1

INTERPOLATING SEQUENCES FOR THE DIRICHLET SPACE AND ITS MULTIPLIERS

CHRISTOPHER J. BISHOP

ABSTRACT. We characterize the universal interpolating sequences for the Dirichlet space, $\mathcal{D}$, and reduce the characterization of the usual interpolating sequences to a question about zero sets for $\mathcal{D}$. We answer some questions of Axler by giving sufficient conditions for a sequence to be interpolating for $M(\mathcal{D})$, the multipliers of $\mathcal{D}$, and characterize the interpolating sequences for $\mathcal{D}$ and $M(\mathcal{D})$ which lie on a radius. For example, we show $\{1 - \exp(-a^n)\}$ is interpolating for any $a > 1$ and that this is essentially best possible. We also give some sufficient conditions for interpolation in $\mathcal{D} \cap \mathcal{H}^\infty$ and $M(D_\alpha)$, $0 < \alpha < 1/2$.

---

[1]This is a copy of my 1994 preprint. Since that time I have learned of independent work of Marshall and Sundberg which includes much of this material (and resolves some problems left open here). I have not updated this preprint yet to properly acknowledge their work and so this preprint is not intended for general distribution.

1991 *Mathematics Subject Classification.* 30H05; Secondary 46E20, 46J15, 30C85.

Key words and phrases. Interpolating sequences, multipliers, Dirichlet space, logarithmic capacity, Bessel capacity, extremal length.

This work is partially supported by NSF Grant DMS 92-04092 and an Alfred P. Sloan research fellowship.

1. **Introduction**

The Dirichlet space $\mathcal{D}$ on the unit disk consists of those analytic functions on the disk such that the Dirichlet norm

$$\|f\|_{\mathcal{D}} = (|f(0)|^2 + \iint_{\mathbb{D}} |f'(z)|^2 dxdy)^{1/2},$$

is finite. A function φ on the unit disk is called a multiplier if $\varphi\mathcal{D} \subset \mathcal{D}$. The closed graph theorem implies that multiplication by such a φ is a bounded operator from $\mathcal{D}$ to itself, so the space of multipliers, $M(\mathcal{D})$, becomes a Banach space using the operator norm, $\|\varphi\|_{M(\mathcal{D})}$. Its easy to check that $M(\mathcal{D}) \subset \mathcal{D} \cap \mathcal{H}^\infty(\mathbb{D})$ and that this inclusion is strict.

We call a sequence $\{z_n\} \subset \mathbb{D}$ interpolating for a space of bounded functions X, if for any bounded sequence $\{a_n\}$ of complex numbers, there is an element $f \in X$ such that $f(z_n) = a_n$. Another way to state this is to define the operator $T : X \to \ell^\infty$ by $f \to \{f(z_n)\}$. Then $\{z_n\}$ is interpolating if T is onto. By the closed graph theorem, if this is always possible then there is such an f satisfying $\|f\|_X \leq C\|a_n\|_\infty$ for some C depending only on $\{z_n\}$. For $X = H^\infty(\mathbb{D})$, the bounded analytic functions on $\mathbb{D}$, the interpolating sequences were completely characterized by Carleson and play an important role in function theory on the disk. Carleson [8] showed that a sequence $\{z_n\}$ in interpolating for $H^\infty(\mathbb{D})$ iff

$$\inf_k \prod_{j\neq k} |\frac{z_k - z_j}{1 - \bar{z}_k z_j}| \geq \delta > 0. \tag{1.1}$$

Using some simple facts about Blaschke products and $H^\infty(\mathbb{D})$ this just says $\{z_n\}$ is interpolating iff for each n there is a function $h_n \in H^\infty(\mathbb{D})$

with $h_n(z_m) = \delta_{nm}$ (δ_{nm} is the Kronecker delta) and $\|h_n\|_\infty$ uniformly bounded. Condition (1.1) can be equivalently expressed as two conditions; first that the sequence is separated with respect to the hyperbolic metric ρ, i.e.,

$$\inf_n \rho_n \equiv \inf_n [\inf_{m:m\neq n} \rho(z_n, z_m)] = \delta > 0, \tag{1.2}$$

(i.e., ρ_n is the hyperbolic distance from z_n to the closest distinct point in the sequence) and satisfies Carleson's condition

$$\sum_{z_n \in S(I)} (1 - |z_n|) \leq C|I|, \tag{1.3}$$

for some $C < \infty$ and all arcs I on the circle. Here $S(I)$ denotes the region bounded by I and the hyperbolic geodesic J with the same endpoints as I. The second condition says that if δ_z denotes the unit point mass at z, then $\sum_n (1 - |z_n|)\delta_{z_n}$ is a Carleson measure. These are also characterized by the property that

$$\iint_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C\|f\|_p^p,$$

for every function f in the Hardy space H^p, defined by the norm

$$\|f\|_p^p = \sup_r \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

For spaces, such as H^2 or $\mathcal{D}$, which contain unbounded functions it is more natural to consider a weighted interpolation problem. Following [16], if H is a Hilbert space of analytic functions on the unit disk and K_w is the reproducing kernel for the point $w \in \mathbb{D}$, we define the operator $T : H \to \ell^\infty$ by $f \to \{f(z_n)/K_{z_n}(z_n)\}$. Following the notation in [9] we say $\{z_n\}$ is a universal interpolating sequence for H if $T(H) = \ell^2$. For $H = H^2$, the Hardy space, Shapiro and Shields [16] showed that that

$\{z_n\}$ was a universal interpolating sequence iff it was interpolating for $H^\infty(\mathbb{D})$.

Given $z \in \mathbb{D}\backslash\{0\}$ let I_z denote the arc on the circle centered at $z/|z|$ of length $2(1-|z|)$; we let I_0 be the whole circle. Given a set E on the circle we let $\operatorname{cap}(E)$ denote its logarithmic capacity. If $E = \cup_j I_j$ is a disjoint union of intervals, then we set $S(E) = \cup_j S(I_j)$. For a point z_n let

$$d_n = \rho(0, z_n) \sim \log(1-|z_n|)^{-1},$$

$$\rho_n = \inf_{m \neq n} \rho(z_n, z_m).$$

We say that a positive measure μ is a Carleson measure for $\mathcal{D}$ if

$$\iint_{\mathbb{D}} |f(z)|^2 d\mu(z) \leq C\|f\|_{\mathcal{D}}^2.$$

We shall prove

Theorem 1.1. *$\{z_n\}$ is a universal interpolating sequence for $\mathcal{D}$ (i.e., $T(\mathcal{D}) = \ell^{\in}$) iff there is a $\delta > 0$ such that*

$$\rho_n \geq \delta d_n, \tag{1.4}$$

for all n and

$$\sum_n d_n^{-1} \delta_{z_n} \tag{1.5}$$

is a Carleson measure for $\mathcal{D}$.

The first condition is just the minimum possible separation an interpolating sequence for $\mathcal{D}$ can have. The second condition can be made more concrete because Stegenga [19] has characterized the Carleson measures for $\mathcal{D}$. In this case his result says (1.2) is equivalent to the following: for every finite union of disjoint intervals $E = \cup_j I_j$ on the circle,

$$\sum_{z_n \in S(E)} d_n^{-1} \leq C(\log 2\mathrm{cap}(E)^{-1})^{-1}. \tag{1.6}$$

The problem of determining when $T(H) = \ell^2$ breaks into two questions: when is $T(H) \subset \ell^2$ and when is $\ell^2 \subset T(H)$? In the first case we will say $\{z_n\}$ is a Carleson sequence for H and in the second that it is interpolating for H. As described in the last paragraph, Stegenga has characterized the Carleson sequences for $\mathcal{D}$, but I do not know any geometric characterization of the interpolating sequences. In the case of the Hardy space, H^2, Shapiro and Shields showed that there is no distinction between interpolating sequences and universal interpolating sequences, i.e., between the questions $\ell^2 \subset T(H^2)$ and $\ell^2 = T(H^2)$. However, for the Dirichlet space we shall see later that there are sequences for which $T(\mathcal{D})$ strictly contains ℓ^2 (Lemma 10.1).

Theorem 1.2. *A sequence $\{z_n\}$ is interpolating for $\mathcal{D}$ iff for every n there is a function $h_n \in \mathcal{D}$ such that $\|h_n\|_{\mathcal{D}} \leq C d_n^{-1}$ and $\|h_n\|_\infty \leq C$ and $h_n(z_m) = \delta_{mn}$.*

This is the same as the characterization of H^∞ interpolating sequences, except that we are lacking a good geometric characterization of the zero sets for the Dirichlet class. The condition implies that $\{z_n\}$ satisfies (1.4). It also implies that $(\log \mathrm{cap}(E_n)^{-1})^{-1} \leq C d_n^{-1}$, where $E_n = \cup_{z_m \in S(I_n)} I_m$. I suspect that there is a necessary and sufficient condition in terms of such capacity estimates, but have not been able to verify this. The best result I know is due to Shapiro and Shields [17]: if $\sum_n d_n^{-1} < \infty$ then there is a $f \in \mathcal{D}$ with $f(0) = 1$ and equal zero at the other z_n's. Moreover, for any $h(t) = o(t)$, there is a set of uniqueness $\{z_n\}$ for $\mathcal{D}$ with $\sum_n h(d_n^{-1}) < \infty$. A sufficient condition for interpolation analogous to the Shapiro-Shields results is

Theorem 1.3. *Suppose $\{z_n\}$ is a sequence so that there exists $\delta > 0$, $0 < \eta < 1$ and $C < \infty$ such that*

$$\rho_n \geq \delta d_n \quad \textit{for all } n,$$

$$\sum_n d_n^{-1} \leq C,$$

and

$$\sum_{z_m \in S(K_n)} d_m^{-1} \leq C d_n^{-1},$$

where K_n is the interval centered at $z_n/|z_n|$ of length $(1-|z_n|)^\eta$. Then $\{z_n\}$ is an interpolating sequence for $\mathcal{D}$.

Our proof of this does not depend on Theorem 1.2 or the result of Shapiro and Shields on zero sets, but follows from a direct construction of the interpolating functions. The third condition is similar to Stegenga's condition, but we only need it for certain arcs, rather than all possible finite unions of intervals. Because of this we can construct a sequence which is interpolating for $\mathcal{D}$, but which is not Carleson for $\mathcal{D}$ (see Section 10). If the d_n grow geometrically, then they satisfy the summability condition above. Thus

Corollary 1.4. *Suppose* $|z_1| < |z_2| < \dots$ *and*

$$\sup_n \frac{\log(1-|z_n|)}{\log(1-|z_{n+1}|)} < 1.$$

Then $\{z_n\}$ *is a weighted* ℓ^2 *interpolating sequence for* $\mathcal{D}$ *(i.e.,* $\ell^2 \subset T(\mathcal{D})$*).*

This improves a result of Rosenbaum where 1 is replaced by 1/9. See [15] or [9]. For sequences on a radius, this condition is both necessary and sufficient,

Corollary 1.5. *Suppose* $z_1 < z_2 < \dots \subset [0,1)$*. Then* $\{z_n\}$ *is interpolating for* $\mathcal{D}$ *iff*

$$\sup_n \frac{\log(1-|z_n|)}{\log(1-|z_{n+1}|)} < 1. \tag{1.7}$$

Two equivalent ways of expressing this condition are

$$\inf_n \rho(z_n, z_{n+1}) \geq \delta \rho(0, z_n) \tag{1.8}$$

for some $\delta > 0$ or

$$\inf_n \rho(z_n, z_{n+1}) a^{-n} > 0 \tag{1.9}$$

for some $a > 1$.

