EQUI-TRIANGULATION OF POLYGONS

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Abstract. We prove that any two polygons of the same area can be triangulated using the same set of triangles. This strengthens the Wallace–Bolyai–Gerwien theorem from dissections to triangulations.

1. Introduction

A polygonal region Ω is a bounded, closed set in the plane whose boundary $P = \partial\Omega$ is a simple closed polygon. A dissection of Ω is a finite collection of polygonal sub-regions that have disjoint interiors and whose union is Ω. The Wallace-Bolyai-Gerwien theorem says any two polygonal regions Ω_1, Ω_2 with the same area have an equi-dissection, i.e., Ω_1 has a dissection, whose pieces can be rotated and translated to form a dissection of Ω_2. In this case, we also say Ω_1 and Ω_2 are equi-dissectable, or dissection equivalent, or scissors congruent.

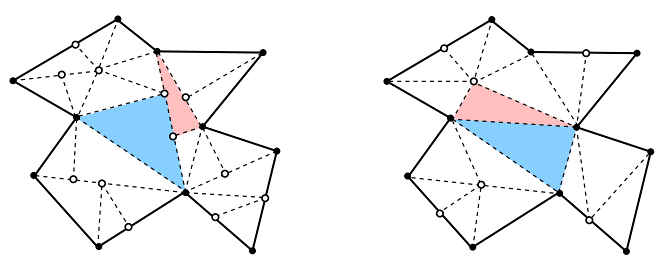

Figure 1. On the left is a triangular dissection of a polygon and on the right a triangulation. The white dots are called Steiner points, i.e., vertices of the that are not vertices of the original polygon.

A triangular dissection is a dissection where all the pieces are triangles. Since any polygon can be triangulated, it follows from the WBG theorem that any two regions

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with a equi-dissection have a triangular equi-dissection. A triangular dissection is a triangulation if any two pieces that intersect either intersect a one point that is a vertex of both pieces or a segment than is a full closed edge of both triangles. In other words, the triangles form a simplicial complex. Although triangulations satisfy much more stringent conditions than dissections, we will show that the Wallace-Bolyai-Gerwien theorem still holds in this more restrictive context.

Theorem 1.1. *Any two polygonal regions of the same area have triangulations that use the same set of triangles (up to rotation and translation).*

We call this an equi-triangulation of the domains. The equi-triangulation we build is a refinement of the equi-dissection we are given, i.e., each dissection piece will be the union of the triangulation pieces it contains. See Figure 2.

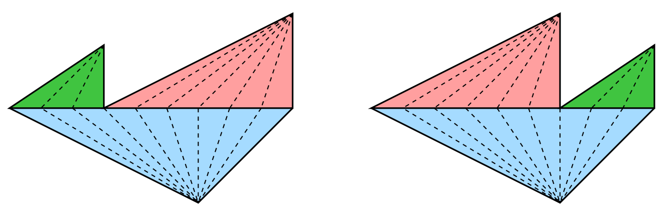

FIGURE 2. The solid triangles show triangular equi-dissections of two polygons and the dashed lines give a refinement to an equi-triangulation. This works here because the horizontal edges have lengths that are integer multiples of a common value. The general case requires a more complicated construction.

Despite a large literature on optimal meshing and triangulation, the question of whether two polygons always have an equi-triangulation seems not to have been previously considered. My interest in this problem was motivated by results in [9] that showed the optimal angle bounds for dissecting a polygon are the same as for triangulating the polygon. This led to the question of what other results for dissections also hold for triangulations, and the Wallace-Bolyai-Gerwien theorem seemed like an obvious candidate to consider.

The basic idea behind Theorem 1.1 is to convert an equi-dissection of two polygonal regions into a simple dynamical system $\Phi : X \to X$ that contains a finite number of special points (vertices of one dissection piece that are interior edge points of another piece). It will be easy to see that we can refine the equi-dissection to an equi-triangulation if and only if each of these points has a finite orbit (is pre-periodic). Simple examples show that, in general, these orbits can be infinite, but we will prove that any equi-dissection can be modified (on arbitrarily small area) to give another equi-dissection of the same two regions for which the orbits are all finite.

2. The Wallace–Bolyai–Gerwien theorem

According to Stewart [21] the Wallace-Bolyai-Gerwien theorem seems to have been proven by William Wallace around 1808, and independently by Paul Gerwien in 1833 in response to a question of Wolfgang (or Wolfgang) Bolyai (father of the hyperbolic geometry Janos Bolyai). Giovannini [15] credits John Lowry with a solution in 1814, in response to question of Wallace, with independent proofs by Bolyai (somewhat sketchy) in 1831 and by Gerwien (very detailed) in 1833.

The proof of the theorem is elementary and well known, but we sketch it here for the convenience of the reader. The first observation is that if the pairs (Ω_1, Ω_2) and (Ω_2, Ω_3) both have equi-dissections, then so does (Ω_1, Ω_3): intersecting the two dissections of Ω_2 gives a polygonal refinement of both dissections that can be transferred via the two correspondences to equivalent dissections of Ω_1 and Ω_2. Thus dissection equivalence is a transitive relation.

Next recall that any polygon can be triangulated. Each triangle of area a has a equi-dissection with a rectangle (see the top of Figure 3), and any rectangle is dissection equivalent to another rectangle with one side twice as long and one side half as long (center of Figure 3). Repeating this, we can obtain a rectangle with one side length s between 1 and 2 and the other between $a/2$ and a. The three-piece dissection at the bottom of Figure 3 shows this is equivalent to a $1 \times a$ rectangle. Stacking such rectangles shows that any polygon of area A is equivalent to a $1 \times A$ rectangle, so by transitivity, we are done.

