QUASICONFORMAL MAPS WITH DILATATIONS OF SMALL SUPPORT

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Abstract. We give an estimate that quantifies the fact that a normalized quasi-conformal map whose dilatation is non-zero only on a set of small area approximates the identity uniformly on the whole plane. The precise statement is motivated by an application to the iteration of entire functions.
1. Introduction

If the dilatation $\mu$ of a quasiconformal map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is small, then we expect $f$ to be close to conformal, hence close to linear. There are at least two reasonable senses in which we can ask $\mu$ to be small: that $\|\mu\|_\infty$ is small or that $\{z : \mu(z) \neq 0\}$ is small. In this note we consider the latter possibility. To be more precise, we say a measurable set $E \subset \mathbb{R}^2$ is $(\epsilon, \varphi)$-thin if $\epsilon > 0$ and
\[
\text{area}(E \cap D(z, 1)) \leq \epsilon \varphi(|z|)
\]
where $\varphi : [0, \infty) \to [0, \pi]$ is a bounded, decreasing function, such that
\[
\int_0^\infty \varphi(r)r^n dr < \infty,
\]
for every $n > 1$. If $a > 0$, the function $\varphi(r) = \exp(-ar)$ satisfies this condition, and this example suffices for many applications.

Recall that a quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ is often normalized by post-composing by a conformal linear map in one of two ways. First, we can assume $f(0) = 0$ and $f(1) = 1$. We call this the 2-point normalization. Second, if the dilatation of $f$ is supported on a bounded set, then $f$ is conformal in a neighborhood of $\infty$ and then we can choose $R$ large and post-compose with a linear conformal map so that
\[
|f(z) - z| = O\left(\frac{1}{|z|}\right),
\]
for $|z| > R/2$. We say that such an $f$ is normalized at $\infty$. This is also called the hydrodynamical normalization of $f$. We will first prove an estimate for the hydrodynamical normalization and then deduce one for the 2-point normalization.

**Theorem 1.1.** Suppose $F : \mathbb{C} \to \mathbb{C}$ is $K$-quasiconformal, and $E = \{z : \mu(z) \neq 0\}$ is bounded (so $F$ is conformal near $\infty$) and $F$ is normalized so
\[
|F(z) - z| \leq M/|z|,
\]
near $\infty$. Assume $E$ is $(\epsilon, \varphi)$-thin. Then for all $z \in \mathbb{C}$,
\[
|F(z) - z| \leq \frac{\epsilon^\beta}{|z| + 1},
\]
where $\beta$ depends only on $K$ and $\varphi$. In particular, as $\epsilon \to 0$, $F$ converges uniformly to the identity on the whole plane.
Corollary 1.2. Suppose \( f : \mathbb{C} \to \mathbb{C} \) is \( K \)-quasiconformal, \( F(0) = 0, F(1) = 1 \), and \( E = \{ z : \mu(z) \neq 0 \} \) is \((\epsilon, \varphi)\)-thin. Then

\[
(1 - Ce^\beta)|z - w| - Ce^\beta \leq |f(z) - f(w)| \leq (1 + Ce^\beta)|z - w| + Ce^\beta,
\]

where \( C \) and \( \beta \) only depend on \( k = \|\mu\|_\infty \) and \( \varphi \).

Similar estimates are known, e.g., compare to the well known result of Teichmüller and Wittich (e.g., Theorem 7.3.1 of [5], [6], [7]) or estimates of Dyn’kin [3]. The version stated above is intended for specific applications to holomorphic dynamics, as in [2] and [4] (a particular consequence used in the latter paper is given as Lemma 1.12). Because the quasiconformal maps used in these references satisfy the strong \( \epsilon \)-thin condition, it seemed desirable to have a self-contained proof of the estimate above.

We will use the following facts about quasiconformal maps which may be found in Ahlfors’ book [1]:

**Lemma 1.3** (Characterization of quasicircles). For each \( K \geq 1 \) there is a \( C = C(K) < \infty \) so that the following holds. If \( f : \mathbb{C} \to \mathbb{C} \) is \( K \)-quasiconformal and \( r > 0 \) so that \( f(\gamma) \subset \{ z : r \leq |z - w| \leq Cr \} \).