In [3] Axler considered interpolating sequences for $M(\mathcal{D})$ and proved that any sequence $\{z_n\}$ in the disk with $|z_n| \to 1$ has a subsequence which is interpolating for $M(\mathcal{D})$. His argument is an application of the Rosenthal-Dor theorem from the theory of Banach spaces and an operator inequality of von Neumann, and does not give any explicit examples. We shall prove that $1 - e^{-e^n}$ is interpolating for $M(\mathcal{D})$ and give an alternate proof of Axler's result based on our example.

Theorem 1.6. *If $\{z_n\}$ is a universal interpolating sequence for $\mathcal{D}$ then it is interpolating for $M(\mathcal{D})$.*

This says that $\{z_n\}$ is interpolating for $M(\mathcal{D})$, i.e., for all $(a_n) \in \ell^\infty$ there is a $\varphi \in M(\mathcal{D})$ such that $\varphi(z_n) = a_n$ and $\|\varphi\|_{M(\mathcal{D})} \leq C\|(a_n)\|_\infty$, if there is a $\delta > 0$ such that

$$\rho_n \geq \delta d_n, \text{ for all } n, \tag{1.10}$$

and $\sum_n d_n^{-1}\delta_{z_n}$ is Carleson for $\mathcal{D}$, i.e., for every finite union of disjoint intervals $E = \cup_j I_j$,

$$\sum_{z_n \in S(E)} d_n^{-1} \leq C(\log 2\mathrm{cap}(E)^{-1})^{-1}. \tag{1.11}$$

The first condition is necessary, but I have not been able to prove the second one is, except in special cases, e.g., if there is $\lambda < 1$ so that $\rho_n < \lambda d_n$ for every n. I would be surprised, however, if these two conditions did not characterize the interpolating sequences for $M(\mathcal{D})$, i.e., $\{z_n\}$ should be interpolating for $M(\mathcal{D})$ iff it is a universal interpolating sequence for $\mathcal{D}$. This is exactly what happens for H^2 and $M(H^2) = H^\infty$.

Given a space of functions A on the disk we can define the Gleason distance

$$d_A(z,w) = \sup\{|f(z) - f(w)| : f \in A, \|f\|_A \leq 1\}.$$

An interpolating sequence (for ℓ^∞) must obviously be separated with respect to this distance. We shall see that

$$d_{M(\mathcal{D})}(w,z) \sim \frac{\rho(z,w)}{\rho(z,0) + \rho(w,0)},$$

which gives (1.10).

If the sequences lies on a radius, say $\{z_n\} \subset [0,1)$, then the separation condition implies the Carleson condition automatically. Thus the interpolating sequences for $M(\mathcal{D})$ on a radius are the same as for $\mathcal{D}$, i.e,

Corollary 1.7. *$z_1 < z_2 < \cdots \subset [0,1)$ is interpolating for $M(\mathcal{D})$ iff*

$$\sup_n \frac{\log(1-|z_n|)}{\log(1-|z_{n+1}|)} < 1.$$

As before this is equivalent to the two conditions (1.8) and (1.9). Thus if $z_1 < z_2 < \dots$ is interpolating for $H^\infty(\mathbb{D})$, then $\rho(0, z_n)$ must grow linearly, while if it is interpolating for $\mathcal{D}$ or $M(\mathcal{D})$ these distances must grow exponentially. If $z_n = 1 - r_n$, then the condition becomes

$$\sup_n \frac{\log r_n}{\log r_{n+1}} < 1.$$

For example, if $r_n = e^{-e^n}$, then

$$\log r_n / \log r_{n+1} = e^{-1} < 1.$$

Thus $1 - e^{-e^n}$ is interpolating for $M(\mathcal{D})$. In [3] Axler asked if $\{1 - n!^{-n!}\}$ was interpolating. Using the previous corollary we simply note

$$\frac{n \log n}{(n+1)\log(n+1)} \leq \frac{1}{n+1} \to 0,$$

as $n \to \infty$, so Axler's sequence has more than enough decay to be an interpolating sequence for $M(\mathcal{D})$.

It is easy to see that Theorem 1.6 implies that if $\{|z_n|\}$ is interpolating for $M(\mathcal{D})$, then so is $\{z_n\}$. This, plus our example of a radial interpolating sequence, gives an alternate proof of Axler's result: if

$|z_n| \to 1$ then $\{z_n\}$ contains an interpolating subsequence for $M(\mathcal{D})$. It also shows that if $E \subset \{|z| = 1\}$ is any closed subset of the unit circle, there is an interpolating sequence for $M(\mathcal{D})$ which accumulates exactly on E.

The similarity between the results for $H^\infty(\mathbb{D})$ and $M(\mathcal{D})$ is not coincidental, since these can be viewed as special cases of a single result, indeed, as the two endpoints of an "interval" of results. Denote by D_α the analytic functions $f = \sum_n a_n z^n$ on the disk for which

$$\|f\|_{D_\alpha} = [\sum_n (1+n^2)^\alpha |a_n|^2]^{1/2} < \infty.$$

Then $D_{1/2} = \mathcal{D}$ is the Dirichlet space and $D_0 = H^2$ is the Hardy space. Since $H^\infty(\mathbb{D})$ is the space of multipliers for H^2 we see that both Theorem 1.6 and Carleson's interpolation theorem are theorems about interpolating sequences for multiplier spaces. For $\alpha < 0$, $M(D_\alpha) = H^\infty(\mathbb{D})$, and for $\alpha > 1/2$, $M(D_\alpha) = D_\alpha$ consists of the analytic functions with (continuous) boundary values in various smoothness classes. Thus $[0, 1/2]$ is the interesting range for bounded interpolation. This also partially explains why the conditions on a $M(\mathcal{D})$ interpolating sequence are so stringent; if α was any larger, then interpolation would be impossible.

A positive measure μ on the disk is called an α-Carleson measure if there is a $C < \infty$ such that $\iint |f|^2 d\mu \le C\|f\|^2_{D_\alpha}$. Both Carleson's theorem and Theorem 1.6 are special cases of

Theorem 1.8. *A sequence $\{z_n\}$ is interpolating for $M(D_\alpha)$, $0 \le \alpha \le 1/2$, if*

$$\inf_n \inf_{m \neq n} d_{M(D_\alpha)}(z_n, z_m) = \delta > 0, \tag{1.12}$$

and

$$\mu = \sum_n B_\alpha(I_{z_n}) \delta_{z_n}, \tag{1.13}$$

is a α-Carleson measure. Here δ_z denotes the point mass at z and B_α denotes the Bessel capacity and satisfies

$$B_\alpha(I_z) \sim \begin{cases} |I|^{1-2\alpha}, & 0 \le \alpha < 1/2 \\ (\log 2|I|^{-1})^{-1}, & \alpha = 1/2 \end{cases}.$$

For $0 < \alpha < 1/2$, this is essentially due to Verbitskii [23]. In this case the separation condition turns out to be the same as for $\alpha = 0$; the sequence must be separated in the hyperbolic metric. We have stated it in the form above merely to include $\alpha = 1/2$ (the Dirichlet space) as well. Stegenga has characterized the Carleson measures for D_α in terms of Bessel capacities, so that the second condition can be made more concrete. Using his criterion, $\{z_n\}$ is interpolating for $M(D_\alpha)$, $0 \le \alpha < 1/2$ if it is separated with respect to the hyperbolic metric and

$$\sum_{z_n \in S(E)} (1 - |z_n|)^{1-2\alpha} \le C B_\alpha(E),$$

for every finite union of disjoint intervals, $E = \cup_j I_j$.

In [19], Stegenga shows that $M(\mathcal{D}) \subset \mathcal{D} \cap \mathcal{H}^\infty(\mathbb{D}) \subset \mathcal{M}(\mathcal{D}_\alpha)$ for all $\alpha < 1/2$, so in some sense, bounded functions in $\mathcal{D}$ are almost multipliers of D. We can consider the problem of interpolating all bounded sequences by elements of $\mathcal{D} \cap \mathcal{H}^\infty(\mathbb{D})$, where we give this space the norm $\|\cdot\|_\infty + \|\cdot\|_{\mathcal{D}}$. Our proof of Theorem 1.6 can be adapted to show

Theorem 1.9. *$\{z_n\}$ is interpolating for $\mathcal{D} \cap \mathcal{H}^\infty(\mathbb{D})$ if $\sum_n \rho_n^{-1} < \infty$.*

Note that the hypothesis implies the sequence is separated in the hyperbolic metric, but does not necessarily imply (1.1). The corresponding result for D_α says that for $0 \leq \alpha < 1/2$, $\{z_n\}$ is interpolating for $D_\alpha \cap H^\infty$ iff it is interpolating for $H^\infty(\mathbb{D})$ and $\sum_n (1-|z_n|)^{1-2\alpha} < \infty$. The proof follows immediately from the proof of Theorem 1.8.

One final comparison is in order. Let QA denote the $H^\infty(\mathbb{D}) \cap \mathrm{VMO}$, the algebra of bounded analytic functions such that

$$\iint_{S(I)} |f'(z)|^2 (1-|z|)\,dxdy = o(|I|),$$

as $|I| \to 0$. Elements of $M(\mathcal{D})$ satisfy

$$\iint_{S(I)} |f'(z)|^2 (1-|z|)\,dxdy = |I|(\log |I|^{-1})^{-1},$$

so $M(\mathcal{D}) \subset$ QA. In fact, we have the stronger containment $\mathcal{D} \cap \mathcal{H}^\infty(\mathbb{D}) \subset$ QA (I thank Sheldon Axler for pointing this out to me). The interpolating sequences for QA have been characterized by Sundberg and Wolff [21] as the "thin" sequences, i.e., those such that

$$\lim_j \Big| \prod_{k \neq j} \frac{z_j - z_k}{1 - \bar{z}_k z_j} \Big| = 1.$$

Since $M(\mathcal{D}) \subset$ QA, interpolating sequences for $M(\mathcal{D})$ must be thin, and in Section 11 we will verify that condition (1.4) by itself implies $\{z_n\}$ is thin. The sequence $\{1 - \exp(-n^2)\}$ is an example which is thin, but not interpolating for $M(\mathcal{D})$.

