Dimension in Transcendental Dynamics
4: A Julia set of dimension one

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Dimensions of sets – revision: 1

f is a transcendental entire function.

Proposition 1

$dim_H F(f) \in \{0, 2\}$.

Proposition 2

$\dim_H J(f) \in [1, 2]$.

Dimensions of sets – revision: 2

Theorem 1 (Stallard, 1997, 2000)

For each $p \in (1, 2]$ *there is a transcendental entire function* f *such that* $\dim_H J(f) = p$.

Theorem 2 (Stallard, 1996)

If $f \in \mathcal{B}$, *then* $\dim_H J(f) > 1$.

Theorem 3 (Stallard, 1994)

$\dim_H K(f) > 0$.

What about dimension equal to one?

Theorem 4 (Bishop, 2012)

There is a transcendental entire function f such that:

1. $\dim_H J(f) = \dim_H J(f) \cap A(f) = 1.$
2. $\dim_H (I(f) \setminus A(f)) = 0.$
3. *Given $\alpha > 0$, f can be constructed such that*

$$\dim_H K(f) = \dim_H (J(f) \setminus A(f)) < \alpha.$$

4. ** $\dim_P J(f) = 1.$
5. ** *$J(f)$ has locally finite 1-dimensional Hausdorff measure.*
6. ** *f can be constructed to have arbitrarily slow growth.*

'In our example, $J(f)$, $J(f) \setminus A(f)$ and $(J(f) \cap I(f)) \setminus A(f)$ are each as small as is possible for a transcendental entire function; in some sense, our example is the "least chaotic" or "most normal" transcendental entire function'.

This talk

- In this talk we sketch the construction and proof of Bishop's result (excluding asterised items).
- All errors and omissions are mine; in particular I may have omitted important elements of the proof in my attempt to present only the basic structure.
- We write $f \approx g$ to indicate that, in some domain, the functions f and g are very close to being equal, in a way which is intuitively obvious and can be made precise. We will only worry about the intuition.
- Since the construction involves a multiply connected Fatou component, we will consider first a simpler example of Baker.

A multiply connected wandering domain

Baker 1963, 1976

Define a transcendental entire function g by

$$g(z) = cz^2 \prod_{k=1}^{\infty} \left(1 + \frac{z}{a_k}\right).$$

Here $c > 0$ is small, $a_1 > 0$ is large, and we set

$$a_{n+1} = ca_n^2 \prod_{k=1}^{n} \left(1 + \frac{a_n}{a_k}\right).$$

Note that, for large n, we have $a_{n+1} \approx g(a_n)$.

Behaviour near the origin

If $|z|$ is small then $g(z) \approx cz^2$.
Hence g has an attracting Fatou component near the origin.

Behaviour far from the origin

- Set $A_n = \{z : \sqrt{a_n} \leq |z| \leq a_n^2\}$.
- Set $B_n = \{z : a_n^2 < |z| < \sqrt{a_{n+1}}\}$.
- If n is sufficiently large ....
  - In A_n we have $g(z) \approx \text{const} \cdot z^{n+1} \left(1 + \frac{z}{a_n}\right)$.
  - In B_n we have $g(z) \approx \text{const} \cdot z^{n+2}$.
  - If $|z| = \sqrt{a_n}$, then $|g(z)| < \sqrt{a_{n+1}}$.
  - If $|z| = a_n^2$, then $|g(z)| > a_{n+1}^2$.
  - Hence $g(A_n) \supset A_{n+1}$ and $g(B_n) \subset B_{n+1}$.
  - Hence $B_n \subset F(g)$, and B_n must be contained in a multiply connected Fatou component.

