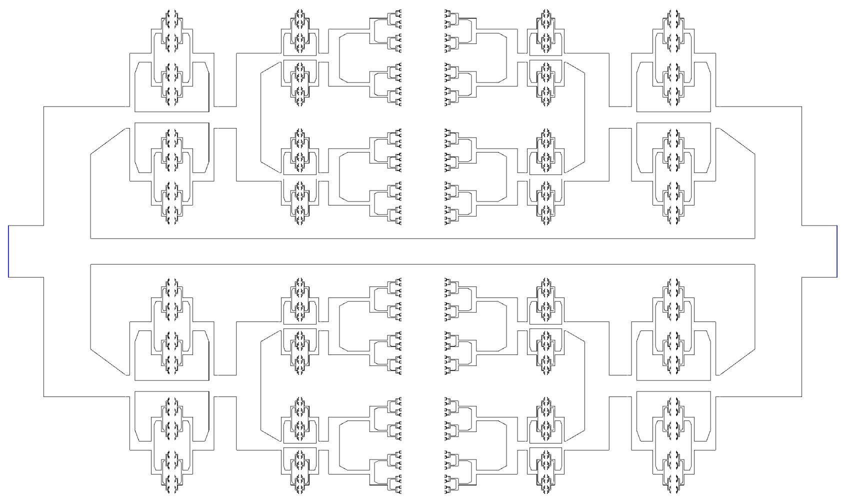

FIGURE 6. The curve corresponding to the tree with vertices $\{\emptyset, 1, 2\}$. There are countably many segment and 10 copies of the linear Cantor set K.

a countable union of line segments and linear Cantor sets and hence is A-removable by one direction of Carleson's theorem. If T has an infinite branch then the curve contains a copy of $E \times K$, where E is a Cantor set depending on the branch, and thus it is non-A-removable by other direction of Carleson's theorem.

The map from trees to curves is continuous from the product topology to the Hausdorff metric because if two trees have the same set of vertices in $[1, \ldots, N]^N$ then the two curves will agree except on a union of rectangles with small diameter (tending uniformly to zero with N) and each contains at least one point inside each of these rectangles; thus the curves are close in the Hausdorff metric. Therefore the set of well-founded trees is the preimage of the set of A-removable curves under a continuous map from trees into the hyperspace of $[-1, 1]$. Hence this collection of A-removable curves is co-analytic complete and, in particular, it is not Borel. $\square$

8. CH-removable sets are co-analytic complete

The following is due to Fred Gehring [21] in 1960. We include a proof for the reader's convenience.

Lemma 8.1. *For compact sets $E \subset [0,1]$, $E \times [0,1]$ is CH-non-removable if and only if E is uncountable.*

Proof. If E is compact and uncountable then it supports positive, finite, non-atomic measure μ. By restricting μ to an appropriate subset set E_0 of zero Lebesgue measure and multiplying by an appropriate constant we may assume μ is singular to Lebesgue measure, is supported in an interval $J = [a,b] \subset [0,1]$, has total mass equal to half the length of J. Fix a constant $c \in [0,1]$ and define $h_c(x) = x$ outside J and

$$h_c(x) = x + c\left(\int_0^x d\mu(t) - \frac{x-a}{2}\right),$$

inside J. It is easy to check this is a homeomorphism that is linear with slope $1 - \frac{c}{2}$ on each component of $J \setminus E_0$. On the other hand, h_c maps E_0 to a set of length $c\ell(J)/2 > 0$. Let $g(y) = \max(0, \frac{1}{2} - |x - \frac{1}{2}|)$ and define

