Conformal Distortion in Space

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Abstract

We study the conformality problems for quasiregular mappings in space. Our approach is based on some new Grötzsch – Teichmüller type modulus estimates that are expressed in terms of the mean value of the dilatation coefficients.

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1. Introduction

Let G be an open set in $\mathbb{R}^n$. A continuous mapping $f : G \to \mathbb{R}^n$ is called K – quasiregular, $K \geq 1$, if $f \in W^{1,n}_{loc}(G)$ and if

$$||f'(x)||^n \leq K\, J_f(x) \quad a.e. \tag{1.1}$$

where $J_f(x)$ stands for the Jacobian determinant of $f'(x)$ and $||f'(x)|| = \sup |f'(x)h|$ where the supremum is taken over all unit vectors $h \in \mathbb{R}^n$. A homeomorphic K – quasiregular mapping is called K – quasiconformal. We shall employ the following distortion coefficients

$$K_f(x) = \frac{||f'(x)||^n}{J_f(x)},\ \ L_f(x) = \frac{J_f(x)}{\ell(f'(x))^n},\ \ H_f(x) = \frac{\|f'(x)\|}{\ell(f'(x))}, \tag{1.2}$$

that are called the outer, inner and linear dilatation of f at x, respectively. Here $\ell(f'(x)) = \inf |f'(x)h|$. These dilatation coefficients are well–defined at regular points of f and, by convention, we let $K_f(x) = L_f(x) = H_f(x) = 1$ at the nonregular points and for a constant mapping.

It is well–known that if $n \geq 3$ and one of the dilatation coefficients of a quasiregular mapping f, say $L_f(x)$, is close to 1, then f is close to a Möbius transformation. In spite of this Liouville's phenomenon the pointwise condition $L_f(x) \to 1$ as $x \to y$, $y \in G$, does not imply in general neither conformality for f at y nor the properties typical for the conformal mappings. The mapping

$$f(x) = x(1 - \log|x|),\ \ f(0) = 0, \tag{1.3}$$

shows that $|f(x)|/|x| \to \infty$ as $x \to 0$ although $L_f(x) = (1 - 1/\log|x|)^{n-1} \to 1$. Nevertheless, the conformal behavior of f at a point can be studied using another measures of closeness of the distortion coefficient to 1. The first such result is due to Teichmüller [T] and Wittich [W]. They proved that if $f : G \to \mathbb{R}^2$ is a quasiconformal homeomorphism such that

$$\int\limits_{|x-y|<r} \frac{L_f(x) - 1}{|x-y|^2} dx \to 0 \quad \text{as} \quad r \to 0, \tag{1.4}$$

for some $y \in G$ then $|f'(y)| = \lambda$, where $\lambda > 0$. In what follows we will call such λ the conformal distortion coefficient of f at y. Similar problems have been studied by Belinski [B], Shabat [SH], Lehto [L], Reich and Walczak [RW], Brakalova and Jenkins [BJ] in plane and by Reshetnyak [R_2] and Suominen [Su] in space. Another approach to the investigation of the pointwise behavior of the quasiconformal mappings based on the Beltrami equation is due to Bojarski [BO] (see, also [Sch], [Iw]).

Consider the class of space radial mappings $f : B \to B$ defined on the unit ball B in $\mathbb{R}^n$ centered at the origin as

$$f(x) = xe^{-\alpha(|x|)}, \ \ \alpha(|x|) = \int\limits_{|x|}^{1} \frac{L_f(t) - 1}{t} dt, \quad f(0) = 0, \tag{1.5}$$

where $L_f(t)$ stands for an arbitrary locally integrable function on $[0, 1]$ such that $L_f(t) \geq 1$ for almost all $t \in [0, 1]$. It follows from (1.5) that $L_f(x) = J_f(x)/\ell(f'(x))^n$ a.e. and therefore $L_f(x)$ agrees with the inner dilatation coefficient of f at x. A simple observation shows that f is conformally differentiable at the origin iff the integral in (1.5) converges as $x \to 0$. For an arbitrary quasiregular mapping $f : B \to B$, $f(0) = 0$, the latter convergence assumption can be written in the form

$$\int\limits_{|x|<r} \frac{L_f(x) - 1}{|x|^n} dx \to 0 \ \text{ as } \ r \to 0 \tag{1.6}$$

and one can expect that the condition (1.6) is necessary for $f(x)$ to be conformal at $x = 0$.

In this paper we derive Grötzsch type modulus inequalities for quasiregular mappings in $\mathbb{R}^n$, $n \geq 2$, where integrals similar to (1.6) control the distortion. Then we make use of such estimates to prove that a space version of the Teichmüller – Wittich result for nonconstant quasiregular mappings holds if we replace the assumption (1.4) by (1.6). Finally we give a condition that guarantees the existence of the conformal distortion coefficient for f at every point of a compact set $E \subset G$ and apply the latter result to study Carleson's rectifiability problem for quasispheres, see [CA], [BP], [ABL]. For convenience we will prove the main statements only for the inner dilatation coefficient $L_f(x)$ because for the other dilatations the corresponding results will follow from the well–known relations (see, e.g., [V], p. 44)

$$L_f(x) \leq K_f^{n-1}(x), \ \ K_f(x) \leq L_f^{n-1}(x), \ \ H_f^n(x) = K_f(x) L_f(x) \tag{1.7}$$

that hold for every $n \geq 2$.

The following standard notations will be used in this paper. The norm of a vector $x \in \mathbb{R}^n$ is written as $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ where $x_1, \ldots, x_n$ are the coordinates of

x. If $0 < a < b < \infty$, the domain $R(a,b) = B(b) \setminus \overline{B}(a)$ is called a spherical annulus, where $B(r)$ is the ball $\{x \in \mathbb{R}^n \mid |x| < r\}$.

2. Modulus Estimates

Let $\mathcal{E}$ be a family of Jordan arcs or curves in space $\mathbb{R}^n$. A nonnegative and Borel measurable function ρ defined in $\mathbb{R}^n$ is called admissible for the family $\mathcal{E}$ if the relation

$$\int\limits_{\gamma} \rho ds \geq 1 \tag{2.1}$$

holds for every locally rectifiable $\gamma \in \mathcal{E}$. The quantity

$$\mathsf{M}(\mathcal{E}) = \inf_{\rho} \int\limits_{\mathbb{R}^n} \rho^n dx, \tag{2.2}$$

where the infimum is taken over all ρ admissible with respect to the family $\mathcal{E}$ is called the modulus of the family $\mathcal{E}$ (see, e.g, [V], p.16, [G$_2$]). This quantity is a conformal invariant and possesses the monotonicity property which says, in particular, that if $\mathcal{E}_1 < \mathcal{E}_2$, that is every $\gamma \in \mathcal{E}_2$ has a subcurve which belongs to $\mathcal{E}_1$, then (see, e.g., [V], p. 16)

$$\mathsf{M}(\mathcal{E}_1) \geq \mathsf{M}(\mathcal{E}_2). \tag{2.3}$$

A space ring $\mathcal{R}$ is defined as a finite domain in $\mathbb{R}^n$ whose complement consists of two components C_0 and C_1. A curve γ is said to join the boundary components in $\mathcal{R}$ if γ lies in $\mathcal{R}$, except for its endpoints that lie in different boundary components of $\mathcal{R}$.