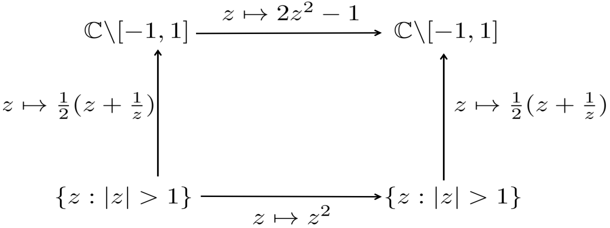
Bishop's approach

Bishop modifies Baker's function so that:

- There is no 'gap' between small modulus behaviour and large modulus behaviour.
- The 'error' between the function and its approximation is very small.
- The Julia set can be partitioned into three subsets, the size of each of which can be controlled:
  - Points whose orbit – eventually – stays near the origin.
  - Points whose orbit – eventually – always 'jumps' up an annulus.
  - Points which 'jump' down annuli infinitely often.

Near the origin: the dynamics of T_2

- Define a function $T_2(z) = 2z^2 - 1$.
- The following diagram commutes:



- Hence $J(T_2) = [-1, 1]$, and all other points iterate to infinity.

Near the origin: the dynamics of p_λ

- Define a function $p_\lambda(z) = \lambda T_2(z)$, for $\lambda > 1$.
- p_λ maps two small intervals to $[-1, 1]$; all other points iterate to infinity.
- $p_\lambda^{\circ k}$ maps 2^k small intervals to $[-1, 1]$.
- A simple calculation based on the size and number of these intervals shows that $\dim_H J(p_\lambda) \to 0$ as $\lambda \to \infty$.

The definition of the function

- Choose $\lambda > 1$ arbitrarily large, so that $\dim_H J(p_\lambda) < \alpha$.
- Choose $R_1 > 0$ large and $K_0 \in \mathbb{N}$ large.
- Define a sequence of (large, increasing) integers (m_k) (which depend on K_0) – to be specified later.
- Define a transcendental entire function f by

$$f(z) = p_\lambda(z)^{\circ K_0} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2}\left(\frac{z}{R_k}\right)^{m_k}\right).$$

- Here we set

$$R_{n+1} = p_\lambda(2R_n)^{\circ K_0} \prod_{k=1}^{n} \left(1 - \frac{1}{2}\left(\frac{2R_n}{R_k}\right)^{m_k}\right),$$

- so that $R_{n+1} \approx f(2R_n)$.

To get an idea ....

- Choose $K_0 = R_1 = 10$.
- Then $m_1 \approx 10^3$, $m_2 \approx 2^{1000}$, $R_2 \approx 10^{300}$.
- $$f(z) \approx p_\lambda(z)^{\circ 10} \left(1 - \frac{1}{2}\left(\frac{z}{10}\right)^{1000}\right)\left(1 - \frac{1}{2}\left(\frac{z}{10^{300}}\right)^{2^{1000}}\right)\cdots .$$

Approximating the function: 1

- If $|z| < R_1/2$, then $f(z) \approx p_\lambda(z)^{\circ K_0}$.
- Hence there is a Cantor repeller $E \subset \{z : |z| \le R_1/2\}$ with $\dim_H E < \alpha$.

Approximating the function: 2

- If $n \in \mathbb{N}$ and $R_n/2 \le |z| \le R_{n+1}/2$, then

$$f(z) \approx p_\lambda(z)^{\circ K_0} \prod_{k=1}^{n} \left(1 - \frac{1}{2}\left(\frac{z}{R_k}\right)^{m_k}\right) \tag{1}$$

$$\approx \text{const} \cdot z^{2^{K_0} + \sum_{k=1}^{n-1} m_k} \left(1 - \frac{1}{2}\left(\frac{z}{R_n}\right)^{m_n}\right) \tag{2}$$

$$= \text{const} \cdot \left(\frac{z}{R_n}\right)^{m_n} \left(2 - \left(\frac{z}{R_n}\right)^{m_n}\right). \tag{3}$$

- Note that the (m_n) are chosen to give equality in (3).
- If $n \in \mathbb{N}$ and $3R_n/2 \le |z| \le R_{n+1}/2$, then

$$f(z) \approx \text{const} \cdot z^{2m_n}. \tag{4}$$

The geometry of T_2: part 1

Recall $T_2(z) = 2z^2 - 1$.

