QUASICONFORMAL MAPS WITH THIN DILATATIONS

CHRISTOPHER J. BISHOP

Abstract. We give an estimate that quantifies the fact that a normalized quasiconformal map whose dilatation is non-zero only on a set of small area approximates the identity uniformly on the whole plane. The precise statement is motivated by applications of the author's quasiconformal folding method for constructing entire functions; in particular a construction of transcendental wandering domains given by Fagella, Godillon and Jarque [7].

1. Introduction

A quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ is a a homeomorphism that is absolutely continuous on almost all horizontal and vertical lines, and whose partial derivatives satisfy $f_{\overline{z}} = \mu f_z$ for some complex valued, measurable function μ with $\|\mu\|_\infty = k < 1$, called the dilatation of f. For the basic properties of quasiconformal maps, see Ahlfors' book [1].

If $\mu = 0$ then f is a conformal homeomorphism of $\mathbb{C}$ to itself, and hence it is linear. If the dilatation μ is small, then we expect f to be close to linear. There are at least two reasonable senses in which we can ask μ to be small: that $\|\mu\|_\infty$ is small or that the set $\{z : \mu(z) \neq 0\}$ is small. In this note we consider the latter possibility.

To be more precise, we say a measurable set $E \subset \mathbb{R}^2$ is (ϵ, φ)-thin if $\epsilon > 0$ and

$$\text{area}(E \cap D(z, 1)) \leq \epsilon \varphi(|z|)$$

where $\varphi : [0, \infty) \to [0, \pi]$ is a bounded, decreasing function, such that

$$\int_0^\infty \varphi(r) r^n dr < \infty,$$

for every $n > 1$. If $a > 0$, the function $\varphi(r) = \exp(-ar)$ satisfies this condition, and this example suffices for many applications.

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Recall that a quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ is often normalized by post-composing by a conformal linear map in one of two ways. First, we can assume $f(0) = 0$ and $f(1) = 1$. We call this the 2**-point normalization**. Second, if the dilatation of f is supported on a bounded set, then f is conformal in a neighborhood of ∞ and then we can choose R large and post-compose with a linear conformal map so that

$$|f(z) - z| = O(\frac{1}{|z|}),$$

for $|z| > R/2$. We say that such an f is normalized at ∞. This is also called the **hydrodynamical normalization** of f. We will first prove an estimate for the hydrodynamical normalization and then deduce one for the 2-point normalization.

Theorem 1.1. *Suppose $F : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, and $E = \{z : \mu(z) \neq 0\}$ is bounded (so F is conformal near ∞) and F is normalized so*

$$|F(z) - z| \leq M/|z|,$$

near ∞. Assume E is (ϵ, φ)-thin. Then for all $z \in \mathbb{C}$,

$$|F(z) - z| \leq \frac{\epsilon^\beta}{|z| + 1},$$

where β depends only on K and φ. In particular, as $\epsilon \to 0$, F converges uniformly to the identity on the whole plane.

Corollary 1.2. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, $F(0) = 0$, $F(1) = 1$, and $E = \{z : \mu(z) \neq 0\}$ is (ϵ, φ)-thin. Then*

$$(1.1) \quad (1 - C\epsilon^\beta)|z - w| - C\epsilon^\beta \leq |f(z) - f(w)| \leq (1 + C\epsilon^\beta)|z - w| + C\epsilon^\beta,$$

where C and β only depend on $k = \|\mu\|_\infty$ and φ.

