THE GEOMETRY OF BOUNDED TYPE ENTIRE FUNCTIONS

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Abstract. We construct functions in the Eremenko-Lyubich class (transcendental entire functions with bounded singular set) whose level-sets have prescribed geometry. We also give a related result for the Speiser class (finite singular set) and discuss some differences between these two classes. The construction may be considered as an approximation result using functions in class $\mathcal{B}$ that complements recent approximation results of Lasse Rempe-Gillen.

Date: Dec 2012.
1991 *Mathematics Subject Classification.* Primary: 30C62 Secondary:
Key words and phrases.
The author is partially supported by NSF Grant DMS 10-06309.

1. Introduction

The singular set of a entire function f is the closure of its critical values and finite asymptotic values and will be denoted $S(f)$. The Eremenko-Lyubich class $\mathcal{B}$ consists of functions such that $S(f)$ is a bounded set (such functions are also called bounded type). In [3] Eremenko and Lyubich showed that if $S(f) \subset \mathbb{D}_R = \{z : |z| < R\}$, then the inverse image Ω of $\mathbb{D}_R^* = \{z : |z| > R\}$ under f is a disjoint union of analytic, unbounded simply connected domains and that f acts a covering map $f : \Omega_j \to \mathbb{D}_R^*$ on each component Ω_j of Ω. Which disjoint unions of analytic, unbounded simply connected domains can arise in this way? The purpose of this note is to show that, essentially, they all do.

If $f \in \mathcal{B}$ and $S(f) \subset \mathbb{D}_R$, we call $\Omega = \{z : |f(z)| > R\}$ a $\mathcal{B}$-level-set and each connected component is called a tract of f. By normalizing f, we will assume for the rest of the paper that $R = 1$. On each tract there is a conformal map $\tau_j : \Omega_j \to \mathbb{H}_r = \{x + iy : x > 0\}$ so that $f(z) = \exp(\tau_j(z))$ on Ω_j. The collection of these conformal maps defines a holomorphic map $\tau : \Omega \to \mathbb{H}_r$. See Figure 1.

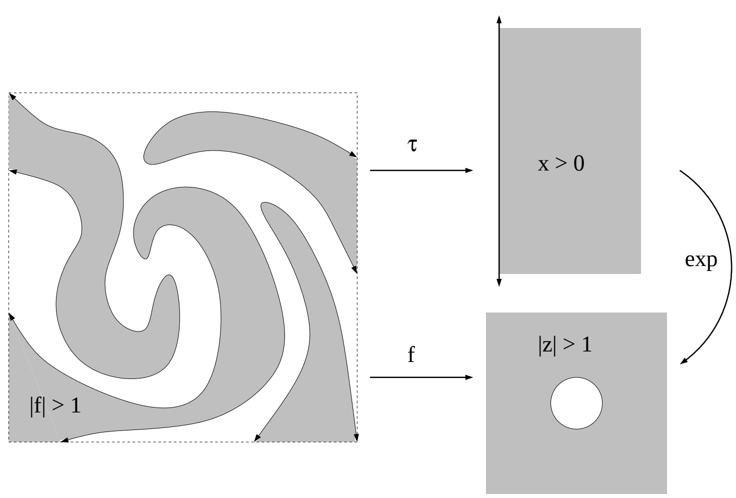

Figure 1. The level-set $\Omega = \{z : |f| > 1\}$ is a union of unbounded, smooth, simply connected tracts and f acts as a universal cover on each tract to $\mathbb{D}^* = \{z : |z| > 1\}$. On each tract $f(z) = \exp(\tau(z))$ where τ is a conformal map from the tract to the right half-plane.

Since $S(f)$ is compact, there is a $\rho > 0$, $S(f) \subset \{z : |z| \leq e^{-\rho}\}$ and hence $\Omega' = \{z : |f| > e^{-\rho}\}$ contains Ω and also consists of simply connected components.

It is locally finite (only a finite number of components meet any compact set) and on each component τ is continuous and 1-to-1 at infinity ($z_n \to \infty$ in Ω_j iff $\tau_j(z_n) \to \infty$). Conversely, we claim these conditions essentially characterize $\mathcal{B}$-level-sets, at least in a quasiconformal sense:

Theorem 1.1. *Suppose $\rho > 0$ and Ω' is a union of disjoint, locally finite, unbounded simply connected regions and $\tau : \Omega' \to \mathbb{H}_r - \rho = \{x + iy : x > -\rho\}$ is conformal and continuous and 1-to-1 at ∞ on each component of Ω'. Then there is a quasi-regular function g that equals e^τ on $\Omega = \tau^{-1}(\mathbb{H}_r)$ and $|g| \le 1$ off Ω. In particular, $\Omega = \{z : |g(z)| > 1\}$ is the level-set of a quasi-regular function of bounded type.*

Instead of defining a quasi-regular function of bounded type directly, we simply note that the measurable Riemann mapping theorem implies that any quasi-regular function g is of the form $g = f \circ \phi$ for some entire function f and some quasiconformal map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$. We say that g has bounded type if f does. Thus every Ω in Theorem 1.1 is the QC image of some $\mathcal{B}$-level-set. This is what we meant above when we said that this condition "essentially" characterizes bounded type level-sets.

In fact, we can be much more precise about the quasiconformal map ϕ that takes Ω to a $\mathcal{B}$-level-set. Note that the points $2\pi i\mathbb{Z} \subset \partial\mathbb{H}_r$ partition the boundary of $\mathbb{H}_r$ into equal sized segments. Thus the points $f^{-1}(1) = \tau^{-1}(2\pi i\mathbb{Z})$ partition $\partial\Omega$ into arcs. We call this a conformal partition of $\partial\Omega$, or the partition induced by τ. Given an arc J in the partition, let

$$J(r) = \{z : \text{dist}(z, J) < r \cdot \text{diam}(J)\}.$$

We call this an r-neighborhood of J. The union of r-neighborhoods over all partition arcs defines an open neighborhood of $\partial\Omega$ that we denote $T_\Omega(r)$. We just write $T(r)$ if the set Ω is clear from context.

Since τ extends to Ω' and maps each component conformally to $\mathbb{H}_r - \rho$, the usual distortion theorems for conformal maps imply all the arcs in the conformal partition of $\partial\Omega$ have bounded geometry with uniform bounds that depend only on ρ. Moreover, adjacent arcs have comparable lengths (again with a constant depending only on ρ). If a component of Ω is "large" compared to a half-plane, say $\Omega_0 = \{z : 0 < \arg(z) < \pi + \epsilon\}$ then the partition elements have lengths that increase to ∞. If

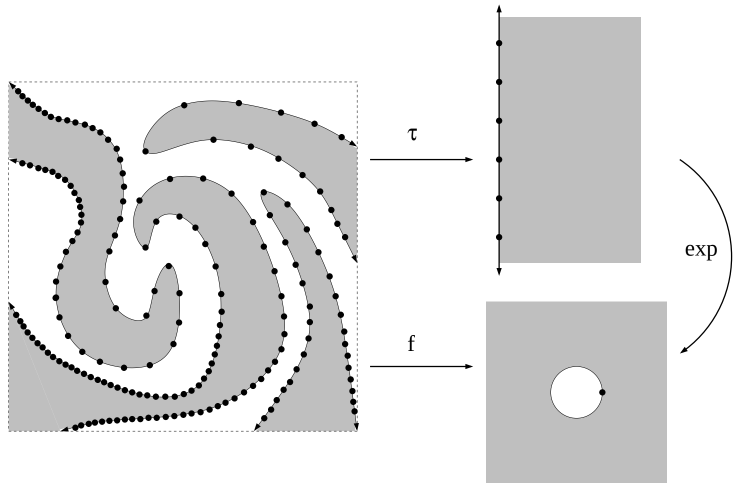

FIGURE 2. The points $f^{-1}(1) \subset \partial\Omega$ partition $\partial\Omega$ into arcs corresponding to equal length segments on $\partial\mathbb{H}_r$ via τ. Because τ extends past $\partial\Omega$ to a map of Ω' onto $\{x+iy : x > -\rho\}$, every arc of $\partial\Omega$ has uniformly bounded geometry with bounds depending only on ρ. The figure is not a computation; the points on $\partial\Omega$ are not exactly placed, but illustrate that adjacent arcs have comparable lengths.

$\Omega_0 = \{z : 0 < \arg(z) < \pi - \epsilon\}$, the partitions lengths tend to zero. If Ω_0 is a half-strip, they tend to zero exponentially fast.