**Theorem 1.4** (Borjarki’s theorem). If \( 1 \leq K < \infty \), there is a \( p > 2 \) and \( A, B < \infty \) so that the following holds. If \( f : \mathbb{C} \to \mathbb{C} \) is \( K \)-quasiconformal, and \( Q \subset \mathbb{C} \) is a square, then

\[
\frac{1}{\text{area}(Q)} \int_Q |f_z|^p dx \, dy \leq A \left( \frac{1}{\text{area}(Q)} \int_Q |f_z|^2 dx \, dy \right)^{1/2} \leq B \frac{\text{diam}(f(Q))}{\text{diam}(Q)}
\]

**Lemma 1.5** (Pompeiu’s formula). If \( \Omega \) has a piecewise \( C^1 \) boundary and \( f \) is quasiconformal on \( \Omega \), then

\[
f(w) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(z)}{z-w} dz - \frac{1}{\pi} \int_{\Omega} \frac{f_z}{z-w} dx \, dy.
\]

If \( \Omega \) is a topological annulus in the plane with boundary components \( \gamma_1, \gamma_2 \) that are closed Jordan curves, then \( \text{mod} (\Omega) \) refers to the modulus of the path family in \( \Omega \) that separates the boundary components. This is the same as the extremal length of the path family that connects the boundary components (also called the extremal
distance between the boundary components). If \( A(a, b) \equiv \{ z : a < |z| < b \} \) then it is a standard fact that \( \text{mod} (A) = \frac{1}{2\pi} \log \frac{b}{a} \). Let

\[
D_f = \frac{|f_z| - |f_{\overline{z}}|}{|f_z| + |f_{\overline{z}}|},
\]

\[
J_f = |f_z|^2 - |f_{\overline{z}}|^2 = (|f_z| - |f_{\overline{z}}|)(|f_z| + |f_{\overline{z}}|),
\]
denote the distortion and Jacobian of \( f \) respectively. Note that \( D_f \geq 1 \) and \( f \) is conformal if and only if \( D_f \equiv 1 \).

**Lemma 1.6.** Suppose \( f \) is a \( K \)-quasiconformal map from \( A_m = A(1, e^m) \) to \( A_M = A(1, e^M) \). Then

\[
M \geq m - \frac{1}{2\pi} \int_{A(1,e^m)} (D_f(z) - 1) \frac{dxdy}{r^2}.
\]

**Proof.** Let \( \Gamma_M \) be the path family connecting the boundary components of \( A_M \). If \( \tilde{\rho} \) is admissible for this family then

\[
\rho(z) = \tilde{\rho}(f(z))(|f_z| + |f_{\overline{z}}|)
\]
is admissible for \( \Gamma_m \), the path family connecting the boundary components of \( A_m \). Therefore the modulus of \( \Gamma_m \) satisfies

\[
\text{mod} (\Gamma_m) \leq \int_{A_m} \tilde{\rho}(f(z))^2(|f_z| + |f_{\overline{z}}|)^2 dxdy.
\]

Applying this formula to the inverse of \( f \) shows that for any admissible \( \rho \) for \( \Gamma_m \),

\[
\text{mod} (f(\Gamma_m)) \leq \int_{A_m} \rho(z)^2 \frac{1}{(|f_z| - |f_{\overline{z}}|)^2} J_f dxdy
\]

\[
\leq \int_{A_m} \rho(z)^2 \frac{1}{(|f_z| - |f_{\overline{z}}|)^2} (|f_z|^2 - |f_{\overline{z}}|^2) dxdy
\]

\[
\leq \int_{A_m} \rho(z)^2 \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} dxdy
\]

\[
\leq \int_{A_m} \rho(z)^2 D_f(z) dxdy.
\]
Applying this with the admissible metric \( \rho(z) = \frac{1}{m|z|} \), we get

\[
\frac{2\pi}{M} = \text{mod}(f(\Gamma_m)) \leq \frac{1}{m^2} \int_{A_m} \frac{D_f(z)}{|z|^2} dx dy
= \frac{1}{m^2} \left[ \int_{A_m} \frac{D_f(z) - 1}{|z|^2} dx dy + \int_{A_m} \frac{1}{|z|^2} dx dy \right]
= \frac{1}{m^2} \int_{A_m} \frac{D_f(z) - 1}{|z|^2} dx dy + \frac{2\pi}{m}.
\]

Rearranging gives

\[
m - M \leq \frac{M}{2\pi m} \int_{A_m} \frac{D_f(z) - 1}{|z|^2} dx dy,
\]

or

\[
M \geq m - \frac{M}{2\pi m} \int_{A_m} \frac{D_f(z) - 1}{|z|^2} dx dy.
\]