Similar estimates are known, e.g., compare to the well known result of Teichmüller and Wittich (e.g., Theorem 7.3.1 of [8], [12], [13]) or estimates of Dyn'kin [6]. The version stated above is intended for specific applications to holomorphic dynamics involving the author's quasiconformal folding technique of constructing entire functions, introduced in [4]. Given an infinite tree T in the plane satisfying certain geometric conditions, this method constructs a quasiregular function g on the whole plane that is holomorphic outside a (usually small) neighborhood U of the tree. Then $f = g \circ \varphi^{-1}$ is entire where φ is the quasiconformal map whose dilatation is given by $\mu = g_{\bar{z}}/g_z$ and is supported in U; such a φ exists by the measurable Riemann mapping theorem. In applications, g is been constructed to have certain properties and we want $f = g \circ \varphi^{-1}$ to have the same or similar properties. Thus we usually want φ to be close to the identity. In many applications of quasiconformal folding the neighborhood U can be chosen to be very small, e.g., (ϵ, e^{-r}) thin, which is why the estimates above are helpful. The folding method has been used to construct various examples in complex analysis and holomorphic dynamics, e.g., [2], [5], [9], [10], [11], [3], [14]. Often estimating the correction map φ is the hardest part of applying the folding method, and these papers sometimes use weaker versions of the estimates given here, or leave various details to the reader. The goal of this note is to provide a complete proof of the estimates needed for many applications of the folding theorem. For example, Lemma 3.3 of the current paper was stated as Theorem 2.5 of [7] by Fagella, Godillon and Jarque, based on "personal communication" from myself. Their paper constructs two entire functions, neither of which has a wandering domain, but whose composition does have a wandering domain, as well as providing addition information about the wandering domains constructed in my paper [4]. This note gives a concrete citation for the result they quote.

2. Quasiconformal maps and conformal modulus

Here we review a few basic facts about quasiconformal maps and conformal modulus that we will need. All these results can be found in Ahlfors' book [1].

Lemma 2.1 (Characterization of quasicircles). *For each $K \geq 1$ there is a $C = C(K) < \infty$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal and $r > 0$ so that $f(\gamma) \subset \{z : r \leq |z - w| \leq Cr\}$.*

Theorem 2.2 (Borjarski's theorem). *If $1 \leq K < \infty$, there is a $p > 2$ and $A, B < \infty$ so that the following holds. If $f : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal, and $Q \subset \mathbb{C}$ is a square, then*

$$(\frac{1}{\text{area}(Q)} \iint_Q |f_z|^p dxdy)^{1/p} \leq A(\frac{1}{\text{area}(Q)} \int_Q |f_z|^2 dxdy)^{1/2} \leq B\frac{\text{diam}(f(Q))}{\text{diam}(Q)}$$

Lemma 2.3 (Pompeiu's formula)**.** *If Ω has a piecewise C^1 boundary and f is quasiconformal on Ω, then*

$$f(w) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)}{z-w}dz - \frac{1}{\pi}\iint_{\Omega} \frac{f_{\overline{z}}}{z-w}dxdy. \tag{2.1}$$

If Γ is a collection of locally rectifiable curves in a planar domain Ω, then a non-negative Borel function ρ is called admissible for Γ if $\int_\gamma \rho ds \geq 1$ for every curve $\gamma \in \Gamma$. The modulus of Γ, denoted $\mathrm{mod}(\Gamma)$, is the infimum of $\int_\Omega \rho^2 dxdy$ over all admissible ρ. The reciprocal of the modulus is called the extremal length of Γ. A quasiconformal map f of Ω with dilatation satisfying $\|\mu\|_\infty = k < 1$ has the property that it can change conformal modulus of a path family in Ω by at most a factor of $K = (k+1)/(k-1)$.

If Ω is a topological annulus in the plane with boundary components γ_1, γ_2 that are closed Jordan curves, then $\mathrm{mod}(\Omega)$ refers to the modulus of the path family in Ω that separates the boundary components. This is the same as the extremal length of the path family that connects the boundary components (also called the extremal distance between the boundary components). If $A(a,b) \equiv \{z : a < |z| < b\}$ then it is standard fact that $\mathrm{mod}(A) = \frac{1}{2\pi}\log\frac{b}{a}$. Let

$$D_f = \frac{|f_z| - |f_{\overline{z}}|}{|f_z| + |f_{\overline{z}}|},$$

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2 = (|f_z| - |f_{\overline{z}}|)(|f_z| + |f_{\overline{z}}|),$$

denote the distortion and Jacobian of f respectively. Note that $D_f \geq 1$ and f is conformal if and only if $D_f \equiv 1$. If $D_f \leq K$, then f can distort the modulus of an annulus by a factor of most K, and hence for a map between round annuli, the ratio of radii changes by a most a power K. In the rest of this section we show that better estimates are possible of $D_f \leq K$ everywhere, but $D_f \approx 1$ "most places".