The following implies and refines Theorem 1.1.

Theorem 1.2. *Suppose Ω is as in Theorem 1.1. Then there is a $f \in \mathcal{B}$ and a K-quasiconformal map ϕ of the plane so that $f \circ \phi = e^\tau$ on Ω, $f \circ \phi$ is bounded off Ω and ϕ is conformal off $T(r) \setminus \Omega$ (in particular, it is conformal on Ω). The constants $K, r < \infty$ depend on ρ but are otherwise independent of Ω and τ.*

For each positive integer n we can find $f_n \in \mathcal{B}$ so that $f_n \circ \phi_n = e^{n\tau}$ on Ω. Since $n\tau$ maps Ω onto $\mathbb{H}_r - n\rho \supset \mathbb{H}_r - \rho$, then the quasiconstant K of ϕ_n is uniformly bounded, but the support of ϕ_n shrinks down to $\partial\Omega$ as $n \to \infty$. Thus ϕ_n tends to the identity on compact sets of $\mathbb{R}^2$ and we get:

Corollary 1.3. *Suppose Ω is as in Theorem 1.1. Then there is a sequence $\{f_n\} \in \mathcal{B}$ and quasiconformal maps $\{\phi_n\}$ with uniformly bounded quasiconstant K so that $\Omega_n = \{z : |f_n(z)| > 1\} = \phi_n(\Omega)$ converges to Ω in the Hausdorff metric on any bounded subset of the plane.*

Under certain circumstances, one can actually prove Ω_n converges to Ω in the Hausdorff metric on the whole plane. For example, a result of Dyn'kin [2] on pointwise differentiability of quasiconformal maps implies this is true if $\text{area}(T(r) \cap \mathbb{D}_t^*) = O(t^{2-4K-\epsilon})$ for some $\epsilon > 0$. Estimates like this can often be proven with explicit calculations if Ω is "thin" near infinity. For example, the tracts in Figure 3 have finite in-radius, and we can use this to prove that $\text{area}(T(r) \cap \mathbb{D}_t^*)$ tends to zero exponentially fast in t (see [1] for details of this calculation). Hence these domains can be uniformly approximated (on the whole plane) by $\mathcal{B}$-level-sets.

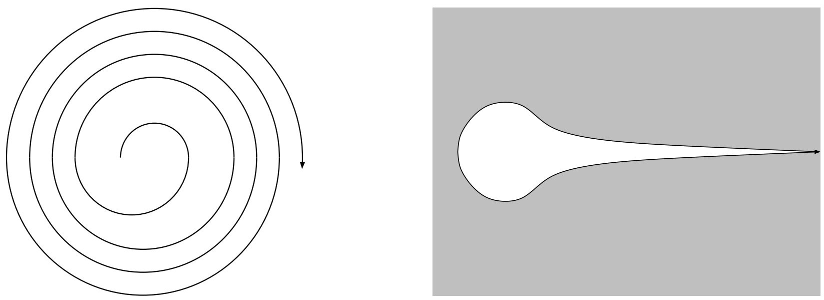

FIGURE 3. Examples where Theorem 1.2 applies and we have good estimates for ϕ. If we take Ω' to be the complement of the spiral curve on the left, we get a tract Ω for class $\mathcal{B}$ that spirals to ∞ as quickly as we wish. Taking Ω' to be the cusp region on the right we obtain $f \in \mathcal{B}$ that grows as quickly on $\mathbb{R}^+$ as we wish; moreover, $\text{area}(\{z : |z| > r, |f(z)| > 1\})$ tends to zero as quickly as we wish.

Theorem 1.2 makes it very simple to construct functions in class $\mathcal{B}$ and can reduce certain constructions in transcendental dynamics to simply drawing a picture of an appropriate tract or level-set. In particular, Theorem 1.2 above and Theorem 3.1 from [9] imply

Corollary 1.4. *Suppose that Ω and τ are as in Theorem 1.1 and that $\overline{\Omega} \subset \mathbb{D}^* = \{z : |z| > 1\}$. Then there is $f \in \mathcal{B}$ such that f and $g = e^\tau$ are quasiconformally conjugate on $\mathbb{D}^*$, which contains the Julia sets of both maps.*

The assumption that $\{z : |f(z)| \geq 1\} = \overline{\Omega} \subset \mathbb{D}^*$, says that f is "disjoint type" (see Proposition 2.8 of [8] for several equivalent formulations of this condition). Since f is disjoint type, one can show its Julia set consists exactly of those points whose iterates stay inside $\overline{\Omega}$ forever. Similarly, the Julia set of g is defined to be the set of points whose iterates stay inside $\overline{\Omega}$ forever. I thank Lasse Rempe-Gillen for pointing out this corollary and for allowing me to include it here. He uses the corollary (and related results from [1]) in his paper [10] to build Julia sets for $\mathcal{B}$ and $\mathcal{S}$ with exotic properties, e.g. connected components that are pseudo-arcs.

The proof of Theorem 1.2 is fairly simple; we sketch it here, leaving the details for later. Let W be the interior of $\mathbb{C} \setminus \Omega$. It is simply connected, non-empty and not the whole plane, so there is a conformal map $\Psi : W \to \mathbb{D}$. This map sends each component of $\partial\Omega$ to an open arc $I_j \subset \mathbb{T}$ and $E = \mathbb{T} \setminus \cup_j I_j$ is a closed set of zero Lebesgue measure (it is the conformal preimage of the single point ∞). Next we construct a Blaschke product B on the disk so that $B \circ \Psi$ approximates e^τ on $\partial\Omega$. This is a standard exercise involving Carleson measures and interpolating sequences in the disk. We then use a quasiconformal "glueing" to create a quasiregular function g on the plane that agrees with e^τ on $\Omega \setminus T(r)$ and agrees with $B \circ \Psi$ in $W \setminus T(r)$, and then apply the measurable Riemann mapping theorem to build a quasiconformal map ϕ so that $f = g \circ \phi^{-1}$ is entire. The only critical points of f correspond to critical points of B and hence have image inside $\mathbb{D}$. Thus $f \in \mathcal{B}$.

The class $\mathcal{S} \subset \mathcal{B}$ consists of those functions for which $S(f)$ is a finite set. $\mathcal{S}$ was named after Andreas Speiser by Eremenko and Lyubich (however, Eremenko has pointed out that Teichmüller [11] was the first to consider special properties of entire functions with finite singular set). Class $\mathcal{S}$ is more restrictive than $\mathcal{B}$ in some ways (e.g., functions in $\mathcal{S}$ can't have wandering domains, whereas those in $\mathcal{B}$ can, [1], [3], [6]), and it is an interesting problem to understand the differences between the classes more clearly. Here is the version of Theorem 1.2 for class $\mathcal{S}$.

Theorem 1.5. *Suppose Ω is as in Theorem 1.1. Then there is a $f \in \mathcal{S}$ and a K-quasiconformal map ϕ of the plane so that $f \circ \phi = e^\tau$ on Ω and ϕ is conformal on Ω^c. The constants $K, r < \infty$ depend on ρ but are otherwise independent of Ω and τ. We may take f to have no finite asymptotic values, exactly two critical values, $\pm \exp(-\rho/2)$, and so that every critical point has degree ≤ 4.*

This is very similar to Theorem 1.2, but with two important differences. First, the dilatation of ϕ is now supported on $\mathbb{C} \setminus \Omega$ instead of $T(r) \setminus \Omega$. Second, Theorem 1.5 omits the phrase "and $f \circ \phi$ is bounded off Ω". Thus Ω need not be the entire level-set of f; it is merely a union of connected components of $\{z : |f| > 1\}$. Thus any $\mathcal{B}$-tract is the QC image of a $\mathcal{S}$-tract, but (as we shall see below) not every $\mathcal{B}$-level-set is the QC image of a $\mathcal{S}$-level-set. In other words, functions in $\mathcal{S}$ and $\mathcal{B}$ do not differ because of the geometry of individual tracts, but because of how the tracts "fit together" to form a level-set.