In these terms the modulus of a space ring has the representation (see, e.g., [G$_2$], [H])

$$\operatorname{mod} \mathcal{R} = \left(\frac{\omega_{n-1}}{\mathsf{M}(\Gamma)} \right)^{1/(n-1)} \tag{2.4}$$

where Γ is the family of curves joining the boundary components in $\mathcal{R}$ and ω_{n-1} is the $(n-1)$–dimensional surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$ (see, e.g., [Z], [G$_2$]).

Note also, that the modulus $\mathsf{M}(\Gamma)$ coincides with the conformal capacity of the space ring $\mathcal{R}$ by a result of Löwner [LC] (see, e.g., [G$_2$]).

In the sequel we will employ only the following two families of curves, lying in the spherical annulus $R(a,b)$, and its images under quasiconformal mappings. The first one, that we denote by $\Gamma_{R(a,b)}$, consists of all locally rectifiable curves γ that join the boundary components in $R(a,b)$. The second family $\Gamma^{\nu}_{R(a,b)}$, with $\nu \in S^{n-1}$ fixed, consists of all locally rectifiable curves γ that join in $R(a,b)$ the two components of $L \cap R(a,b)$ where $L = \{t\nu : t \in R\}$.

In order to derive the desired estimates we need the following two statements.

2.5. **Lemma.** *Let $f : G \to G'$ be a quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then for each curve family Γ in G*

$$\mathsf{M}(f(\Gamma)) \le \int\limits_G \rho^n L_f(x)dx \tag{2.6}$$

for every ρ admissible for Γ.

Proof. To prove (2.6) we first recall the following Väisälä's inequality

$$\mathsf{M}(\Gamma) \le \int\limits_G \rho^{*n}(f(x))\|f'(x)\|^n dx \tag{2.7}$$

that holds for every curve family Γ in G and every ρ^* admissible for $\Gamma' = f(\Gamma)$. This inequality is contained in the proof of Theorem 32.3 from [V].

Indeed, let Γ_0 denote the family of all locally rectifiable curves $\gamma \in \Gamma$ such that f is absolutely continuous on every closed subcurve of γ. Since f is ACL^n, it follows from Fuglede's theorem (see, e.g., [V], p. 95) that $\mathsf{M}(\Gamma \setminus \Gamma_0) = 0$. Hence $\mathsf{M}(\Gamma) = \mathsf{M}(\Gamma_0)$.

Let ρ^* be admissible for Γ'. Define $\rho : \mathbb{R}^n \to \mathbb{R}$ by

$$\rho(x) = \rho^*(f(x))L(x, f) \tag{2.8}$$

for $x \in G$ and $\rho(x) = 0$ for $x \notin G$ where

$$L(x, f) = \limsup_{h\to 0} \frac{|f(x+h) - f(x)|}{|h|}. \tag{2.9}$$

If $\gamma \in \Gamma_0$ then Theorem 5.3 from [V] yields

$$\int\limits_\gamma \rho ds \ge \int\limits_{f\circ\gamma} \rho^* ds \ge 1 \tag{2.10}$$

Thus, ρ is admissible for Γ_0 and therefore

$$\mathsf{M}(\Gamma) = \mathsf{M}(\Gamma_0) \le \int\limits_G \rho^n dx = \int\limits_G \rho^{*n}(f(x))L^n(x, f)dx = \tag{2.11}$$

$$= \int\limits_G \rho^{*n}(f(x))\|f'(x)\|^n dx,$$

since f is differentiable almost everywhere in G and $L(x, f) = \|f'(x)\|$ at every point of differentiability.

Applying formula (2.7) to the inverse of f yields

$$\mathsf{M}(f(\Gamma)) \le \int\limits_G \rho^n L_f(x)dx \tag{2.12}$$

for every ρ admissible for Γ.

2.13. **Lemma.** *Let $\mathcal{R}$ be a space ring that contains the spherical annulus $R(a,b)$ and let E_1, E_2 be two disjoint subsets of $\mathcal{R}$ such that each sphere $S^{n-1}(t)$, $a<t<b$, meets both E_1 and E_2. If $\mathcal{E}$ is the family of all curves joining E_1 and E_2 in $\mathcal{R}\setminus\{E_1\cup E_2\}$ then*

$$\mathsf{M}(\mathcal{E}) \geq c_n \log\frac{b}{a} \tag{2.14}$$

where

$$c_n = \frac{1}{2}\omega_{n-2}\left(\int_0^\infty t^{\frac{2-n}{n-1}}(1+t^2)^{\frac{1}{1-n}}\right)^{1-n}. \tag{2.15}$$

If $\mathcal{R}=R(a,b)$ and E_1, E_2 are the components of $L\cap R(a,b)$, where L is a line through the origin in the direction of a unit vector ν, then

$$\mathsf{M}(\mathcal{E}) = c_n \log\frac{b}{a}. \tag{2.16}$$

This useful result, the proof of which is based on the combination of the space moduli technique and Hardy–Littlewood–Polya's symmetrization principle, is due to Gehring [G$_1$] (see, also, [V], p. 27, [C], p. 58, [R$_1$], p. 108).

Let $f:\mathbb{R}^n\to\mathbb{R}^n$, $f(0)=0$, $n\geq 2$, be a quasiconformal mapping. We will use the following standard notations

$$M_f(r)=\max_{|x|=r}|f(x)|, \quad m_f(r)=\min_{|x|=r}|f(x)|. \tag{2.17}$$

2.18. **Theorem.** *Let $f:\mathbb{R}^n\to\mathbb{R}^n$, $f(0)=0$, $n\geq 2$, be a quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then for every spherical annulus $R(a,b)$*

$$\log\frac{b}{a} - \operatorname{mod} f(R(a,b)) \leq \tag{2.19}$$

$$\leq \frac{\operatorname{mod}^n f(R(a,b))}{\sum_{k=1}^{n-1}\log^{n-k}\frac{b}{a}\operatorname{mod}^k f(R(a,b))}\frac{1}{\omega_{n-1}}\int_{R(a,b)}\frac{L_f(x)-1}{|x|^n}dx.$$

Proof. Let $R(a,b)$ be an arbitrary spherical annulus in $\mathbb{R}^n$ and let $\Gamma_{R(a,b)}$ be the family of curves which join the boundary components of $R(a,b)$. Then (2.6) yields

$$\mathsf{M}(f(\Gamma_{R(a,b)})) \leq \int_{R(a,b)} \rho^n L_f(x)dx \tag{2.20}$$

for every ρ admissible with respect to a family $\Gamma_{R(a,b)}$.

