CONFORMAL IMAGES OF CARLESON CURVES

CHRISTOPHER J. BISHOP

ABSTRACT. We show that if \( \gamma \) is a curve in the unit disk, then arclength on \( \gamma \) is a Carleson measure iff the image of \( \gamma \) has finite length under every conformal map onto a domain with rectifiable boundary.


1991 \textit{Mathematics Subject Classification}. Primary: 30C62 Secondary: 30C45
\textit{Key words and phrases.}

The author is partially supported by NSF Grant DMS 10-06309.
1. Curves with rectifiable image

In this note we prove the following result, answering a question asked by Percy Deift. The question is a natural one and the proof follows standard techniques, but I have not been able to find this result in the literature. Recall that a positive measure $\mu$ on the open unit disk, $\mathbb{D}$, is called a Carleson measure if

$$
\|\mu\|_C = \sup_{|z|=1, r>0} \frac{\mu(D(z, r))}{r} < \infty.
$$

The left hand side is called the Carleson norm of the measure.

**Theorem 1.1.** If $\gamma$ is a curve in the unit disk, then arclength on $\gamma$ is a Carleson measure iff the image of $\gamma$ has finite length under every conformal map onto a bounded domain with rectifiable boundary.

**Proof.** One direction is an easy consequence of known facts. If $f$ is a conformal map on a rectifiable domain, then the F. and M. Riesz theorem says that its derivative is in the Hardy space $H^1$. For a Jordan domain, the Hardy space norm of $f'$ is the length of the image’s boundary. If the boundary is not a Jordan curve then we may replace “length” by “1-dimensional Hausdorff measure” (also denoted by $\ell$) and get

$$
\ell(\partial \Omega) \leq \|f'\|_{H^1} \leq 2\ell(\partial \Omega).
$$

For any $H^p$ function $g$ on the unit disk

$$
\int gd\mu \leq C_p \|\mu\|_C \|g\|_{H^p},
$$

where $\| \cdot \|_{H^p}$ is the Hardy space norm. Thus taking $g = f'$ we see that

$$
\ell(f(\gamma)) = \int_{\gamma} |f'| ds \leq C_1 \|\mu\|_C \ell(\partial f(\mathbb{D})),
$$

where $\mu$ denotes arclength measure on $\gamma$.

The other direction is less standard. Let $\mu$ denote arclength measure on a curve $\gamma$ and suppose this is not a Carleson measure (for the purposes of this proof any finite, positive measure will do); we are going to show that for any non-Carleson measure $\mu$ there is a conformal map onto a rectifiable so that $\int |f'|d\mu = \infty$. Then there must be sequence of disks centered on the unit circle where

$$
\mu(D(x_n, r_n)) \geq nr_n,
$$
and $|x_n| \not\to 1$. Note these conditions still hold when we pass to a subsequence.

Fix one such disk $D = D(x, r)$ and let $W_t = D \cap \{|z| < t\}$. Since $D$ is the union of the $W_t$'s as $t \not\to 1$, we can choose a $t$ so that $\mu(W_t) \geq \frac{1}{2} \mu(D)$. For each disk in our sequence, make such a choice and let $W_n$ denote the corresponding sequence of sets. By passing to a subsequence we can assume these sets are disjoint, indeed, that $\text{diam}(W_{n+1}) \ll \text{dist}(W_n, \mathbb{T})$. Let $d_n = \text{diam}(W_n)$ (Euclidean metric).

We will define a conformal map $h$ onto a rectifiable domain such that $|h'| \geq b_n/d_n$ on $W_{k_n}$ where $b_n \not\to 0$ is a fixed sequence depending on $\{d_n\}$, but $\{k_n\}$ is any subsequence of the positive integers which increases sufficiently quickly. Thus

$$\int |f'|d\mu \geq \sum_n k_n b_n = \infty$$

if we choose $k_n \geq 1/b_n$.

The basic building block is the conformal map from the unit disk, $\mathbb{D}$, to the union of $\mathbb{D}$ and $D(1 + a, a(1 + \epsilon))$. This is a union of two overlapping disks that has a small “gap” joining them and the size of this gap can be made as small as we wish by taking $\epsilon \not\to 0$. We will denote this domain $\Omega_{a,\epsilon}$. See Figure 1.

The conformal map onto this domain is a composition of Möbius transformations and power functions, but we will not need the explicit formula. We will only use the following facts:

1. We can choose the map $f$ to be symmetric with respect to the real line, and fix 0.
2. Fix $a > 0$. There is an $r_a > 0$ so that for any $0 < r < r_a$, we can choose $\epsilon > 0$ so that $f(1 - r) = a$.
3. There is a $c > 0$ (independent of $a$ and $\epsilon$) so that for this choice of $\epsilon$, $|f'| \geq c/a$ on $D(1, r)$.
4. By replacing $f(z)$ by $f(rz)$ for $r$ very close to 1, we can assume $f$ has a conformal extension across $\mathbb{T}$ and the previous conditions still hold.
5. As $\epsilon \not\to 0$, the map $f$ from $\mathbb{D}$ to $\Omega_{a,\epsilon}$ converges uniformly on $\mathbb{D} \setminus D(1, \delta)$ ($\delta > a$ fixed) to the identity function and its derivative converges uniformly to 1.