Suppose $f \in \mathcal{S}$ and $S(f) \subset \mathbb{D}$. As before, assume $\mathrm{dist}(S(f), \partial\mathbb{D}) = 1 - e^{-\rho}$ and let

$$\delta = \min\{|a-b| : a, b \in S(f), a \neq b\},$$

and $\eta = \min(1 - e^{-\rho}, \delta)$. For $\epsilon < \eta/4$ the disks of radius ϵ centered at points of $S(f)$ are pairwise disjoint (even have disjoint doubles) and all lie inside $\mathbb{D}$. Thus the pre-image of such a disk is disjoint from $\Omega = \{z : |f(z)| > 1\}$ and consists of simply connected components. If $a \in S(f)$ let $\Omega(a, \epsilon) = f^{-1}(D(a, \epsilon))$ be such a pre-image. A component of $\Omega(a, \epsilon)$ is either bounded and contains a critical point with critical value a, or is unbounded and has asymptotic value a along some unbounded path γ in the component. The points $f^{-1}(a + \epsilon)$ partition $\partial\Omega(a, \epsilon)$ into arcs with uniformly bounded geometry, just as $f^{-1}(1)$ partitions $\partial\Omega$. Let $X = \overline{\mathbb{D}} \setminus \bigcup_a D(a, \epsilon)$, where the union is over $a \in S(f)$. Then X is a "Swiss cheese", i.e., disk with finitely many disjoint subdisks removed. For functions $f \in \mathcal{S}$, the preimage of this set must be "small" in the sense that is lies close to $\partial\Omega = \{z : |f(z)| = 1\}$:

Theorem 1.6. *For any $\epsilon < \eta/4$, there is a $r < \infty$ so that $f^{-1}(X) \subset T_\Omega(r)$. For each partition arc I of $\partial\Omega(a, \epsilon)$ there is a partition arc J of $\partial\Omega$ so that $I \subset J(r)$ and $J \subset I(r)$; thus $|I| \simeq |J| \simeq \mathrm{dist}(I, J)$.*

Here $|I|$ denotes the diameter of I (since these arcs have uniformly bounded geometry this is also comparable to their length). The theorem says that each complementary component W of $T(r) \cup \Omega$ is contained in some component of some $\Omega(a, \epsilon)$. If W is unbounded, then a must be an asymptotic value of f. From this it is easy to see that the half-strip $S = \{x + iy : x > 0, |y| < 1\}$ cannot be the QC image of any $\mathcal{S}$-level-set for a function with no finite asymptotic values. In Section 1.7 we will also eliminate functions with asymptotic values and prove S is not the QC image of any $\mathcal{S}$-level-set. On the other hand, Theorem 1.2 implies S is the QC image of some $\mathcal{B}$-level-set. Thus

Theorem 1.7. *There is a $\mathcal{B}$-level-set that is not the QC image of any $\mathcal{S}$-level-set.*

Very roughly, our results say that the tracts of a function in $\mathcal{B}$ can be separated by large gaps and only need be disjoint and locally finite. Moreover, τ on each tract can be chosen independently of the choices on other tracts. However, for class $\mathcal{S}$, the choice of τ on different tracts must be related to each other and the corresponding partitions of the tract boundaries must have nearby elements with comparable sizes.

Our proof of Theorem 1.2 is self-contained, but our proof of Theorem 1.5 depends on a construction of Speiser class functions from [1]. The results in this paper were motivated by the approximation results of Lasse Rempe-Gillen in [9]. He proves a stronger form of approximation, but for a domain Ω with a single connected component and satisfying greater regularity than used here (he assumes $\Omega = \tau^{-1}(\mathbb{H}_r)$ where τ is a conformal map of Ω' to $\{x + iy : x > -1 - c\log(1 + |y|)\}$).

The construction in [1] produces functions in class $\mathcal{S}$ without extra tracts, but requires Ω to satisfy certain geometric properties that are usually easy to verify in particular applications. These properties are another reflection of the difference between $\mathcal{B}$ and $\mathcal{S}$. [1] can be considered as a more intricate version of the current paper that gives more precise control over the constructed function.

The remaining sections of the paper are as follows:

Section 2: Construct simple folding maps.
Section 3: Prove a simple estimate on interpolating Blaschke products.
Section 4: Prove Theorem 1.2.
Section 5: Prove Theorem 1.5.
Section 6: Prove Theorem 1.6.
Section 7: Prove Theorem 1.7.

2. Simple foldings

In this section we construct quasiconformal self-maps of $\mathbb{H}_u$ that we call "simple foldings". These will be used in the proof of Theorem 1.2. The analogous step in the proof of Theorem 1.5 is a much more complicated folding map, whose construction takes up most of [1].

Lemma 2.1. *Suppose n is odd and $I_1, \dots, I_n$ is the partition of $I = [0, n]$ into n unit intervals. Let $Q = I \times [0, 1]$. Then there is a quasiconformal map $\phi : Q \to W \subset Q$ so that*

(1) *ϕ is the identity on $\partial Q \setminus I$.*
(2) *ϕ is linear on each I_j.*
(3) *$\phi(I_1) = I$.*
(4) *$\phi(I_k) = \phi(I_{n-k+2})$ for $k = 2, \dots, (n+1)/2$.*

Proof. The proof is a picture, namely Figure 4. For clarity we vertically stretch Q into a square Q', and then compress it back to a rectangle. We define the map $\phi : Q'' \to W$ by giving compatible finite triangulations $\mathcal{T}_1$ of Q' and $\mathcal{T}_2$ of W. This means that there is a 1-to-1 correspondence between the triangles in $\mathcal{T}_1$ and $\mathcal{T}_2$ such that two triangles in $\mathcal{T}_1$ share an edge iff the corresponding triangles in $\mathcal{T}$ do. If this is the case, then on each triangle in $\mathcal{T}_1$ we define ϕ to be the unique affine map to the corresponding triangle in $\mathcal{T}_2$. This gives a piecewise linear map of Q' to W. Thus all that remains to do is to draw the triangulations, verify compatibility, and verify the boundary conditions in Lemma 2.1. These are all apparent from Figure 4.

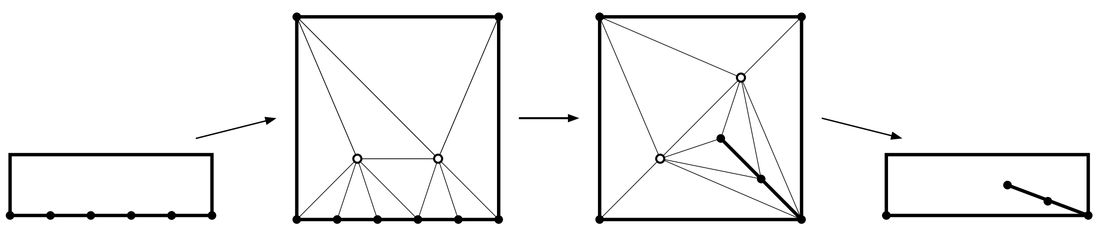

FIGURE 4. The pictorial proof of Lemma 2.1 for $n = 5$.

□

3. An estimate for certain interpolating Blaschke products

Recall that the pseudo-hyperbolic metric on $\mathbb{D}$ is given by

$$\rho(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

The usual hyperbolic metric ψ can be written in terms of ρ as

$$\psi = \log \frac{1+\rho}{1-\rho},$$

(see page 5 of [7]), but ρ is sometimes easier to compute with. We shall use the observation that if $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then $f(z) = 0$ and $\rho(z,w) \le \lambda$ imply $|f(w)| \le \lambda$ (the Schwarz lemma). Also, if z, w are on the same radius of $\mathbb{D}$ and $1 - |z| > \frac{1-|w|}{1-\lambda}$, then $\rho(z,w) > \lambda$.

Suppose $E \subset \mathbb{T}$ is compact and $\mathcal{I} = \{I_j\}$ are the connected components of $\mathbb{T} \setminus E$. Suppose that each I_j is partitioned into intervals so that adjacent intervals have comparable lengths, that the length of any interval is less than the distance to E and the lengths only tend to zero as we approach E. This gives a partition of $\mathbb{T} \setminus E$ that we denote $\mathcal{J} = \{J_k\}$.

Suppose $0 < \lambda < 1$ (a precise value of will be chosen below in order to satisfy certain estimates; think of λ near 1, so $1 - \lambda$ is small). For each element J_k of $\mathcal{J}$ let $z_k \in \mathbb{T}$ be its center point and define

$$w_k = z_k(1 - \frac{\ell(J_k)}{1-\lambda}).$$

This point is the vertex of a "tent" with base J_k and that is approximately $(1-\lambda)^{-1}$ times higher than it is wide. Thus viewed from w_k, the interval J_k has harmonic measure about $1 - \lambda$.