Using the formula (2.4), we obtain from (2.20)

$$(\operatorname{mod} f(R(a,b)))^{1-n} \leq \frac{1}{\omega_{n-1}}\int_{R(a,b)}\rho^n L_f(x)dx. \tag{2.21}$$

On the other hand the function

$$\rho_0(x) = \frac{1}{|x| \log \frac{b}{a}}, \tag{2.22}$$

is admissible with respect to $\Gamma_{R(a,b)}$ since for every curve $\gamma \in \Gamma_{R(a,b)}$

$$\int_\gamma \rho_0 ds \geq \int_a^b \frac{1}{r \log \frac{b}{a}} dr = 1. \tag{2.23}$$

Substituting ρ_0 in (2.21) and noting that

$$\frac{1}{\omega_{n-1}} \int_{R(a,b)} \rho_0^n(x) dx = \left(\log \frac{b}{a}\right)^{1-n} \tag{2.24}$$

we arrive at the inequality

$$(\operatorname{mod} f(R(a,b)))^{1-n} - \left(\log \frac{b}{a}\right)^{1-n} \leq \frac{1}{\omega_{n-1} (\log \frac{b}{a})^n} \int_{R(a,b)} \frac{L_f(x) - 1}{|x|^n} dx \tag{2.25}$$

that can be rewritten in the form (2.19). The proof is completed.

2.26. **Corollary.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $n \geq 2$, be a quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then for every spherical annulus $R(a,b)$*

$$\log \frac{b}{a} - \operatorname{mod} f(R(a,b)) \leq \frac{1}{\omega_{n-1}} \int_{R(a,b)} \frac{L_f(x) - 1}{|x|^n} dx. \tag{2.27}$$

Proof. If $\log(b/a) \leq \operatorname{mod} f(R(a,b))$, then the inequality (2.27) is trivial. If $\log(b/a) > \operatorname{mod} f(R(a,b))$, then (2.25) can be rewritten as

$$\left(\frac{\beta}{\alpha}\right)^{n-1} - 1 \leq \frac{M}{\beta}$$

where $\beta = \log(b/a)$, $\alpha = \operatorname{mod} f(R(a,b))$ and M is the right hand side of (2.27). Now

$$\frac{\beta}{\alpha} - 1 \leq \left(\frac{\beta}{\alpha}\right)^{n-1} - 1 \leq \frac{M}{\beta} \leq \frac{M}{\alpha}$$

and this gives (2.27).

2.28. **Corollary.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $n \geq 2$, be a quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then for every spherical annulus $R(a,b)$*

$$\log \frac{b}{a} - \log \frac{M_f(b)}{m_f(a)} \leq \frac{1}{\omega_{n-1}} \int_{R(a,b)} \frac{L_f(x) - 1}{|x|^n} dx. \tag{2.29}$$

Proof. Since the space ring $f(R(a,b))$ is contained in the spherical annulus $R(m_f(a),\, M_f(b))$, the monotonicity principle for the modulus yields

$$\operatorname{mod} f(R(a,b)) \le \operatorname{mod} R(m_f(a), M_f(b)) = \log \frac{M_f(b)}{m_f(a)} \tag{2.30}$$

because for every annulus $R(a,b)$

$$\operatorname{mod} R(a,b) = \log \frac{b}{a} \tag{2.31}$$

(see, e.g., [V], p. 22).

2.32. **Theorem.** *Let $f:\mathbb{R}^n \to \mathbb{R}^n$, $f(0)=0$, $n \ge 2$, be a quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then for every spherical annulus $R(a,b)$*

$$\mathsf{M}(f(\Gamma^{\nu}_{R(a,b)})) - c_n \log \frac{b}{a} \le \int\limits_{R(a,b)} \rho_0^n(x,\nu) \frac{L_f(x)-1}{|x|^n} dx \tag{2.33}$$

where

$$\rho_0(x,y) = (c_n/\omega_{n-2})^{\frac{1}{n-1}} \left(1 - \langle x/|x|, y/|y|\rangle^2\right)^{\frac{2-n}{2(n-1)}} \tag{2.34}$$

and c_n is the constant defined by (2.15).

Proof. Fix a unit vector $\nu = y/|y| \in \mathbb{R}^n$ and consider the family $\Gamma^{\nu}_{R(a,b)}$ of curves which join $\{t\nu:\ -b<t<-a\}$ to $\{t\nu:\ a<t<b\}$ in $R(a,b)$. By Lemma 2.5

$$\mathsf{M}(f(\Gamma^{\nu}_{R(a,b)})) \le \int\limits_{R(a,b)} \rho^n L_f(x) dx \tag{2.35}$$

for each admissible ρ with respect to $\Gamma^{\nu}_{R(a,b)}$.

Now we are going to show that the function

$$\rho_\nu(x) = \frac{1}{|x|}\rho_0(x,y) \tag{2.36}$$

is admissible for the family $\Gamma^{\nu}_{R(a,b)}$ for every fixed $\nu = y/|y|$.

Indeed, let γ be a rectifiable curve in $\Gamma^{\nu}_{R(a,b)}$ and let $\varphi(x) = x/|x|$. Then $\varphi\circ\gamma$ is a curve on S^{n-1} and γ joins the antipodal points $\pm y/|y|$. Since $|\varphi'(x)| = 1/|x|$ then (see [V], Cor. 5.4)

$$\int\limits_{\gamma} \rho_\nu(x) ds = \int\limits_{\gamma} \rho_0(\varphi(x),y)|\varphi'(x)||dx| \ge \int\limits_{\varphi\circ\gamma} \rho_0(x,y) ds. \tag{2.37}$$

In order to continue the estimation of the above integral, let us rewrite $\rho_0(x,y)$ in the form

$$\rho_0(x,y) = p_n^{-1} \left(\frac{(1-\langle x/|x|, y/|y|\rangle^2)^{\frac{1}{2}}}{2} \right)^{\frac{2-n}{n-1}} \tag{2.38}$$

with

$$p_n = 2\int_0^\infty r^{\frac{2-n}{n-1}}(1+r^2)^{\frac{1}{1-n}}dr \tag{2.39}$$

and introduce a certain coordinate system on the sphere S^{n-1}.

Denote by $\mathbb{V}^{n-1}$ a hyperplane passing through the origin and orthogonal to the vector $y/|y|$. Let $t = P(x) : S^{n-1} \to \mathbb{V}^{n-1}$ be the stereographic projection with the pole at the point $y/|y|$ and $F(t)$ be the inverse mapping. Provide the sphere S^{n-1} with the spherical coordinates $\alpha_1, ..., \alpha_{n-1}$ in such a way that α_1 stands for the angle between the radius vectors going from the origin to the points x and $-y/|y|$ of the unit sphere. In these terms $|t| = \tan(\alpha_1/2)$ and therefore, $\sin\alpha_1 = 2|t|/(1+|t|^2)$. On the other hand, $1 - \langle x/|x|, y/|y|\rangle^2 = \sin^2\alpha_1$, so

$$\rho_0(\cdot, y)\circ F(t) = p_n^{-1}\left(\frac{1+|t|^2}{|t|}\right)^{\frac{n-2}{n-1}}. \tag{2.40}$$

Since $x = F(t)$ is conformal and $|F'(t)| = 2/(1+|t|^2)$ we get

$$\int_{\varphi\circ\gamma}\rho_0 ds = \int_{F\circ\varphi\circ\gamma}\rho_0\circ F|F'(t)||dt| \geq 2p_n^{-1}\int_0^\infty |t|^{\frac{2-n}{n-1}}(1+|t|^2)^{\frac{1}{1-n}}d|t| = 1 \tag{2.41}$$

having completed the verification of the admissibility.