If \( M > m \), the lemma is trivially true. If \( M \leq m \), then because the integral in non-negative, the inequality above becomes

\[
M \geq m - \frac{1}{2\pi} \int_{A_m} \frac{D_f(z) - 1}{|z|^2} dx dy.
\]

Thus in either case the lemma holds. \( \square \)

**Lemma 1.7.** Suppose \( f \) is a \( K \)-quasiconformal map from \( A_m = A(1, e^m) \) to \( A_M = A(1, e^M) \). Then

\[
M \leq m + \frac{1}{2\pi} \int_{A_m} (D_f - 1) \frac{dx dy}{r^2}.
\]

**Proof.** If we cut \( A_m \) with a radial slit and let \( g = \log(f) \), then \( g \) maps \( A_m \) to a quadrilateral with its vertical sides on \( \{ x = 0 \} \) and \( \{ x = M \} \). This quadrilateral has area \( 2\pi M \). If we integrate over the radial segments in \( A_m \), we get

\[
M \leq \int_{1}^{\exp(m)} (|g_z| + |g_{\bar{z}}|) dr
\]

so integrating over all angles and using \( r dr d\theta = dx dy \) gives

\[
2\pi M \leq \int_{0}^{2\pi} \int_{1}^{\exp(m)} (|g_z| + |g_{\bar{z}}|) dr d\theta \leq \int_{A_m} (|g_z| + |g_{\bar{z}}|) \frac{dx dy}{r}.
\]
Thus by Cauchy-Schwarz,

\[
(2\pi M)^2 \leq \left( \int_{A_m} (|g_z| + |g_{\bar{z}}|)(|g_z| - |g_{\bar{z}}|) \, dx \, dy \right) \left( \int_{A_m} \frac{|g_z| + |g_{\bar{z}}|}{|g_z| - |g_{\bar{z}}|} \, dx \, dy \right)
\]

\[
\leq \left( \int_{A_m} J_g \, dx \, dy \right) \left( \int_{A_m} D_g \, dx \, dy \frac{1}{r^2} \right)
\]

\[
\leq 2\pi M \left( \int_{A_m} D_f \, dx \, dy \frac{1}{r^2} \right),
\]

where in the last line we have used the facts that \( g(A_m) \) has area \( 2\pi M \) and \( D_g = D_f \) (since \( \log z \) is conformal on the slit annulus). Thus

\[
M \leq \frac{1}{2\pi} \int_{A_m} 1 + (D_f(z) - 1) \frac{dx \, dy}{|z - w|^2}
\]

\[
= m + \frac{1}{2\pi} \int_{A_m} (D_f(z) - 1) \frac{dx \, dy}{|z - w|^2}.
\]

□

The following simply combines the last two results.

**Corollary 1.8.** Suppose \( f \) is a \( K \)-quasiconformal map from \( A_m = A(1, e^m) \) to \( A_M = A(1, e^M) \). Then

\[
M = m + O\left( \frac{1}{2\pi} \int_{A_m} \frac{D_f(z) - 1}{r^2} \, dx \, dy. \right)
\]

A special case of this is:

**Corollary 1.9.** Suppose \( f \) is a \( K \)-quasiconformal map from \( A_m = A(1, e^m) \) to \( A_M = A(1, e^M) \). Suppose \( D_f(z) \leq D \) on \( A_m \). Suppose \( \mu \) is the dilatation of \( f \), that \( E = \{ z : \mu(z) \neq 0 \} \) and that \( E_k = E \cap \{ 2^{k-1} < |z| < 2^k \} \). If we choose an integer \( n \) so that \( m \leq 2^n \), then

\[
M = m + O((D - 1) \sum_{k=0}^{n} 2^{-2k} \text{area}(E_k)).
\]

Next we apply these estimates to quasiconformal maps with dilatations that have small support in a precise sense.

**Lemma 1.10.** Suppose \( F \) is a \( K \)-quasiconformal map with dilatation \( \mu \), that \( \mu \) has bounded support, and that \( F \) has the hydrodynamical normalization at \( \infty \). Let \( E = \)
\( \{ z : \mu(z) \neq 0 \} \) and suppose for some \( t > 0 \), \( E \) satisfies
\[
\int_{E \setminus D(w,t)} \frac{dxdy}{|z-w|^2} \leq a,
\]
for every \( w \in \mathbb{C} \). Then there is a \( C - C(K,a) < \infty \), depending only on \( K \) and \( a \), so that for every \( w \in \mathbb{R}^2 \) and \( r \geq t \),
\[
\frac{1}{C} \leq \frac{1}{r} \text{diam}(F(D(w,r))) \leq C.
\]

**Proof.** We need only prove this for \( r = t \) since for \( r > t \), we can simply apply the lemma after setting \( t = r \) (the integral just gets smaller).