Joining the points $\{w_k\}$ for all the intervals J_k in the partition of a single interval $I \in \mathcal{I}$ defines a curve γ_I that joins the endpoints of I through $\mathbb{D}$. The subdomain of $\mathbb{D}$ bounded by I and γ_I will be denoted by V_I and the union of these subdomains over all $I \in \mathcal{I}$ will be denoted V. Let $V' \subset V$ be the image of V under the map $z \to z/\sqrt{|z|}$. Near the boundary this map approximately halves the distance to the boundary, so $\partial V_I' \cap \mathbb{D}$ is a curve that lies approximately halfway between γ_I and I.

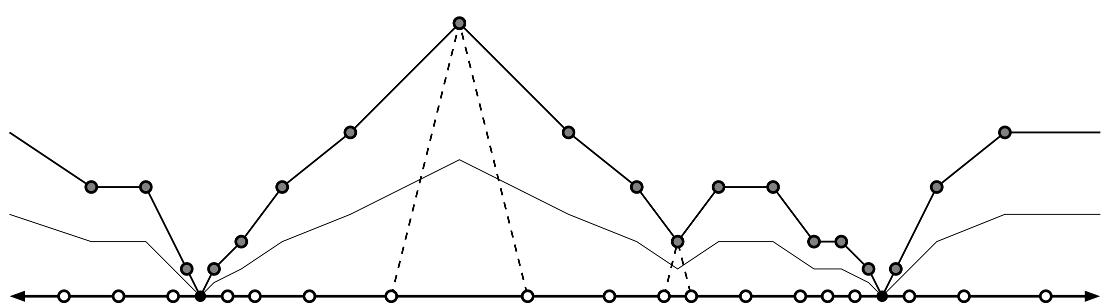

FIGURE 5. Joining the points w_k creates the curve γ and the region V between γ and $\partial\mathbb{D}$. In this figure $\lambda = 1/2$; Each point w_k is centered above an interval J_k at height twice J_k's length. Black dots indicate the endpoints of $\mathcal{I}$ and white dots the endpoints for $\mathcal{J}$. The region V' is bounded by a curve about halfway between γ and $\partial\mathbb{D}$.

For each component γ_I of $\partial V \cap \mathbb{D}$, choose points $\{a_n\}$ along $\partial V \cap \mathbb{D}$ that are spaced approximately unit distance apart in the pseudo-hyperbolic metric (since the points

w_n are spaced about distance $1-\lambda$ apart, there will be about one point a_n for every $(1-\lambda)^{-1}$ w_n's). If K is any arc on $\mathbb{T}$ and Q is the Carleson square with base K, then

$$\sum_{a_k \in Q} 1 - |a_k| \leq C\ell(K).$$

This implies the sequence is an interpolating sequence (see Chapter VII of [7]) and hence the corresponding Blaschke product

$$B(z) = \prod \frac{|a_n|}{a_n} \frac{z - a_n}{1 - \bar{a}_n z},$$

converges and is an interpolating Blaschke product. In particular, there is a $\delta > 0$ so that $|B(z)| > \delta$ whenever $\inf_k \rho(z, a_k) > \delta$.

Lemma 3.1. *There is $C < \infty$ so that for B as above,*

$$\frac{1}{C} \frac{(1-\lambda)}{\ell(J_k)} \leq |B'(z)| \leq C \frac{(1-\lambda)}{\ell(J_k)}$$

for all $z \in J_k \in \mathcal{J}$.

Proof. This is a standard estimate involving Poisson kernels (e.g., see Chapter VII of [4]), but we give the proof for completeness.

B extends to be holomorphic in the region obtained by reflecting V across the circle and is uniformly bounded on the region V' and its reflection across the circle. The region V' contains a disk of radius $\simeq \ell(J_j)/(1-\lambda)$ centered at $z_k \in J_k$, so the Cauchy estimate implies $|B'| \leq C(1-\lambda)/\ell(J_k)$ for some uniform C.

Since the Blaschke product B maps each arc in $\mathcal{I}$ into the circle $\mathbb{T}$,

$$|B'(z)| = \frac{\partial}{\partial \theta} \arg(B(re^{i\theta})),$$

for $z = re^{i\theta}$ on such an arc. Moreover,

$$\frac{\partial}{\partial \theta} \arg(B(e^{i\theta})) = \sum_k P_{a_k}(e^{i\theta}),$$

where P_a denotes the Poisson kernel with respect to the point a. This kernel satisfies the estimates

$$\frac{1}{C} \frac{1}{1-|a|} \leq P_a(e^{i\theta}) \leq C \frac{1}{1-|a|}$$

for $e^{i\theta}$ on the interval K_a of length $(1-|a|)$ centered at $a/|a|$. This gives the lower bound and finishes the proof of the lemma. □

4. Proof of Theorem 1.2

Let notation be as in the introduction and let $\mathcal{K}$ denote the partition edges on $\partial\Omega$ induced by τ. Then $W = \mathbb{C} \setminus \cup_j \overline{\Omega_j}$ is a simply connected domain, so it is the image of the unit disk under a Riemann map $\Phi : \mathbb{D} \to W$. Each preimage $I_j = \Phi^{-1}(\partial\Omega_j)$ is an open arc on the unit circle, $\mathbb{T} = \partial\mathbb{D}$ and the elements of $\mathcal{K}$ map to intervals $\{J_k\}$ that partition each of the I_j's. Moreover, adjacent J's have comparable length, with a uniform constant (depends only on ρ). We let $\mathcal{I}$ be the collection of intervals $\{I_j\}$ and let $\mathcal{J}$ denote the J_k's.

We now apply Lemma 3.1. Suppose λ has be chosen close enough to 1 so that

$$\frac{\epsilon}{\ell(J_k)} \leq |B'(z)| \leq \pi\ell(J_k),$$

for all k and for some fixed $\epsilon > 0$. This means that the image of J_k under B is less than a half-circle but more than a fixed fraction of the unit circle. Let $X \subset \cup_j I_j$ be the countable set of points where $B = 1$. This partitions each arc I_j into subarcs. Since B maps each of these subarcs onto a full circle, each subarc must hit at least 3 of the J_k, and hits at most a bounded number of the J_k's.

Let $Y = \Phi(X)$. These are points on $\partial\Omega$ that partition each component $\partial\Omega_j$ into arcs. We call this partition $\mathcal{L}$. Each element of $\mathcal{L}$ hits at least three elements of $\mathcal{K}$ (since these correspond to the arcs $\{J_k\}$ on the circle) but at most a bounded number of such elements. We can construct a quasiconformal map $\psi : \mathbb{D} \to \mathbb{D}$ that is the identity on X and off V and so that $\varphi = \phi \circ \Phi^{-1}$ is length respecting on $\mathcal{L}$ (length respecting means that $|\varphi'|$ is constant on elements of $\mathcal{L}$, i.e., lengths of subsets of element of $\mathcal{L}$ are multiplied by a constant that may depend on the element).

Thus each curve $\partial\Omega_j$ is partitioned in two ways: the partition induced by the conformal map τ of Ω_j to a half-plane and the partition induced using the map Φ from the unit disk to W. Elements of the two partitions are "similar in size" in the sense that each element hits several only a uniformly bounded number of elements from the other partition. In fact, we can perturb $\mathcal{K}$ slightly so that each element of $\mathcal{L}$ is actually a finite union of elements from $\mathcal{K}$. This is possible by a simple lemma from [1]:

Lemma 4.1. *Suppose $\mathcal{I} = \{I_j\}$ is a partition of the real numbers such that every interval has length ≥ 1 and there is an $M < \infty$ so that adjacent intervals have lengths* *within a factor of M of each other. Then there is a bi-Lipschitz map of the real line that sends every element of the partition to an interval with odd integer length (the bi-Lipschitz constant only depends on M). This map has a quasiconformal extension to the upper half-plane that is the identity outside the strip $\{x+iy : 0 < y < 1\}$.*

By definition, $i\tau$ maps $\mathcal{K}$ to the integer partition of $\mathbb{R}$ and we define $\mathcal{I}$ to be the image of $\mathcal{L}$. Now apply the lemma to define a map ψ that send the partition $\mathcal{I}$ to a partition $\mathcal{L}$ with integer endpoints and odd, uniformly bounded, lengths. We can now apply the construction in Section 2 to construct a quasiconformal folding map ψ_j of each Ω_j into a subdomain W_j of itself that either identifies an element of $\mathcal{K}$ with another element of $\mathcal{K}$ or with an element of $\mathcal{L}$. Moreover, we can take ψ_j to be length respecting on each element of $\mathcal{K}$.