$$F(x,y) = (h_{g(y)}(x), y).$$

See Figure 7. This is a homeomorphism of the plane that is the identity off $J \times [0,1]$, and for any component K of $J \setminus E_0$ F is a skew linear map on $J \times [0, \frac{1}{2}]$ and $J \times [\frac{1}{2}, 1]$ with uniformly bounded dilatation. Thus F is quasiconformal off $E_0 \times [0,1]$. It is not quasiconformal on the whole plane because the zero length set $E_0 \times \{y\}$ is mapped to a set of positive length for each $0 < y < 1$, and thus $E_0 \times [0,1]$ is a set of zero area that is mapped to positive area; this is impossible for quasiconformal maps, see e.g., [2]. Using the measurable Riemann mapping theorem, we can find a quasiconformal mapping φ of the whole plane so that $\varphi \circ F$ is conformal off $E \times [0,1]$ but not quasiconformal everywhere, hence not conformal everywhere. Thus $E \times [0,1]$ is CH-non-removable.

If E is C-non-removable with witness f and if $z_0 \notin E$, then

$$g(z) = f(z) - f(z_0)/(z - z_0)$$

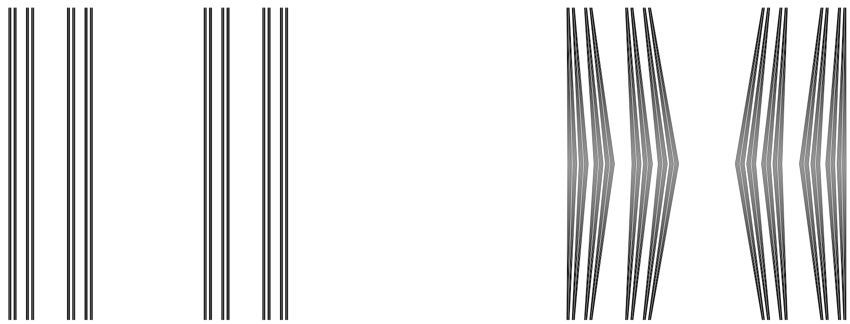

FIGURE 7. If E is a Cantor set there is homeomorphism h of $\mathbb{C}$ that is quasiconformal off $E \times [0,1]$ and maps $E \times [0,1]$ to a set of positive area. This can't happen if E has zero length and h is quasiconformal on the whole plane.

is continuous and bounded on the plane and holomorphic off E, so E is also A-non-removable. If E is compact and countable, then Carleson's Theorem 6.2 show that $E \times [0,1]$ is A-removable, and the previous sentence implies E is CH-removable. $\square$

Corollary 8.2. *CH-removable sets in $S = [0,1]^2$, are co-analytic complete in 2^S.*

The proof is the same as for A-removable sets, except using Gehring's result in place of Carleson's. On the other hand, I have been unable to give an analogous construction to Theorem 7.1:

Question 1. *Are the CH-removable curves co-analytic complete?*

One approach to this question would be to use a theorem of Robert Kaufman [29], who proved that whenever $E \subset [0,1]$ is compact and uncountable, $E \times [0,1]$ contains the graph of a continuous function f defined on E that is a CH-non-removable set. Extending f to be continuous on $\mathbb{R}$ and linear on the complementary intervals of E, gives a graph that is Jordan curve containing a CH-nonremovable graph, and hence is non-removable itself. Thus we might try to prove CH-removable curves are co-analytic complete by mapping trees to graphs of continuous functions (instead of product sets) and using Kaufman's theorem (instead of Carleson's or Gehring's). However, I have not yet seen how to make this work.

The difficulty is that Kaufman's construction starts by choosing a positive, non-atomic measure μ (all points have mass zero) on the uncountable set $E \subset I = [0,1]$.

So it seems that we need a Borel map from trees to probability measures so that the non-well-founded trees are the preimage of the non-atomic measures. However, it is easy to see that the non-atomic measures are co-analytic in $P([0,1])$, so such a map is impossible, since the preimage of a co-analytic set under a Borel map must be co-analytic. The space $P(I)$ of probability measures on $I = [0,1]$ can be made into a Polish space using the dual Lipschitz metric

$$d(\mu,\nu) = \sup_f \left| \int f d\mu - \int f d\nu \right|,$$

where the supremum is over all 1-Lipschitz functions. This metrizes the weak* topology on measures; see Appendix A.3 of [11].