The paper is organized as follows. In the next section we will review some results about $M(\mathcal{D})$ and the hyperbolic metric. In section 3 we will show how to prove directly that $1 - e^{-e^n}$ is interpolating for $M(\mathcal{D})$ and we give a proof of Axler's result. We will also indicate how to give a proof of Corollary 1.7 along these lines and give a similar proof of Corollary 1.5 in Section 4. In Section 5 we will prove Theorem 1.6. In Sections 6 we prove Theorem 1.9 and in Section 7 we prove Theorem 1.8. We then turn to Theorem 1.3 and use it to deduce Theorem 1.1 in Section 8. In Section 9 we prove Theorem 1.2. In Section 10 we give an example of a sequence with is interpolating for $\mathcal{D}$, but not a universal interpolating sequence. In Section 11, we give some geometric consequences of the separation condition $\rho_n \geq \delta d_n$, and we finish with some remarks in Section 12.

I started thinking about this problem because of Axler's paper [3] and I thank him for providing addition information about the topic. The idea and proof of our example of an explicit interpolating sequence for $M(\mathcal{D})$ follow easily from Smith and Stegenga's paper [18] and the proof of the interpolating criteria for $M(\mathcal{D})$ was motivated by Stegenga's characterization of $M(\mathcal{D})$ in [19].

2. **Preliminaries**

In this section we review a variety of results and definitions involving hyperbolic geometry, interpolating sequences, capacity and the Dirichlet space. The reader may wish to skip this section and only refer to it when necessary.

The hyperbolic metric ρ on the disk is given by

$$\rho(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{2|dz|}{1-|z|^2}$$

where the infimum is over all rectifiable curves γ joining z_1 and z_2 in the disk. If $z_1 = 0$, $z_2 = r > 0$ this equals

$$\rho(0, r) = \log \frac{1+r}{1-r}.$$

The geodesics for the hyperbolic metric are well known to consist of the diameters of the disk and all circular arcs in the disk which are orthogonal to the boundary. For a simply connected domain the hyperbolic metric is defined via the Riemann mapping $\Phi : \mathbb{D} \to \Omega$; $\rho_\Omega(z_1, z_2) = \rho(\Phi^{-1}(z_1), \Phi^{-1}(z_2))$. For a multiply connected domain, we do the same thing but replace the Riemann map by a universal covering map $\Phi : \mathbb{D} \to \Omega$. A few basic properties we need are:

Proposition 2.1. (Schwarz's Lemma) If $f : \Omega_1 \to \Omega_2$ is analytic then

$$\rho_{\Omega_2}(f(z_1), f(z_2)) \le \rho_{\Omega_1}(z_1, z_2),$$

with equality iff f is a covering map.

Proposition 2.2. *Suppose $z_1, z_2 \in \Omega$ with $Re(z_1) = a$ and $Re(z_2) = b$. Assume that for each $a < x < b$ the segment $I_x \subset \{Re(z) = x\}$ separates z_2 from z_1 in Ω and assume that $\theta(x) \equiv |I_x|$ is measurable. Then*

$$\rho_\Omega(z_1, z_2) \ge C + \pi \int_a^b \frac{dx}{\theta(x)}.$$

Proposition 2.3. *Suppose $\Omega = \{(x, y) : |y - m(x)| \le \frac{1}{2}\theta(x), a < x < b\}$. Suppose further that $\{(x, y) : |x - x_1| < \delta, |y - y_1| < \delta\} \subset \Omega$. Let*

$z_1 = x_1 + iy_1$ and $z_1 = b + m(b)$. Then

$$\rho_\Omega(z_1, z_2) \leq C(\delta) + \pi \int_a^b \frac{dx}{\theta(x)} + \pi \int_a^b \frac{m'(x)^2 + \frac{1}{12}\theta'(x)^2}{\theta(x)} dx,$$

where

$$C(\delta) = \frac{\pi}{\delta^2} \int_{x_1-\delta}^{x_1+\delta} \theta(x) dx.$$

The last two propositions are in the text of Garnett and Marshall [12]. The first is usually referred to as the Ahlfors distortion theorem and can be found in several other sources. The second is due to Beurling and can be found in his collected works [6]. We will use the following results on interpolating sequences. The first is a simple exercise; the second a famous result of Carleson [8]. Both are in Chapter VII of [11].

Proposition 2.4. *Suppose X is a Banach space and $\{z_n\}$ is a sequence of linear functionals on X with $\|z_j\| = 1$. Suppose that there is a $C < \infty$ so that for every $(a_n) \in \ell^p$, there is an $x \in X$ such that $\|x\|_X \leq K\|(a_n)\|_p$ and $|z_j(x) - a_j| \leq \frac{1}{2}\|(a_n)\|_p$ for every j. Then $\{z_j\}$ is ℓ^p-interpolating.*

Proposition 2.5. *The sequence $\{z_n\}$ is interpolating for $H^\infty(\mathbb{D})$ iff there are $\delta > 0$ and $C < \infty$ such that*

$$\inf_{n \neq m} \rho(z_n, z_m) \geq \delta,$$

$$\sum_{z_n \in S(I)} (1 - |z_n|) \leq C|I|.$$

An equivalent way of stating these two conditions is the single condition

$$\inf_k \prod_{j \neq k} |\frac{z_k - z_j}{1 - \bar{z}_j z_k}| = \eta > 0.$$

The usual Cauchy estimate implies the first condition is the minimum possible separation for an interpolating sequence. The second says that $\sum_n (1-|z_n|)\delta_{z_n}$ is a Carleson measure for H^2. We could make it look more like the condition in Theorem 1.1 by replacing I by a finite union of intervals E. Since length is additive, this would be an equivalent formulation.

It is easy to check that if $\{z_n\} \subset [0,1)$, then it is interpolating iff $\sup_k (1-z_{k+1})/(1-z_k) = \lambda < 1$ (i.e., we have geometric decay to the boundary). We will use the following version of this fact: if $\Omega = \{(x,y) : |y| < a\}$ is a strip and $z_1 < z_2 < \dots$ are on the real axis, then $\{z_n\}$ is interpolating for $H^\infty(\Omega)$ iff

$$\inf_n (z_{n+1} - z_n) = \delta > 0.$$

This can be proved by via a conformal mapping of the strip to the disk. Interpolating sequences for $H^\infty(\mathbb{D})$ are also weighted interpolating sequences for the Hardy space H^2, i.e.,

Proposition 2.6. [16] *For every $(c_n) \in \ell^2$ there is function $f \in H^2$ with $\|f\|_2 \le C$ and $f(z_n) = c_n(1-|z_n|)^{-1/2}$ iff $\{z_n\}$ is interpolating for $H^\infty(\mathbb{D})$.*

We will also use the following estimates (e.g., Exercise II.5 in [11])

Proposition 2.7. *If $f \in H^2$, then*

$$|f(z)| \le [(1+|z|)/(1-|z|)]^{1/2}\|f\|_2,$$

and

$$|f'(z)| \le C(1-|z|)^{3/2}\|f\|_2.$$

Suppose Ω is a domain with finite area, area(Ω). Ω is called an analytic Poincaré domain if

$$K_\Omega(z_0) = \sup_{f \in H(\Omega)} \frac{\iint_\Omega |f - f(z_0)|^2 dxdy}{\iint_\Omega |f'|^2 dxdy} < \infty.$$

Combining results of Axler and Shields [4] and Stegenga [19] gives

Proposition 2.8. *Suppose Ω has finite area and $\varphi : \mathbb{D} \to \Omega$ is a Riemann mapping. Then the following are equivalent:*

1. *Ω is an analytic Poincaré domain.*
2. *There is a $C < \infty$ so that $\iint_{\mathbb{D}} |\varphi'|^2 |f| dxdy \le C \|f\|_{\mathcal{D}}$ for every $f \in \mathcal{D}$.*
3. *There is a $C < \infty$ so that given any finite collection of disjoint arcs $E = \cup_j I_j$ on the unit circle,*

$$\iint_{S(E)} |\varphi'|^2 dxdy \le C[\log \frac{2}{cap(E)}]^{-1},$$

where $D(E) = \sup_j S(I_j)$ and $S(I)$ is the region in $\mathbb{D}$ bounded by I and the hyperbolic geodesic with the same endpoints as I and $cap(E)$ denotes the logarithmic capacity of E. (We will refer to this as Stegenga's condition.)

We have $\varphi \in M(\mathcal{D})$ if and only if φ is bounded and any of the above hold. The norm of φ as an operator on $\mathcal{D}$ is bounded by a multiple of $\|\varphi\|_\infty + C$, where C is any of the constants appearing in (1)-(3).

Here, $S(I)$ is the region in the disk bounded by I and the hyperbolic geodesic with the same endpoints. For an arc I on the circle, its logarithmic capacity is (approximately) its length, so Stegenga's condition

(5) above in the case of a single arc says there must be a $C < \infty$ such that

$$\iint_{S(I)} |\varphi'|^2 dxdy \leq \frac{C}{\log 2|I|^{-1}}.$$

(The "2" is present just to keep us from dividing from zero in the case that I is the whole circle.) Let J denote the hyperbolic geodesic with the same endpoints as I. Note that $\rho(0, J)$, the hyperbolic distance of J from the origin, equals (up to an additive constant) $\log 2|I|^{-1}$. Furthermore, the integral

$$\iint_{S(I)} |\varphi'|^2 dxdy$$

equals area$(f(S(I)))$. Thus Stegenga's condition for arcs can be rewritten in terms of Ω instead of φ as follows: let $w_0 = \varphi(0)$ and given an hyperbolic geodesic J in Ω not containing w_0, let $S(J)$ be the component of $\Omega \backslash J$ not containing w_0 and let area$(S(J))$ denote its area. Then Stegenga's condition says

$$\rho(\Omega, w_0) \equiv \sup_J \rho_\Omega(w_0, J)\text{area}(S(J)) < \infty.$$

It is not sufficient in Proposition 2.8 to have Stegenga's condition just for arcs (see [19] or [18] for counterexamples). However, it does suffice in special cases. For example,

Proposition 2.9. [18] *Suppose* $\theta : [0, \infty) \to [0, \infty)$ *is a nonincreasing function which is continuous from the right and satisfies* $\theta(x) = \theta_0$ *for* $0 \leq x < \theta_0$. *Define*

$$\Omega = \{(x, y) : |y| < \theta(x), 0 \leq x < \infty\} \cup [(-\theta_0, 0) \times (-\theta_0, \theta_0)].$$

Then $K_\Omega(w_0) \sim \rho(\Omega, w_0)$.