Noting that

$$\int_{R(a,b)}\rho_\nu^n dx = \int_a^b\left(\int_{S^{n-1}(r)}\rho_\nu^n dm_{n-1}\right)dr = \tag{2.42}$$

$$\int_a^b\left[\int_{S^{n-1}}\rho_\nu^n(ru)r^{n-1}dm_{n-1}(u)\right]dr = \int_a^b\frac{dr}{r}\int_{S^{n-1}}\rho_0^n dm_{n-1}(x) = c_n\log\frac{b}{a},$$

since

$$\int_{S^{n-1}}\rho_0^n dm_{n-1}(x) = c_n \tag{2.43}$$

and substituting $\rho_\nu(x)$ in the inequality (2.35) we get

$$\mathsf{M}(f(\Gamma^\nu_{R(a,b)})) - c_n\log\frac{b}{a} \leq \int_{R(a,b)}\rho_0^n(x,y)\frac{L_f(x)-1}{|x|^n}dx \tag{2.44}$$

and arrive at the stated conclusion. Here we have also used the following relation

$$\int_{S^{n-1}}\rho_0^n dm_{n-1}(x) = 2^{n-1}\int_{\mathbb{V}^{n-1}}\frac{\rho_0^n(F(t),y)}{(1+|t|^2)^{n-1}}dm_{n-1}(t) = \tag{2.45}$$

$$p_n^{-n}2^{n-1}\int_{\mathbb{V}^{n-1}}|t|^{\frac{n(2-n)}{n-1}}(1+|t|^2)^{\frac{n(n-2)}{n-1}}(1+|t|^2)^{1-n}dm_{n-1}(t) =$$

$$\omega_{n-2}p_n^{-n}2^{n-1}\frac{p_n}{2} = \omega_{n-2}p_n^{1-n}2^{n-2} = c_n.$$

The proof is completed.

2.46. **Corollary.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $n \geq 2$, be a K – quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then*

$$\int\limits_{S^{n-1}} \mathsf{M}(f(\Gamma^{\nu}_{R(a,b)}))dm_{n-1}(\nu) \leq c_n \int\limits_{R(a,b)} \frac{L_f(x)dx}{|x|^n}. \tag{2.47}$$

Proof. The function $\rho_0(x,y)$ is symmetric in the sense that $\rho_0(x,y) = \rho_0(y,x)$, $x, y \in S^{n-1}$, and therefore

$$\int\limits_{S^{n-1}} \rho_0^n dm_{n-1}(y) = \int\limits_{S^{n-1}} \rho_0^n dm_{n-1}(x) = c_n. \tag{2.48}$$

If we integrate the inequality (2.33) with respect to the parameter y over the sphere S^{n-1} then, by Fubini's theorem and relation (2.48), we get

$$\int\limits_{S^{n-1}} \mathsf{M}(f(\Gamma^{\nu}_{R(a,b)}))dm_{n-1}(\nu) \leq c_n \int\limits_{R(a,b)} \frac{L_f(x)dx}{|x|^n}. \tag{2.49}$$

The proof is complete.

2.50. **Corollary.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $n \geq 2$, be a K – quasiconformal mapping with the inner dilatation coefficient $L_f(x)$. Then*

$$\log\frac{m_f(b)}{M_f(a)} - \log\frac{b}{a} \leq \frac{1}{\omega_{n-1}} \int\limits_{R(a,b)} \frac{L_f(x) - 1}{|x|^n} dx. \tag{2.51}$$

Proof. If $m_f(b) \leq M_f(a)$ then the inequality (2.51) is trivial. Assume that $m_f(b) > M_f(a)$. Then the space ring $\mathcal{R} = f(R(a,b))$ contains the spherical annulus $R(M_f(a),\, m_f(b))$. The curve family $f(\Gamma^{\nu}_{R(a,b)})$ satisfies all the assumptions of Lemma 2.13 with the spherical annulus $R(M_f(a), m_f(b))$ as a subset of $f(R(a,b))$. Therefore, inequality (2.14) and (2.16) imply that

$$\mathsf{M}(f(\Gamma^{\nu}_{R(a,b)})) \geq \mathsf{M}(\Gamma^{\nu}_{R(M_f(a),m_f(b))}) = c_n \log\frac{m_f(b)}{M_f(a)}. \tag{2.52}$$

This together with (2.47) yields (2.51).

The following statements may be of independent interest.

2.53. **Theorem.** *Let f be a K – quasiconformal mapping of a spherical annulus $R(a,b)$ onto another spherical annulus $R(c,d)$ with the inner dilatation coefficient $L_f(x)$. Then*

$$-\frac{1}{\omega_{n-1}}\int\limits_{R(a,b)}\frac{L_f(x)-1}{|x|^n}dx \le \log\frac{b}{a}-\log\frac{d}{c}\le \tag{2.54}$$

$$\le \frac{\log^n(d/c)}{\sum_{k=1}^{n-1}\log^{n-k}(b/a)\cdot\log^k(d/c)}\cdot\frac{1}{\omega_{n-1}}\int\limits_{R(a,b)}\frac{L_f(x)-1}{|x|^n}dx.$$

Proof. The left inequality follows from Corollary 2.50 and the right one is a consequence of Theorem 2.18.

If f is a K – quasiconformal mapping in the plane, then (2.54) yields

$$\left(\frac{b}{a}\right)^{\frac{1}{K}}\le\frac{d}{c}\le\left(\frac{b}{a}\right)^K \tag{2.55}$$

and we recognize the classical Grötzsch inequality for annuli (see, e.g., [LV], p. 38).

2.56. **Corollary.** *Let f be a K – quasiconformal mapping of a spherical annulus $R(a,b)$ onto another spherical annulus $R(c,d)$ with the inner dilatation coefficient $L_f(x)$. Then*

$$\left|\log\frac{d}{c}-\log\frac{b}{a}\right|\le\frac{1}{\omega_{n-1}}\int\limits_{R(a,b)}\frac{L_f(x)-1}{|x|^n}dx. \tag{2.57}$$

Indeed, if $\log(d/c) > \log(b/a)$ then (2.57) follows from the inequality (2.51). If $\log(d/c) < \log(b/a)$ then (2.57) follows from the inequality (2.29).

For $n=2$ we arrive at the modulus estimations under quasiconformal mappings in the plane with the variable dilatation coefficient established by Belinski [B].