Later it will be convenient to assume that we are given an equi-dissection so that there are corresponding pieces, say Q_1^1 and Q_1^2 that have corresponding sides $S_1 \subset \partial\Omega_1$ and $S_2 \subset \partial\Omega_2$. If Ω_1 and Ω_2 are equal area polygonal regions, this is easy to accomplish. Choose Q_1^1, Q_1^2 to be small squares of equal size that each have exactly one side lying on $\partial\Omega_1$ and $\partial\Omega_2$. Then $\Omega_1 \setminus Q_1^1$ and $\Omega_2 \setminus Q_1^2$ are equal area polygonal

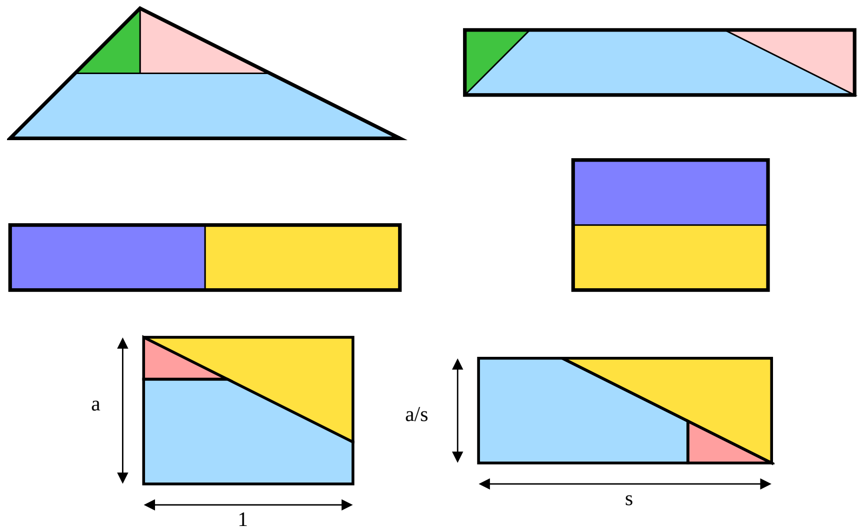

FIGURE 3. The proof of the Wallace-Bolyai-Gerwien theorem: (1) a triangle is always dissection equivalent to a rectangle, (2) we can change the rectangle's side length by powers of 2, (3) when one side has length $1 < s \leq 2$$ we can make it equal to 1. Thus every polygon of area A is dissection equivalent to a $1 \times A$ rectangle.

regions, so they have an equi-dissection by the Wallace–Bolyai–Gerwien theorem. Adding Q_1^1 and Q_1^2 back in gives an equi-dissection of Ω_1 and Ω_2 (which can be made triangular by triangulating the square). See Figure 6 where the white squares denote one possible choice of such a Q_1^1 and Q_2^2.

3. Swaps and flips

Suppose $\mathcal{P} = \{P_k\}_1^N$ is finite collection of disjoint simple polygons. By a polygon we mean a simple closed Jordan curve consisting on a finite number of line segments. Let V_k denote the (finite) vertex set of P_k. The points of $E_k = P_k \setminus V_k$ are called the edge points of P_k (or the interior edge points, to be very precise). We define the polygonal region Ω_k to be the union of P_k and its bounded complementary component.

An arrangement of $\mathcal{P}$ is a set of planar isometries $\mathcal{F} = \{f_k\}_1^N$ so that the image regions $\{Q_k\} = \{f_k(\Omega_k)\}_1^N$ have disjoint interiors. We say this arrangement is a dissection of $\Omega = \cup_{k=1}^N f_k(\Omega_k)$ into the pieces $\{Q_k\}$. If all the maps $\{f_k\}$ preserve orientation (i.e., they are made up from rotations and translations) then we say the dissection preserves orientation or has "no reflections" or "no flips". Otherwise we say the dissection allows reflections. If all the polygons $\{P_k\}$ are triangles, we call this a triangular dissection of Ω.

Given an arrangement $\mathcal{F}$ of $\mathcal{P}$ there is an associated "flip" map ϕ defined on the disjoint union $\sqcup_k E_k$ of the edge points of all the P_k. The images of such points will either be points other points in $\sqcup E_k$ or one of two special points v and b that we add. We define the map ϕ (for "flip") as follows:

• First, given a point $x \in E_k$, set $\phi(x) = y$ if $y \in E_j$, $j \neq k$ and $f_j(y) = f_k(x)$. Geometrically, $p = f_k(x) = f_j(y)$ is an edge point for two different dissection pieces.
• Next, set $\phi(x) = v$ if there is a $j \in [1, N]$ and $y \in V_j$ so that $f_j(y) = f_k(x)$. These correspond to edge points of one dissection element that are vertices of a different element. Since $f_k(x)$ could be a vertex for several pieces, y is not well defined and so we set $\phi(x) = v$ to record this fact.
• Finally, set $\phi(x) = b$ if there is no j so that $f_(x) \in f_j(P_j)$. These are interior edge points of one dissection element that lie on the boundary of the region Ω being dissected. Again, there is no point in any polygon that x should map to, so we send it to the special point b.