Lemma 2.4. *Suppose f is a K-quasiconformal map from $A_m = A(1, e^m)$ to $A_M = A(1, e^M)$. Then*

$$M \geq m - \frac{1}{2\pi}\int_{A(1,e^m)} (D_f(z) - 1)\frac{dxdy}{r^2}.$$

Proof. Let Γ_M be the path family connecting the boundary components of A_M. If $\tilde{\rho}$ is admissible for this family then

$$\rho(z) = \tilde{\rho}(f(z))(|f_z| + |f_{\overline{z}}|)$$

is admissible for Γ_m, the path family connecting the boundary components of A_m. Therefore the modulus of Γ_m satisfies

$$\mathrm{mod}(\Gamma_m) \leq \int_{A_m} \tilde{\rho}(f(z))^2 (|f_z| + |f_{\overline{z}}|)^2 dxdy.$$

Applying this formula to the inverse of f shows that for any admissible ρ for Γ_m,

$$\begin{aligned}
\mathrm{mod}(f(\Gamma_m)) &\leq \int_{A_m} \rho(z)^2 \frac{1}{(|f_z| - |f_{\overline{z}}|)^2} J_f dxdy \\
&\leq \int_{A_m} \rho(z)^2 \frac{1}{(|f_z| - |f_{\overline{z}}|)^2} (|f_z|^2 - |f_{\overline{z}}|^2) dxdy \\
&\leq \int_{A_m} \rho(z)^2 \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} dxdy \\
&\leq \int_{A_m} \rho(z)^2 D_f(z) dxdy.
\end{aligned}$$

Applying this with the admissible metric $\rho(z) = \frac{1}{m|z|}$, we get

$$\begin{aligned}
\frac{2\pi}{M} = \mathrm{mod}(f(\Gamma_m)) &\leq \frac{1}{m^2} \int_{A_m} \frac{D_f(z)}{|z|^2} dxdy \\
&= \frac{1}{m^2} [\int_{A_m} \frac{D_f(z) - 1}{|z|^2} dxdy + \int_{A_m} \frac{1}{|z|^2} dxdy] \\
&= \frac{1}{m^2} \int_{A_m} \frac{D_f(z) - 1}{|z|^2} dxdy + \frac{2\pi}{m}.
\end{aligned}$$

Rearranging gives

$$m - M \leq \frac{M}{2\pi m} \int_{A_m} \frac{D_f(z) - 1}{|z|^2} dxdy,$$

or

$$M \geq m - \frac{M}{2\pi m} \int_{A_m} \frac{D_f(z) - 1}{|z|^2} dxdy.$$

If $M > m$, the lemma is trivially true. If $M \leq m$, then because the integral is non-negative, the inequality above becomes

$$M \geq m - \frac{1}{2\pi} \int_{A_m} \frac{D_f(z) - 1}{|z|^2} dxdy.$$

Thus in either case the lemma holds. □

Lemma 2.5. *Suppose f is a K-quasiconformal map from $A_m = A(1, e^m)$ to $A_M = A(1, e^M)$. Then*

$$M \leq m + \frac{1}{2\pi} \int_{A_m)} (D_f - 1) \frac{dxdy}{r^2}.$$

Proof. If we cut A_m with a radial slit and let $g = \log(f)$, then g maps A_m to a quadrilateral with its vertical sides on $\{x = 0\}$ and $\{x = M\}$. This quadrilateral has area $2\pi M$. If we integrate over the radial segments in A_m, we get

$$M \leq \int_1^{\exp(m)} (|g_z| + |g_{\overline{z}}|)dr$$

so integrating over all angles and using $rdrd\theta = dxdy$ gives

$$2\pi M \leq \int_0^{2\pi}\int_1^{\exp(m)} (|g_z| + |g_{\overline{z}}|)drd\theta \leq \int_{A_m} (|g_z| + |g_{\overline{z}}|)\frac{dxdy}{r}.$$