Question 2. *Are CH-removable continuous graphs co-analytic complete in* 2^S*?*

We also recall some questions from the introduction:

Question 3. *Do CH-removable sets form a* σ*-algebra? Is the union of two CH-removable sets removable?*

Recently, Dimitrios Ntalampekos [39] has suggested a characterization of CH-removable sets that is closely related to the characterization of S-removable sets due to Ahlfors and Beurling. Given two continua F_1, F_2 inside an open planar domain Ω, we consider the family Γ of rectifiable paths connecting F_1 to F_2. Given set $E \subset \mathbb{C}$, we can consider the sub-family Γ_E of Γ consisting of paths that miss E. If for every Ω, F_1, F_2 as above, the extremal length of Γ_E is the same as the extremal length of Γ then we say E is negligible for extremal distances, or "NED" for brevity. Ahlfors and Beurling [1] proved that a compact set E is S-removable if and only if it is NED.

Ntalampekos calls a set CNED (countably negligible for extremal distances) if Γ always has the same extremal length as the sub-family consisting of paths that hit E in at most countably many distinct points (we do not care how often each point of E is hit by a path). In [39] he shows that several known families of CH-removable sets are special cases of CNED sets, and conjectures that closed CNED sets are the same as CH-removable sets.

9. How hard is conformal welding?

We recall some definitions from the introduction. If Γ is a closed Jordan curve in the plane, the Riemann mapping theorem gives conformal maps f and g from the inside and outside of the unit circle to the inside and outside of Γ. By Carathéodory's theorem these maps extend to be homeomorphisms of $\mathbb{T}$ to Γ (this was actually first proven by his student Marie Torhorst in her 1918 doctoral dissertation using Carathéodory's theory of prime ends, so perhaps it is more appropriate to call it the Carathéodory-Torhorst theorem; see [45] for some of the history). Thus $h = g^{-1} \circ f : \mathbb{T} \to \mathbb{T}$ is a homeomorphism, and circle homeomorphisms that arise in this way are called conformal weldings.

Not every homeomorphism is a welding. Consider the graph of $\sin(1/x)$ for $x \neq 0$, together with the limiting segment $[-i, i]$. See Figure 8. This is closed set X dividing the plane into two simply connected domains and one can show that the conformal maps form either side of $\mathbb{T}$ to either side of X still define a circle homeomorphism h. However, h cannot correspond to any Jordan curve Γ; if it did, one could conformally map the two sides of X to the two sides of Γ so that the maps agree along the graph of $\sin(1/x)$. Since this smooth curve is removable for conformal homeomorphisms the map extends to be conformal from the complement $[-i, i]$ to the complement of a point. Since the complement of the segment is conformally equivalent to the unit disk, we would get conformal map between the disk and the plane, which would violate Liouville's theorem. Thus this homeomorphism is not a conformal welding.

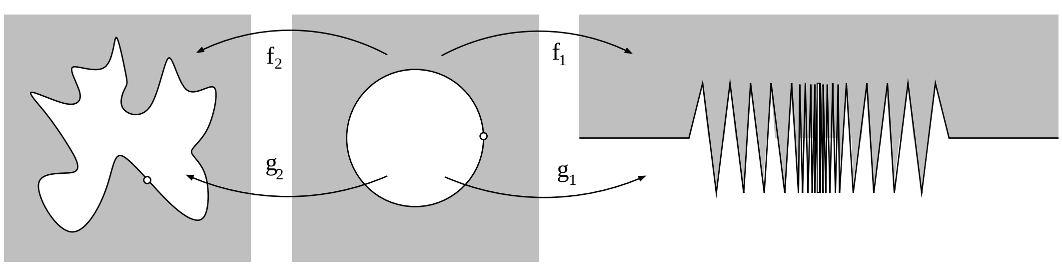

Figure 8. An example of a non-welding homeomorphism. If f_1, g_1 map the two sides of $\mathbb{T}$ to the two sides of a $\sin(1/x)$ curve γ, then $h = g_1^{-1} \circ f_1$ is a homeomorphism, but is not a conformal welding, as explained in the text.