Note that the dissection is actually a mesh if and only if $\phi^{-1}(v) = \emptyset$, i.e., there are no edge points of one dissection element that correspond to a vertex of a different element. In particular, a triangular dissection is a triangulation if and only if $\phi^{-1}(v) = \emptyset$. Points of $E = \sqcup E_k$ such that $\phi(x) = v$ will be called exceptional vertices; these are edge points of one dissection piece that are vertices of another piece. We are interesting in taking a dissection where such points occur and replacing it by another dissection where they do not.

To make ϕ into a map from space into itself, we set $X = (\sqcup_k E_k) \sqcup \{v\} \sqcup \{b\}$ and set $\phi(v) = v$ and $\phi(b)$. Thus b, v are "cemetery states": when iterating ϕ, we never leave these states after reaching them. Thus ϕ is a map of X to itself, although the dynamics of this map are not very complicated: each point x either satisfies $x = \phi^2(x)$, or $\phi^2(x) = \phi(x) = v$, or $\phi^2(x) = \phi(x) = b$. Here $\phi^2(x)$ is defined to be the iteration $\phi(\phi(x))$ and, more generally, we inductively set $\phi^{n+1}(x) = \phi(\phi^n(x))$. However, things becomes more interesting when we consider two dissections, and allow the two flip maps to interact.

Suppose $\mathcal{F}_1 = \{f_k^1\}_1^N$ and $\mathcal{F}_2 = \{f_k^2\}_1^N$ are two arrangements of $\mathcal{P}$ giving an equi-dissection of polygonal regions Ω_1 and Ω_2. The images $Q_k^j = f_k^j(P_k)$ for $1 \leq k \leq N$ and $j \in 1, 2$ are called the dissection pieces and the pair Q_k^1 and Q_k^2 are called corresponding pieces. We denote the isometry that "swaps" two corresponding pieces Q_k^1, Q_k^2 by $\sigma_k^1 = f_k^2 \circ (f_k^1)^{-1}$. This map is orientation preserving if the arrangements are. There may be other isometries between the pieces due to symmetries, but we assume we have fixed a single choice of σ_k. Sub-pieces (e.g., vertices or edges) of corresponding dissection pieces are called corresponding if they are images under this isometry. If corresponding pieces are not already triangles, clearly we can triangulate one of them, and then take the corresponding triangulation of the other to get equi-triangulations of the two pieces.

Let $X = \left(\sqcup_{k=1}^N E_k\right) \sqcup \left(\sqcup_{k=1}^N E_k\right) \sqcup \{v\} \sqcup \{b\}$, be the disjoint union of two copies of each edge of $\{\mathcal{P}\}$ together with our two special states v and b. Define a map by "swap then flip". More precisely

$$\Phi(x) = \begin{cases} \phi_2(\sigma(x)), & x \in X_1 \\ \phi_1(\sigma^{-1}, (x)) & x \in X_2 \\ v, & x = v \\ b, & x = b \end{cases},$$

so that we have a well defined map $\Phi : X \to X$. An example is shown in Figure 4. We start at the point labeled 1. This is an exceptional point where the flip map ϕ_1 is not defined, but the swap map σ is defined and sends it to the point 2. Then ϕ_2 maps it to 3. Thus the first application of $\Phi = \phi_2 \circ \sigma$ sends 1 to 3. Another application of σ sends 3 to 4, and then ϕ_1 sends this to the point b.

The same dynamics are expressed more concisely in Figure 5, where we simply show x, $\Phi(x)$ and $\Phi^2(x)$ as the points labeled $1, 2, 3$ respectively. Because we have removed the gaps between dissection pieces, we have added arrows to show which piece each piece belongs to. These arrows are needed in general, but in this case, the initial point can only belong to one piece; it is a vertex of the other pieces that it belongs to. Hence each of its iterates is forced to belong to a particular piece. Later in this paper, we will only draw orbits of exceptional points, and we will omit the arrows since we can deduce their directions by examining the orbit.

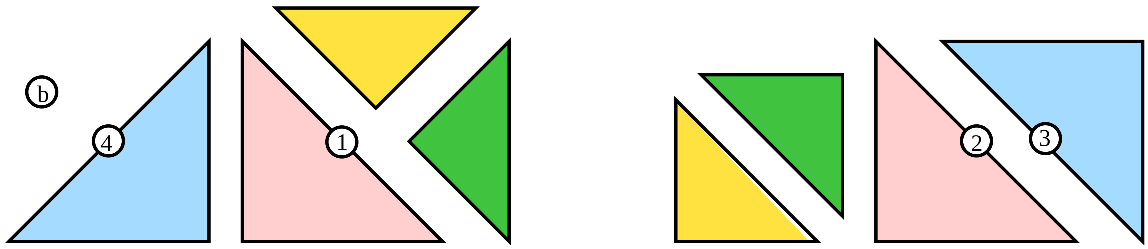

FIGURE 4. Two iterates of Φ. The orbit ends at the special point b.

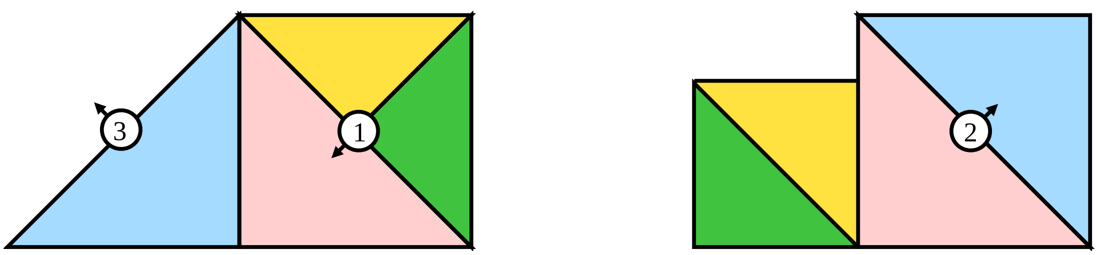

FIGURE 5. A more concise version of Figure 4. The arrows are redundant since iterates of exceptional vertices are forced to belong to a particular piece in each dissection. Also any point on the boundary of the dissected region is actually equal to the special point b.