Thus setting $g = \exp \circ \tau_j \circ \psi_j^{-1}$ on W_j and $g = B \circ \psi \circ \Phi$ on W defines a continuous quasi-regular function on the plane. Thus there is a quasiconformal map ϕ of the plane so that $f = g \circ \phi$ is entire. Clearly f has no critical values in $\{|f| > e^R\}$ and hence is in $\mathcal{B}$. Moreover, the strip $\{x+iy : 0 < y < 1\}$ and the set V both map into $T(r)$ (under τ^{-1} and Φ respectively) so ϕ is only non-conformal in $T(r)$.

This is not quite what was claimed in the theorem, since the dilatation of ϕ is supported in $T(r)$, instead of $T(r) \setminus \Omega$. However, we can fix this with a simple trick. Take $\Omega'' = \tau^{-1}(\mathbb{H}_r - \rho/2)$. Then $\Omega \subset \Omega'' \subset \Omega'$ and $\partial\Omega''$ has a partition that corresponds 1-to-1 with the partition of $\partial\Omega$. Corresponding partition elements have comparable sizes that are also comparable to their distance apart, so for any $s > 0$ we can choose an $r > 0$ so that $T''(s) = T_{\Omega''}(s) \subset T_\Omega(r)$. So if we construct ϕ with dilatation supported in $T''(s) \setminus \Omega$ then it is also supported in $T(r) \setminus \Omega$.

To do this, we apply the construction given above to Ω'' with one small change. Fix a number $M > 1$ and define a horizontal stretch

$$\nu(x, y) = (\min(\frac{2M}{\rho}(x + \frac{\rho}{2}), x + M), y).$$

This takes the vertical strip $\{-\frac{\rho}{2} < x < 0\}$ to the strip $\{0 < x < M\}$ quasiconformally, and translates $\mathbb{H}_r$ to $\mathbb{H}_r + M$. Then $\sigma = \tau^{-1} \circ \nu \circ \tau$ is a QC map of $\Omega'' \to \Omega$ and $\Omega'' \setminus \Omega$ is mapped to a region that covers $T(r) \cap \Omega$ if M is large enough (depending only on r). Defining $g = \exp \circ \tau_j \circ \psi_j \circ \sigma$, gives a quasiregular function on the plane (the boundary values of g on $\partial\Omega''$ are not changed by σ since σ fixes $\partial\Omega$), but now the dilatation is supported in $T_\Omega(r) \setminus \Omega$, as desired.

To prove the final statement in the theorem, we use a theorem of Walsh [12] that all the critical points of a Blaschke product are inside the hyperbolic convex hull of the zeros. For the product we constructed, this means they all lie in $\mathbb{D} \setminus V$. It easy to see that $|B| < 1-\eta$ on ∂V. By the maximum principle this means $|B| \leq 1-\eta$ on $\mathbb{D} \setminus V$ and hence at every critical point. The lower bound follows because the zeros of our Blaschke product form an interpolating sequence with a uniformly bounded constant. This implies there are positive ϵ and δ, depending only on our choice of λ, so that

(1) $|B| > \delta$ outside a ϵ-neighborhood of the zero set
(2) B' never vanishes inside this ϵ-neighborhood.

Thus $|B|$ is bounded below on its critical set, and this completes the proof of Theorem 1.2.

5. Proof of Theorem 1.5

We start by recalling a result from [1] that contains most of the work needed to prove Theorem 1.5.

Suppose $f \in \mathcal{S}$ and the critical values of f are exactly $\{-1, 1\}$. Let $T = f^{-1}([-1, 1])$. Let $U = \mathbb{C} \setminus [-1, 1]$ and let $\Omega = f^{-1}(U)$. Then each component of Ω is siply connected and f acts as a covering map from each component of Ω to U. The boundary of Ω is an infinite tree where the vertices are the preimages of $\{-1, 1\}$. Given $r > 0$ and and edge e on $\partial\Omega$ we define a neighborhood

$$e(r) = \{z : \text{dist}(z, e) < \text{diam}(e)\},$$

and define a neighborhood of $\partial\Omega$ by takeing the union over all edges. As before this neighborhood will be denoted $T(r)$ or $T_\Omega(r)$. For each connected component of Ω there is a conformal map τ to $\mathbb{H}_r$ so that $f = \cosh \circ \tau$. The edges of $\partial\Omega$ are mapped to intervals of length π on $\partial\mathbb{H}_r$.

Now suppose we start with an infinite tree T and want to construct an f so that $T = \partial\Omega = \partial f^{-1}(U)$. We say the graph T has "bounded geometry" if:

(1) every edge is twice differentiable with uniform bounds.
(2) edges meet at angles uniformly bounded away from zero.

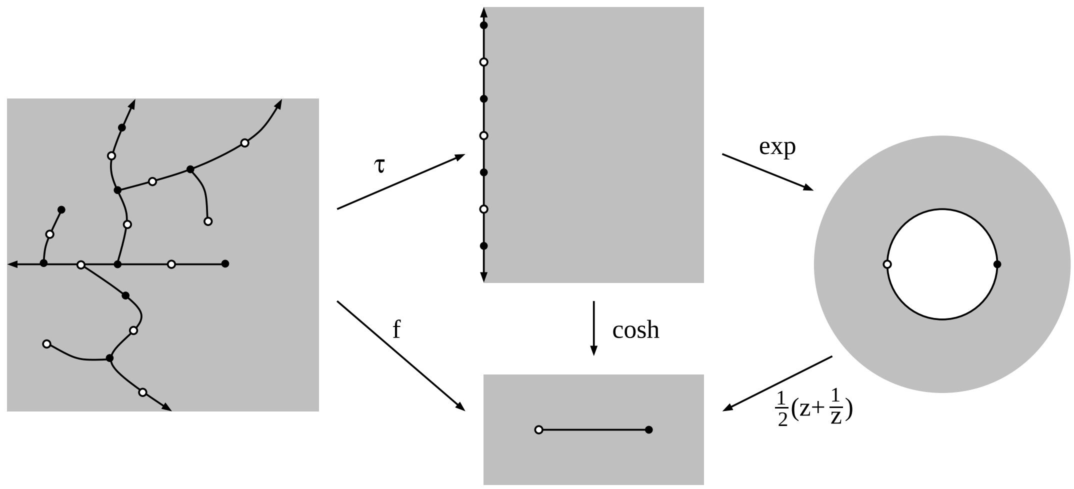

FIGURE 6. A function with two critical values at $\{-1, 1\}$ and no finite asymptotic values. $T = f^{-1}([-1, 1])$ is a tree with vertices labeled ± 1 (shown as black and white dots). τ is a conformal map from each complementary component of T to the right half-plane. $f = \cosh \circ \tau$.

(3) adjacent edges have uniformly comparable lengths.
(4) non-adjacent edges e, f satisfy $\text{dist}(e, f)/\text{diam}(e) > 0$ with a uniform bound.

The following is the main result from [1] for constructing functions in $\mathcal{S}$.

Theorem 5.1. *Suppose T is a bounded geometry infinite tree and each component of $\Omega = \mathbb{C} \setminus T$ has a conformal map $\tau : \Omega \to \mathbb{H}_r$ that maps each edge to an interval of length $\geq \pi$ on $\partial \mathbb{H}_r$. Then there exists $f \in \mathcal{S}$ with critical values ± 1, a $r > 0$ and a K-quasiconformal ϕ so that $f \circ \phi = \cosh \circ \tau$ off $T(r)$ and ϕ is conformal off $T(r)$. The constants r, K only depend on the bounded geometry constants of T.*

The proof of Theorem 1.5 from Theorem 5.1 starts just like the proof of Theorem 1.2. Let $W = \mathbb{C} \setminus \cup_j \overline{\Omega_j}$. This is a proper simply connected domain in the plane so by the Riemann mapping theorem there is a conformal map $\Phi : \mathbb{D} \to W$. Each $\partial \Omega_j$ has an open arc $I_j \subset \mathbb{T}$ as a preimage The partition $\mathcal{K}$ of $\partial \Omega_j$ induced by τ corresponds via Φ to a partition $\mathcal{J}$ of I_j such that adjacent intervals in $\mathcal{J}$ have comparable lengths (with a fixed constant, independent of j). In particular, we can choose a point v_j for each I_j so that the distances from v_j to each endpoint of I_j are comparable. We call this the "approximate center" of I_j.