Thus by the earlier proposition, if such a φ satisfies Stegenga's condition for all arcs, it satisfies his condition in general. Define the Bessel capacity B_α for subsets of $\mathbb{R}$ as

$$B_\alpha(E) = \inf\{\|f\|_2^2 : f \geq 0, g_\alpha * f \geq 1 \text{ on } E\},$$

where $g_\alpha(x)$ is the Bessel kernel. For x near 0, $g_\alpha(x) \sim |x|^{\alpha-1} \equiv k_\alpha(x)$ but has exponential decay near ∞. See [2], [13], [20]. Although Bessel capacities are defined for subsets of the line we will frequently refer to the capacity of a subset of the unit circle, using the usual identification of the circle with $[-\pi, \pi]$. Also, since we are only interested in subsets of $[-\pi, \pi]$, the capacities defined by g_α and k_α are equivalent. For an interval I

$$B_\alpha(I) \sim \begin{cases} |I|^{1-2\alpha}, & 0 \leq \alpha < 1/2 \\ (\log 2|I|^{-1})^{-1}, & \alpha = 1/2 \end{cases}.$$

For $\alpha = 1/2$, the Bessel capacity is essentially the logarithmic capacity, i.e.,

$$B_{1/2}(E) \sim [\log 2\text{cap}(E)^{-1}]^{-1}.$$

It is easy to check (using the fact that convolution by Bessel kernels induces isomorphisms of the Dirichlet classes) that

Proposition 2.10. *Suppose E is a closed set in $\{|z| = 1\}$ and $0 \leq \alpha \leq 1/2$. Then*

$$B_\alpha(E) \sim \inf\{\iint_{\mathbb{D}} |\nabla u(z)|^2 (1-|z|)^{1-2\alpha} dxdy : u(0) = 0, u(z) \geq 1, z \in E\}.$$

There is an alternate version of this for the case $\alpha = 1/2$ which we will need. It can be deduced from the previous result, using standard estimates on conformal mappings.

Proposition 2.11. *Suppose $f \in \mathcal{D}$ satisfies $f(0) = 1$ and let $\{z_k\}$ be its zero set. Let $I_k = I_{z_k}$ be the interval centered at $z_k/|z_k|$ of length $2(1 - |z_k|)$ and $E = \cup_k E_k$. Then $B_{1/2}(E) \leq C\|f\|_{\mathcal{D}}$.*

There is a more precise form of this for $\alpha = 1/2$ which we will use, giving the form of the optimal u. It is "standard" and may be found in e.g., [12]

Proposition 2.12. *Let E and F be disjoint closed sets on the unit circle, each consisting of finitely many closed arcs. Suppose there is an arc σ such that $E \subset \sigma$ and $F \cap \sigma = \emptyset$. Then there is a conformal map φ of the disk onto a rectangle, R, with a finite number of horizontal line segments removed, such that $\varphi(E)$, $\varphi(F)$ are the vertical sides of the rectangle. The function $u = Re(\varphi)$ is the function of minimum Dirichlet integral with values 0 on E and 1 on F and $D(u) = \iint |\nabla u|^2 dxdy = area(R)$. Furthermore, u is the unique solution of the mixed boundary value problem with $u = 0$ on E, $u = 1$ on F and $\partial u/\partial n = 0$ on the rest of the boundary.*

We will apply this in the case that F is a quarter-circle and E is contained in the opposite quadrant. Given such an F and E, take φ as in the proposition. Let $R = [0, 1] \times [-\gamma/2, \gamma/2]$ denote the rectangle given by proposition. Then $\gamma \sim B_{1/2}(E)$. Set $\psi = (1 - \gamma)\phi + \gamma$ and

$$f(z) = \exp(A(1 - 1/\psi(z)).$$

An easy calculation shows the Dirichlet integral of f is bounded by $C\gamma$, $\|f\|_\infty = 1$, $|f(0)| \geq 1 - CA\gamma$, $|f| \leq \exp(-C\gamma^{-1})$ on E, and

$$\iint_{S(E)} |f'(z)|^2 dxdy \leq \iint_{\{\gamma<|z|<C\gamma\}\cap R} e^{A(1-1/z)} dxdy \leq C\exp(A(1-C/\gamma)).$$

Thus f has Dirichlet norm about the same as φ, but is small on $S(E)$ (whereas $\mathrm{Re}(\varphi)$ was small on E).

Given any E on the circle we can write it as the union of sets contained in quarter circles, and take each F to be the opposing quarter circle. The product of the corresponding functions satisfies

Lemma 2.13. *Suppose E is a finite union of closed arcs on the circle. Given a $A > 1$, there is an analytic f on the disk so that*

$$\|f\|_\infty \leq 1,$$

$$|f(0)| \geq 1 - CAB_{1/2}(E)$$

$$|f(z)| \leq \exp(-ACB_{1/2}), \quad z \in S(E)$$

$$\iint_{\mathbb{D}} |f'(z)|^2 dxdy \leq CB_{1/2}(E),$$

and

$$\iint_{S(E)} |f'(z)|^2 dxdy \leq C\exp(-A(1 - C/B_{1/2}(E))).$$

Another fact about capacity which we will need is a little less standard.

Lemma 2.14. *Suppose $E = \cup_j I_j$ is a finite union of disjoint intervals and suppose $\beta > 0$. Let $\tilde{I}_j$ be the interval concentric with I but of length I^β. Then if $\tilde{E} = \cup_j \tilde{I}_j$, $B_{1/2}(\tilde{E}) \leq CB_{1/2}(E)$ where C depends only on β.*

Proof. This is obvious for intervals. In general, take a disjoint subcollection of the intervals $\tilde{I}_j$ so that $B_{1/2}(\cup_k \tilde{I}_k) \geq \frac{1}{2} B_{1/2}(\tilde{E})$ (this can be done via a covering lemma, e.g. Lemma I.4.4 of [11] applied to $\{\tilde{I}_j\}$ and the equilibrium measure $\tilde{\mu}$ for $\tilde{E}$). Now define a measure on each I_k by $\mu(I_k) = \tilde{\mu}(\tilde{I}_k)$, and equal to a multiple of Lebesgue measure on each interval. It is easy to see that $\|\mu * \log \frac{1}{|x|}\|_\infty \leq C \frac{1}{\beta} \|\tilde{\mu} * \log \frac{1}{|x|}\|_\infty$, which proves the claim since $\cup I_k \subset E$. $\square$

Recall the definition of the spaces D_α from the introduction. In [19] Stegenga proves

Proposition 2.15. *For $0 \leq \alpha \leq 1/2$, $f \in M(D_\alpha)$ iff $f \in H^\infty(\mathbb{D})$ and $|f'|^2(1-|z|)^{1-2\alpha} dxdy$ is an α-Carleson measure.*

Proposition 2.16. *A positive Borel measure μ on the disk is a α-Carleson measure for $0 \leq \alpha \leq 1/2$ iff there is a constant $C < \infty$ such that*

$$\mu(S(E)) \leq C B_\alpha(E),$$

for every finite, disjoint union of intervals $E = \cup_j I_j$.

Thus $f \in M(D_\alpha)$ iff $f \in H^\infty(\mathbb{D})$ and satisfies Stegenga's condition

$$\iint_{S(E)} |f'|^2 (1-|z|)^{1-2\alpha} dxdy \leq C B_{\alpha,2}(E),$$

for every finite, disjoint union of intervals $E = \cup_j I_j$.

3. Radial interpolating sequences for $M(\mathcal{D})$

We will now use some of the results described in the previous section to construct an interpolating sequence for $M(\mathcal{D})$. Let $\theta(x) = e^{-x}$. We wish to apply Propositions 2.2, 2.3 and 2.9 to the domain

$$\Omega = \{(x,y) : |y| < \frac{1}{2}e^{-x}, 0 \leq x < \infty\} \cup [(-1,0)\times(-1,1)].$$

Let $\varphi : \mathbb{D} \to \Omega$ be the Riemann map which maps 0 to 0 and the interval $[0,1)$ to the ray $[0,\infty)$. By Proposition 2.2,

$$\rho_\Omega(0,r) \geq \pi \int_0^r e^x dx - C_1 = \pi e^r - C_1.$$

To obtain an upper estimate we take $m \equiv 0$ in Proposition 2.3 and note that

$$\int_0^r \frac{\theta'(x)^2}{\theta(x)} = \int_0^r e^{-x} dx \leq 1,$$

and hence,

$$\rho_\Omega(0,r) \leq C + \pi \int_0^r e^x dx + 1 \leq \pi e^r + C_2.$$

Now suppose J is a geodesic with endpoints $(x, \pm\theta(x)/2)$, $x > 1$. It is easy to check that this curve crosses the axis at a point r with $|r-x| \leq Ce^{-x}$, and this is the closest point to 0, so $\rho_\Omega(0,J) \leq \pi e^r + C$. The area of $S(J)$ is $\leq Ce^{-r}$, so

$$\rho_\Omega(0,J)\text{area}(S(J)) \leq C,$$

with a constant independent of J. To check this for other geodesics is a simple exercise (just compare it to an appropriate geodesic of the form above). Thus by Proposition 2.9 φ satisfies Stegenga's condition.

Now suppose $\{z_n\}$ is any sequence in the disk (converging to 1) such that

$$\text{Re}(\varphi(z_1)) < \text{Re}(\varphi(z_2)) < \dots,$$

with

$$\text{Re}(\varphi(z_n)) + \delta < \text{Re}(\varphi(z_{n+1})),$$

for every $n = 1, 2, \dots$ and some fixed $\delta > 0$. Then by our remarks in the previous section, $\{\varphi(z_n)\}$ is an interpolating sequence for $H^\infty(S)$, $S = \{(x, y) : |y| < \pi/2\}$. Thus given any bounded sequence $\{a_n\}$ there is an $F \in H^\infty(S)$ so that $F(\varphi(z_n)) = a_n$. By the Cauchy estimates, $|F'(z)| \le C\|F\|_\infty$ on all of Ω. Therefore $\psi = F \circ \varphi$ satisfies

$$\|\psi\|_\infty \le \|F\|_\infty < \infty,$$

and by the chain rule

$$|\psi'(z)| \le |F'(\varphi(z))||\varphi'(z)| \le C|\varphi'(z)|,$$

so ψ satisfies Stegenga's condition (since φ does). Thus $\psi \in M(\mathcal{D})$ and

$$\psi(z_n) = a_n, \quad n = 1, 2, \dots.$$

Thus $\{z_n\}$ is interpolating for $M(\mathcal{D})$.

If $z_n = 1 - e^{-e^n}$ then $|\rho(0, z_n) - e^n| \le C$. Thus $|\varphi(z_n) - n/\pi| \le Ce^{-n}$, and so the sequence is certainly separated in the sense above. Thus $\{1 - e^{-e^n}\}$ is an interpolating sequence for $M(\mathcal{D})$.

We can also prove Axler's result. Suppose $\{z_n\}$ is a sequence in $\mathbb{D}$ with $|z_n| \to 1$. Choose a subsequence (which we will call $\{z_n^1\}$) so that $z_n^1 \to e^{i\theta_0}$. Then $\varphi(z_n^1 e^{-i\theta_0}) \to \infty$, so we can choose another subsequence $\{z_n^2\}$ so that

$$\mathrm{Re}(\varphi(z_n^2 e^{-\theta_0})) + 1 < \mathrm{Re}(\varphi(z_{n+1}^2 e^{-i\theta_0}))$$

for every $n = 1, 2, \dots$. Then $\{\varphi(z_n^2 e^{-i\theta_0})\}$ is interpolating for $H^\infty(S)$, as above, and so the subsequence $\{z_n^2\}$ is interpolating for $M(\mathcal{D})$.