4. ORBITS AND TRIANGULATIONS

We say an edge point of one dissection piece is an exceptional vertex if it is the vertex of another dissection piece, i.e., the flip map ϕ sends it to the special point v. It is easy to see that a triangular dissection is a triangulation if and only if there are no exceptional vertices, i.e., iff $\Phi^{-1}(v) \neq \{v\}$. The orbit of such a point under Φ is called an exceptional orbit.

If $EV_1 = \phi_1^{-1}(v) \setminus \{v\}$ is not empty, we can modify the $\mathcal{F}_1$ by adding $EV_1 \cap P_k$ to the vertex set of V_k of P_k and triangulating P_k using this new, larger, vertex set. This replaces P_k by a finite set of new triangles. Doing this for each triangle P_k gives two new triangular dissections of Ω and a new maps ϕ_1, ϕ_2. The new ϕ_1 satisfies $\phi_1^{-1}(v) = \{v\}$ because we have removed all the previous preimage points from the domain of definition of ϕ_1 (we converted them from edge points to vertex points). Thus the first dissection has been refined to a triangulation.

However, because we have increased the set of vertices in both dissections, we may have increased the set $EV_2 = \phi_2^{-1}(v) \setminus \{v\}$ by adding the set $\sigma(EV_1)$ to the previously defined set of exceptional vertices EV_2. We can apply the same procedure to the second dissection, triangulating each dissection piece with the extra vertices. This converts the second dissection to a triangulation, but may make the first one (which had been a triangulation) back into a dissection. If this back-and-forth procedure terminates, then we obtain an equi-triangulation, as desired.

A case where every exceptional vertex has finite orbit is illustrated in Figures 6 and 7. More generally, this happens whenever all internal edges of the dissection have rational length ratios. In this case, there are only a finite number of potential places the orbits can land, so they must all terminate, as in the proof of Lemma 4.1.

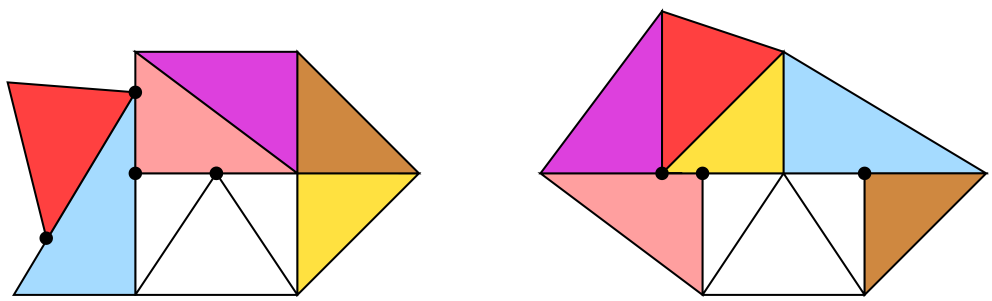

FIGURE 6. An equi-dissection of two polygons with seven exceptional points.

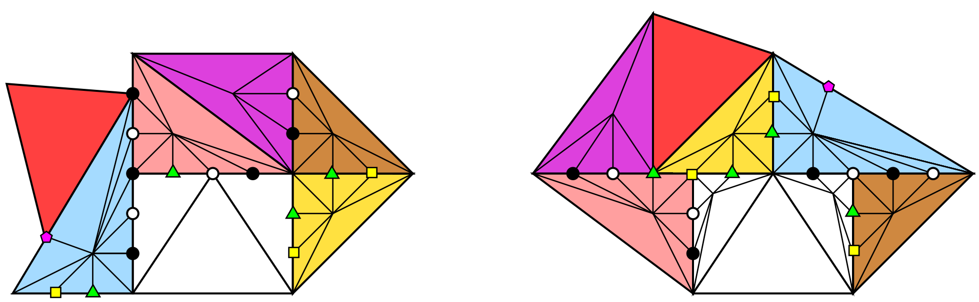

FIGURE 7. The orbits of the seven exceptional points: two orbits (black and white dots) join pairs of exceptional points. The other three orbits (colored polygons) start at exceptional points and end on the boundary. Also shown is a corresponding equi-triangulation.

Lemma 4.1. *Suppose we have a equi-dissection $(\mathcal{F}_1, \mathcal{F}_2)$ of polygonal regions (Ω_1, Ω_2). Then every exceptional orbit is finite if and only if there is a equi-triangulation of (Ω_1, Ω_2) that refines the given equi-dissection.*

Proof. Suppose the orbit condition holds. Take all the orbit points as vertices of the polygons in $\mathcal{P}$ and triangulate using exactly these vertices. Transferring these triangulation to the dissections, we get triangulations of the dissection pieces so that there are no exceptional vertices. Thus we have an equi-dissection.