Note that all the inequalities proved in this section remain valid also for ACL^n homeomorphisms in $\mathbb{R}^n$ with locally integrable dilatation coefficients. Moreover, the estimates (2.54) and (2.57) are sharp. For instance, the radial mappings of the type (1.5) provide the equality in (2.57).

3. Conformal Distortion

We apply estimates proved in Section 2 to a space version of the regularity problem studied by Teichmüller [T] and Wittich [W].

3.1. **Theorem.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$, $f(0) = 0$, be a nonconstant K – quasiregular mapping with the inner dilatation coefficient $L_f(x)$ and*

$$I(r) = \frac{1}{\omega_{n-1}} \int\limits_{|x|<r} \frac{L_f(x) - 1}{|x|^n} dx \to 0 \quad as \ r \to 0. \tag{3.2}$$

Then the radius of injetivity of f at 0, $R_f(0)$, satisfies $R_f(0) > 0$ and there exists a constant C,

$$\min_{|x|=R} |f(x)| \frac{e^{-I(R)}}{R} \leq C \leq \max_{|x|=R} |f(x)| \frac{e^{I(R)}}{R}, \quad 0 < R \leq R_f(0), \tag{3.3}$$

such that

$$\frac{|f(x)|}{|x|} \to C \quad as \ x \to 0. \tag{3.4}$$

3.5. **Remark.** *The proof of Theorem 3.1 is also valid if $n = 2$ and f is a homeomorphism.*

3.6. **Corollary.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $n \geq 2$, be a K – quasiconformal mapping satisfying (3.2). Then $|f(x)| \sim C|x|$ as $x \to 0$ and inequalities (3.3) can be replaced by*

$$\min_{|x|=1} |f(x)| e^{-I(1)} \leq C \leq \max_{|x|=1} |f(x)| e^{I(1)}. \tag{3.7}$$

In the case $n = 2$ we arrive at the Teichmüller – Wittich result for K – quasiconformal mappings in the plane (see, also, [LV], Lemma 6.1). For $n \geq 3$ the asymptotic behavior of f described in Corollary 3.6 has been proved by Suominen [Su] for K – quasiconformal mapping in Riemannian manifolds.

3.8. **Remark.** *The above statements hold if we replace the inner dilatation $L_f(x)$ by the outer dilatation $K_f(x)$ or linear dilatation $H_f(x)$, respectively.*

It is well-known that a sense-preserving locally L-bilipschitz mapping $f : G \to \mathbb{R}^n$ is $L^{2(n-1)}$ - quasiregular; a locally L-bilipschitz mapping f satisfies for each $L' > L$, $x \in G$, and for some $\delta > 0$ the double inequality

$$1/L' \leq |f(y) - f(z)|/|y - z| \leq L' \tag{3.9}$$

whenever $y, z \in B(x, \delta)$. A more general class than sense - preserving locally bilipschitz mappings is provided by the class of mappings of bounded length distortion (BLD), see [MV]. These mappings form also a subclass of quasiregular mappings as well.

3.10. **Corollary.** *Let $f : G \to \mathbb{R}^n$, be a bilipschitz mapping with the coefficient of quasiisometry $L(x)$. If $y \in G$ and*

$$\frac{1}{\omega_{n-1}} \int\limits_{|x-y|<r} \frac{L(x)-1}{|x-y|^n} dx \to 0 \quad as \ r \to 0, \tag{3.11}$$

then there is a constant $C > 0$ such that

$$\frac{|f(x)-f(y)|}{|x-y|} \to C \quad as \ x \to y. \tag{3.12}$$

This statement was proved recently in [K].

3.13. **Remark.** *If we replace (3.2) by the following stronger requirement*

$$\int\limits_0^1 \frac{\delta_f(t)}{t} dt < \infty, \tag{3.14}$$

where

$$\delta_f(t) = \operatorname{ess} \sup_{|x|<t} (K_f(x) - 1), \tag{3.15}$$

then, by the well–known Reshetnyak theorem (see [R_2], p. 204), $f(x)$ will be conformally differentiable at the origin.

The well – known Liouville's theorem in space states that if the dilatation coefficient of a quasiregular mapping is close to 1, then f is close to a Möbius transformation. The next lemma, that gives a weak integral condition for this phenomenon, will be used for the proof of Theorem 3.1. Before its statement, let us recall some basic notions from the space infinitesimal geometry studied in [GMRV_2].

Let $f : G \to \mathbb{R}^n, n \geq 2$, be a nonconstant K–quasiregular mapping, $y \in G$, $t_0 = \operatorname{dist}(y, \partial G)$, $R(t) = t_0/t$, $t > 0$. For $x \in B(0, R(t))$ we set

$$F_t(x) = \frac{f(tx+y) - f(y)}{\tau(y,f,t)} \ , \tag{3.16}$$

where

$$\tau(y,f,t) = \left(\frac{\operatorname{meas} f(B(y,t))}{\Omega_n} \right)^{\frac{1}{n}} . \tag{3.17}$$

Here Ω_n denotes the volume of the unit ball B in $\mathbb{R}^n$. Let $T(y,f)$ be a class of all the limit functions for the family of the mappings F_t as $t \to 0$, where the limit is taken in terms of the locally uniform convergence. The set $T(y,f)$ is called the infinitesimal space for the mapping f at the point y. The elements of $T(y,f)$ are called infinitesimal mappings and the family (3.16) is called an approximating family for f at y. $T(y,f)$ is not empty and consists only of nonconstant K – quasiregular mappings $F : \mathbb{R}^n \to \mathbb{R}^n$ normalized by $F(0) = 0$, $F(\infty) = \infty$, $\operatorname{meas} F(B) = \Omega_n$, see [$\mathrm{GMRV}_2$], Th. 2.7.

3.18. **Lemma.** *Let $f : G \to \mathbb{R}^n$, $n \geq 2$, be a nonconstant K – quasiregular mapping with the inner dilatation coefficient $L_f(x)$ and let E be a compact subset of G. If*

$$\frac{1}{\Omega_n t^n} \int\limits_{|x-y|<t} (L_f(x) - 1)dx \to 0 \quad as \ t \to 0 \tag{3.19}$$

uniformly in $y \in E$ then:

i) The infinitesimal space $T(y, f)$ consists of the linear isometric mappings only;

ii) For $n \geq 3$ the mapping f is locally homeomorphic in E;

iii) The mapping f preserves infinitesimal spheres and spherical annuli centered at y in the sense that

$$\frac{\max_{|x-y|=r} |f(x) - f(y)|}{\min_{|x-y|=r} |f(x) - f(y)|} \to 1 \quad as \ r \to 0, \tag{3.20}$$

and for each $c \geq 1$, $c^{-1} \leq |x|/|z| \leq c$,

$$\frac{|f(x+y) - f(y)|}{|f(z+y) - f(y)|} - \frac{|x|}{|z|} \to 0 \tag{3.21}$$

as $x, z \to 0$ uniformly in $y \in E$.