The mapping \( G(z) = F(tx)/t \), satisfies the same estimates as \( F \), but with \( t \) replaced by 1. If we prove the lemma for \( G \), it follows for \( F \), so it suffices to assume \( t = 1 \).

By assumption we can choose \( R > 100 \) so that \(|f(z) - z| \leq 1\), for \(|z| > R/8\). The result is clear if \(|w| > R/2\), so we may assume \(|w| \leq R/2\). Fix such a \( w \). Let \( m = \log R \), so \( R = e^m \), and consider the annulus \( A = \{ z : 1 < |z-w| < e^m \} \). \( F(A) \) is a topological annulus and can be conformally mapped to \( A_M = \{ 1 < |z| < e^M \} \) for some \( M > 1 \). By Corollary 1.8,
\[
M = m + O\left( \int_{A_m} \frac{Df - 1}{|z-w|^2} dxdy \right).
\]

By our assumptions, this becomes
\[
M = m + O\left( \frac{K - 1}{2\pi} \int_{A_m} 1_E(z) \frac{dxdy}{|z-w|^2} \right) = m + O(Ka),
\]
where \( 1_E \) denotes the indicator function of \( E \) (the function that is one on \( E \) and zero off \( E \)) and we have used the fact that \( E \) has finite planar area and \(|z-w|^{-1} \leq 1\) on \( A_m \) (recall \( w \) is the center of the annulus and the inner radius is at least 1.).

By Corollary 1.3, the boundary components of \( f(A_m) \) are each closed curves that are contained in round annuli (with concentric circles) of bounded modulus (depending on \( K \)). Thus \( f(A_m) \) is contained in a topological annulus \( A' \) with circular boundaries \( \gamma_1, \gamma_2 \) (not necessarily concentric) whose diameters are comparable to the diameters of the boundary components of \( f(A_m) \). By monotonicity of modulus, the modulus of the annulus \( A' \) (denoted \( M'/2\pi \)) is larger than the modulus \( M/2\pi \) of \( f(A) \), hence \( M' \geq M \). Moreover, we claim
\[
M' \leq \log \frac{\text{diam}(\gamma_2)}{\text{diam}(\gamma_1)}.
\]
This is well known to hold with equality if the circles $\gamma_1, \gamma_2$ are concentric. If they are not, then we can apply a Möbius transformation that maps the outer circle, $\gamma_2$, to itself and moves the inner circle, $\gamma_1$ to circle concentric with $\gamma_2$. This makes the Euclidean diameter of $\gamma_1$ larger and preserves the modulus between the circles, and this proves the claimed inequality. Thus

$$M \leq M' \leq \log \frac{\text{diam}(\gamma_2)}{\text{diam}(\gamma_1)},$$

or

$$\text{diam}(\gamma_1) \leq \text{diam}(\gamma_2) \cdot e^{-M} = \text{diam}(\gamma_2) \cdot e^{-m+O(KA)}.$$

Since $|f(z) - z| \leq 1$ on $\{|z| = R\}$ we know $\text{diam}(\gamma_2) \simeq R = e^m$. Using this and the fact $M = m + O(Ka)$ prove above gives

$$\text{diam}(f(\{|z-w|=1\})) \simeq \text{diam}(\gamma_1) = O(e^{Ka}).$$

To get the other direction, we choose $\gamma_1, \gamma_2$ to be circles that bound an annulus inside $f(A_m)$, again with diameters comparable to the diameters of the corresponding components of $\partial f(A_m)$. We then use monotonicity again, and argue as before, but now we note that since $f$ is close the identity for $|z| > R/2$, the curve $\gamma_1$ is not too close to $\gamma_2$, i.e., the distance between them is comparable to $R$. Thus in the argument above, where we moved $\gamma_1$ be be concentric with $\gamma_2$, its Euclidean diameter was only changed by a bounded factor. Thus

$$\text{diam}(\gamma_1) \gtrsim \text{diam}(\gamma_2) \cdot e^{-M} = \text{diam}(\gamma_2) \cdot e^{-m-O(KA)} \gtrsim e^{-O(KA)}.$$

This proves the lemma. \qed

If $F$ is as above, Bojarski’s theorem (Theorem 1.4) says there is a $p = p(K) > 2$ so that the $L^p$ norm of $F_z$ is uniformly bounded on every unit radius disk. If a region can be covered by $n$ such disks then the $L^p$ norm is $O(n^{1/p})$ with a uniform constant, i.e.,