Conversely, if the equi-dissection has a refinement that is an equi-triangulation, then the orbits of the exceptional points must lie among the vertices of the triangulation. This is a finite set, and since no orbit can be periodic (exceptional vertices have mo pre-images) or pre-periodic (every point has at most one pre-image), we deduce every orbit terminates after a finite number of steps. □

5. Terminating an orbit

On the other hand, Figure 8 shows an example (a equi-dissection of two different rectangles) with two exceptional points that both have infinite orbits. In this section, we will show how such infinite orbits can be terminated by altering the dissection so that Φ eventually maps the orbit onto b.

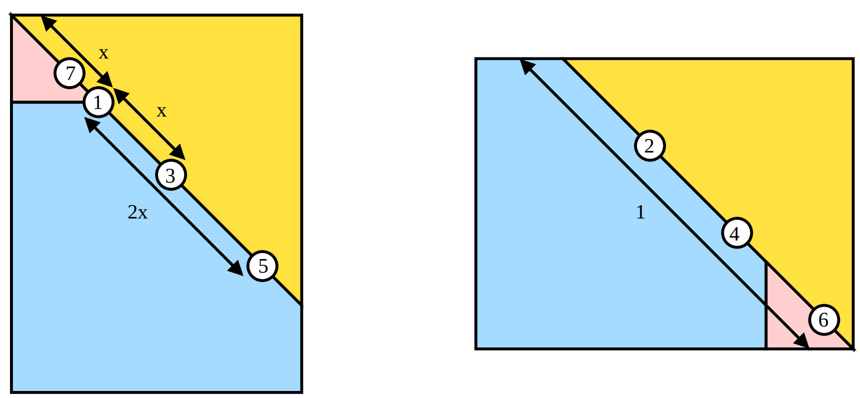

FIGURE 8. An equi-dissection of two rectangles of the same area, normalized so the hypotenuse of the larger right triangle is 1 and the of the smaller is $0 < x < 1$. Iterating the exceptional point from the left dissection gives an orbit of the form $nx \mod 1$, so is infinite if x is irrational.

Consider the dissection of two equal area rectangles illustrated in Figure 8. There are two exception vertices: one in each dissection where the tree pieces meet. In Figure 9 one exception vertex is marked with a black dot (the one in the left dissection) and the other with a white dot.

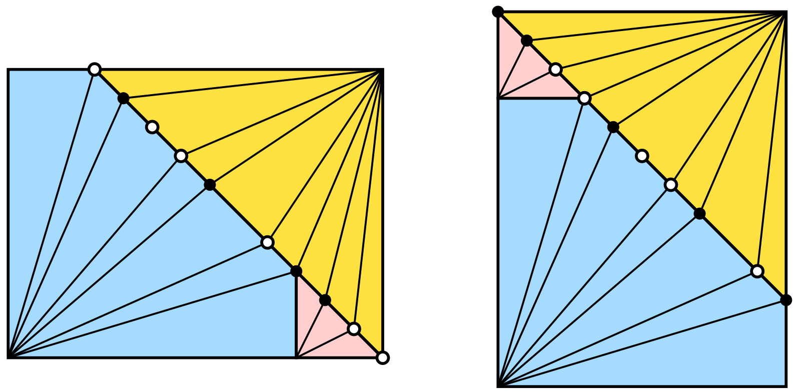

FIGURE 9. After iterating each of the two initial exceptional points three times, two exceptional points remain.

We then take three iterations of each and triangulate with these vertices. We triangulate the pieces using these new vertices, and then take one more iteration of the white vertex, as shown in Figure 9. The fourth iterate of the white exceptional vertex landed in a short segment I in the righthand dissection whose endpoints are earlier iterates of exceptional vertices. We take a triangle T with I as its base and connect the orbit point in I to the opposite vertex of T. In the other dissection, we place a copy of T, but with the orbit point and connecting segment are omitted. These two copies of T are mapped to triangles in the opposite dissections so that the base edge is mapped onto the boundaries of the region. See Figure 11. With this new dissection, the orbit of the white exceptional vertex lands on the boundary after five steps, but the black orbit still needs to be dealt with.

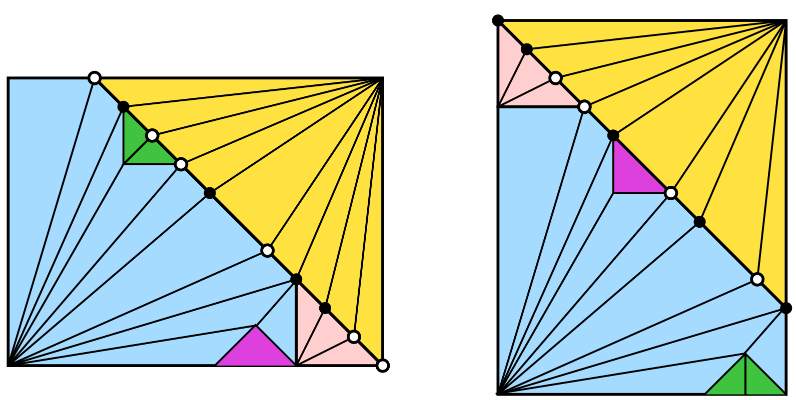

FIGURE 10. Terminating the white exceptional point.

In Figure 11 we take the fourth iterate of the black exceptional vertex, and perform a similar operation: create a triangle with one side containing this point and a copy of the triangle in the other dissection without the point. Then map these triangles to copies where the base side maps to the boundary of the regions being dissected.