Proof of Lemma 3.18. i) Let F_t be the approximating family for f at y. Assume that $t_j \to 0$ as $j \to \infty$ and $F_{t_j}(x) \to F(x)$ locally uniformly as $j \to \infty$. By formula (3.16) we get that

$$K_{F_{t_j}}(x) = K_f(t_j x + y) \ \ a.e. \tag{3.22}$$

and hence (3.19) can be written as

$$\int\limits_{|x|<R} (K_{F_{t_j}}(x) - 1)dx \to 0 \ \text{ as } \ j \to \infty \tag{3.23}$$

for every positive constant R. The latter limit implies that $K_{F_{t_j}}(x) \to 1$ as $j \to \infty$ in measure in $\mathbb{R}^n$. Without loss of generality we may assume that $K_{F_{t_j}}(x) \to 1$ almost everywhere and $F_{t_j}(x) \to F(x)$ locally uniformly as $j \to \infty$. This can be achieved by passing to a subsequence. By Theorem 3.1 from [GMRV$_1$], the limit mapping $F(x)$ is a nonconstant 1 – quasiregular mapping. Applying Liouville's theorem we see that f is a Möbius mapping. Because of the above normalization, $F(x)$ is a linear isomerty.

ii) By Lemma 4.5 from [MRV] we see that

$$\limsup_{j \to \infty} i_{F_{t_j}}(0) \leq i_F(0) = 1, \tag{3.24}$$

where $i_f(x)$ denotes the local topological index of f at x. Thus all the mappings $F_{t_j}(x)$ are locally injective at 0 for $j > j_0$. By (3.16) we deduce that f is locally injective at y, too.

iii) Let us assume the converse. Then there exist $c \geq 1$, sequences $y_j \in E$, $x_j, z_j \to 0$ as $j \to \infty$ satisfying the condition $c^{-1} \leq |x_j|/|z_j| \leq c$, such that

$$\left| \frac{|f(x_j+y_j)-f(y_j)|}{|f(y_j+y_j)-f(y_j)|} - \frac{|x_j|}{|y_j|} \right| \geq \varepsilon > 0. \tag{3.25}$$

By analogy with the preceding considerations, we introduce the following auxiliary family of nonconstant K – quasiregular mappings

$$F_j(x) = \frac{f(|x_j|x+y_j)-f(y_j)}{\tau(y_j,f,|x_j|)} \tag{3.26}$$

with the distortion coefficients $K_{F_j}(x) = K_f(|x_j|x+y_j)$. Then the convergence

$$\frac{1}{\Omega_n t^n} \int\limits_{|x-y_j|<t} (K_f(x)-1)dx \to 0 \quad \text{as} \quad t \to 0 \tag{3.27}$$

uniform in $y \in E$ with $t = |x_j|R$, $R > 0$, implies that

$$\int\limits_{|x|<R} (K_{F_j}(x)-1)dx \to 0 \quad \text{as} \quad j \to \infty \tag{3.28}$$

for every positive R. Since E is a compact subset of G, then we can repeat the corresponding sequential arguments to show that every limit function for the family of the mappings $F_j(x)$ as $j \to \infty$, is a linear isometry $F(x)$. Without loss of generality we may assume that $F_j(x) \to F(x)$ as $j \to \infty$.

Set $\zeta_j = x_j/|x_j|$, $w_j = z_j/|x_j|$. We may assume that $\zeta_j \to \zeta_0$, $|\zeta_0| = 1$, and $w_j \to w_0$, $c^{-1} \leq |w_0| \leq c$, as $j \to \infty$. Otherwise we can pass to some appropriate subsequences. Since $F_j(\zeta_j) = (f(x_j+y_j)-f(y_j))/\tau(y_j,f,|x_j|) \to F(\zeta_0)$ and $F_j(w_j) = (f(z_j+y_j)-f(y_j))/\tau(y_j,f,|x_j|) \to F(w_0)$ and F is linear isometry it follows that

$$0 = \frac{|F(\zeta_0)|}{|F(w_0)|} - \frac{|\zeta_0|}{|w_0|} = \lim_{j\to\infty} \left| \frac{|F_j(\zeta_j)|}{|F_j(w_j)|} - \frac{|x_j|}{|z_j|} \right| = \tag{3.29}$$

$$\lim_{j\to\infty} \left| \frac{|f(x_j+y_j)-f(y_j)|}{|f(z_j+y_j)-f(y_j)|} - \frac{|x_j|}{|z_j|} \right|.$$

Formula (3.29) provides a contradiction to the inequality (3.25). The relation (3.21) is a simple consequence of (3.20).

Proof of Theorem 3.1 Let $f : G \to \mathbb{R}^n$, $n \geq 3$, be a nonconstant K – quasiregular mapping. For every such mapping $f(x)$ and every $y \in G$ we define the radius of injectivity $R_f(y)$ of f at y as a supremum over all $\rho > 0$ such that $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$ in the ball $|x-y| < \rho$ in G, see [MRV].

Let us now assume that the integral

$$I(r) = \frac{1}{\omega_{n-1}} \int\limits_{|x|<r} \frac{L_f(x)-1}{|x|^n} dx \tag{3.30}$$

converges. The evident inequality

$$\frac{1}{r^n} \int\limits_{|x|<r} (L_f(x)-1)dx \leq \int\limits_{|x|<r} \frac{L_f(x)-1}{|x|^n} dx \tag{3.31}$$

yields

$$\frac{1}{\Omega_n r^n} \int\limits_{|x|<r} (L_f(x) - 1)dx \to 0 \quad \text{as} \quad r \to 0, \tag{3.32}$$

and we make use of the weak conformality result, stated in Lemma 3.18. It provides us, in particular, with the information that the mapping f is locally homeomorphic at the origin, $R_f(0) > 0$, and that

$$\lim_{r\to 0} \log \frac{M_f(r)}{m_f(r)} = 0. \tag{3.33}$$

Hence, in order to deduce (3.4) it suffices to show that

$$\lim_{r\to 0} \log \frac{M_f(r)}{r} = a \tag{3.34}$$

and for this we use the Cauchy criterion

$$-\varepsilon < \log \frac{M_f(r_2)}{M_f(r_1)} - \log \frac{r_2}{r_1} < \varepsilon. \tag{3.35}$$

Let us fix a positive number R, $0 < R < R_f(0)$, and first prove the left inequality in (3.35).

The convergence of the integral (3.30) implies that given $\varepsilon > 0$ there exists $\delta > 0$ such that $I(\delta) < \varepsilon/2$. Therefore, for every $0 < r_1 < r_2 < \delta$ by Corollary 2.28

$$\log \frac{r_2}{r_1} - \log \frac{M_f(r_2)}{m_f(r_1)} \leq I(\delta) < \varepsilon/2. \tag{3.36}$$

On the other hand, without loss of generality, we may assume that the relation (3.33) yields

$$\log \frac{M_f(r_2)}{m_f(r_1)} = \log \frac{M_f(r_2)}{M_f(r_1)} + \log \frac{M_f(r_1)}{m_f(r_1)} \leq \log \frac{M_f(r_2)}{M_f(r_1)} + \varepsilon/2. \tag{3.37}$$

From (3.36) and (3.37) we derive the left inequality in (3.34).

