CONFORMAL IMAGES OF CARLESON CURVES

CHRISTOPHER J. BISHOP

Abstract. We show that if γ is a curve in the unit disk, then arclength on γ is a Carleson measure iff the image of γ has finite length under every conformal map of the disk onto a bounded domain with a rectifiable boundary.

In this note we characterize curves in $\mathbb{D}$ for which arclength is a Carleson measure, in terms of conformal maps onto rectifiable domains, answering a question asked by Percy Deift (personal communication) arising from his work on Riemann-Hilbert problems. The question seems natural and the proof follows from standard techniques, but I have not been able to locate this result in the literature.

Recall that a positive measure μ on the open unit disk, $\mathbb{D}$, is called a Carleson measure if

$$\|\mu\|_C = \sup_{|z|=1, r>0} \frac{\mu(D(z,r))}{r} < \infty.$$

The left hand side is called the Carleson norm of the measure.

Theorem 1. *If γ is a curve in the unit disk, then arclength on γ is a Carleson measure iff the image of γ has finite length under every conformal map onto a bounded domain with rectifiable boundary.*

Proof. One direction is an easy consequence of known facts. If f is a conformal map onto a rectifiable domain, then the F. and M. Riesz theorem (e.g., Theorem VI.1.2 of [2]) says that its derivative is in the Hardy space H^1. For a Jordan domain, the H^1 norm of f' is the length of the image's boundary. If the boundary is not a Jordan curve then we may replace "length" by "1-dimensional Hausdorff measure" (also denoted by ℓ) and get $\ell(\partial\Omega) \leq \|f'\|_{H^1} \leq 2\ell(\partial\Omega)$. For any H^p function g on the

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unit disk

$$\int |g|^p d\mu \leq C_p \|\mu\|_C \|g\|_{H^p},$$

(e.g., Theorem II.3.9 of [1]) where $\|\cdot\|_{H^p}$ is the Hardy space norm. Thus taking $g = f'$ we see that

$$\ell(f(\gamma)) = \int_\gamma |f'| ds \leq C_1 \|\mu\|_C \cdot \ell(\partial f(\mathbb{D})),$$

where μ denotes arclength measure on γ.

The converse requires more work. Theorem II.3.9 of [1] implies that if μ is not Carleson, then there is a $g \in H^1$ so that $\int |g| d\mu = \infty$. By the usual factorization theorems for Hardy spaces (e.g., Corollary II.5.7 of [1]), we can assume g never vanishes in $\mathbb{D}$, but this is not quite enough to deduce that $g = h'$ for some conformal map h. Instead, we will explicitly construct a conformal map h onto a rectifiable domain so that $\int |h'| d\mu = \infty$.

Our conformal map h will be built as a limit of compositions from a collection of conformal maps defined as follows. Suppose $0 < a < 1$ and let $\Omega_{a,\epsilon} = \mathbb{D} \cup D(1 + a, (1+\epsilon)a)$ be the overlapping union of the unit disk $\mathbb{D}$ and a smaller disk centered outside of $\mathbb{D}$. See Figure 1. The conformal map $\mathbb{D} \to \Omega_{a,\epsilon}$ is a composition of Möbius transformations and power functions, but we will not need the explicit formula. We will only use the following facts.

Lemma 2. *There is a constant $0 < c < 1$ so that given any $0 < a < 1$ and $0 < \delta < 1/2$, there exists an $0 < r_a < 1$ so that the following holds. For any $0 < r < r_a$ there is an $\epsilon > 0$ and a conformal map $f : \mathbb{D} \to \Omega_{a,\epsilon}$ such that:*

(1) *$f(0) = 0$ and f is symmetric with respect to $\mathbb{R}$,*
(2) *$f(1-r) = 1 + a$,*
(3) *$|f'| \geq ca/r$ on $D(1, r)$.*
(4) *f has a conformal extension across $\mathbb{T}$,*
(5) *$|f(z) - z| < \delta$ and $|f'(z) - 1| < \delta$ on $\overline{\mathbb{D}} \setminus D(1, \delta)$.*

The lemma be proven by an explicit calculation of f, or by applying symmetry and distortion properties of conformal maps (e.g., Koebe's $\frac{1}{4}$-theorem). The idea for (2) is that the hyperbolic distance between 0 and a is a continuous function of ϵ and it goes to ∞ as ϵ goes to zero. For a given a, ϵ we can choose r so the image is $> 1 + a$,