It is a long standing, and apparently very difficult, problem to characterize conformal weldings among circle homeomorphisms. We explained in Section 7 that circle homeomorphisms are a G_δ set in $C(\mathbb{T},\mathbb{T})$, and hence a Polish space.

Question 4. *Are conformal weldings Borel in the space of circle homeomorphisms?*

It's not hard to show

Lemma 9.1. *The set of conformal weldings is analytic in* $\mathrm{Homeo}(\mathbb{T},\mathbb{T})$.

Proof. Briefly, each closed curve gives a conformal welding and the mapping is continuous, so conformal weldings are the continuous image of a Polish space, hence analytic. However, there is actually a family of conformal weldings associated to each curve, so we have to be slightly more careful.

For each 1-to-1 map $\gamma : \mathbb{T} \to \mathbb{C}$ (the parameterization of a closed Jordan curve), let $\mathcal{F}_\gamma \subset C(\mathbb{C},\mathbb{C})$ be the homeomorphisms of the plane that are holomorphic on $\mathbb{D}$ and map $\mathbb{T}$ to $\Gamma = \gamma(\mathbb{T})$, and let $\mathcal{G}_\gamma \subset C(\mathbb{C},\mathbb{C})$ be the homeomorphisms that are analytic outside Γ and map Γ to $\mathbb{T}$. Then, since uniform limits of holomorphic functions are holomorphic, $\{(\gamma,\mathcal{F}_\gamma,\mathcal{G}_\gamma)\}$ is a closed set inside the product $\mathrm{Homeo}(\mathbb{T},\mathbb{C}) \times \mathrm{Homeo}(\mathbb{C},\mathbb{C}) \times \mathrm{Homeo}(\mathbb{C},\mathbb{C})$. Map this closed set into $\mathbb{T}\times\mathbb{T}\times\mathbb{T}$ by $(\gamma, f, g) \mapsto (z, f(\gamma(z)), g(\gamma(x)))$. The projection of the image onto the latter two coordinate is the graph of a conformal welding homeomorphism, and every welding occurs for some choice of (γ, f, g) so the set of conformal weldings is the continuous image of a closed set in a product of Polish spaces, hence is analytic. □

The best known sufficient condition for being a conformal welding (due to Pfluger [42]) is quasisymmetry: h is M-quasisymmetric if

$$\frac{1}{M} \leq \frac{|f(I)|}{|f(J)|} \leq M,$$

whenever I, J are adjacent arcs on $\mathbb{T}$ of the same length, and $|I|$ denotes the length of an arc. For a fixed M, this is clearly a closed condition, so taking $M \to \infty$ along the integers shows quasisymmetric homeomorphisms are a F_σ set inside $\mathrm{Homeo}(\mathbb{T},\mathbb{T})$. Quasisymmetric weldings correspond precisely to closed curves that are quasicircles. i.e., images of the unit circle under quasiconformal maps of the plane. There are numerous characterizations of this class of curves, e.g., any two points $z, w \in \gamma$

are connected by a subarc with diameter bounded by by $O(|z-w|)$. See [2]. It is easy to see M-quasisymmetric maps are nowhere dense, so the set of quasisymmetric homeomorphisms is first category in the space of all circle homeomorphisms.

A more recent (and somewhat more obscure) sufficient condition is that h be log-singular, i.e., that there exist a set $E \subset \mathbb{T}$ of logarithmic capacity zero so that $\mathbb{T} \setminus f(E)$ also has logarithmic capacity zero. In [10] it is proven that h is log-singular if and only if the curve is flexible; this implies that the set of curves corresponding to h is dense in the space of all closed curves with the Hausdorff metric. See [10] for the precise definition. Quasisymmetric and log-singular circle homeomorphisms are easily seen to be disjoint sets (QS homeomorphisms preserve sets of zero logarithmic capacity). Recently, Alex Rodriguez proved that any circle homeomorphism is the composition of two log-singular homeomorphisms, and hence any circle homeomorphism is the composition of two conformal weldings [46]. However, his proof decomposes even "nice" homeomorphisms as the composition of two highly singular maps. Is this necessary? Can a homeomorphisms with some given modulus of continuity be decomposed into welding with similar estimates?