For proving the right inequality in (3.34) we first note that by Corollary 2.50 we may assume that

$$\log \frac{m_f(r_2)}{M_f(r_1)} - \log \frac{r_2}{r_1} \leq I(\delta) < \varepsilon/2. \tag{3.38}$$

Applying (3.33) we see that

$$\log \frac{M_f(r_2)}{M_f(r_1)} - \log \frac{m_f(r_2)}{M_f(r_1)} = \log \frac{M_f(r_2)}{m_f(r_2)} < \varepsilon/2. \tag{3.39}$$

Combining (3.38) with (3.39) we obtain the right side inequality (3.34) and therefore, the Cauchy criterion (3.35).

In order to prove inequalities (3.3) let us first note that by Corollary 2.28

$$\log \frac{R}{r} - \log \frac{M_f(R)}{m_f(r)} < I(R) \tag{3.40}$$

for every $0 < r \le R$. Using relation (3.33) we deduce that

$$\log \frac{M_f(r)}{r} < \log \frac{M_f(R)}{R} + I(R) + O(r). \tag{3.41}$$

Thus

$$\lim_{r\to 0} \log \frac{M_f(r)}{r} \le \log \frac{M_f(R)}{R} + I(R). \tag{3.42}$$

Next, by Corollary 2.50

$$\log \frac{m_f(R)}{M_f(r)} - \log \frac{R}{r} < I(R). \tag{3.43}$$

Since (3.39) implies that

$$\log \frac{M_f(r)}{r} > -I(R) + \log \frac{m_f(R)}{R} + O(r) \tag{3.44}$$

we get

$$\lim_{r\to 0} \log \frac{M_f(r)}{r} \ge \log \frac{m_f(R)}{R} - I(R) \tag{3.45}$$

and thus complete the proof.

The following statement is a strengthened version of Theorem 3.1.

3.46. **Theorem.** *Let $\varphi : G \to \mathbb{R}^n$, $n \ge 3$, be a nonconstant K – quasiregular mapping and let E be a compact set in G. If*

$$I(r) = \frac{1}{\omega_{n-1}} \int\limits_{|x-y|<r} \frac{L_\varphi(x) - 1}{|x-y|^n} dx \to 0 \quad \textit{as } r \to 0, \tag{3.47}$$

uniformly in $y \in E$, then there exists a positive continuous function $C(y)$, $y \in E$, such that

$$\frac{|f(x) - f(y)|}{|x-y|} \to C(y) \quad \textit{as } x \to y \tag{3.48}$$

uniformly in $y \in E$ and for $0 < R < R_f(y)$

$$\min_{|x-y|=R} |f(x) - f(y)| \frac{e^{-I(R)}}{R} \le C(y) \le \max_{|x-y|=R} |f(x) - f(y)| \frac{e^{I(R)}}{R}. \tag{3.49}$$

Here $R_f(y)$ stands for the radius of injectivity of f at y.

Proof. For each fixed $y \in G$ we will consider the following auxiliary K – quasiregular mappings

$$f(x) = \varphi(x+y) - \varphi(y) \tag{3.50}$$

defined for $|x-y| < \operatorname{dist}(y, \partial G)$. Denoting by $L_f(x,y)$ the inner dilatation coefficient for f we see that $L_f(x,y) = L_\varphi(x+y)$ a.e. in a neighborhood of the point $y \in G$. Then (3.47) implies that

$$\int_{|x|<r} \frac{L_f(x,y)-1}{|x|^n}dx \to 0 \quad \text{as } r \to 0 \tag{3.51}$$

uniformly in $y \in E$.

So, the mapping f satisfies all the conditions of Theorem 3.1 and hence

$$\frac{|f(x)|}{|x|} = \frac{|\varphi(x+y)-\varphi(y)|}{|x|} \to C(y) \quad \text{as } x \to 0 \tag{3.52}$$

for every fixed $y \in E$.

In order to show that the limit (3.52) is uniform with respect to $y \in E$ we have to analyze the proof of Theorem 3.1. It is based on the following two distortion estimates of Corollary 2.50 and Corollary 2.28

$$\log\frac{r_2}{r_1} - \log\frac{M_f(r_2)}{m_f(r_1)} \le \frac{1}{\omega_{n-1}} \int_{R(r_1,r_2)} \frac{L_f(x,y)-1}{|x|^n}dx, \tag{3.53}$$

$$\log\frac{m_f(r_2)}{M_f(r_1)} - \log\frac{r_2}{r_1} \le \frac{1}{\omega_{n-1}} \int_{R(r_1,r_2)} \frac{L_f(x,y)-1}{|x|^n}dx, \tag{3.54}$$

and the weak conformality consequence

$$\log\frac{M_f(r)}{m_f(r)} \to 0 \quad \text{as } r \to 0, \tag{3.55}$$

provided by Lemma 3.18. Lemma 3.18 states also that the uniform convergence (3.47) with respect to the parameter y implies the uniform convergence (3.55). Hence from (3.53) – (3.55) and the uniform convergence (3.47) we obtain that for every $\varepsilon > 0$ there is $\delta > 0$ such that $0 < r_1 < r_2 < \delta$ implies

$$\left|\log\frac{M_f(r_2)}{M_f(r_1)} - \log\frac{r_2}{r_1}\right| < \varepsilon \tag{3.56}$$

for every $y \in E$ where

$$M_f(r) = \max_{|x|=r}|f(x)| = \max_{|x|=r}|\varphi(x+y)-\varphi(y)|. \tag{3.57}$$

Thus, we have arrived at the Cauchy criterion for the function $M_f(r)/r$ to converge to a nonzero limit uniformly in $y \in E$. The proof is complete.

3.58. **Corollary.** *Let $f : G \to \mathbb{R}^n$, be a locally bilipschitz mapping with the coefficient of quasiisometry $L(x)$ and let E be a compact set in G. If*

$$\int_{|x-y|<r} \frac{L(x)-1}{|x-y|^n}dx \to 0 \quad \textit{as } r \to 0, \tag{3.59}$$

locally uniformly in $y \in E$, then there exists a positive continuous function $C(y)$, $y \in E$, such that

$$\frac{|f(x)-f(y)|}{|x-y|} \to C(y) \quad \textit{as } x \to y \tag{3.60}$$

uniformly in $y \in E$.

This statement follows immediately from Theorem 3.46 if we recall that every locally L-bilipschitz mapping in G is K - quasiregular with $K \le L^{2(n-1)}$.

3.61. **Corollary.** *Let* $f : G \to \mathbb{R}^n$, $n \ge 2$, *be a* K – *quasiconformal mapping and let* E *be a compact subset of* G. *If*

$$\frac{1}{\omega_{n-1}} \int\limits_{|x-y|<r} \frac{L_f(x) - 1}{|x-y|^n} dx \to 0 \quad \text{as } r \to 0 \tag{3.62}$$

uniformly in $y \in E$, *then there exists a positive constant* L *such that*

$$\frac{1}{L}|x - z| \le |f(x) - f(z)| \le L|x - z| \tag{3.63}$$

whenever $x, z \in E$.