One can also use the map φ to prove Corollary 1.7. First suppose $\{z_n\} \subset [0, 1)$ satisfies $\rho(z_n, z_{n+1}) \ge \delta\rho(0, z_n)$ and let $x_n = \varphi(z_n)$. By

Proposition 2.2 and Proposition 2.3

$$\delta e^{-x_n} \leq C + \delta\rho(0, z_n) \leq C + \pi \int_{x_n}^{x_{n+1}} e^{-x} dx = C + \pi e^{-x_n} - e^{x_{n+1}},$$

For large enough n this implies $x_{n+1} \geq x_n + \eta$ for some $\eta > 0$ and all n. Thus $\{z_n\}$ is interpolating by the argument given above.

Now suppose $\{z_n\}$ is interpolating for $M(\mathcal{D})$, so then there is an $f \in M(\mathcal{D})$ with $f(z_n) = 0$, $f(z_{n+1}) = 0$, and $\|f\|_{M(\mathcal{D})} \leq C$. Let $W = \mathbb{D} \cap B(1, 2(1-|z_n|)) \setminus B(1, (1-|z_{n+1}|)/2)$. By Stegenga's condition the area of $f(W)$ is at most $C\rho(z_n, 0)^{-1}$. If θ is as in Proposition 2.2, then the Cauchy-Schwarz inequality implies

$$1 \leq (\int_0^1 \theta(x) dx)(\int_0^1 \frac{dx}{\theta(x)}).$$

Thus

$$\rho(z_n, z_{n+1}) \geq C/\text{area}(f(W)) \geq C\rho(0, z_n),$$

as desired. This proves Corollary 1.7

More generally we can show

Lemma 3.1. *If $\{z_n\} \subset \mathbb{D}$ is interpolating for $M(\mathcal{D})$ then there is a $\delta > 0$ such that $\rho_n \geq \delta d_n$ for all n.*

Proof. Suppose $f \in M(\mathcal{D})$, $z \in \mathbb{D}$. Then Stegenga's condition applied to the interval $I = 2I_z$ gives

$$\iint_{S(I)} |f'|^2 dx dy \leq C[\log(1-|z|)^{-1}]^{-1} \sim B_{1/2}(E),$$

which implies

$$|f'(z)| \leq C\|f\|_{M(\mathcal{D})}(1-|z|)^{-1}\rho(0, z)^{-1}.$$

Suppose z_n, z_m are two points and γ the geodesic segment between them. Let $x \in \gamma$ be the point closest to the origin. Since the sequence is interpolating there is a function $f \in M(\mathcal{D})$ with $f(z_n) = 0$, $f(z_m) = 1$ and $\|f\|_{M(\mathcal{D})} \leq C$. Either $|f(x)| \geq 1/2$ or $|f(x)-1| \geq 1/2$, and without loss of generality we assume the former. Then

$$\begin{aligned}
\frac{1}{2} \leq |f(x) - f(z_n)| &\leq \int_\gamma |f'(z)||dz| \\
&\leq -C \int_{1-|z_n|}^{1-|x|} (r \log r)^{-1} dr \\
&\leq C[\log\log(1-|z_n|)^{-1} - \log\log(1-|x|)^{-1}]
\end{aligned}$$

Therefore

$$\log(1-|z_n|)^{-1} \geq e^{1/2C} \log(1-|x|)^{-1},$$

or $\rho(0, z_n) \geq e^{1/2C} \rho(0, x)$. Thus

$$\rho(z_n, z_m) \geq \rho(z_n, x) \geq \rho(z_n, 0) - \rho(x, 0) \geq (1 - e^{-1/2C}) \rho(0, z_n).$$

Thus $\rho_n \geq \delta d_n$, as desired. $\square$

4. **Radial interpolating sequences for $\mathcal{D}$**

One can also use the function φ in the previous section to produce examples of interpolating sequences for the Dirichlet space. Given a sequence $(c_n) \in \ell^2$ we want to find a $f \in \mathcal{D}$ so that $f(z_n) = c_n d_n^{1/2}$. Given a sequence $r_n \in \Omega$ with $r_n + \delta \leq r_{n+1}$, for every $n = 1, 2, \ldots$ and some fixed $\delta > 0$, this becomes $f(r_n) = c_n \sqrt{\pi} e^{r_n/2}$. By the characterization of weighted interpolating sequences for the Hardy space H^2, there is an analytic function F on the strip so that

$$F(r_n) = c_n\sqrt{\exp(-\tilde{\rho}(0, r_n))} = c_n e^{r_n/2},$$

for any $(c_n) \in \ell^2$. Here $\tilde{\rho}$ denotes the hyperbolic metric in the strip. Moreover, Proposition 2.7 implies

$$|F'(z)| \leq C\sqrt{\exp(-\tilde{\rho}(0, |z|))} = Ce^{-|z|},$$

for $\mathrm{Im}(z) \leq \pi/4$. If we set $\Omega_n = \Omega \cap \{z : n \leq \mathrm{Re}(z) \leq n+1\}$, then the area of $F(\Omega) = \cup F(\Omega_n)$ is bounded by $\sum_n e^{-n} e^{n/2} < \infty$. Thus F restricted to Ω is in the Dirichlet class and accomplishes the desired interpolation. Hence for points on a radius the separation condition $\rho(z_n, z_{n+1}) \geq \delta\rho(0, z_n)$ for some $\delta > 0$, suffices for weighted interpolation in the Dirichlet space. This condition is also necessary, as can be seen by the following

Lemma 4.1. *If $f \in \mathcal{D}$, $f(z) = 0$, $f(w) = A$, then $\rho(z, w) \geq C(A/\|f\|_{\mathcal{D}})^2$*

Proof. The area of $f(\mathbb{D})$ is at most $\|f\|_{\mathcal{D}}$, so taking θ is in Proposition 2.2, the Cauchy-Schwarz inequality implies

$$A^2 = (\int_0^A 1 dx)^2 \leq (\int_0^A \theta(x)dx)(\int_0^A \frac{dx}{\theta(x)}) \leq (\|f\|^2_{\mathcal{D}})(\tilde{\rho}(0, A) - C).$$

Thus $\tilde{\rho}(0, A) \geq A^2 - C$, where $\tilde{\rho}$ denotes the hyperbolic metric in $f(\mathbb{D})$. By Schwarz's lemma, this proves the desired inequality, since it is evidently true for small A. □

So suppose $\{z_n\}$ is interpolating for $\mathcal{D}$ and fix some n. Let z_m satisfy $\rho(z_m, z_n) = \rho_n$ and choose $f \in \mathcal{D}$ so that $f(z_m) = 0$, $f(z_n) = d_n^{1/2}$ and $\|f\|_{\mathcal{D}} \leq C$. Then by the lemma $\rho_n \geq C^{-1} d_n$, as desired.

5. Constructing interpolating functions for $M(\mathcal{D})$

In this section we will prove that the condition in Theorem 1.6 implies $\{z_n\}$ is interpolating for $M(\mathcal{D})$. This construction need only be slightly modified to give proofs of Theorem 1.9 and Theorem 1.3.

So suppose $\{z_n\}$ satisfies

$$\rho_n \equiv \inf_{m\neq n} \rho(z_n, z_m) \geq \delta d_n$$

and

$$\sum_{z_n \in S(E)} d_n^{-1} \leq C_1(\log 2\mathrm{cap}(E)^{-1})^{-1}.$$

Let $\epsilon_n = \rho_n^{-1}$. Our second hypothesis clearly implies $\sum_n \epsilon_n < \infty$.

To each point z_n associate the hyperbolic disk D_n centered at z_n with hyperbolic radius $\frac{1}{2}\rho_n$. Note that these disks are all disjoint. The disk D_n is also a disk in the Euclidean metric of center x_n and radius r_n. Note that the hyperbolic distance from z_n to x_n is at least $\frac{1}{4}\rho_n$. Let w_n be the point midway between x_n and z_n in the hyperbolic metric. Let K_n be the arc on the unit circle concentric with $z_n/|z_n|$ and of length $2(1-|x_n|)$. Let f_n be the conformal map of the unit disk onto the region

$$R_n = ([C_2\epsilon_n, 1] \times [-C_2\epsilon_n, C_2\epsilon_n]) \cup D(C_2\epsilon_n, C_2\epsilon_n) \cup D(1, C_2\epsilon_n)$$

which maps z_n to 1 and $-z_n/|z_n|$ to 0. The constant C_2 is chosen so that w_n is mapped to ϵ_n (i.e., $C_2 \sim 1$). This implies the point x_n is mapped to $\exp(-\rho_n/4)$. The map f can be expressed as

$$f(z) = \varphi(C_2\epsilon_n \log \frac{z-1}{z+1}),$$

where φ is the conformal map of the strip $\{|y| < C_2\epsilon_n\}$ onto R_n. It's easy to see that $|\varphi'| \le 1$ (with ~ 1 on $\{0 \le x \le 1\}$), and so

$$|f'(z)| \le \begin{cases} C\epsilon_n, & |z-1| \ge 1-w_n \\ C\epsilon_n|1-z|^{-1}, & 1-|z_n| \le 1-|z| \le 1-w_n \\ C\epsilon_n(1-|z_n|)^{-1}, & |1-z| \le 1-|z_n| \end{cases}.$$

We can easily verify the each of the following properties:

1. $\iint_{D_n} |f_n'(z)|^2 dxdy \le (C_2+2)\epsilon_n \le \text{area}(R_n)$.
2. For any $z \notin S(K_n)$,

$$|f(z)| \le C\exp(-\rho_n/4)\epsilon_n \equiv C_3(n)\epsilon_n.$$

3. Let $D_n(N) = D_\rho(z_n, N)$ denote the hyperbolic disk of radius $N \ge \rho_n$ centered at z_n. Then

$$\iint_{\mathbb{D}\backslash D_n(N)} |f_n'(z)|^2 dxdy \le C\epsilon_n^2 \exp(-(N-\rho_n/2)).$$

Using hyperbolic geometry we can check that the image under $F(\mathbb{D}\backslash D_n(N)) \cap \{k\epsilon_n \le x \le (k+1)\epsilon_n\}$ lies within $\epsilon_n \exp(-(N - \rho_n/2) - k)$ of the boundary of $R_n = f(\mathbb{D})$. Thus summing over k shows the area of the image is less than $\epsilon_n^2 \exp(-N+\rho_n/2)$, as claimed.

Order the points so that $|z_1| \le |z_2| \le \dots$. Given a sequence $\{a_n\}$ with $\|(a_n)\|_\infty \le 1$, we will construct a sequence of functions of the form

$$F_n(z) = \sum_{k=1}^{n} b_n f_n(z).$$

We will prove that these functions satisfy

1. $|F_n(z_k) - a_k| \le \sum_{j=k}^n C_3(j)\epsilon_j, k = 1, \dots, n-1$
2. $\|F_n\|_\infty \le C + \|F_{n-1}\|_\infty + CC_2\epsilon_n$.

3. $|F_n'(z)|^2 dxdy$ is a 1/2-Carleson measure with estimates independent of n.

Let $F = \lim_n F_n$. Since $\sum_n \epsilon_n < \infty$ the third condition implies F is bounded and the last condition implies $F \in M(\mathcal{D})$. Note that for d_n large, $C_3(n)$ is small, so by omitting a bounded number of points which lie within a bounded distance of the origin, we get

$$|F(z_n) - a_n| \leq \sum_n C_3(n)\epsilon_n \leq 1/2.$$

Thus the remaining points $\{z_n\}$ are interpolating by Proposition 2.4.