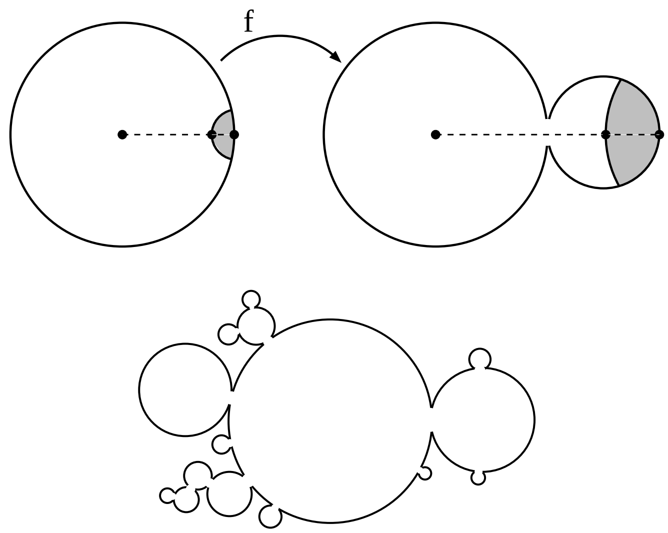

FIGURE 1. The top picture shows the domain $\Omega_{a,\epsilon}$ which is a small disk attached to the unit disk. A properly placed Carleson region is expanded by this map to a size comparable to the added "bubble" and $|f'|$ is comparable to the ratio the diameters of the region and its image. By composing maps of this form, we get build a sequence of domains that look like the lower picture, except that in the proof the sizes of the "bubbles" shrink much more dramatically.

but the image tends to 1 as $\epsilon \searrow 0$, so there is an intermediate choice of ϵ where r maps to $1+a$. By replacing $f(z)$ by $f(sz)$ for s very close to 1, we can assume f has a conformal extension across $\mathbb{T}$ and the previous conditions still hold. We leave the details to the reader.

By conjugating f with a rotation of $\mathbb{D}$ (i.e., replace $f(z)$ by $f(\lambda z)/\lambda$, $|\lambda| = 1$), we can clearly make $|f'|$ large on any sufficiently small Carleson disk, not just those centered at 1.

Let μ denote arclength measure on a curve γ and suppose this is not a Carleson measure. Then there must be sequence of disks centered at points $\{x_n\}$ on the unit circle and radii $\rho_n \to 0$ so that

$$\mu(D(x_n, \rho_n)) \geq n\rho_n.$$

Fix one such disk $D = D(x, r)$ and let $W_t = D \cap \{|z| < t\}$. Since $D \cap \mathbb{D}$ is the union of the W_t's as $t \nearrow 1$, we can choose a t so that $\mu(W_t) \geq \frac{1}{2}\mu(D)$. For each disk in our sequence, make such a choice and inductively define a subsequence of sets $\{W_n\}$

so that $\mu(W_n) \geq nd_n$ and $d_{n+1}2^{-n} \cdot \text{dist}(W_n, \mathbb{T})$, where $d_n = \text{diam}(W_n)$ (Euclidean diameter). We now proceed by induction to construct a sequence of conformal maps $\{h_j\}$ on $\mathbb{D}$ that map our non-Carleson curve γ to curves with longer and longer length. The limiting map h will map γ to a curve of infinite length.

Start with $a = \delta = 1/2$ and let r_a be as in the lemma. Choose k_1 so large that the region $W_{k_1} \subset D(x_{k_1}, \rho_{k_1})$ has diameter less than r_a. By the lemma, we can choose a point $a_1 = a \cdot x_{k_1}$ outside $\mathbb{D}$, an $\epsilon_1 > 0$, and a conformal map $f_1 : \mathbb{D} \to \Omega_{a_1,\epsilon_1}$ so that $|f_1'| \geq ca_1/\rho_{k_1}$ on W_{k_1}, and f_1 extends to be analytic on $\{|z| < 1 + s_1\}$ for some positive s_1. Let $h_1 = f_1$.

In general, assume we have used the lemma to choose conformal maps $f_1, \dots, f_{n-1}$ and that they and all have a conformal extension to $\{|z| < 1+s_{n-1}\}$ for some positive s_{n-1}. Let $h_{n-1} = f_1 \circ \cdots \circ f_{n-1}$. Let $M_{n-1} = \max |h_{n-1}'|$ over the closed unit disk (since h_{n-1} has a holomorphic extension across the boundary, this maximum is certainly finite). Similarly, let $m_n = \min |h_{n-1}'| > 0$. Choose $0 < a_n < s_{n-1}$ and $\epsilon_n > 0$ so small that $a_n M_{n-1} \leq 2^{-n}$ and so that the conformal map f_n given by the lemma satisfies both

$$|f_n(z) - z| \leq s_{n-1}/2, \text{ and } |f_n' - 1| \leq 2^{-n},$$

on $\overline{\mathbb{D}} \setminus D(1, s_{n-1})$. Moreover, $|f_n'| \geq c/(a_n\rho_{k_n})$ on $D(1, r_n)$, where $r_n = r_{a_n}$ as given by the lemma.