Proof. We first show that

$$M = \sup_{x,z \in E,\, x \ne z} \frac{|f(x) - f(z)|}{|x - z|} < \infty. \tag{3.64}$$

Let us assume the converse. Then there exist sequences $x_j, z_j \in E$ such that

$$\lim_{j \to \infty} \frac{|f(x_j) - f(z_j)|}{|x_j - z_j|} = \infty. \tag{3.65}$$

Without loss of generality we may assume that $x_j \to x_0$, $z_j \to z_0$. Since E is a compact set then $x_0, z_0 \in E$. If $x_0 \ne z_0$ then

$$\lim_{j \to \infty} \frac{|f(x_j) - f(z_j)|}{|x_j - z_j|} = \frac{|f(x_0) - f(z_0)|}{|x_0 - z_0|} \ne \infty. \tag{3.66}$$

If $x_0 = z_0 = y$ then

$$\lim_{j \to \infty} \frac{|f(x_j) - f(z_j)|}{|x_j - z_j|} = C(y), \tag{3.67}$$

by Theorem 3.46. Since $C(y) < \infty$ then (3.67) provides a contradiction to the relation (3.65).

Repeating the preceding arguments and taking into account both the injectivity of f in G and the inequality $C(y) > 0$, $y \in E$, we get that

$$N = \inf_{x,z \in E,\, x \ne z} \frac{|f(x) - f(z)|}{|x - z|} > 0. \tag{3.68}$$

The inequalities (3.64) and (3.68) imply the existence of a positive constant L such that (3.63) holds whenever $x, z \in E$.

Next we will apply Theorem 3.46 to a space version of the rectifiability problem for quasiconformal mappings studied by Carleson [CA]. It is well-known that a quasiconformal mapping $f : G \to \mathbb{R}^n$ being an ACL^n homeomorphism need not be absolutely continuous on some subsets E of G of a smaller dimension than n. Hence the image $f(\gamma)$ of a rectifiable curve $\gamma \subset G$ under quasiconformal mapping f may fail to be rectifiable. The following statement provides a sufficient condition that guarantees the rectifiability of $f(\gamma)$.

3.69. **Corollary.** *Let $f : G \to \mathbb{R}^n$, $n \geq 2$, be a K – quasiconformal mapping and let γ be a rectifiable curve in G. If*

$$\frac{1}{\omega_{n-1}} \int\limits_{|x-y|<r} \frac{L_f(x) - 1}{|x-y|^n} dx \to 0 \quad as \ r \to 0 \tag{3.70}$$

uniformly in $y \in \gamma$, then $\Gamma = f(\gamma)$ is rectifiable and moreover,

$$\int\limits_{f(\gamma)} dS = \int\limits_{\gamma} C(y) ds, \tag{3.71}$$

where $C(y)$ is defined by (3.48).

Proof. The following double inequality is trivial

$$\ell(f'(y))ds \leq dS \leq ||f'(y)||ds \tag{3.72}$$

where ds and dS stand for the element of the length of the curve γ at the point $y \in \gamma$ and its image under the mapping f, respectively. On the other hand, Theorem 3.46 provides the explicit representation for the conformal distortion coefficient of f at y and hence

$$\ell(f'(y)) = ||f'(y)|| = \lim_{h \to 0} \frac{|f(y+h) - f(y)|}{|h|} = C(y). \tag{3.73}$$

From (3.72) we deduce that the line elements of γ and $f(\gamma)$ are connected by the relation $dS = C(y)ds$ and thus, we arrive at the formula (3.71). Rectifiability of $f(\gamma)$ now follows from Corollary 3.61 because γ is compact and therefore $C(y) \leq L$, $y \in \gamma$.

Note that formula (3.71) provides the following double inequality

$$\frac{1}{L} \leq \frac{\text{length}\, f(\gamma)}{\text{length}\, \gamma} \leq L \tag{3.74}$$

and the constant L can be also estimated by means of formula (3.49).

Using the preceding approach we can apply Theorem 3.46 to the study of some geometric properties of K - quasispheres, that is, images of the unit sphere S^{n-1} of $\mathbb{R}^n$ under K – quasiconformal mappings of $\mathbb{R}^n$. When $n = 2$, they are called quasicircles or quasiconformal curves and studied in details in a number of the well-known papers, see, e.g., [ABL], [BP], [BG]. The problem concerns sufficient conditions which guarantee the rectifiability of a quasisphere.

For a set $E \subset \mathbb{R}^n$ and for $\delta > 0$ let

$$\Lambda_\alpha^\delta(E) = \gamma_{n,\alpha} \inf_{\{B_j\}} \sum_j d(B_j)^\alpha, \tag{3.75}$$

where the infimum is taken over all countable coverings $\{B_j\}$ of E with $d(B_j) < \delta$. Here the B_j are balls of $\mathbb{R}^n$ and $d(B_j)$ is the diameter of B_j (see, e.g., [F], p. 7). The quantity

$$\Lambda_\alpha(E) = \lim_{\delta\to 0} \Lambda_\alpha^\delta(E), \tag{3.76}$$

finite or infinite, is called the α-dimensional normalized Hausdorff measure of the set E.

P. Mattila and M. Vuorinen [MVM] proved that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is K – quasiconformal, $K(t) = (f| \cup_{|x|=1} B(x,t))$, $\alpha(t) = K(t)^{1/(n-1)}$, then the Dini condition

$$\int_0^1 \frac{1-\alpha(t)}{t} dt < \infty \tag{3.77}$$

implies that $\Lambda_{n-1}(f(S^{n-1})) < \infty$.

This result can be strengthened in the following directions. First, the well – known Reshetnyak's theorem states that the Dini condition (3.77) implies the uniform conformal differentiability of the mapping f in S^{n-1} (see [R$_2$], p. 378). Hence (3.77) gives a sufficient condition for the quasisphere $f(S^{n-1})$ to be smooth. On the other hand, the following statement provides a condition weaker then (3.77) for $f(S^{n-1})$ to be rectifiable.

3.78. **Corollary.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, be a K – quasiconformal mapping and let*

$$\int_{|x-y|<r} \frac{L_f(x) - 1}{|x-y|^n} dx \to 0 \quad \text{as } r \to 0 \tag{3.79}$$

uniformly in $y \in S^{n-1}$. Then

$$\Lambda_\alpha(f(S^{n-1})) = \int_{f(S^{n-1})} d\sigma = \int_{S^{n-1}} C^{n-1}(y) d\sigma \leq L^{n-1}\omega_{n-1} \tag{3.80}$$

where $L = \max_{y\in E} C(y)$. Here $d\sigma$ stands for the $(n-1)$ - dimensional surface area element for S^{n-1} and $C(y)$ is defined by (3.48).

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