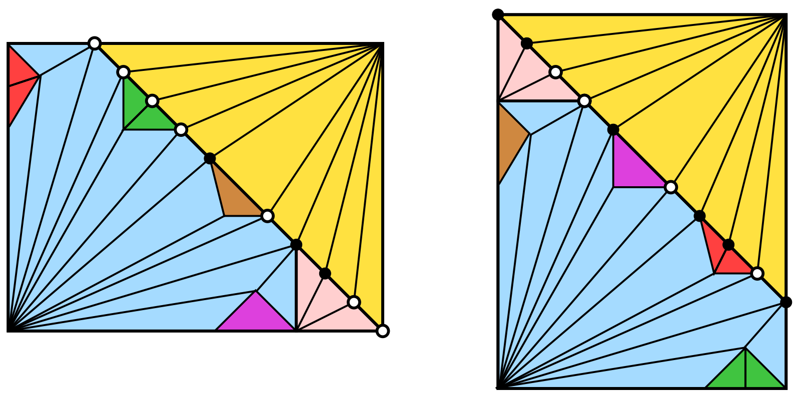

FIGURE 11. Terminating the black exceptional point.

In general, the idea is to iterate an exceptional vertex to obtain a sequence $v_1, v_2, v_3, \ldots$. If this sequence eventually lands on another exceptional vertex or lands on the boundary of one of the two dissected regions, then simply add this finite orbit to our set of vertices. If this never happens, then the sequence can be continued forever. We will prove later that in this case, for any $\epsilon > 0$ there is an iterate v_n that lands inside a segment I of length $< \epsilon$ whose endpoints are either earlier iterates in the same orbit, or elements of finite orbits that terminate (see Lemma 6.1 for the precise statement). Assume for the moment that this is true.

Take a triangle T that has I for its base and is contained in the dissection piece Q associated to v_n. Without loss of generality we assume A is dissection element of Ω_1. Let w be the vertex of T that is not on I (we may assume it is contained in the interior of the dissection piece Q). Map Q to the corresponding piece Q' of the other dissection and let T' be the image, with base I' and vertex w' being the images of I and w respectively. In T we connect v_n to w by a segment; this cuts T into two sub-triangles. We do not place the image of v_n in T' or connect it to w'. Thus T' remains undivided. Note however, that $Q \setminus T$ and $Q' \setminus T'$ are congruent polygons and so have identical triangulations using only the given boundary vertices.

Next we need to assume that our initial equi-dissection contains corresponding pieces Q_1, Q_2 that have corresponding sides S_1, S_2 that are on the boundaries of Ω_1 and Ω_2 respectively. We proved in Section 2 that any two equal area polygonal regions have such an equi-dissections. By taking ϵ small enough we may assume we can place a copy $\widetilde{T}$ of T in Q_2 with the base edge I mapping into $S_2 \subset \partial\Omega_2$, and a copy $\widetilde{T}'$ of T' in Q_1 with the base edge equal to the corresponding subinterval of $S_1 \subset \partial\Omega_1$. Note that $Q_1 \setminus \widetilde{T}'$ is isometric to $Q_2 \setminus \widetilde{T}$, hence these polygons have corresponding triangulations using only the given boundary vertices.

6. All orbits can be terminated

In the previous section we saw that an infinite forward orbit can be terminated by altering the dissection so as to map the next iterate onto the boundary of one of the dissected regions. This required the orbit to land on some interval I with endpoints that where themselves orbit points of exceptional vertices and that that I can be mapped to a boundary arc. Doing this for different exceptional orbits requires disjoint arcs on the boundary, and to make sure there is enough space, we want I to be sufficiently short. The following lemma says this is always possible.

Lemma 6.1. *Suppose Ω_1, Ω_2 are polygonal regions with an equi-dissection, and let (Φ, X) be as above. Suppose EV is the set of exceptional vertices and $v_1 = v, v_2, v_3, \dots$ is an exceptional orbit that is defined for all $n \in \mathbb{N}$. Let $EV_n = EV \cup \{v_1, \dots, v_n\}$. Then given any $\epsilon > 0$ there is an interval I on the boundary of some dissection piece so that*

1. $v_n \in I$
2. *I has length $< \epsilon$,*
3. *the endpoints of I are in EV_{n-1}.*

Proof. The basic idea is that an infinite sequence in a bounded set must get close to itself eventually. The only difficulty is that we need v_n to be close to earlier orbit points on both sides on v_n. We will assume this fails and derive a contradiction.

First of all, there must be $m < n$ so that v_m and v_n are on the same edge of the same dissection piece and within ϵ of each other; otherwise some edge would contain infinitely many orbit points all distance ϵ apart, which is impossible.

Let J be the segment connecting v_m and v_n. Replacing v_m by another orbit point if necessary, we may assume v_m is the only orbit point in J with index $< n$. Now iterate J forward one step to get an interval J' of the same length with endpoints v_{m+1}, v_{n+1}. J' is not on the boundary; otherwise v_n would be on the boundary too, and we have assumed this never happens.

If J' contains an exceptional vertex w, then w must be an interior point of J', since we assumed the orbit $\{v_k\}$ never hits an exceptional vertex other than v_1. Thus the orbit of w can be terminated using the interval J' and the procedure in Section 5. We then replace J' by the sub-segment between w and v_{n+1}. We then iterate this new interval forward. At each stage, we either get an image interval of the same length which can then be iterated again, or we encounter an exceptional vertex as a interior point. In the latter case, we terminate that exceptional orbit and replace the interval with a smaller one (still of positive length).