Thus by Cauchy-Schwarz,

$$\begin{aligned}
(2\pi M)^2 &\leq (\int_{A_m} (|g_z| + |g_{\overline{z}}|)(|g_z| - |g_{\overline{z}}|)dxdy)(\int_{A_m} \frac{|g_z| + |g_{\overline{z}}|}{|g_z| - |g_{\overline{z}}|}\frac{dxdy}{r^2}) \\
&\leq (\int_{A_m} J_g dxdy)(\int_{A_m} D_g \frac{dxdy}{r^2}) \\
&\leq 2\pi M(\int_{A_m} D_f \frac{dxdy}{r^2}),
\end{aligned}$$

where in the last line we have used the facts that $g(A_m)$ has area $2\pi M$ and $D_g = D_f$ (since $\log z$ is conformal on the slit annulus). Thus

$$\begin{aligned}
M &\leq \frac{1}{2\pi}\int_{A_m} 1 + (D_f(z) - 1)\frac{dxdy}{|z-w|^2} \\
&= m + \frac{1}{2\pi}\int_{A_m} (D_f(z) - 1)\frac{dxdy}{|z-w|^2}.
\end{aligned}$$

□

The following simply combines the last two results.

Corollary 2.6. *Suppose f is a K-quasiconformal map from $A_m = A(1, e^m)$ to $A_M = A(1, e^M)$. Then*

$$M = m + O(\frac{1}{2\pi}\int_{A_m} \frac{D_f(z) - 1}{r^2} dxdy.)$$

A special case of this is:

Corollary 2.7. *Suppose f is a K-quasiconformal map from $A_m = A(1, e^m)$ to $A_M = A(1, e^M)$. Suppose $D_f(z) \leq D$ on A_m. Suppose μ is the dilatation of f, that $E =$*

$\{z : \mu(z) \neq 0\}$ *and that* $E_k = E \cap \{e^{k-1} < |z| < e^k\}$. *If we choose an integer* n *so that* $m \leq 2^n$, *then*

$$M = m + O((D-1)\sum_{k=0}^{n} e^{-2k}\text{area}(E_k)).$$

3. Dilatations with thin support

Next we apply these estimates to quasiconformal maps with dilatations that have small support in a precise sense.

Lemma 3.1. *Suppose* F *is a* K*-quasiconformal map with dilatation* μ, *that* μ *has bounded support, and that* F *has the hydrodynamical normalization at* ∞. *Let* $E = \{z : \mu(z) \neq 0\}$ *and suppose for some* $t > 0$, E *satisfies*

$$\int_{E\setminus D(w,t)} \frac{dxdy}{|z-w|^2} \leq a,$$

for every $w \in \mathbb{C}$. *Then there is a* $C = C(K, a) < \infty$, *depending only on* K *and* a, *so that for every* $w \in \mathbb{R}^2$ *and* $r \geq t$,

$$\frac{1}{C} \leq \frac{\text{diam}(F(D(w,r))}{r} \leq C.$$

Proof. We need only prove this for $r = t$ since for $r > t$, we can simply apply the lemma after setting $t = r$ (the integral just gets smaller).

The mapping $G(z) = F(tx)/t$, satisfies the same estimates as F, but with t replaced by 1. If we prove the lemma for G, it follows for F, so it suffices to assume $t = 1$.