If $\{w_1, \dots w_N\}$ are the omitted points, let p be a polynomial interpolating the correct values at these points and B the Blaschke product with these zeros. Then B is bounded away from zero on the rest of the sequence so there is a function $f \in M(\mathcal{D})$ with $f(z_n) = a_n/B(z_n)$, by the argument above. Thus $p + Bf \in M(\mathcal{D})$ interpolates correctly on the whole sequence. Therefore $\{z_n\}$ is interpolating for $M(\mathcal{D})$.

Let $F_1(z) = a_1 f_1(x)$. Then $F(z_1) = a_1$ and it clearly satisfies all the desired conditions. In general, suppose we have already defined F_{n-1}. Let $b_n = a_n - F_{n-1}(z_n)$ and set $F_n = F_{n-1} + b_n f_n$. Then trivially, $F_n(z_n) = a_n$. If $k < n$ then $|z_k| \leq |z_n|$ and the disks D_k, D_n are disjoint, so $z_k \notin S(K_n)$. Thus by the remark above, $|f_n(z_k)| \leq \epsilon_n$. Thus

$$|F_n(z_k) - a_k| \leq \sum_{j=k+1}^{n} C_3(j)\epsilon_j, \quad k = 1, \dots, n-1.$$

Similarly,

$$|F_n(z)| \leq |F_{n-1}(z)| + C_3(n)\epsilon_n, \quad z \notin S(K_n).$$

On $S(K_n)$, F_n takes values within ϵ_n of the convex hull of the union of $F_{n-1}(S(K_n))$ and $D(a_n, C_2\epsilon_n)$. Therefore

$$\|F_n\|_\infty \le 1 + \|F_{n-1}\|_\infty + CC_2\epsilon_n,$$

as desired.

This argument also shows something that we will need later, namely,

$$|F_j(z_n)| \le |F_{j-1}(z_n)| + \begin{cases} C|a_k|, & z_n \in S(K_j) \\ C\epsilon_n|a_n|, & z \notin S(K_j) \end{cases}.$$

Thus

$$|F_{n-1}(z_n)| \le C \sum_{k: z_n \in S(K_k)} |a_k| + \sum_{k<n} \epsilon_n |a_n|. \tag{5.1}$$

Moreover, this holds even if the a_n's are not uniformly bounded. The argument using convexity actually shows something stronger: for each n there is a non-negative sequence $\{p_{nk}\}$ with $\sum_k p_{nk} = 1$ so that

$$|F_{n-1}(z_n)| \le C \sum_{k: z_n \in S(K_k)} p_{nk}|a_k| + \sum_{k<n} \epsilon_n |a_n|. \tag{5.2}$$

However, we will not use this stronger estimate in the current paper.

Finally, we must verify the Carleson measure condition. Suppose we are given a finite union of disjoint intervals $E = E_0 = \cup_j I_j^0$. Choose $\beta > 0$ (depending on c_0) so that $\rho(z, S(I)) \le c_0\rho(0, z)$ implies $z \in S(\tilde{I})$ where $\tilde{I}$ is the interval concentric with I, but of length $|I|^\beta$. Let $E_1 = \cup_j \tilde{I}_j^0$. Write E_1 as a union of disjoint closed intervals $\cup I_j^1$ and let $E_2 = \cup \tilde{I}_j^2$. Continue in this way creating sets $E \subset E_1 \subset \dots$. By Lemma 2.14 we have $B_{1/2}(E_n) \le A^n B_{1/2}(E)$ for some A depending only on c_0. Partition the integers in collections $\mathcal{C}_\|$ by putting $n \in \mathcal{C}_\|$ if $z_n \in S(E_k)$ but $z_n \notin S(E_{k-1})$. If $n \in \mathcal{C}_\|$, then $\rho(z_n, S(E)) \ge a^k$ for some $a > 1$. Let χ_{D_n} denote the characteristic function of D_n and $\chi_{D_n^c}$ denote the characteristic function of $\mathbb{D}\backslash D_n$. Then

$$|F'| \leq \sum_n |b_n f_n'| \chi_{D_n} + \sum_n |b_n f_n'| \chi_{D_n^c}.$$

Since the disks $\{D_n\}$ are disjoint,

$$\begin{aligned} \iint_{S(E)} |F'(z)|^2 dxdy &\leq C \sum_n \iint_{D_n \cap S(E)} |f_n'(z)|^2 dxdy \\ &\quad + \iint_{S(E)} |\sum_{k>0} \sum_{n \in \mathcal{C}_{\|}} f_n'(z) \chi_{D_n^c}(z)|^2 dxdy. \end{aligned}$$

The first term is bounded by

$$\sum_{n \in \mathcal{C}_\infty} \iint_{D_n} |f_n'(z)|^2 dxdy \leq \sum_{n \in \mathcal{C}_\infty} \epsilon_n \leq C B_{1/2}(E).$$

The second term is smaller and can be handled by Minkowski's inequality

$$\begin{aligned} (\int_{S(E)} |\sum_{k>0} \sum_{n \in \mathcal{C}_{\|}} f_n'(z) \chi_{D_n^c}(z)|^2 dxdy)^{1/2} &\leq C \sum_k \sum_{n \in \mathcal{C}_{\|}} (\iint_{S(E)} |f_n'(z)|^2 \chi_{D_n^c}(z) dxdy)^{1/2} \\ &\leq C \sum_k \sum_{n \in \mathcal{C}_{\|}} (\iint_{\mathbb{D}\backslash D_n(a^k)} |f_n'(z)|^2 dxdy)^{1/2} \\ &\leq C \sum_k \sum_{n \in \mathcal{C}_{\|}} \epsilon_n e^{-a^k/2} \\ &\leq C \sum_k e^{-a^k/2} B_{1/2}(E_k) \\ &\leq C \sum_k e^{-a^k} A^k B_{1/2}(E) \\ &\leq C B_{1/2}(E). \end{aligned}$$

This proves that $|F'|^2 dxdy$ is a Carleson measure for $\mathcal{D}$, and completes the proof that $F \in M(\mathcal{D})$ and hence the proof of sufficiency in Theorem 1.6.

6. Interpolation in $\mathcal{D} \cap \mathcal{H}^\infty(\mathbb{D})$

To prove Theorem 1.9, we follow the proof of the previous section. Define the functions $\{f_n\}$ as before and set $F_n(z) = F_{n-1}(z) + b_n f_n(z)$ where $b_n = a_n - F_{n-1}(z_n)$. The proof that the functions $\{F_n\}$ are uniformly bounded is the same as before. The proof that

$$|F(z_n) - a_n| \leq 1/2,$$

also still works, as long as $\inf_n \rho_n$ is sufficiently large, i.e., the points are sufficiently far apart. Since the $\{F_n\}$ are uniformly bounded, then so are the $\{b_n\}$. To show F is in the Dirichlet space we repeat the verification of Stegenga's condition, but now we only need the case $S(E) = \mathbb{D}$.

Since $\sum \rho_n < \infty$, there can only be a finite number of n with ρ_n less than a given M, so omitting a finite subset gives an interpolating sequence. Let p be a polynomial solving the interpolation problem on this finite set and q a polynomial vanishing only on this set. Then if f interpolates the values $(a_n/q(z_n)) - p(z_n)$ at the remaining points, $p + qf \in \mathcal{D} \cap \mathcal{H}^\infty(\mathbb{D})$ interpolates on the whole sequence.

7. Interpolation in $M(D_\alpha)$

In this section we will sketch the proof of Theorem 1.8 for $0 < \alpha < 1/2$. Suppose $\{z_n\}$ is satisfies

$$\rho(z_n, z_m) \geq \delta, \tag{7.1}$$

for all $m \neq n$ and

$$\sum_{z_n \in \cup S(I_j)} (1-|z_n|)^{1-2\alpha} \leq B_\alpha(\cup_j I_j). \tag{7.2}$$

Define the Blaschke product associated to the sequence

$$B(z) = \prod_n \frac{z - z_n}{1 - \bar{z}_n z} \frac{|z_n|}{z_n}.$$

Using results of Stegenga and Maz'ya, Verbitskii showed

Lemma 7.1. [23] *(7.1) and (7.2) are necessary and sufficient for* $B \in M(D_\alpha)$.

Now that we have $B \in M(\mathcal{D}_\alpha)$, the proof that $\{z_n\}$ is interpolating for $M(\mathcal{D}_\alpha)$ can be completed in any of several ways. For example, Earls' proof of Carleson's theorem ([10], Theorem VII.5.1 of [11]) shows that if $\{z_n\}$ is interpolating for $H^\infty(\mathbb{D})$, i.e.,

$$\inf_k \prod_{j \neq k} |\frac{z_k - z_j}{1 - \bar{z}_j z_k}| = \eta > 0.$$

and $\{a_n\} \in \ell^\infty$ then there is sequence $\{w_n\}$ with $\rho(w_n, z_n) \leq \eta/2$ such that the Blaschke product

$$\tilde{B}(z) = \prod_n \frac{w_n - z}{1 - \bar{w}_n z} \frac{|w_n|}{w_n}$$

satisfies

$$C\tilde{B}(z_n) = a_n, \quad n = 1, 2, \ldots,$$

for some constant C. Since $\{w_n\}$ is close to $\{z_n\}$ it satisfies similar estimates to $\{z_n\}$, so $\tilde{B} \in M(D_\alpha)$ with estimates depending only on the $\{z_n\}$. This completes the proof of Theorem 1.8.