Consider a Whitney decomposition of the disk, as illustrated in Figure 7. This consists of a central disk of radius 1/2. The annulus $\{\frac{1}{2} < |z| < \frac{3}{4}\}$ is divided into eight equal sectors, the annulus $\{\frac{3}{4} < |z| < \frac{7}{8}\}$ into sixteen sectors, and so on. Each box has two radial sides and two circular arc sides concentric with the origin. The arc closer to the origin is called the top of the box and the arc further from the origin is called the bottom. Each bottom arc is divided into two pieces by the tops of the Whitney boxes below it. We call these the left and right sides of the bottom arc (left is the one further clockwise).

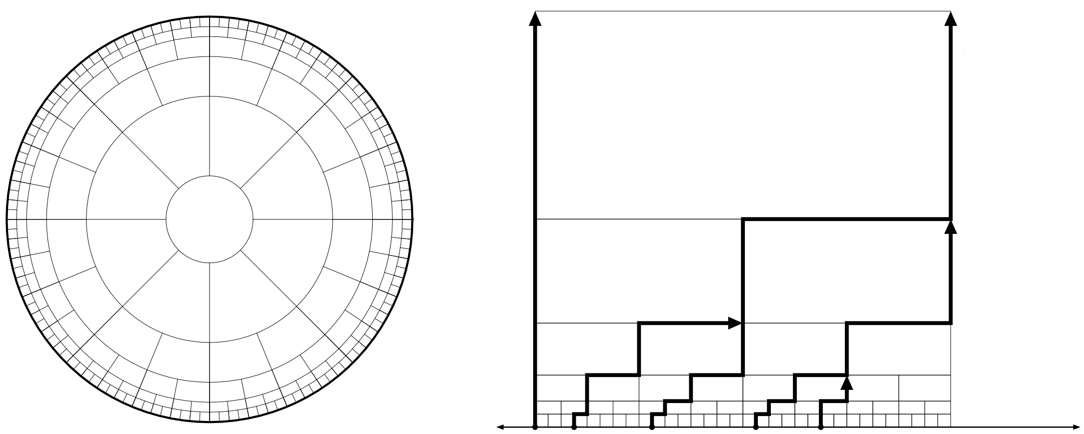

FIGURE 7. On the left is the Whitney decomposition of the disk. On the right is an enlargement near the boundary. Any boundary point can be joined to the central disk by a path moving along edges of Whitney boxes: move radially towards the origin whenever possible, and move counterclockwise (right in the picture) otherwise.

Each point on the unit circle can be connected to the central disk by a path lying on the decomposition boundaries that moves towards the origin whenever possible and moves counterclockwise otherwise. See Figure 7. Note that such a path never contains the "left-half" of the bottom of any Whitney box. For each arc I_j we connect the approximate center of I_j to the central disk by such a path. The union of all such paths is a closed set and divides the disk into countably many simply connected subdomains. Every such subdomain is an infinite union of Whitney boxes; a finite union would contain a box closest to the unit circle and the bottom of this box would be on a path, which is impossible since the left side of the bottom can't be on any path.

Thus every subdomain W has a boundary that hits $\mathbb{T}$, and $\partial W \cap \mathbb{T}$ must be a closed interval (otherwise there is a component of $\partial W \setminus \mathbb{T}$ that doesn't intersect the central disk, which is impossible by construction). This interval must hit E (otherwise two paths where generated in the same component of $\mathbb{T} \setminus E$, contrary to the construction), and it must hit E in a single point (otherwise W separates some component I of $\mathbb{T} \setminus E$ from the central disk, contradicting the fact that the approximate center of I_k is connected to the central disk). Thus we can choose a conformal map of each subdomain W to $\mathbb{H}_r$ with the single point $\partial W \cap E$ mapping to ∞.

If we add the vertices of the Whitney graph to ∂W there will be infinitely many of them accumulating to some partition point v_j, and this prevents ∂W from being a bounded geometry tree. To fix this, we choose a dyadic Carleson box Q_j containing v_j with base $J_j \subset I_j$ and

$$\frac{1}{8}\ell(I_j) \leq \ell(J_j) \leq \frac{1}{4}\ell(I_j).$$

We then replace the arc of $\partial W_j \cap Q_j$ by a line segment with the same endpoints. The length of this segment is comparable to the distance between v_j and its neighboring partition points and is also comparable to the adjacent segment in the path $\partial W_j \setminus Q_j$.

Now ∂W is a bounded geometry tree when we add the partition points of $\mathcal{J}$. If we had a conformal map from each component W to $\mathbb{H}_r$ that maps each edge to length $\geq \pi$ then we would could apply Theorem 5.1 and would be done. However, this need not be the case. In particular, some of the elements of $\mathcal{J}$ might be so short that they have small image. However, we can fix this by subdividing each W. Let Ψ be a conformal map of W to $\mathbb{H}_r$ sending $E \cap \partial W$ to ∞ and let $\mathcal{P}$ be the partition of $\partial \mathbb{H}_r$ induced by Ψ and the partition of ∂W. Decompose $\mathbb{H}_r$ into countably many horizontal half-strips corresponding to the partition of the boundary. Adjacent half-strips have comparable width, so if we add vertices to the horizontal edges whose spacing is equal to the thinner of the two adjacent strips, we get a bounded geometry tree. Moreover, each half-strip has a conformal maps to $\mathbb{H}_r$ of the form $\sigma = z \to \lambda \cosh(\pi \frac{z}{w_j} - y_j))$. If we choose $\lambda \gg 1$ appropriately on each half-strip, we can insure that every edge of the tree get has as large σ-length as we want.

The map Ψ^{-1} transfers this decomposition of $\mathbb{H}_r$ to a decomposition of W and $\sigma_j \circ \Psi$ are the desired conformal maps to $\mathbb{H}_r$. Thus Theorem 5.1 can be applied and gives

Theorem 5.2. *Suppose Ω is as in Theorem 1.1. Then there is a $f \in \mathcal{S}$ and a K-quasiconformal map ϕ of the plane so that $f \circ \phi = \cosh \circ \tau$ on Ω and ϕ is conformal on $\Omega \setminus T(r)$. The constants $K, r < \infty$ depend on ρ but are otherwise independent of Ω and τ. f has no finite asymptotic values, exactly two critical values, $\pm \exp(-\rho/2)$, and every critical point has degree ≤ 4.*

This is Theorem 1.5 except that the exponential function has been replaced by cosh and ϕ is conformal on $\Omega \setminus T(r)$ instead of Ω.

We can fix the first problem as follows. Let Ω' be as in Theorem 1.1 and let $\Omega'' = \tau^{-1}(\mathbb{H}_r - \frac{1}{2}\rho)$. Then $\Omega \subset \Omega'' \subset \Omega'$ and we can apply Theorem 1.5 to Ω'' with $\tau(z)$ replaced by $\tilde{\tau}(z) = \tau(z) + \rho/2$. Let E denote the ellipse that is the image of the vertical line $L = \{x + iy : y = \rho/2\}$ under cosh and let C be the image of this line under e^z (C is just the circle of radius $e^{\rho/2}$ around the origin). Let σ be a quasiconformal map of the plane that maps E to C, is conformal outside E, fixes each point of $[-1, 1]$ and is symmetric with respect to both the real and imaginary axes. Then $e^z = \sigma(\cosh(z))$ to the right of the line L. Thus $\sigma \circ f$ is quasiregular and by the measurable Riemann mapping theorem there is a quasiconformal ψ so that $f \circ \psi = \sigma \circ f$. Moreover, our choices give $\sigma \circ f = e^\tau$ on Ω,

To fix the fact that the dilatation of ϕ is supported in $\Omega^c \cup T(r)$ and not $T(r)$, we use the same trick of replacing Ω by Ω'' as we did in the proof of Theorem 1.2 (see the end of Section 4). However, one additional fact is needed from [1]: in Theorem 5.1, if Ω_j is a component of Ω so that τ maps every edge of $\partial\Omega_j$ to length π on $\mathbb{H}_r$, then the dilatation of $\phi \circ \tau^{-1}$ (this is a QC map on $\mathbb{H}_r$) is supported the vertical strip $\{x + iy : 0 < x < 1\}$. In our application, the partition of Ω is defined to have this property (it is only the new components we add where the τ-length might be large). This completes the proof of Theorem 1.5.

6. Proof of Theorem 1.6

For $a \in S(f)$ and $\epsilon < \eta/4$, let $D_a = D(a, \epsilon)$ and let $2D_a = D(a, 2\epsilon)$. For each a, the set $\Omega(a, \epsilon) = f^{-1}(D_a)$ has simply connected components, and on each such component W the map $f : W \to D_j$ acts either as

(1) a 1-to-1 map onto D_j,

(2) a $(d + 1)$-to-1 branched cover of D_j with a single critical value at a or

(3) a ∞-to -1 cover of $D_a \setminus \{a\}$.