**Corollary 1.11.** *If $F$ satisfies the conditions of Lemma 1.10, and $p = p(K) > 2$ is as above, then $\|F_z \cdot 1_{D(z,r)}\|^p = O(r^{2/p})$ uniformly for all $z \in \mathbb{C}$.***

**Proof of Theorem 1.1.** Suppose the support of $\mu$ is contained in $D(0, R)$. The main idea is to use the Pompeiu formula

$$F(w) = \frac{1}{2\pi i} \int_{|z|=r} \frac{F(z)}{z-w} \, dz - \frac{1}{\pi} \iint_{|z|<r} \frac{F_z}{z-w} \, dx \, dy. \quad (1.3)$$
Because of our assumptions on $F$, the first integral is
\[
\frac{1}{2\pi i} \int_{|z|=r} \frac{z + O(1/|z|)}{z - w} \, dz = w + O(1/r).
\]
The left-hand side of (1.3) and the second integral are both constant for $r > R$, so the first integral must equal $w$ for all $r > R$. Thus
\[
F(w) = w - \frac{1}{\pi} \iint_{|z|<r} \frac{F_z}{z - w} \, dx \, dy = w - \frac{1}{\pi} \iint_{|z|<r} \frac{\mu F_z}{z - w} \, dx \, dy.
\]
Since $|F_z| = |\mu F_z| \leq k|F_z|$, we get
\[
|F(w) - w| \leq \frac{k}{\pi} \int_{|z|<r} \frac{|F_z|}{|z - w|} \, dx \, dy.
\]
where $k = (K - 1)/(K + 1)$ is our upper bound for $|\mu|$.

The estimate in the theorem already holds if $|w| \geq R$, so assume $|w| < R$. Let $r = \max(1, |w|/2)$. We will estimate the integral
\[
\int_{E} \frac{F_z}{z - w} \, dx \, dy,
\]
by cutting $D(0, R)$ into three pieces:
\[
D_1 = \{ z : |z - w| \leq 1 \}
\]
\[
A = \{ z : 1 \leq |z - w| \leq r \}
\]
\[
X = D(0, R) \setminus (D_1 \cup A),
\]
and showing the integral over each piece is $O(\epsilon^\beta/|w|)$ for some $\beta > 0$ depending only on $K$.

First consider $D_1$. With $p$ as in Corollary 1.4, the $L^p$ norm of $F_z$ over $D_1$ is uniformly bounded, so using Hölder’s inequality with the conjugate exponents, we get
\[
(1.4) \int_{D_1} \frac{F_z}{z - w} \, dx \, dy = O(\|1_{E\cap D(w,1)}\|_q).
\]
Since $E \cap D(w,1)$ has area at most $\varphi(|w|) \leq \varphi(r)$, the $L^q$ norm on the right side of (1.4) is bounded above by what happens when $E \cap D(w,1)$ is a disk of radius $s \simeq (\epsilon \varphi(r))^{1/2}$ centered at $w$. In this case we get the bound (using polar coordinates and recalling $1 < q < 2$)
\[
O\left(\int_{0}^{s} r^{-q} r \, dr\right)^{1/q} = O(s^{(2-q)/q}) = O((\epsilon \varphi(r))^{1-\frac{1}{2}}).
\]
Since \( \varphi \) tends to zero faster than any polynomial, this is \( = O(\epsilon^{\frac{1}{q} - \frac{1}{2}}) \). This is the desired estimate with \( \beta = \frac{1}{q} - \frac{1}{2} > 0 \).

Next consider the integral over \( A \):

\[
\int_{A} \left| \frac{F_z}{z - w} \right| dxdy = \int_{A} 1_{E}(z)|F_z|dxdy
\]

\[
= (\int_{A} 1_{E}(z)^q dxdy)^{1/q}(\int_{A} |F_z|^p dxdy)^{1/p}
\]

\[
= O(\text{area}(E \cap A))^{1/q}\|F_z 1_A\|_p
\]

\[
= O((\epsilon r^2 \varphi(r))^{1/q} r^{2/p})
\]

\[
= O(\epsilon^{1/q} \frac{1}{|w|}),
\]

since \( \varphi \) decays faster than any power.