Question 5. *Is any bi-Hölder circle homeomorphism the composition of bi-Hölder welding maps?*

If γ is a closed curve with complementary components Ω_1, Ω_2, we say $x \in \gamma$ is rectifiable accessible from Ω_k, $k = 1, 2$ if it is the endpoint of a rectifiable curve in Ω_k. By a result of Gehring and Hayman ([22] or Exercise III.16 of [20]) this occurs iff the hyperbolic geodesic ending at x has finite Euclidean length. A result of Charles Pugh and Conan Wu [43] says there is a residual set of closed curves γ so that no point on γ is rectifiably accessible from both sides at once. In their terminology, γ in not pierced by any rectifiable arc. By a result of Beurling the set of points that are not rectifiably accessible from Ω_k, $k = 1, 2$ is the image of zero logarithmic capacity set on $\mathbb{T}$ under any conformal map $\mathbb{D} \to \Omega_k$; see [7], Exercise III.23 of [20], or [5]. If follows that every curve in this G_δ set has a conformal welding that is log-singular.

Theorem 9.2. *The collection of CH-non-removable closed curves is residual in the space of all closed Jordan curves.*

Question 6. *Is the set of log-singular homeomorphisms residual in the space of all circle homeomorphisms?*

Question 7. *What is the Borel complexity of the log-singular homeomorphisms?*

It is not hard to show that they are at least analytic: h is log-singular if for every $n \in \mathbb{N}$ there is a compact set such that both E and $h(E^c)$ have logarithmic capacity less than $1/n$ (Lemma 11 of [10]). Thus the log-singular maps are a countable intersection of projections of the Borel sets $\{(h, E) : \text{cap}(E), \text{cap}(h(E^c)) < 1/n\}$ in $\text{Homeo}(\mathbb{T}, \mathbb{T}) \times 2^{\mathbb{T}}$. Can analytic be improved to Borel?

Recall that we say Γ' is a CH-image of Γ if $\Gamma' = f(\Gamma)$ where f is a homeomorphism of the sphere that is conformal off Γ. We will say this is a strict CH-image if f is not a Möbius transformation, and say it is a very strict CH-image if $f(\Gamma)$ is not a Möbius image of Γ. It is tempting to say that a strict image is also very strict, but this is not true. Maxime Fortier Bourque pointed out that the image of Γ under a non-Möbius homeomorphism of the sphere might coincidentally agree with its image under some Möbius map. Moreover, Malik Younsi [53] has constructed a curve with a strict CH-image that agrees with itself. In Younsi's example, there are also very strict CH-images that are not Möbius images, so it is possible that a very strict image exits whenever a strict image does.

Question 8. *Is the map from (equivalence classes of) curves to (equivalence classes of) conformal weldings* 1*-to-*1 *exactly on the CH-removable curves?*

I expect this is true. The following is a stronger version.

Question 9. *Does every CH-non-removable curve have a CH-image of positive area?*

The following special case may be easier.

Question 10. *Does every log-singular circle homeomorphism have an associated curve of positive area?*

Question 11. *Is the map from equivalence classes of curves to equivalence classes of weldings always either* 1*-to-*1 *or uncountable-to-*1*?*

Question 12. *Are CH-images of a curve a connected set in the Hausdorff metric?*

If a curve has positive area, then by scaling a non-zero dilatation supported on the curve, we can produce a 1-parameter family of non-removable curves, none of which is a Möbius image of the others.