In the first two cases W is bounded and in the third it is unbounded and contains a path to ∞ along which f as asymptotic value a.

As noted in the introduction, the preimages $f^{-1}(a+\epsilon)$ partition $\partial\Omega(a,\epsilon)$ into arcs. For each D_a we can choose a curve γ_a connecting it to $\partial\mathbb{D}_R$ inside $X = \mathbb{D}_R \setminus \cup_{a\in S(f)} D_a$. The family of paths homotopic to γ_a has positive, finite extremal length, and since there are only finitely many families, there is a maximum such extremal length, say Λ. By conformal invariance, any partition arc I of any $\partial\Omega(a,\epsilon)$ can be joined to a partition arc J of $\partial\Omega$ by a path family of extremal length at most Λ. This implies that

$$\mathrm{dist}(I,J) = O(\mathrm{diam}(J)),$$

with a constant that depends on Λ.

Moreover, since f^{-1} is univalent on any disk of radius $\epsilon/2$ centered on ∂D_a, the Koebe distortion theorem implies that $\mathrm{diam}(I) = O(\mathrm{dist}(I,J))$, with an absolute constant. Hence $I \subset J(r)$ for some r that depends only on Λ. Similarly $J \subset I(r)$. Thus $\partial\Omega(a,\epsilon) \subset T(r)$ and this implies $f^{-1}(X) \subset T(r)$, as claimed. This completes the proof of the theorem.

If the critical points of f have uniformly bounded degree D, then the components of $\Omega(a,\epsilon)$ containing critical points have boundaries with at most D partition arcs, each with diameter comparable to the whole component (the constant depending only on D). Since one of these arcs in contained in some $J(r)$ the whole component will be contained in $J(Cr)$ for some C depending only of D. Thus

Corollary 6.1. *If $f \in \mathcal{S}$ has no finite asymptotic values and every critical point has uniformly bounded degree, then there is a $r > 0$ so that $\mathbb{C} = \Omega \cup T(r)$.*

This is the type of function produced by Theorem 1.1 in [1]. For example, [1] contains the construction of $f \in \mathcal{S}$ so that area$(\{z : |f(z)| > 1\}) < \infty$ using high degree crticial points and claims a similar construction is possible using asymptotic values. The corollary above implies one of these two devices is needed to build such an example.

7. The half-strip is not the QC image of any $\mathcal{S}$-level-set

We will show that there is not any global quasiconformal map of the plane that takes the half-strip $S = \{x+iy : x > 0, |y| < 1\}$ to any $\mathcal{S}$-level-set, i.e., there can't be an $f \in \mathcal{S}$ that has a single tract that is this "narrow" near infinity. The main idea is that if we partition ∂S using conformal maps for S and its complement, we get very different behaviors: the "inside" partition has diameters decaying exponentially, and the "outside" has diameters growing like a square root. If S were a $\mathcal{S}$-level-set, its complement would have to approximate a component of $\Omega(a, \epsilon)$ for some $a \in S(f)$ (in fact, for some asymptotic value a), and by Theorem 1.6, the partition for the outside of S would have to be comparable to the partition for the inside, which it clearly false. To make this argument precise, and to apply it to any quasiconformal image of S we need to state a few technical lemmas regarding harmonic measure and quasicircles.

Lemma 7.1. *Suppose Ω is as in Theorem 1.1. If ϕ is a K-quasiconformal map of the plane that is conformal on Ω, then*

$$T_{\phi(\Omega)}(t) \subset \phi(T_\Omega(r)) \subset T_{\phi(\Omega)}(s),$$

where t, s depend only on r and K.

This is immediate from the definitions and the fact that quasiconformal maps are also quasisymmetric.

Lemma 7.2. *Suppose Ω is bounded by a Jordan curve through ∞ and $\{\mathcal{J}\}$ is a conformal partition of $\partial\Omega$. Suppose Ω contains an unbounded quasidisk W. Then there is a sequence of partition elements $J_j \in \mathcal{J}$ and numbers $R_j \nearrow \infty$ so that J_j hits the circle $\{|z| = R_j\}$ and*

$$\operatorname{diam}(J_j) \geq C R_j^{-\beta}$$

for some $C, \beta < \infty$.

Proof. If there is a sequence of partition elements so that $\operatorname{diam}(J_j) \geq 1$, we are done by taking $\alpha = 0$, so we may assume that that $\operatorname{diam}(J_j)$ is bounded.

We use the notation $\omega(z, E, \Omega)$ for the harmonic measure of $E \cap \partial\Omega$ in the domain Ω with respect to the base point z. This is the value at z of the harmonic function on

Ω with boundary values 1 on E and 0 elsewhere. See [5] for an excellent introduction to harmonic measure.

Choose a base point for harmonic measure $z_0 \in W \subset \Omega$. Note that for $R \geq |z_0|$,

$$\omega(z_0, \mathbb{D}_R^*, W) \geq C_1 R^{-\alpha},$$

since W is a quasidisk. Again by properties of quasidisks, we can choose a point $w \in W$ with $|w| = 3R$ and $\mathrm{dist}(w, \partial W) \geq cR$ so that

$$\omega(z_0, D(w, cR/2), W \setminus D(w, cR/2)) \geq C_2 R^{-\alpha}.$$

See Figure 8.

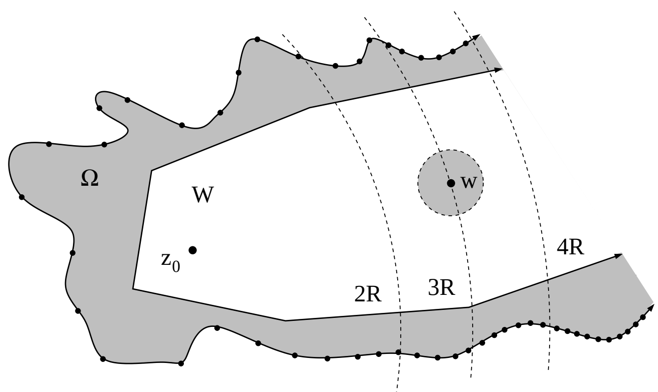

FIGURE 8. This illustrates the proof that if a tract Ω is "big" enough to contain an unbounded quasicircle, then there is a polynomial lower bound for how quickly partition elements can shrink.

If $R > \min_{z \in \partial\Omega} |z|$, then a Brownian motion started at w has a positive chance of hitting $\partial\Omega$ without leaving $\{2R < |z| < 4R\}$, so

$$\omega(w, \mathbb{D}_{4R} \setminus \mathbb{D}_{2R}, \Omega) \geq C_3 > 0.$$

Thus by the Markov property of harmonic measure, and the fact that $W \subset \Omega$ we see

$$\omega(z_0, \mathbb{D}_{4R} \setminus \mathbb{D}_{2R}, \Omega) \geq C_4 R^{-\alpha}.$$

Moreover, by taking R large enough, we may assume that any partition arc of $\partial\Omega$ that hits the annulus $\{2R < |z| < 4R\}$ is contained in the annulus $\{R < |z| < 5R\}$ (recall we can assume partition arcs have bounded diameters). Because

$$\omega(z_0, \mathbb{D}_{5R}^*, \Omega) \geq C_5 R^{-\alpha},$$

there can be at most $C_6 R^\alpha$ such partition arcs. The arcs can be labeled so that the nth arcs has harmonic measure $\simeq n^{-2}$ with respect to z_0 (this is true for the integer partition of $\partial\mathbb{H}_r$ and harmonic mesure is a conformal invariant. Hence at least one of these arcs, say J, has harmonic measure $\geq (C_3/C_6)R^{-\alpha}$ with respect to w (since the union of these arcs covers $\partial\Omega \cap \{2R < |z| < 4R\}$ and hence the union has measure $\geq C_3 > 0$ with respect to w). By Beurling's projection theorem (e.g., Corollary 9.3 of [5]),

$$\omega(w, J, \Omega) \leq C\sqrt{\text{diam}(J)/\text{dist}(w, \partial\Omega)},$$

which leads to

$$\text{diam}(J) \gtrsim \omega(w, J, \Omega)^2 \cdot \text{dist}(w, \partial\Omega) \gtrsim R^{1-2\alpha},$$

as desired with $\beta = 1 - 2\alpha$. □

Lemma 7.3. *Suppose Ω is the image of the half-strip $S = \{x + iy : x > 0, |y| < 1\}$ under a quasiconformal map ϕ of the plane and $\mathcal{J}$ is a conformal partition of $\partial\Omega$. Then all the partition elements satisfy*

$$\text{diam}(J) \leq C \text{dist}(J, 0)^{-\gamma},$$

for every $\gamma > 0$, i.e., the diameters tend to zero faster than any power.