By assumption we can choose $R > 100$ so that $|f(z) - z| \leq 1$, for $|z| > R/8$. The result is clear if $|w| > R/2$, so we may assume $|w| \leq R/2$. Fix such a w. Let $m = \log R$, so $R = e^m$, and consider the annulus $A = \{z : 1 < |z - w| < e^m\}$. $F(A)$ is a topological annulus and can be conformally mapped to $A_M = \{1 < |z| < e^M\}$ for some $M > 1$. By Corollary 2.6,

$$M = m + O(\int_{A_m} \frac{D_f - 1}{|z-w|^2} dxdy).$$

By our assumptions, this becomes

$$M = m + O(\frac{K-1}{2\pi}\int_{A_m} \mathbf{1}_E(z)\frac{dxdy}{|z-w|^2}) = m + O(Ka),$$

where $\mathbf{1}_E$ denotes the indicator function of E (the function that is one on E and zero off E) and we have used the fact that E has finite planar area and $|z-w|^{-1} \leq 1$ on A_m (recall w is the center of the annulus and the inner radius is at least 1.).

By Lemma 2.1, the boundary components of $f(A_m)$ are each closed curves that are contained in round annuli (with concentric circles) of bounded modulus (depending on K). Thus $f(A_m)$ is contained in a topological annulus A' with circular boundaries γ_1, γ_2 (not necessarily concentric) whose diameters are comparable to the diameters of the boundary components of $f(A_m)$. By monotonicity of modulus, the modulus of the annulus A' (denoted $M'/2\pi$) is larger than the modulus $M/2\pi$ of $f(A)$, hence $M' \geq M$. Moreover, we claim

$$M' \leq \log \frac{\operatorname{diam}(\gamma_2)}{\operatorname{diam}(\gamma_1)}.$$

This is well known to hold with equality if the circles γ_1, γ_2 are concentric. If they are not, then we can apply a Möbius transformation that maps the outer circle, γ_2, to itself and moves the inner circle, γ_1 to circle concentric with γ_2. This make the Euclidean diameter of γ_1 larger and preserves the modulus between the circles, and this proves the claimed inequality. Thus

$$M \leq M' \leq \log \frac{\operatorname{diam}(\gamma_2)}{\operatorname{diam}(\gamma_1)},$$

or

$$\operatorname{diam}(\gamma_1) \leq \operatorname{diam}(\gamma_2) \cdot e^{-M} = \operatorname{diam}(\gamma_2) \cdot e^{-m+O(KA)}.$$

Since $|f(z) - z| \leq 1$ on $\{|z| = R\}$ we know $\operatorname{diam}(\gamma_2) \simeq R = e^m$. Using this and the fact $M = m + O(Ka)$ prove above gives

$$\operatorname{diam}(f(\{|z-w| = 1\})) \simeq \operatorname{diam}(\gamma_1) = O(e^{Ka}).$$

To get the other direction, we choose γ_1, γ_2 to be circles that bound an annulus inside $f(A_m)$, again with diameters comparable to the diameters of the corresponding components of $\partial f(A_m)$. We then use monotonicity again, and argue as before, but now we note that since f is close the identity for $|z| > R/2$, the curve γ_1 is not too close to γ_2, i.e., the distance between them is comparable to R. Thus in the argument above, where we moved γ_1 be be concentric with γ_2, its Euclidean diameter was only changed by a bounded factor. Thus

$$\operatorname{diam}(\gamma_1) \gtrsim \operatorname{diam}(\gamma_2) \cdot e^{-M} = \operatorname{diam}(\gamma_2) \cdot e^{-m-O(KA)} \gtrsim e^{-O(KA)}.$$

This proves the lemma. □

If F is as above, then Theorem 2.2 says there is a $p = p(K) > 2$ so that $\|F_z\|_p$ is uniformly bounded on every unit radius disk. If a region can be covered by n such disks then the L^p norm is $O(n^{1/p})$ with a uniform constant, i.e.,

Corollary 3.2. *If F satisfies the conditions of Lemma 3.1, and $p = p(K) > 2$ is as above, then $\|F_z \cdot \mathbf{1}_{D(z,r)}\|^p = O(r^{2/p})$ uniformly for all $z \in \mathbb{C}$.*

Proof of Theorem 1.1. Suppose the support of μ is contained in $D(0, R)$. The main idea is to use the Pompeiu formula

$$F(w) = \frac{1}{2\pi i}\int_{|z|=r} \frac{F(z)}{z-w}dz - \frac{1}{\pi}\iint_{|z|<r} \frac{F_{\overline{z}}}{z-w}dxdy. \tag{3.1}$$