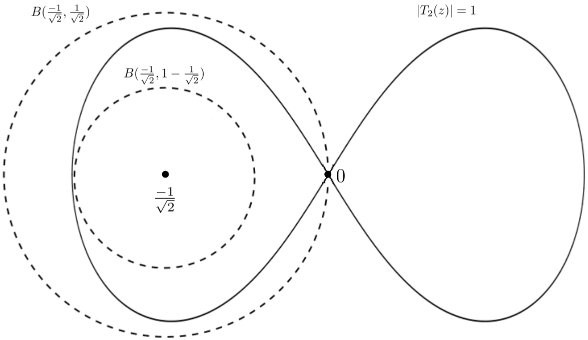


Image (part): Bishop (2012)

The geometry of T_2: part 2

- Define a function $H_n(z) = z^n(2 - z^n) = -T_2\left(\frac{z^n - 1}{\sqrt{2}}\right)$.

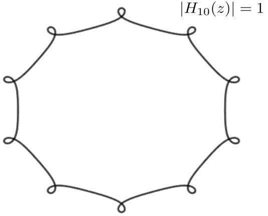


Image (part): Bishop (2012)

- H_n is conformal in the 'petals'.
- H_n is $2n$-1 elsewhere.
- These facts will be used later when counting preimages.

The geometry of T_2: part 3

- Note that we have shown that $f(z) \approx \text{const} \cdot H_n(z/R_n)$, where the constant is comparable to R_{n+1}.
- Indeed, this fact motivated our choice of the (m_k) and the structure of the polynomials in f.

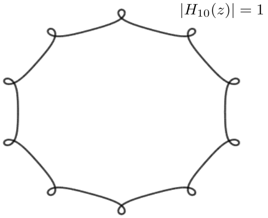


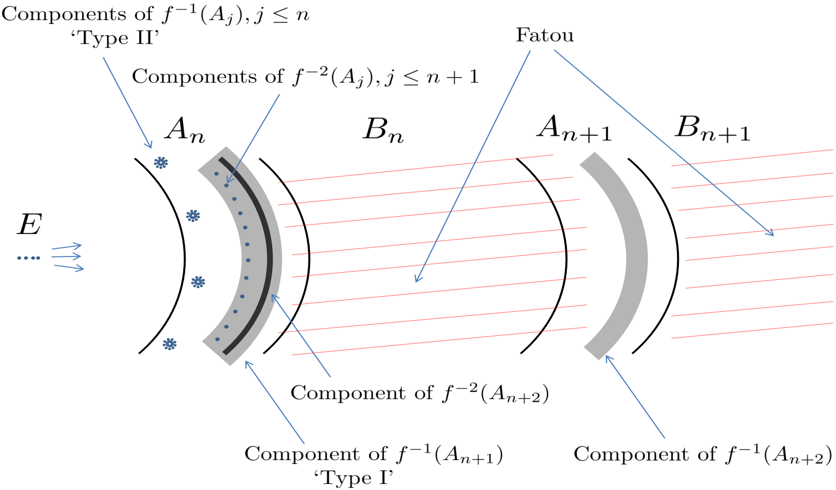
Image (part): Bishop (2012)

- Hence we have good control on the behaviour of f.

Behaviour far from the origin, i.e. $|z| \geq R_1/2$

- Set $A_n = \{z : 1/2R_n \leq |z| \leq 4R_n\}$ (includes petals).
- Set $A'_n = \{z : 3/2R_n \leq |z| \leq 5/2R_n\}$ (outside petals).
- Set $B_n = \{z : 4R_n \leq |z| \leq 1/2R_{n+1}\}$ (far from petals).
- From the previous approximations, it is straightforward to show that $f(A'_n) \supset A_{n+1}$ and hence $f(B_n) \subset B_{n+1}$.
- Hence $B_n \subset F(f)$, and B_n must be in a multiply connected Fatou component.