These can be proven by an explicit calculation of $f$, or by applying symmetry and distortion properties of conformal maps (e.g., Koebe’s $\frac{1}{4}$-theorem). The idea for (2)
is that the hyperbolic distance between 0 and a is a continuous function of \( \epsilon \) and it goes to \( \infty \) as \( \epsilon \) goes to zero. For a given \( a, \epsilon \) we can choose \( r \) so the image is \( > 1 + a \), but the image tends to 1 as \( \epsilon \searrow 0 \), so there is an intermediate choice of \( \epsilon \) where \( r \) maps to \( 1 + a \).

By rotating (replace \( f(z) \) by \( f(\lambda z)/\lambda, |\lambda| = 1 \)), we can clearly make \( |f'| \) large on any sufficiently small Carleson disk, not just those centered at 1.

We proceed by induction. Start with \( a_1 = 1 \) and choose \( f_1 \) and \( k_1 \) so that \( |f'_1| \geq 1 \) on \( W_{k_1} \) and \( f_1 \) extends to be analytic on \( \{|z| < 1 + r_1\} \) for some positive \( r_1 \).

In general, assume we have chosen \( f_1, \ldots, f_{n-1} \) and all have a conformal extension to \( \{|z| < 1 + r_n\} \) for some positive \( r_{n-1} \). Let \( h_{n-1} = f_1 \circ \cdots \circ f_{n-1} \). Let \( M_{n-1} = \max |h'_{n-1}| \) over the closed unit disk (since \( h_{n-1} \) has a holomorphic extension across the boundary, this maximum is certainly finite). Similarly, let \( m_n = \min |h'_{n-1}| \).

Choose \( a_n \ll r_n/2 \) so small that \( a_n M_n \leq 2^{-n} \) and so small that any conformal map \( g \) onto \( \Omega_{a,\epsilon} \), satisfies

\[
|g(z) - z| \leq r_{n-1}/2,
\]
on $\overline{D} \setminus D(1, r_{n-1})$. Then choose a region $W_{k_n}$ so far down our sequence that

1. its diameter is smaller than $r_{a_n}$ from condition (2) above,
2. The minimum and maximum of $|h'_{n-1}|$ over $W_{k_n}$ differ by at most a factor of 2 (possible by the distortion theorem for conformal maps if $\text{diam}(W_{k_n}) \ll r_n$).

Let $C_n$ denote the minimum of $|h'_{n-1}|$ over $W_{k_n}$.
3. $\mu(W_n) \geq \text{diam}(W_n)/(m_n a_n)$.

Then choose $\epsilon_n$ and the corresponding conformal map $f_n$ so that $|f_n'| \geq c/a_n$ on $W_n$.

Consider the composition $h_n = h_{n-1} \circ f_n = f_1 \circ \cdots f_{n-1} \circ f_n$. By choice, the image of each map is inside a disk where the next map is conformal so the compositions are all well defined. Moreover, the limiting map is a conformal map on disk and is the uniform limit of $\{h_n\}$ on compact subsets of $\mathbb{D}$.

On each $W_{k_j}$ we have

$$|h_n'| \geq |h_j'| (\prod_{m=j+1}^{n} (1 - 2^{-m})) \geq c|h_j'|.$$

Thus later generations of the construction do not greatly effect the expansion we have already created on earlier regions. In particular

$$\int |h_n'| d\mu \geq c \sum_{j=1}^{n} \int_{W_{k_j}} |h_j'| d\mu \geq c \sum_{j=1}^{n} 1 = cn,$$

for a uniform constant $c$. Hence the same holds for $h$ and so $h$ maps $\gamma$ to a curve of infinite length.

Finally, we have to check that $h$ maps $\mathbb{D}$ to a domain with rectifiable boundary. However, the domain $h_n(\mathbb{D})$ is obtained by taking the union of $\mathbb{D}$ with disk of diameter $a_n$ and composing with the map $h_{n-1}$ and then dilating the map very slightly to make sure it has an analytic extension. Adding the disk adds length $O(a_n)$ and composing with $h_{n-1}$ gives a curve which is in the union of $\partial h_{n-1}(\mathbb{D})$ and the image of the small disk. This image has length $O(M_{n-1} a_n) = O(2^{-n})$. Dilating shortens the length of the boundary curve (since $|f'|$ is subharmonic the length of $f(|z| = r)$ is always less than the length of $f(|z| = 1)$ for any conformal map). Therefore the length of $\partial h_n(\mathbb{D})$ is uniformly bounded by some $L$. 
To deduce that $\partial h(\mathbb{D})$ also has bounded length we can argue as follows. The length of $\partial h(\mathbb{D})$ is

$$\sup_{0<r<1} \int |h'(re^{i\theta})|d\theta.$$  

On the other hand, for any fixed $r$, $h_n$ converges uniformly to $h$ on the compact set $\{|z|=r\}$ and hence its derivative converges uniformly to $h'$ on this set. Thus

$$\int |h'(re^{i\theta})|d\theta \leq \sup_n \int |h'_n(re^{i\theta})|d\theta \leq L.$$  

Taking the sup over $r$ we see $h'$ is in the Hardy space $H^1$ and so $h$ maps onto a rectifiable domain.

□

C.J. Bishop, Mathematics Department, SUNY at Stony Brook, Stony Brook, NY 11794-3651

E-mail address: bishop@math.sunysb.edu