To estimate the integral over \( X \), write

\[
X = \bigcup_{k=1}^{\infty} X_k = \bigcup_{k=1}^{\infty} X \cap A_k = \bigcup_{k=1}^{\infty} X \cap \{ z : k-1 \leq |z| < k \},
\]

Then

\[
\int_{X_k} 1_{E}(z)|F_z|dxdy = (\int_{A_k} 1_{E}(z)^q dxdy)^{1/q}(\int_{A_k} |F_z|^p dxdy)^{1/p}
\]

\[
= (\text{area}(E \cap A_k))^{1/q}(\int_{A_k} |F_z|^p dxdy)^{1/p}
\]

\[
= (\epsilon k \varphi(k))^{1/q}(O(k))^{1/p}
\]

\[
= O(\epsilon^{1/q} \varphi(k))^{1/q} k^{1+1/p}
\]

\[
= O(\epsilon^{1/q} k^{-2}),
\]

again since \( \varphi \) decays faster than any power. Summing over \( k \) gives the desired estimate. This proves the theorem with \( \beta = \frac{1}{q} - \frac{1}{2} > 0 \).

The proof given above shows that the conclusion of Theorem 1.1 still holds if \( \int_{0}^{\infty} \varphi(r)r^ndr < \infty \) for some (large) finite \( n \) that depends on \( K \) (in particular, it depends on the value \( p > 2 \) so that \( F_z \in L^p \) in Bojarski's theorem). Similarly, we can assume less if we simply want a uniform bound on \( |F(w) - w| \), rather than the \( O(1/|z|) \) estimate above. We leave these generalizations to the reader.

**Proof of Corollary 1.2.** First we note that it suffices to prove this with the additional assumption that \( \mu \) has bounded support, for a general quasiconformal \( f \) is the
pointwise limit of such maps (truncate $\mu_f$, apply the measurable Riemann mapping theorem and show the truncated maps converge uniformly on compact subsets to $f$).

So assume $\mu = \mu$ has bounded support, say inside the disk $D(0, R)$. Then $f$ is conformal outside $D(0, R)$, so we can post-compose by a conformal linear map $L$ to get a quasiconformal map

$$F(z) = z + O\left(\frac{1}{z}\right),$$

or

$$|F(z) - z| \leq C/|z|,$$

outside $D(0, 2R)$ with a constant that does not depend on $F$ (this follows from the distortion theorem for conformal maps). We apply Theorem 1.1 to get

$$|F(z) - z| \leq Ce^\beta,$$

for all $z$ with constants $C, \beta$ that depend only on $k$. Note that

$$f(z) = \frac{F(z) - F(0)}{F(1) - F(0)},$$

and that

$$|F(1) - F(0) - 1| \leq C e^\beta,$$

so,

$$|f(z) - f(w)| = \left| \frac{F(z) - F(w)}{F(1) - F(0)} \right| = \frac{|z - w| + O(\epsilon^\beta)}{1 + O(\epsilon^\beta)},$$

and this implies (1.1). \hfill \square

The following consequence of Theorem 1.2 is used in [4].

**Lemma 1.12.** Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is $K$-quasiconformal, it fixes 0 and 1, maps $\mathbb{R}$ to $\mathbb{R}$, and is conformal in the strip $\{x + iy : |y| < 1\}$. Let $E = \{z : \mu(z) \neq 0\}$ and suppose $E$ is $(\epsilon, \varphi)$-thin. If $\epsilon$ is sufficiently small (depending on $k$ and $\varphi$), then $0 < \frac{1}{C} \leq |f'(x)| \leq C < \infty$ for all $x \in \mathbb{R}$, where $C$ depends on $K, \varphi$ and $\epsilon$ is otherwise independent of $f$. If we fix $K$ and $\varphi$ and let $\epsilon \to 0$ then $C \to 1$.

**Proof.** For each $x \in \mathbb{R}$, $f$ is conformal on the disk $D(x, 1) \subset S$, so Koebe’s $\frac{1}{4}$-theorem says that

$$|f'(x)| \simeq \text{dist}(f(x), \partial f(D(x, 1))).$$
However taking $z = x$ and $w \in \partial D(x, 1)$ in (1.1) shows that
\[
\text{dist}(f(x), \partial f(D(x, 1))) \simeq 1.
\]
This gives the first claim. When $\epsilon$ is small, then (1.1) implies that
\[
(1 - \delta)S \subset f(S) \subset (1 + \delta),
\]
where $\delta > 0$ tends to zero with $\epsilon$ (for fixed $k$ and $a$). Thus as $\epsilon \to 0$, $f$ converges uniformly to the identity on $S$. In particular, $f'$ converges uniformly to 1 on $\mathbb{R}$. □

References


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