8. A sufficient condition for interpolation in $\mathcal{D}$

In this section we prove Theorem 1.3. Suppose $\{z_n\}$ satisfies the conditions in Theorem 1.3, i.e.,

$$\rho_n \geq \delta d_n, \tag{8.1}$$

$$\sum_n d_n^{-1} \leq C, \tag{8.2}$$

$$\sum_{z_k \in K_n} d_k^{-1} \leq C d_n^{-1}, \tag{8.3}$$

and suppose $(c_n) \in \ell^2$, $\|c_n\|_2 = 1$ is the sequence of values to be interpolated. Let $a_n = c_n\sqrt{d_n}$ and define the functions $\{F_n\}$ as in the proof of Theorem 1.6. Recall that $b_n = a_n - F_{n-1}(z_n)$. Since $a_n = c_n\sqrt{d_n}$ we have $|b_n| \leq |c_n|\sqrt{d_n} + |F_{n-1}(z_n)|$. In Section 5 (equation (5.1)) the second term was shown to be bounded by

$$C \sum_{k: z_n \in S(K_k)} |a_k| + C \sum_{k<n} \epsilon_k |a_k|.$$

Since $\epsilon_n \sim d_n^{-1}$ and $a_n = c_n\sqrt{d_n}$, the second of these terms is bounded by

$$C \sum_{k<n} |c_k| d_k^{-1/2} \leq C(\sum_n |c_n|^2)^{1/2} (\sum_k d_k^{-1})^{1/2} \leq C.$$

Therefore,

$$|b_n| \leq C(1 + |c_n|\sqrt{d_n} + \sum_{k: z_n \in S(K_k)} |c_k|\sqrt{d_k}),$$

Thus

$$\begin{aligned}
\sum_n |b_n|^2 d_n^{-1} &\leq C\sum_n (d_n^{-1/2} + |c_n| + d_n^{-1/2} \sum_{k:z_n\in S(K_k)} p_{k,n}|c_k|d_k^{1/2})^2 \\
&\leq C\sum_n d_n^{-1} + C\sum_n |c_n|^2 d_n + C\sum_k |c_k|^2 d_k \sum_{n:z_n\in S(K_k)} \\
&\leq C + C\sum_n |c_n|^2 + C\sum_k |c_k|^2 d_k d_k^{-1} \\
&\leq C
\end{aligned}$$

Let $\chi_{D_n}, \chi_{D_n^c}$ denote the characteristic functions of $D_n, \mathbb{D}\backslash D_n$ respectively. Using Minkowski's inequality we get,

$$\begin{aligned}
\|F\|_{\mathcal{D}}^2 &\leq C\iint_{\mathbb{D}} |b_n|^2 |f_n'(z)|^2 \chi_{D_n}(z)dxdy + C\iint_{\mathbb{D}} (\sum_n |b_n||f_n'(z)|\chi_{D_n^c}(z))^2 dxdy \\
&\leq C\sum_n \iint_{D_n} |b_n|^2|f_n'(z)|^2 dxdy + (\sum_n |b_n|^2[\iint_{\mathbb{D}} |f_n'(z)|^2\chi_{D_n^c}(z)dxdy]^{1/2})^2 \\
&\leq \sum_n |b_n|^2 d_n^{-1} + (\sum_n |b_n|[\iint_{\mathbb{D}\backslash D_n} |f_n'(z)|^2 dxdy]^{1/2})^2 \\
&\leq C + C(\sum_n |b_n| d_n^{-1})^2 \\
&\leq C + C(\sum_n |b_n|^2 d_n^{-1})(\sum_n d_n^{-1}) \\
&\leq C.
\end{aligned}$$

Thus $F \in \mathcal{D}$. Using similar estimates,

$$\begin{aligned}
\sum_k |F(z_k)d_k^{-1/2} - c_k|^2 &\leq \sum_k d_k^{-1}(\sum_{n=k}^{\infty} C_3(n)|b_n|\epsilon_n)^2 \\
&\leq \sup_n C_3(n) \sum_k d_k^{-1} (\sum_n |b_n|^2)^{1/2} (\sum_k d_n^{-1})^{1/2} \\
&\leq C\sup_n C_3(n) \sum_k d_k^{-1} \\
&\leq C\sup_n C_3(n).
\end{aligned}$$

This is small if the sequence is sufficiently separated. Therefore, the argument above shows $\{z_n\}$ is interpolating if it is sufficiently separated. The condition $\rho_n \sim d_n$, implies that only a bounded number of points satisfy $\rho_n \leq M$ for any given M, and all of these are within a bounded hyperbolic distance of the origin. But as in the proof of Theorem 1.6, the finitely many points can be added back and the sequence is still interpolating for $\mathcal{D}$.

Now we can prove Theorem 1.1. First suppose $\{z_n\}$ is a universal interpolating sequence. Then $T(\mathcal{D}) \subset \ell^{\in}$, so $\{z_n\}$ is Carleson and so (1.2) is satisfied because of Stegenga's theorem. Since $\ell^2 \subset T(\mathcal{D})$, the separation condition (1.1) holds by Lemma 4.1.

Conversely, suppose (1.1) and (1.2) hold. Then $T(\mathcal{D}) \subset \ell^{\in}$ by Stegenga's theorem again. Stegenga's condition applied just to intervals implies (8.2) and (8.3) so $\ell^2 \subset T(\mathcal{D})$ by Theorem 1.3. This completes the proof of Theorem 1.1.

9. **Interpolating sequences and zero sets**

Suppose $\{z_n\}$ is a weighted interpolating sequence for $\mathcal{D}$. Then for any n there is a function f_n so that $f_n(z_n) = d_n^{1/2}$ and $f_n(z_m) = 0$ for all $m \neq n$. Thus $h_n = f_n d_n^{-1/2}$, has the desired properties, except that it might not be bounded. However, $h_n(\mathbb{D})$ has finite area, so by a result of Nguyen Xuan Uy [22] there is a non-constant, Lipschitz function F on the sphere, analytic on $h_n(\mathbb{D})$. We can easily arrange for $F(0) = 0$ and $F(1) = 1$ so that $F \circ h_n$ is bounded, takes the correct values at the points $\{z_n\}$ and has bounded Dirichlet integral (since $|F'|$ is bounded, $|(F \circ h_n)'| \leq C|h_n'|$). Thus the condition in Theorem 1.2 is necessary.

To prove sufficiency, let $\{h_n\}$ be the sequence of functions with the given properties and let $(c_n) \in \ell^2$ be the sequence to be interpolated. We would like to set $f(z) = \sum_n c_n d_n^{1/2} h_n(z) d_n^{1/2}$; this has the right values at the points $\{z_n\}$, but might not have bounded Dirichlet integral. To bound this, we will replace each h_n by a product $h_n f_n$ where $f_n(z_n) = 1$ and f_n has very small Dirichlet integral far from z_n.

Fix an n and move z_n to the origin by a Möbius transformation τ. Since $h_n \circ \tau$ equals 1 at 0 and 0 at the other points, Proposition 2.11 says that if $E = \cup_{m \neq n} I_{\tau(z_m)}$, then $B_{1/2}(E) \leq C d_n^{-1}$. Now let $f_n \circ \tau$ be the function given by Lemma 2.13 and let $g_n = (h_n f_n)/f_n(0)$ (if d_n is large enough with respect to the number A in the definition of f_n, then $f_n(0) > 1/2$. We will choose A large later and the proof will apply to the sequence with the finitely many n for which $f_n(0) < 1/2$ omitted).

Define $F(z) = \sum_n c_n d_n^{1/2} g_n(z)$. Let $S_n = S(K_n) \setminus \cup_{m \neq n} S(I_n)$. The S_k's have bounded overlap (see Corollary 11.2). Using this and Minkowski's inequality shows

$$\begin{aligned}
\iint_{\mathbb{D}} |F'(z)|^2 dxdy &\leq C \sum_n |c_n|^2 d_n \iint_{S_n} |g_n'(z)|^2 dxdy \\
&\quad + C(\sum_n |c_n| [d_n \iint_{S_n^c} |g_n'(z)|^2 dxdy]^{1/2})^2 \\
&\leq C \sum_n |c_n|^2 + C(\sum_n |c_n| [d_n \exp(-CAd_n)]^{1/2})^2 \\
&\leq C \sum_n |c_n|^2 [1 + \sum_n d_n \exp(-2CAd_n)]^2
\end{aligned}$$

Here we have used Lemma 2.13 to estimate each term of

$$\iint |g'(z)|^2 dxdy \leq \iint |h_n'(z) f_n(z)|^2 dxdy + \iint |h_n(z) f_n'(z)|^2 dxdy.$$

In the first term on the right, $\iint |h'|^2$ is bounded and $|f_n| \leq \exp(-CAd_n)$. In the second term, $|h_n|$ is bounded and $\iint |f'|^2 \leq \exp(-CAd_n)$.

If we choose $A > 3/C$ then $\exp(-CAd_n) \leq (1-|z_n|)^3$. Since $\sum d_n (1-|z_n|)^3$ is summable for any hyperbolically separated sequence, we get $F \in \mathcal{D}$. Thus $\{z_n\}$ is interpolating if we drop a finite number of terms, but these can be added back by the arguments in earlier sections.

10. **Interpolating, but not Carleson**

Lemma 10.1. *There is a sequence $\{z_n\}$ so that $T(\mathcal{D})$ strictly contains ℓ^2, i.e., $\{z_n\}$ is an interpolating sequence for $\mathcal{D}$, but not a universal interpolating sequence.*

To prove this let $r_n = 1 - e^{-e^n}$ and let $N_n = e^{n/4}$. Let $\{I_n\}$ be a collection of disjoint arcs of length $e^{-e^n/4}$. To each arc I_n associate N_n equally spaced points $\{z_j^n\}$ on the arc $r_n I_n$. The arguments of adjacent points differs by at least $e^{-e^n/2}/e^{n/4} \geq e^{-e^n/2}$ and therefore the points satisfy the separation condition

$$\rho(z_j^n, z_k^n) \geq \delta e^n \geq \delta \rho(0, z^n, j).$$

It is easy to check that points associated to different arcs are also satisfy the separation condition. Then if $\{z_k\}$ denotes the union of all the points

$$\sum_k d_k^{-1} = \sum_n N_n e^{-n} \leq \sum_n e^{-3n/4} < \infty,$$

so the second condition of Theorem 1.3 is satisfied. Finally, for each point z_k, there are no other points z_j in $S(K_k)$ (if η is small enough) so the third condition is also satisfied. Thus $\{z_k\}$ is interpolating for $\mathcal{D}$ by Theorem 1.3.

On the other hand, $\{z_k\}$ does not satisfy Stegenga's condition. For if we consider $I = I_n$, then $S(I)$ contains N_n points, so

$$\sum_{z_k \in S(I)} d_k^{-1} = N_n e^{-n} = e^{-3n/4}.$$

However,

$$B_{1/2}(I) \sim (\log |I|^{-1})^{-1} = 4e^{-n}.$$

The previous sum is much larger, so Stegenga's condition fails.

It is possible to construct a sequence $\{z_k\}$ so that Stegenga's condition holds for all intervals, i.e.,

$$\sum_{z_k \in S(I)} d_k^{-1} \leq C B_{1/2}(I),$$

for all intervals, but does not hold uniformly for all finite unions of intervals. Thus even this stronger hypothesis does not imply $\{z_k\}$ is a universal interpolating sequence. Such a sequence can be constructed using the example of a conformal mapping given in Corollary 5.2 of [18]. They build a map $\varphi : \mathbb{D} \to \Omega$ where Ω is a countable union of rectangular "rooms". If we let $\{z_k\}$ be the preimages under φ of the center of each room, then the estimates given there show $\{z_k\}$ satisfies Stegenga's condition uniformly on all arcs, but not on all finite unions of arcs (in addition to the estimates given there one only needs that for each z_k, $d_k^{-1} \sim \rho_k^{-1}$ is comparable to that area of the kth "room").

11. Some consequences of the separation condition

In this section we will show a few simple geometric consequences of equation (1.1), i.e., the assumption that $\rho_n \geq \delta d_n$ for some $\delta > 0$ and all n. In particular, we will show that this assumption by itself implies the sequence is interpolating for H^∞.

Lemma 11.1. *Suppose $\{z_n\}$ satisfies $\rho_n \geq \delta d_n$ for every n. Then*

$$\#\{n : z_n/|z_n| \in I, R \leq d_n \leq (1+\epsilon)R\} \leq 1 + |I|e^{R(1-C\delta)},$$

where $\epsilon = \delta/8$ and C is independent of δ.