Now choose k_n so large that the region W_{k_n} satisfies:

(6) $\text{diam}(W_{k_n}) < r_{a_n}$ (r_a as given by the lemma),
(7) The minimum and maximum of $|h_{n-1}'|$ over W_{k_n} differ by at most a factor of 2 (this is possible by the distortion theorem for conformal maps if $\text{diam}(W_{k_n})$ is small enough).
(8) $k_n \geq c/(m_n a_n)$.

By the definition of W_n, Condition (8) implies

$$\mu(W_{k_n})/\text{diam}(W_{k_n}) \geq k_n \geq c/(m_n a_n)$$

or

$$\mu(W_{k_n}) \geq \frac{c \cdot \text{diam}(W_{k_n})}{m_n a_n}.$$

By conjugating f_n by an appropriate rotation, we get a function (also called f_n) so that $|f_n'| \geq ca_n/\rho_{k_n}$ on W_{k_n}. This implies that the length of σ inside W_{k_n} is expanded to approximately unit length under f_n. We want to show this is also true for the composition $h_n = h_{n-1} \circ f_n = f_1 \circ \cdots f_{n-1} \circ f_n$ and show these maps have a limit h with the same property.

By construction, the image of each map f_j lies inside a disk where the map f_{j-1} is defined and conformal so the composition is well defined and conformal on $\mathbb{D}$. Since the maps f_j converge uniformly to the identity on compact subsets of $\mathbb{D}$ (as rapidly as we wish), the limiting map h exists and is conformal on $\mathbb{D}$. Next we check that $h(\gamma)$ has infinite length and that $h(\mathbb{T})$ is rectifiable.

On each W_{k_j} we have

$$|h_n'| \;\;\geq\;\; |h_j'|(\prod_{m=j+1}^{n}(1-2^{-m})) \geq c|h_j'|.$$

Thus later generations of the construction do not greatly effect the expansion we have already created on earlier regions. Since $h_n \to h$ uniformly on compact sets, we also have $h_n' \to h'$ uniformly on compact sets and hence

$$\int_K |h'|d\mu = \lim_n \int_K |h_n'|d\mu,$$

for any compact $K \subset \mathbb{D}$. In particular,, we can let $K = W_{k_1} \cup \cdots \cup W_{k_n}$ be a finite union of the sets W_{k_j} and note that

$$\int_K |h_n'|d\mu \geq c\sum_{j=1}^{n}\int_{W_{k_j}} |h_j'|d\mu \gtrsim \sum_{j=1}^{n} m_{n-1}\cdot\frac{1}{|a_j|\rho_{k_j}}\cdot\rho_{k_j}k_j \gtrsim \sum_{j=1}^{n} 1 \to \infty$$

by our choice of k_j in Condition (8) above. Thus $h(\gamma)$ has infinite length.

Finally, we have to check that h maps $\mathbb{D}$ to a domain with rectifiable boundary. However, the domain $h_n(\mathbb{D})$ is obtained by taking the union of $\mathbb{D}$ with disk of diameter a_n and composing with the map h_{n-1} and then dilating the map very slightly to make sure it has a conformal extension across the unit circle. Adding the disk adds length $O(a_n)$ and composing with h_{n-1} gives a curve which is in the union of $\partial h_{n-1}(\mathbb{D})$ and the image of the small disk. This image has length $O(M_{n-1}a_n) = O(2^{-n})$. Dilating shortens the length of the boundary curve (since $|f'|$ is subharmonic the length of $f(|z| = r)$ is always less than the length of $f(|z| = 1)$ for any conformal map). Thus we can choose $|a_n| \searrow 0$ so rapidly that the length of $\partial h_n(\mathbb{D})$ is uniformly bounded above by some $L < \infty$.

Next, note that the length of $\partial h(\mathbb{D})$ is equal to

$$\sup_{0<r<1} \int |h'(re^{i\theta})| d\theta.$$

On the other hand, for any fixed r, h_n converges uniformly to h on the compact set $\{|z| = r\}$ and hence its derivative converges uniformly to h' on this set. Thus for a fixed $0 < r < 1$,

$$\int |h'(re^{i\theta})| d\theta \leq \sup_n \int |h_n'(re^{i\theta})| d\theta \leq L.$$

Taking the sup over r we see $h' \in H^1$ and so $h(\mathbb{T})$ is rectifiable. $\square$

Although Deift's question concerned curves, we never used this, and we have actually proven that a positive measure μ on the disk is Carleson iff $\int |f'| d\mu < \infty$ for any conformal map f onto a rectifiable domain.

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References

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C.J. Bishop, Mathematics Department, Stony Brook University, Stony Brook, NY 11794-3651

Email address: bishop@math.sunysb.edu