Note that if we ever encounter the same exceptional vertex w twice, say at v_{n+k} and later at v_{n+j}, then v_{n+j} is strictly closer w than v_{n+k} was. Thus our orbit has landed inside a interval of length $< \epsilon$ with endpoints in EV_{n+j-1}. But we assumed this never happens. Thus iterating intervals forward we only encounter each exceptional vertex once and hence we only encounter them only finitely often. Thus the iterated interval is only shortened finitely many times. Thus, eventually, the lengths stay fixed forever.

This implies that there are two iterates of the interval that overlap. The overlap cannot be the entire interval, for then the endpoints agree and we get a period orbit, which is impossible since we assumed the orbit is infinite. Thus both intervals have some length $\delta > 0$ and one is equal to the other shifted by $\eta < \delta$. If the iterates are M steps apart in the orbit, then taking another M steps gives a third interval that is a η-shift of the second interval and in the same direction, i.e., it is 2η-shift of the first interval. But since the forward iterates never hit the boundary or an exceptional orbit, this process can be continued forever. Thus there is an edge of the equi-dissection that contains infinitely many η-shifts of a fixed point, which is clearly a contradiction, proving the lemma. $\square$

7. Proof of Theorem 1.1

Proof. As noted in Section 2, given two polygonal regions of equal area, we may assume that they have an equi-dissection that contains a pair of corresponding squares that have corresponding sides on the boundary of the respective regions. Let $|S|$ denote the length of these sides. Let M be the number of exceptional vertices in the equi-dissection and choose a positive $\epsilon < |S|/4M$. This means that there is more than enough space in S to perform the terminating construction for every exceptional orbit, if needed.

For every exceptional vertex with finite orbit, add its orbit to the vertex set of the dissections. For each exceptional vertex with infinite orbit, use Lemma 6.1 and the ϵ chosen above to terminate the orbit by modifying the dissection as described in Section 5. Continue until we reach a dissection where every exceptional vertex has finite orbit. Then we are done by Lemma 4.1. □

8. Questions and remarks

Dissections and equi-dissections have been the source of many problems in recreational mathematics, often involving the minimal number of pieces needed to decompose a polygon into shapes of a certain type or needed to equi-decompose two given polygons. Numerous references related to dissection and some equi-dissections of common polygons using the least number of pieces are given in [27].

Burago and Zalgaller [11] proved that every polygon has an acute triangulation. Even more, every dissection of a polygon has a refinement that is an acute triangulation. See also [3], [6], [7], [20]. Thus any two equal area polygons have an equi-dissection in which every element is an acute triangle. Do they have an equi-triangulation in which every element is an acute triangle?

One helpful tool here may be a lemma of Bern, Ruppert and Marshall that says that if we add vertices to a polygon P so that each resulting edge is the diameter of an open disk containing none of the vertices (this is called a Gabriel edge), the polygonal region bounded by P has a triangulation by acute triangles. See [6], [7], [10]. If we can find a finite, invariant set with this property, then we would obtain an equi-triangulation by acute triangles using exactly the given vertices on the boundary. Every acute triangulation is a Delaunay triangulation, so an easier question in theory (but probably not in practice) is whether any two polygons have an equi-triangulations that are both Delaunay triangulations of their vertex sets. (A triangulation is Delaunay if whenever two triangles share an edge, the two angles opposite this edge sum to $\leq 180°$).

A special case of re-arrangements is when we have two different dissections of the same region, or even the same dissection, but with maps that swap congruent elements or map an element to itself using a symmetry of that piece. One such special case was studied in [7], [8] where a polygonal region was dissected into isosceles triangles and trapezoids and each piece was mapped to itself by a reflection across its line of symmetry (i.e., the two non-base sides are swapped). See Figure 12 for an example. This iteration was used to find triangulations and quad-meshes of PSLGs (planar straight line graphs) with good angle bounds. Infinite orbits were terminated by a nonlinear perturbation of the maps between pieces, instead of changing the dissection, as is done in this paper. Any triangular dissection gives rise to such a dynamical system by cutting each triangle into four pieces as shown in Figure 13.

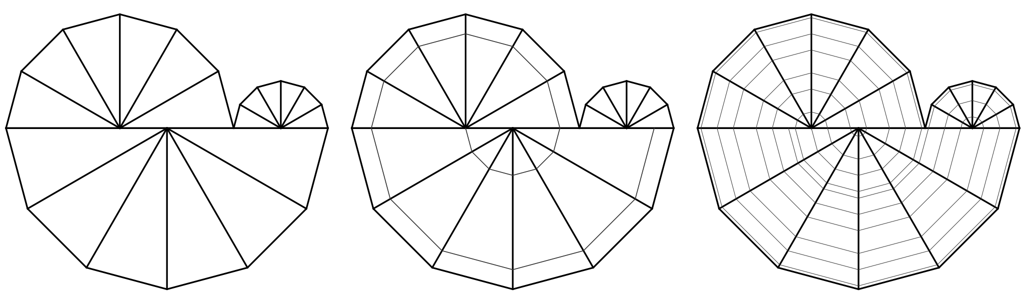

FIGURE 12. An isosceles dissection. Here each triangle is reflected across its line of symmetry, and an orbit can be followed by connecting consecutive iterates by line segments. In this example, no internal edge ever maps to a boundary edge, and there are four exception vertices.