$F(f)$ and $J(f)$



Components of $f^{-1}(A_j), j \leq n$
'Type II'
Components of $f^{-2}(A_j), j \leq n+1$
Fatou
A_n
B_n
A_{n+1}
B_{n+1}
E
Component of $f^{-2}(A_{n+2})$
Component of $f^{-1}(A_{n+1})$
'Type I'
Component of $f^{-1}(A_{n+2})$

Partition the Julia set

- For $k \leq 0$, set $A_k = \{z : |z| \leq R_1/2, f^{k+1}(z) \in A_1\}$.
- Set $A = \bigcup_{k=-\infty}^{\infty} A_k$.
- Set $B = \bigcup_{k=1}^{\infty} B_k$.
- The orbit of a point z must eventually:
  - Land in B in which case $z \in F(f) \cap A(f)$. We have no further interest in these points.
  - Land in E. Let the set of these points be E', and note that $E' \subset J(f) \cap K(f)$.
  - Always lie in A, i.e. $z \in X := \cap_{n=1}^{\infty} f^{-n}(A)$. We further partition this set as follows:
    - $Z \subset X$ consists of those points whose orbit, eventually always 'goes up' an annulus. Note that $Z = J(f) \cap A(f)$.
    - $Y \subset X$ consists of those points whose orbit, fails to 'goes up' an annulus infinitely often. Note that $Y \subset J(f) \backslash A(f)$.

The result follows from the following.

Lemma 5

If $S \subset \mathbb{C}$, *then* $\dim_H f^{-1}(S) = \dim_H f(S) = \dim_H S$.

Lemma 6

$\dim_H E' = \dim_H E < \alpha$.

Lemma 7

$\dim_H Z = 1$.

Lemma 8

$\dim_H Y \cap A_m \leq \alpha$, *for* $m \in \mathbb{Z}$.
Moreover, for $z \in Y$, *let* $m(z) = \min\{m : \exists n \text{ s.t. } f^n(z) \in A_m\}$.
Then $\dim_H \{z \in Y : m(z) \geq m\} \to 0$, *as* $m \to \infty$.

If $S \subset \mathbb{C}$, then $\dim_H f^{-1}(S) = \dim_H f(S) = \dim_H S$.

This follows from standard properties of Hausdorff dimension for any non-constant entire function.

$\dim_H E' = \dim_H E < \alpha$.

This follows from the previous lemma, and the size of E.

$\dim_H Z = 1$.

- By the earlier lemma, we only need to estimate, for each $m \in \mathbb{N}$, the dimension of $\{z \in Z : f^n(z) \in A_{m+n}, \text{for } n \geq 0\}$.
- For $n \geq 0$, consider the nested topological annuli

$$\Gamma_{m,n} = \{z \in A_m : f^j(z) \in A_{m+j}, \text{ for } j = 1, \ldots, n\}.$$

- Recall that f is very closed to a monomial in each A'_n.
- It can be deduced that the widths of the $\Gamma_{m,n}$ decrease to zero uniformly in n, and these sets limit on a smooth Jordan curve.

Penultimate slide, final lemma.

Lemma 9

$\dim_H Y \cap A_m \leq \alpha$, *for* $m \in \mathbb{Z}$.
Moreover, for $z \in Y$, *let* $m(z) = \min\{m : \exists n \text{ s.t. } f^n(z) \in A_m\}$.
Then $\dim_H \{z \in Y : m(z) \geq m\} \to 0$, *as* $m \to \infty$.

- We cover $Y \cap A_m$ with nested collections of sets W, where W is such that $f^{n-1}(W) \subset A_{k'}$ and $f^n(W) = A_k$, where $k < k'$.
- We can count the number of such preimages using the previous comments on the multiplicity of H_n.
- We can estimate the diameters of these preimages using the scaling properties of H_n.
- Both parts of the lemma can be derived from these facts.

Thanks

Thanks to Chris Bishop for his assistance with the preparation of these slides.