Question 13. *Is there a 1-parameter family of zero-area, non-CH-removable curves that is continuous in the Hausdorff metric, so that no element is a Möbius image of any other member of the family?*

Question 14. *The CH-images of a flexible curve are dense in the space of closed Jordan curves, and hence are not a closed set. Is it Borel? (It must be analytic.) Is it connected? Can it be totally disconnected? (Not if the answer to Question 10 is yes.)*

Yet another sufficient condition for being a welding map is given in Guy David's paper [16]. Roughly, it says that h is a welding if it has diffeomorphic extension H to the disk whose dilatation $\mu = H_{\overline{z}}/H_z$ satisfies $|\mu| > 1 - \epsilon$ only on a set of area $O(\exp(-O(1/\epsilon)))$. These are also called trans-quasiconformal homeomorphisms. Are these a Borel subset of all circle homeomorphisms? See also Lehto's solution of certain degenerate Beltrami equations in [35]. Both David's and Lehto's approaches are discussed in [4].

10. What are natural ranks for removable sets?

This section requires greater familiarity with the transfinite ordinals than did earlier sections. Very briefly, each ordinal is a well ordered set (each element has a successor, although some elements have no predecessor). The ordinals themselves are well ordered and there is a first well ordering of an uncountable set, which is denoted ω_1. Every ordinal that becomes before ω_1 is, by definition, the well ordering of some countable set. The continuum hypothesis is the claim that $\omega_1 = c$, where c is the cardinality of $\mathbb{R}$, and is well known to be independent of ZFC.

If X is Polish and $A \subset X$ is co-analytic, then there is always a co-analytic rank on A. This is a function ρ on X that assigns each each point of X to some ordinal $\leq \omega_1$ and such that

(1) $A = \{x \in X : \rho(x) < \omega_1\}$,
(2) $\{(x, y) \in A \times A : \rho(x) < \rho(y)\}$ is co-analytic in $X \times X$,

(3) $\{(x, y) \in A \times A : \rho(x) \leq \rho(y)\}$ is co-analytic in $X \times X$.

Given such a function ρ, one can show that for every countable ordinal α, every set $A_\alpha = \{x \in A : \rho(x) \leq \alpha\}$ is a Borel set and every analytic subset of A is contained in some A_α. Moreover, A is Borel if and only if every co-analytic rank of A is bounded above by some countable ordinal.

The standard example (dating back to Cantor and motivating his invention of transfinite ordinals) involves the derived sets of a compact set in $\mathbb{R}$. Given a compact K, the derived set K' is K with it isolated points removed; this is a compact subset of K, with at most countably many point removed. If K was finite then $K' = \emptyset$, and otherwise we can repeat the process to get the second derived set K''. Continuing, we get a nested sequence of sets that either becomes empty after $n < \infty$ steps (in which case we set $\rho(K) = n$) or we get an infinite, strictly decreasing sequence of nested compact sets whose intersection is a non-empty compact set K^ω. If the derived set of K^ω is empty, then set $\rho(K) = \omega$, and otherwise continue as before. We proceed with this using transfinite induction. If K is countable, then since we remove at least one point at each stage, we must reach the empty set at some countable ordinal, and take this ordinal to be the rank of K. Since we remove only countably many points at each stage, starting with an uncountable sets never gives the empty set at any countable ordinal. For such sets the rank is defined to be ω_1. This defines a rank for the co-analytic set of countable, compact subsets of $[0, 1]$.

In [32] Kechris and Woodin describe a natural rank on the set of everywhere differentiable functions in $C([0, 1])$. See also [33], [34], [44], for comparisons between their rank and other ranks on the same set. A thesis of [32] is that "natural" co-analytic sets should have natural ranks.

Question 15. *What is a natural rank on the space of conformally removable sets?*

For the special case of product sets $E \times [0, 1]$ with E countable, we can just take the usual rank on countable compact sets described above.

Question 16. *Can the derived set rank on* $E \times [0, 1]$ *be extended to a co-analytic rank on all removable sets in* $[0, 1]^2$*?*

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