Because of our assumptions on F, the first integral is

$$\frac{1}{2\pi i}\int_{|z|=r} \frac{z + O(1/|z|)}{z-w}dz = w + O(1/r).$$

The left-hand side of (3.1) and the second integral are both constant for $r > R$, so the first integral must equal w for all $r > R$. Thus

$$F(w) = w - \frac{1}{\pi}\iint_{|z|<r} \frac{F_{\overline{z}}}{z-w}dxdy = w - \frac{1}{\pi}\iint_{|z|<r} \frac{\mu F_z}{z-w}dxdy.$$

Since $|F_{\overline{z}}| = |\mu F_z| \leq k|F_z|$, we get

$$|F(w) - w| \leq \frac{k}{\pi}\int_E |\frac{F_z}{z-w}|dxdy.$$

where $k = (K-1)/(K+1)$ is our upper bound for $|\mu|$.

The estimate in the theorem already holds if $|w| \geq R$, so assume $|w| < R$. Let $r = \max(1, |w|/2)$. We will estimate the integral

$$\int_E |\frac{F_z}{z-w}|dxdy,$$

by cutting $D(0, R)$ into three pieces:

$$D_1 = \{z : |z-w| \leq 1\}$$
$$A = \{z : 1 \leq |z-w| \leq r\}$$
$$X = D(0, R) \setminus (D_1 \cup A),$$

and show integral over each piece is $O(\epsilon^\beta/|w|)$ where $\beta = \beta(K) > 0$.

First consider D_1. With p as in Corollary 2.2, the L^p norm of F_z over D_1 is uniformly bounded, so using Hölder's inequality with the conjugate exponents, we get

$$\int_{D_1} |\frac{F_z}{z-w}|dxdy = O(\|\frac{\mathbf{1}_{E\cap D(w,1)}}{|z-w|}\|_q). \tag{3.2}$$

Since $E \cap D(w,1)$ has area at most $\varphi(|w|) \leq \varphi(r)$, the L^q norm on the right side of (3.2) is bounded above by what happens when $E \cap D(w,1)$ is a disk of radius $s \simeq (\epsilon\varphi(r))^{1/2}$ centered at w. In this case we get the bound (using polar coordinates and recalling $1 < q < 2$)

$$O([\int_0^s r^{-q} r dr]^{1/q}) = O(s^{(2-q)/q}) = O((\epsilon\varphi(r))^{\frac{1}{q}-\frac{1}{2}}).$$

Since φ tends to zero faster than any polynomial, this is $= O(\epsilon^{\frac{1}{q}-\frac{1}{2}}\frac{1}{|w|})$. This is the desired estimate with $\beta = \frac{1}{q} - \frac{1}{2} > 0$.

Next consider the integral over A:

$$\begin{aligned}
\int_A |\frac{F_z}{z-w}|dxdy &= \int_A \mathbf{1}_E(z)|F_z|dxdy \\
&= (\int_A \mathbf{1}_E(z)^q dxdy)^{1/q}(\int_A |F_z|^p dxdy)^{1/p} \\
&= O(\text{area}(E\cap A))^{1/q}\|F_z\mathbf{1}_A\|_p \\
&= O((\epsilon r^2\varphi(r))^{1/q})r^{2/p} \\
&= O(\epsilon^{1/q}\frac{1}{|w|}),
\end{aligned}$$

since φ decays faster than any power.