Given an polygon P, we let $\Phi(P)$ denote the infimum of θ so that P has a triangulation with all angles $\leq \theta$. As proved in [9], this bound is attained except in a few specials cases when $\Phi(P) = 60°$. Therefore any equi-triangulation of polygons P_1, P_2 contains an angle $\geq \max(\Phi(P_1), \Phi(P_2))$. Can we compute $\Phi(P_1, P_2)$ the optimal upper angle bound for any equi-triangulation of P_1 and P_2? Is $\Phi(P_1, P_2) = \max(\Phi(P_1), \Phi(P_2))$? This seems unlikely in general.

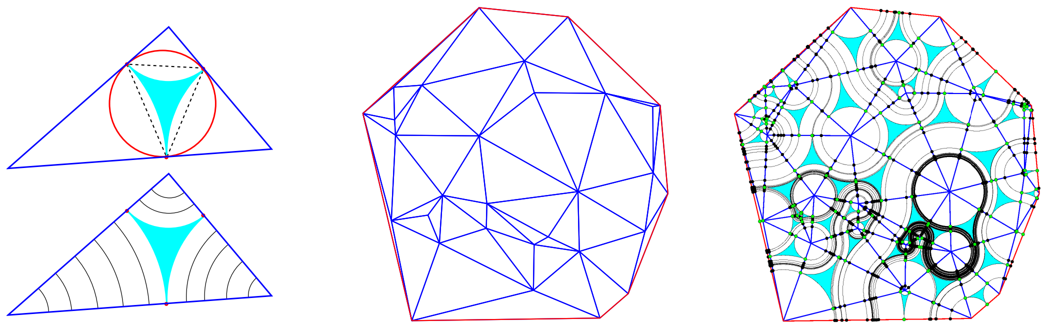

FIGURE 13. The in-circle of a triangle cuts the triangle into three isosceles triangles and a central triangle, and a flip can be applied to each isosceles triangles. This gives an dissection associated to any triangulation and an associated dynamical system on the edges. Here, the orbits are connected by circular arcs instead of segments, and we show a Delaunay triangulation of 30 random points and the corresponding orbits of the exceptional points.

Our proof terminates orbits by altering the dissection to map orbits onto the boundary where they must terminate. What if there is no boundary, i.e., do two polygonal closed surfaces of the same area have an equi-triangulation? It is easy to see that they have an equi-dissection by a very slight modification of the proof of the Wallace–Bolyai–Gerwien theorem. For example, do a cube and a tetrahedron of equal surface area have equi-triangulations? Any two equal area tetrahedrons?

Is computing the minimum number of polygons needed in an equi-dissection of two polygonal regions NP-hard? Triangular dissections? Equi-triangulations? Upper and lower bounds for the equi-dissection problem have been given in terms of the geometry of the polygons, or in various special cases, e.g., by Alfred Tarski [23].

David Hilbert's third problem asked if any three dimensional polyhedra can be dissected into tetrahedra? Max Dehn [12] showed in 1900 that this is only possible for certain polyhedron, and Dehn's necessary conditions (the Dehn invariant is zero) was shown to be sufficient in 1965 by Sydler. See [17], [22]. Any two polyhedra, both with the same volume and both with Dehn invariant zero, have an equi-dissection into tetrahedra. Do they also have a equi-triangulation into tetrahedra? (Any intersecting tetrahedra must intersect in a common vertex, edge or face.) An acute equi-triangulation? Finding an acute triangulation for a polyhedron, even for the unit cube in $\mathbb{R}^3$, is difficult, e.g., [18], [25]. In [5], the authors show that some polyhedron have tetrahedral dissections with far fewer elements than tetrahedral triangulations, and also consider the effect of allowing interior vertices or not.

In recent years hinged dissections have been studied, e.g., [13]. See Figure 14. Every two polygons have a hinged equi-dissection [1], but do they have a hinged equi-triangulation? Can the construction in [1] be modified convert a equi-triangulation into a hinged equi-triangulation?

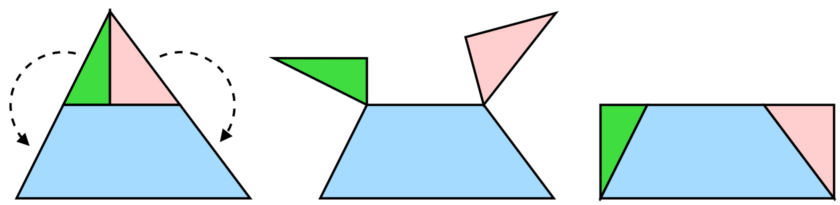

FIGURE 14. The equi-dissection of a triangle and a rectangle is hinged.

Two sets are called "equi-decomposable" if they can each be written as a finite union of disjoint subsets, where the sub-pieces of one decomposition can be rearranged to give the pieces of the other decomposition. This is the concept used in the Banach-Tarski theorem [4]: two polygonal regions are equi-decomposable iff they are equi-dissectable iff they have equal area. The failure of this in higher dimensions leads to famous the Banach-Tarski paradox: any ball of volume V is equi-decomposable with the union of two disjoint balls, each of volume V. See e.g., [14], [24].

The very simple dynamical system studied in this note is a special case of an interval exchange map (see e.g., [19], [26], [28]). It would be interesting if some of the highly developed theory of such maps could be applied to problems of dissection and triangulation (or vice versa). There is a notion of polygon exchange maps that seems even more closely connected to polygonal dissections, but so far this seems mostly to have been studied for translation maps on partitions of rectangles into sub-rectangles (see e.g., [2], [16]).

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