Proof. Without loss of generality we can assume $0 \in \Omega$ and 0 is fixed by ϕ. We can write ϕ as a composition of two K-quasiconformal maps $\phi = \phi_2 \circ \phi_1$, where ϕ_1 has dilatation μ_1 supported in S and ϕ_2 is conformal on $W = \phi_1(S)$. Since

$$\int_S \frac{dxdy}{1 + x^2 + y^2} \leq \infty,$$

a theorem of Teichmüller and Wittich implies that $|\phi_1(z)/z|$ has a limit as $z \to \infty$. Thus by rotating and dilating, we can assume

$$W \cap \mathbb{D}_R^* \subset \{z : \arg(z) < \delta\},$$

for every $\delta > 0$, where R is chosen large enough depending on δ. The Ahlfors distortion theorem then implies

$$\omega(0, \mathbb{D}_R^*, W) = O(R^{-\alpha}),$$

for every $\alpha < \infty$, i.e., harmonic measure in W near ∞ dies faster than any power.

Consider the square S_n inside S between $\{x = n\}$ and $\{x = n + 1\}$. Then $W_n = \phi_1(S_n)$ is a quasidisk and its preimage in $\mathbb{H}_r$ under the conformal map $\tau : \mathbb{H}_r \to W$

is generalized quadrilateral Q_n with two sides on $\partial\mathbb{H}_r$ and modulus bounded above and below. Because of these bounds, the diameters of Q_n must grow exponentially, and hence ∂Q_n hits $\geq ce^{cn}$ partition intervals for some fixed c (depending only on K). Hence W_n hits the same number of partition arcs on ∂W. Because W_n is a quasidisk, each of these partition arcs has diameter bounded by $C\text{diam}(W_n)\exp(-an)$ for another constant a depending only on K.

Let $R_n = \text{dist}(W_n, 0)$. As noted above, $\omega(0, W_n, W) \leq CR_n^{-\alpha}$ and so the same estimate is true for $\omega(1, \partial Q_n, \mathbb{H}_r)$. But we also know that $\omega(1, \partial Q_n, \mathbb{H}_r) \geq ce^{-cn}$. Combining these upper and lower bounds we get

$$c\exp(-cn) \leq CR_n^{-\alpha},$$

and this implies

$$R_n = O(\exp(cn/\alpha)),$$

for any $\alpha < \infty$. Since $\text{diam}(W_n) = O(R_n)$, we deduce that all the partition elements hitting W_n have diameters less than $O(\exp(cn/\alpha - an))$ where a, c are positive constants that depend only on K and α is as large as we wish. Taking α large enough, we see that the partition elements hitting W_n have diameters bounded by

$$\exp(-an/2) = \exp(-(a/2)(cn/\alpha)(\alpha/c)) = O(R_n^{-(a\alpha)/(2c)}) = O(R_n^{-\gamma}),$$

for any $\gamma < \infty$, as desired. See Figure 9.

Since ϕ_2 is conformal on W, the partition for Ω is just the image of the partition for W under ϕ_2, and since ϕ_2 is bi-Hölder, it follows that the partitions still decay faster than any power. □

Lemma 7.4. *There is no $f \in \mathcal{S}$ with a single tract Ω that is the quasiconformal image of the half-strip $S = \{x + iy : x > 0, |y| < 1\}$.*

Proof. Suppose there were a K-quasiconformal map ϕ of the plane taking S to the level-set $\Omega = \{z : |f(z)| > R\}$ of some $f \in \mathcal{S}$. Choose d and ϵ as in Theorem 1.6 and let r be as given by the theorem. Let s be as given by Lemma 7.1.

As in the proof of Lemma 7.3, write $\phi = \phi_2 \circ \phi_1$ where ϕ_1 is conformal off S and set $W = \phi_1(S)$. By our earlier estimates

$$W \cup T_W(s) \subset V = \mathbb{D}_R \cup \{z : |\arg(z)| < \pi/4\},$$

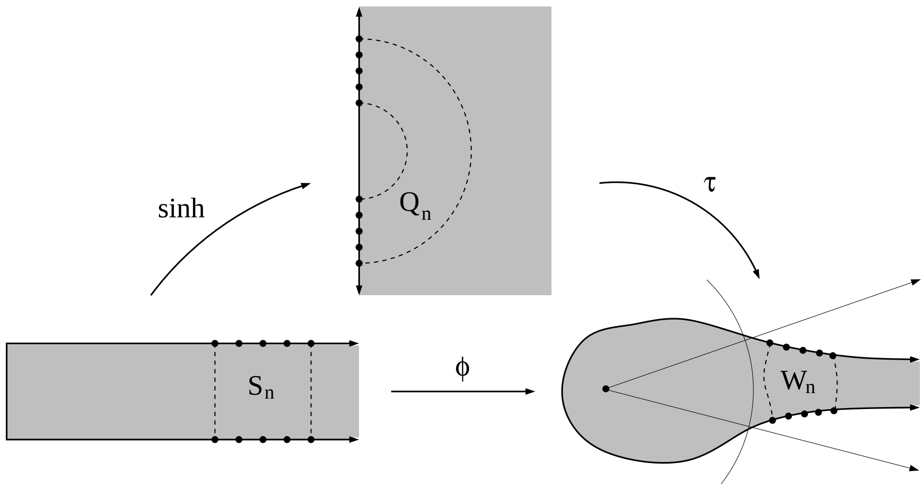

FIGURE 9. Partition elements for a half-strip have lengths that decay exponentially and this is also true for any quasiconformal image of a half-strip. Roughly speaking, this holds since QC maps are Hölder continuous and hence preserve exponential bounds. However, if the map is not conformal on S, then the conformal partition need not be preserved and an extra argument is needed.

If R is chosen large enough depending on s. Note that V^c is a quasidisk and hence so is the image $V' = \phi_2(V^c)$ and that by Lemma 7.1, this domain is contained in the complement of $\Omega \cup T_\Omega(t)$. Therefore v' is contained inside some component U of $\Omega(a, \epsilon)$ for $a \in S(f)$.

Lemma 7.2 applies to U and Lemma 7.3 applies to Ω, giving estimates that contradict the conclusion of Theorem 1.6 that partition elements for ∂U are contained in r-neighborhoods of partition elements for $\partial\Omega$. This proves that Ω could not have been the level-set of any $f \in \mathcal{S}$. $\square$

References

[1] C.J. Bishop. Constructing entire functions by quasiconformal folding. preprint, 2011.

[2] E. M. Dyn′kin. Smoothness of a quasiconformal mapping at a point. *Algebra i Analiz*, 9(3):205–210, 1997.

[3] A. È. Erëmenko and M. Yu. Lyubich. Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier (Grenoble)*, 42(4):989–1020, 1992.

[4] J.B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981.

[5] J.B. Garnett and D.E. Marshall. *Harmonic measure*, volume 2 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2005.

[6] L.R. Goldberg and L. Keen. A finiteness theorem for a dynamical class of entire functions. *Ergodic Theory Dynam. Systems*, 6(2):183–192, 1986.

[7] A. Hinkkanen. Entire functions with no unbounded Fatou components. In *Complex analysis and dynamical systems II*, volume 382 of *Contemp. Math.*, pages 217–226. Amer. Math. Soc., Providence, RI, 2005.

[8] Helena Mihaljević-Brandt. Semiconjugacies, pinched Cantor bouquets and hyperbolic orbifolds. *Trans. Amer. Math. Soc.*, 364(8):4053–4083, 2012.

[9] L. Rempe. Hyperbolic entire functions with full hyperbolic dimension and approximation by functions in the Eremenko-Lyubich class. preprint 2011, arXiv:1106.3439v2.

[10] Lasse Rempe-Gillen. Arc-lke continua, Julia sets of entire functions and Eremenko's conjecture. preprint, 2012.

[11] O. Teichmüller. Eine umkehrung des zweiten hauptsatzes der wertverteilungslehre. *Deutsche Math.*, 2:96–107, 1937.

[12] J. L. Walsh. On the critical points of functions possessing central symmetry on the sphere. *Amer. J. Math.*, 70:11–21, 1948.

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