Proof. Suppose $a = e^{-R}$. The hyperbolic width of the annulus $\{z : a^{1+\epsilon} \leq 1 - |z| \leq a\}$, is $c_0 R/8$, so to each z_n there is a point z_n^* on $\{z : 1-|z| = a\}$, so that $\rho(z_n, z_n^*) \leq \delta R/8$. Therefore given distinct n, m we must have $\rho(z_n^*, z_m^*) \geq \delta R/4$. Thus there are at most $1+|I|e^{R(1-C\delta)}$such points. □

Corollary 11.2. *Suppose $\{z_n\}$ satisfy $\rho_n \geq \delta d_n$ for some $\delta > 0$ and all n. Then for any $\lambda < 1$, the hyperbolic disks $B_n = B_\rho(z_n, \lambda d_n)$ have bounded overlap (with constant depending only on δ and λ).*

Proof. Of course this is not true for $\lambda = 1$, since the closure of every ball then contains the origin. Suppose $z \in B_n$. Then its easy to check that $|z - z_n| \leq C(1-|z|)^\alpha$ and $(1-|z_n|) \geq (1-|z|)^\beta$ for some $\alpha, \beta > 0$. The intersection of these two conditions gives a region S that is covered by finitely many of the annular regions $A_k = \{\exp(-(1+\epsilon)^{k+1} \leq 1-|z| \leq \exp(-(1+\epsilon)^k \leq 1 - |z|\}$ considered in the lemma, and by the lemma, each $S \cap A_k$ contains only a bounded number of points. □

This result shows that the conditions in Theorem 1.6 are necessary if $\rho_n \leq \lambda d_n$ for some $\lambda < 1$. For in that case we choose an interpolating function f so that if z_n^* is the nearest distinct point of the sequence to z_n then $|f(z_n) - f(z_n^*)| \geq 1$. It is easy to see that that there must be an interpolating function such that $\iint_{B_\rho(z_n,\rho_n)} |f'(z)|^2 dxdy \geq C d_n^{-1}$. Using bounded overlap of these balls, if $f \in M(\mathcal{D})$, then Stegenga's condition for f implies it for $\{z_n\}$.

Corollary 11.3. *Suppose $\{z_n\}$ satisfies $\rho_n \geq \delta d_n$ for every n. Then*

$$\sum_{z_n \in S(I)} (1 - |z_n|) \leq |I| + o(|I|),$$

with an estimate that depends only on δ. In particular, $\{z_n\}$ is a thin sequence.

Proof. Let $|I| = e^{-R}$ and $a_k = e^{-R(1+\epsilon)^k}$. Choose k_0 to be the smallest integer such that

$$(1+\epsilon)^k - \delta C k \geq 2 + \frac{1}{2}(1+\epsilon)^k$$

(these are the constants in Lemma 11.1) Then

$$\sum_{z_n \in S(I)} (1 - |z_n|) \;\leq\; \sum_{k=0}^{\infty} \sum_{\{m : a_{k+1} \leq 1 - |z_m| \leq a_k\}} (1 - |z_n|)$$

Break the k-sum into two pieces for $\{k < k_0\}$ and $\{k \geq k_0\}$. The first sum may contain one term of size $|I|$ and a bounded number of terms each smaller than $|I|^{1+\epsilon}$. Therefore this sum is bounded by

$|I|(1+o(|I|))$. To estimate the second sum,

$$\begin{aligned}\sum_{k=0}^{\infty} \sum_{\{m: a_{k+1} \leq 1-|z_m| \leq a_k\}} (1-|z_n|) &\leq \sum_{k=k+0}^{\infty} C \exp(-R(1+\epsilon)^k) \exp(-kR\delta/C)) \\ &\leq \sum_{k=k_0}^{\infty} C \exp(-R(1+\epsilon)^k + Rk\delta/C) \\ &\leq C|I|^2 \sum_{k=k_0}^{\infty} C \exp(-\frac{1}{2}(1+\epsilon)^k) \\ &\leq C(\delta)|I|^2. \quad \square\end{aligned}$$

12. **Remarks**

There are still many unresolved questions. For example, are the conditions in Theorem 1.6 also necessary for interpolation in $M(\mathcal{D})$? What about the conditions in Theorem 1.8? What is the correct characterization of interpolating sequences for $\mathcal{D}$? Is the requirement that $\sum_n d_n^{-1} < \infty$ necessary? If not, construct an example of an interpolating sequence for which this sum diverges. Roughly speaking, if interpolation is always possible by a sum $\sum_n a_n h_n(z)$ where $h_n(z_m) = \delta_{nm}$ and the derivative h_n' is "concentrated" around z_n, then we would expect $\sum d_n^{-1} < \infty$ to be necessary.

If it is not the sum of the d_n's which is important, perhaps it is some sum involving the capacity of intervals associated to the sequence. For example, suppose $\{z_n\}$ is a sequence, not containing the origin. To each point of the sequence we can associate the interval I_n which is the base of maximal dyadic Carleson box containing z_n. These intervals can then be arranged into generations by containment in the usual way. Let E_k be the union of the intervals in the kth generation. Is the condition $\sum_k B_{1/2}(E_k) < \infty$ either necessary or sufficient for $\{z_n\}$ to be a zero sequence? I don't know any reason why this should be, but it is consistent with all known results. Furthermore, if we replace $B_{1/2}$ by B_0 we obtain the (known) characterization for zeros of the Hardy space, H^2.

If this were true, then to characterize interpolating sequences we would move each point z_n to the origin by a Möbius transformation, and define generation sets E_k^n. We would then have that the sequence is interpolating for $\mathcal{D}$ if has the separation condition and $\sum_k B_{1/2}(E_k^n) \leq Cd_n^{-1}$ for some uniform C.

If $\{z_n\}$ is interpolating for $\mathcal{D}$ and if $\sum d_n^{-1} < \infty$, then any bounded sequence can be interpolated. Therefore we can find a Dirichlet class function which fails to have non-tangential limits at every non-tangential limit point of $\{z_n\}$. A result of Beurling says that a Dirichlet function must have non-tangential limits except possibly on a set of zero logarithmic capacity [5] (see also [7]). Thus we must have $B_{1/2}(E_k^n) \to 0$ as $k \to \infty$, at the very least. Much more detailed information is available about the boundary behavior of functions in the Dirichlet (and related spaces), e.g., [1]. Perhaps some of the results or techniques of this type will be useful for the interpolation problem.

In a recent paper [14], Rochberg and Wu give an alternate characterization of the Dirichlet type spaces D_α. They also consider the spaces W_α defined by

$$\|f\|_{W_\alpha}^2 = \sup_{\|g\|_{D_\alpha}=1} \iint_{\mathbb{D}} |g(z)|^2 |f'(z)|^2 (1-|z|)^{1-2\alpha} dxdy.$$

Then $W_0 = \text{BMO}$ and $M(D_\alpha) = W_\alpha \cap H^\infty(\mathbb{D})$. A η lattice is a collection of points $\{w_j\}$ in $\mathbb{D}$ such that $B_\rho(w_j, 5\eta)$ covers $\mathbb{D}$ and the balls $B_\rho(w_j, \eta/5)$ are disjoint. Rochberg and Wu show that $f \in W_\alpha$ iff for all small enough η, any $b > 1$, and any η lattice $\{w_j\}$, f can be written

$$f(z) = \sum_j \lambda_j \left(\frac{1 - |w_j|^2}{1 - \bar{w}_j z}\right)^b,$$

where $\{\lambda_j\}$ satisfy the condition that $\sum |\lambda_j|^2 \delta_{w_j}$ is an α-Carleson measure. Perhaps this result can be used to prove the necessity of our interpolation condition for $M(D_\alpha)$, but I have not seen how to do this.

References

[1] W. Rudin A. Nagel and J. Shapiro. Tangential boundary behavior of functions in Dirichlet-type spaces. *Annals of Math.*, 116:331–360, 1982.

[2] N. Aronszajn and K.T. Smith. Theory of Bessel potentials I. *Ann. Inst. Fourier, Grenoble*, 11:385–475, 1961.

[3] S. Axler. Interpolation by multipliers of the Dirichlet space. *Quart. J. Math. Oxford*, 43:409–419, 1992.

[4] S. Axler and A. Shields. Univalent multipliers on the Dirichlet space. *Michigan Math. J.*, 32:65–80, 1984.

[5] A. Beurling. Ensembles exceptionnels. *Acta. Math.*, 72:1–13, 1940.

[6] A. Beurling. *Collected works of Arne Beurling*. Springer-Verlag, 1990. Edited by L. Carleson.

[7] L. Carleson. *On a class of meromorphic functions and its asssociated exceptional sets*. PhD thesis, Uppsall, 1950.

[8] L. Carleson. An interpolation problem for bounded analytic functions. *American J. Math.*, 80:921–930, 1958.

[9] K.C. Chan and A.L. Shields. Zero sets, interpolating sequences and cyclic vectors for Dirichlet spaces. *Michigan Math. J.*, 39:289–307, 1992.

[10] J.P. Earl. On the interpolation of bounded sequences by bounded functions. *J. London Math. Soc.*, 2:544–548, 1970.

[11] J. B. Garnett. *Bounded Analytic Functions*. Academic Press, 1981.

[12] J.B. Garnett and D.E. Marshall. *Harmonic Measure*. In preparation.

[13] N.G. Meyers. A theory of capacities for functions in Lebesgue classes. *Math. Scand.*, 26:255–292, 1970.

[14] R. Rochberg and Z. Wu. A new characterization of Dirichlet type spaces and applications. *Illinois J. Math.*, 37:101–122, 1993.

[15] J.T. Rosenbaum. *Interpolation in Hilbert spaces of analytic functions*. PhD thesis, University of Michigan, 1965.

[16] H. Shapiro and A. Shields. On some interpolation problems for analytic functions. *Amer. J. Math.*, 83:513–532, 1961.

[17] H. Shapiro and A. Shields. On the zeros of functions with finite Dirichlet integral and some related function spaces. *Math. Z.*, 80:217–229, 1962.

[18] W. Smith and D.A. Stegenga. Poincaré domains in the plane. In *Complex Analysis, Joensuu: Proceedings of R. Nevalinna Colloquium*, Lecture Notes in Math. 1351, pages 312–327. Springer-Verlag, 1988.

[19] D.A. Stegenga. Multipliers of the Dirichlet space. *Illinois J. Math.*, 24:113–139, 1980.

[20] E.M. Stein. *Singular integrals and differentiablity properties of functions.* Princeton University Press, 1970.

[21] C. Sundberg and T. Wolff. Interpolating sequences for QA_B. *Trans. Amer. Math. Soc.*, 276:551–581, 1983.

[22] N.X. Uy. Removable sets of analytic functions satisfying a Lipschitz condition. *Arkiv Mat.*, 17:19–27, 1979.

[23] I.E. Verbitskii. Multipliers in spaces with "fractional" norms, and inner functions. *Sibirskii Mat. Zhurnal*, 26:51–72, 1985.

C. Bishop, Mathematics Department, SUNY Stony Brook, Stony Brook, NY 11794

E-mail address: `bishop@math.sunysb.edu`