To estimate the integral over X, write

$$X = \cup_{k=1}^R X_k = \cup_{k=1}^R X \cap A_k = \cup_{k=1}^R X \cap \{z : k-1 \leq |z| < k\},$$

Then

$$\begin{aligned}\int_{X_k} \mathbf{1}_E(z)|F_z|dxdy &= (\int_{A_k} \mathbf{1}_E(z)^q dxdy)^{1/q}(\int_{A_k} |F_z|^p dxdy)^{1/p} \\ &= (\text{area}(E \cap A_k))^{1/q}(\int_{A_k} |F_z|^p dxdy)^{1/p} \\ &= (\epsilon k\varphi(k))^{1/q}(O(k))^{1/p} \\ &= O(\epsilon^{1/q}\varphi(k))^{1/q}k^{1+1/p}) \\ &= O(\epsilon^{1/q}k^{-2}),\end{aligned}$$

again since φ decays faster than any power. Summing over k gives the desired estimate. This proves the theorem with $\beta = \frac{1}{q} - \frac{1}{2} > 0$. □

The proof given above shows that the conclusion of Theorem 1.1 still holds if $\int_0^\infty \varphi(r)r^n dr < \infty$ for some (large) finite n that depends on K (in particular, it depends on the value $p > 2$ so that $F_z \in L^p$ in Bojarski's theorem). Similarly, we can assume less if we simply want a uniform bound on $|F(w) - w|$, rather than the $O(1/|z|)$ estimate above. We leave these generalizations to the reader.

Proof of Corollary 1.2. First we note that it suffices to prove this with the additional assumption that μ has bounded support, for a general quasiconformal f is the pointwise limit of such maps (truncate μ_f, apply the measurable Riemann mapping theorem and show the truncated maps converge uniformly on compact subsets to f).

So assume $\mu = \mu$ has bounded support, say inside the disk $D(0, R)$. Then f is conformal outside $D(0, R)$, so we can post-compose by a conformal linear map L to get a quasiconformal map so that $|F(z) - z| \leq C/|z|$, outside $D(0, 2R)$ with a constant that does not depend on F (this follows from the distortion theorem for conformal maps). We apply Theorem 1.1 to get $|F(z) - z| \leq C\epsilon^\beta$, for all z with constants C, β that depend only on k. Note that

$$f(z) = \frac{F(z) - F(0)}{F(1) - F(0)},$$

and that $|F(1) - F(0) - 1| \leq C\epsilon^\beta$, so we get

$$|f(z) - f(w)| = |\frac{F(z) - F(w)}{F(1) - F(0)}| = \frac{|z - w| + O(\epsilon^\beta)}{1 + O(\epsilon^\beta)},$$

and this implies (1.1). □

The following consequence of Theorem 1.2 is used in [7].

Lemma 3.3. *Suppose* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *is* K*-quasiconformal, it fixes* 0 *and* 1*, maps* $\mathbb{R}$ *to* $\mathbb{R}$*, and is conformal in the strip* $\{x + iy : |y| < 1\}$*. Let* $E = \{z : \mu(z) \neq 0\}$ *and suppose* E *is* (ϵ, φ)*-thin. If* ϵ *is sufficiently small (depending on* k *and* φ*), then* $0 < \frac{1}{C} \leq |f'(x)| \leq C < \infty$ *for all* $x \in \mathbb{R}$*, where* C *depends on* K*,* φ *and* ϵ *is otherwise independent of* f*. If we fix* K *and* φ *and let* $\epsilon \to 0$ *then* $C \to 1$*.*

Proof. For each $x \in \mathbb{R}$, f is conformal on $D(x, 1) \subset S$, so Koebe's $\frac{1}{4}$-theorem implies

$$|f'(x)| \simeq \mathrm{dist}(f(x), \partial f(D(x, 1))).$$

However taking $z = x$ and $w \in \partial D(x, 1)$ in (1.1) shows that

$$\mathrm{dist}(f(x), \partial f(D(x, 1))) \simeq 1.$$

This gives the first claim. When ϵ is small, then (1.1) implies that

$$(1 - \delta)S \subset f(S) \subset (1 + \delta),$$

where $\delta > 0$ tends to zero with ϵ (for fixed k and a). Thus as $\epsilon \to 0$, f converges uniformly to the identity on S. In particular, f' converges uniformly to 1 on $\mathbb{R}$. □

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C.J. Bishop, Mathematics Department, Stony Brook University, Stony Brook, NY 11794-3651

Email address: `bishop@math.sunysb.edu`