A GEOMETRIC APPROACH TO POLYNOMIAL AND RATIONAL APPROXIMATION

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Abstract. We strengthen the classical approximation theorems of Weierstrass, Runge and Mergelyan by showing the polynomial and rational approximants can be taken to have a simple geometric structure. In particular, when approximating a function f on a compact set K, all the critical points of our approximants lie close to K, and all the critical values lie close to $f(K)$. Our proofs rely on extensions of (1) the quasiconformal folding method of the first author, and (2) a theorem of Carathéodory on approximation of bounded analytic functions by finite Blaschke products.

1. Introduction

The following is Runge's classical theorem on polynomial approximation.

Theorem 1.1. [Run85] *Let f be a function analytic on a neighborhood of a compact set $K \subset \mathbb{C}$, and suppose $\mathbb{C} \setminus K$ is connected. For all $\varepsilon > 0$, there exists a polynomial p so that*

$$||f - p||_K := \sup_{z \in K} |f(z) - p(z)| < \varepsilon.$$

This famous result does not say much about what the polynomial approximant p looks like off the compact set. For various applications, it would be useful to understand the global behavior of p and, in particular, the location of the critical points and values of p. To this end, we state our first result (Theorem A below) after introducing the following notation.

Notation 1.2. For any compact set $K \subset \mathbb{C}$ we denote by $\mathrm{fill}(K)$ the union of K with all bounded components of $\mathbb{C} \setminus K$. We say K is *full* if $\mathbb{C} \setminus K$ is connected. We let $\mathrm{CP}(f)$ denote the set of critical points of an analytic function f, and let $\mathrm{CV}(f) := f(\mathrm{CP}(f))$ denote its critical values. A *domain* in $\widehat{\mathbb{C}}$ is an open, connected subset of $\widehat{\mathbb{C}}$.

Theorem A. (Polynomial Runge+) *Let $K \subset \mathbb{C}$ be compact and full, $\mathcal{D}$ a domain containing K, and suppose f is a function analytic in a neighborhood of K. Then for all $\varepsilon > 0$, there exists a polynomial p so that $||p - f||_K < \varepsilon$ and:*

(1) $\mathrm{CP}(p) \subset \mathcal{D}$,
(2) $\mathrm{CV}(f|_K) \subset \mathrm{CV}(p) \subset \mathrm{fill}\{z : d(z, f(K)) \leq \varepsilon\}$.

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Analogous improvements of the polynomial approximation theorems of Mergelyan and Weierstrass will be stated and proved in Section 9 (see Theorem 9.8 and Corollary 9.9). When K is not full, uniform approximation by polynomials is not always possible, and so we turn to rational approximation. We denote the Hausdorff distance between two sets X, Y, by $\mathrm{d}_H(X, Y)$.

Theorem B. (Rational Runge+) *Let $K \subset \mathbb{C}$ be compact, $\mathcal{D}$ a domain containing K, f a function analytic in a neighborhood of K, and suppose $P \subset \widehat{\mathbb{C}} \setminus K$ contains exactly one point from each component of $\widehat{\mathbb{C}} \setminus K$. Then there exists $P' \subset P$ so that for all $\varepsilon > 0$, there is a rational function r so that $||r - f||_K < \varepsilon$ and:*

(1) $\mathrm{d}_H(r^{-1}(\infty), P') < \varepsilon$ and $|r^{-1}(\infty)| = |P'|$,
(2) $\mathrm{CP}(r) \subset \mathcal{D}$,
(3) $\mathrm{CV}(f|_K) \subset \mathrm{CV}(r) \subset \mathrm{fill}\{z : d(z, f(K)) \leq \varepsilon\}$.

The behavior of p off K is of particular interest in applications, such as in complex dynamics where approximation results have been used to prove the existence of various dynamical behaviors for entire functions (see, for example, [EL87], [BT21], [MRW21], [ERS22], [MRW22], [BEF+22]). However, not understanding the critical points and values of p means it has not been known whether these behaviors can occur within restricted classes of entire functions, such as the well studied Speiser or Eremenko-Lyubich classes (see the survey [Six18]).

Our approach is based on two ideas. The first is to show that on a compact subset $K \subset \mathbb{C}$ of a finitely connected domain $\Omega \subset \mathbb{C}$, any bounded analytic function can be approximated uniformly by an analytic function $B : \Omega \to \mathbb{C}$ having the property that $|B|$ is constant on each component of $\partial\Omega$. This extends a classical theorem of Carathéodory [Car54] concerning finite Blaschke products on the unit disk to more general regions, and may be of independent interest.

The second idea is an extension of quasiconformal folding, a type of quasiconformal surgery introduced in [Bis15], to extend the (generalized) Blaschke product B from Ω to a quasiregular mapping $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with specified poles. The map g may be taken close to holomorphic in a suitable sense, and so the Measurable Riemann Mapping Theorem (MRMT for brevity) will imply there is a quasiconformal mapping ϕ so that $g \circ \phi^{-1}$ is the desired polynomial or rational approximant.

This approach yields not only information on the critical points and values of the approximants as in Theorems A and B, but more broadly a detailed description of the geometric structure of these approximants. We end the introduction by describing this geometric structure in a few cases. First we introduce some more notation.

Notation 1.3. Let $D \subset \widehat{\mathbb{C}}$ be a simply connected domain so that $\infty \notin \partial D$. We let $\psi_D : \mathbf{E} \to D$ denote a Riemann mapping, where $\mathbf{E} = \mathbb{D}$ is D is bounded and $\mathbf{E} = \mathbb{D}^* := \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ if D is unbounded, in which case we specify $\psi_D(\infty) = \infty$.

First consider the case when K is full and connected, and f is holomorphic in a neighborhood of K satisfying $||f||_K < 1$. Let Ω, Ω' be analytic Jordan domains containing K, $f(K)$, respectively, so that f is holomorphic in Ω. Then the mapping

$$F := \psi_{\Omega'}^{-1} \circ f \circ \psi_\Omega : \mathbb{D} \to \mathbb{D}$$

is holomorphic, and by Carathéodory's theorem for the disk (see Theorem 2.1), there is a finite Blaschke product $b : \mathbb{D} \to \mathbb{D}$ that approximates F on the compact set $\psi_\Omega^{-1}(K)$. Therefore

$$B := \psi_{\Omega'} \circ b \circ \psi_\Omega^{-1} : \Omega \to \Omega'$$

is a holomorphic function that approximates f on K, and moreover B restricts to an analytic, finite-to-1 map of $\Gamma := \partial\Omega$ onto $\Gamma' := \partial\Omega'$.

In this paper, we will show that B can be approximated on Ω by a polynomial p so that $p^{-1}(\Gamma')$ is an approximation of Γ. More precisely, $p^{-1}(\Gamma')$ is connected, and consists of a finite union of Jordan curves $\{\gamma_j\}_0^n$ bounding pairwise disjoint Jordan domains $\{\Omega_j\}_0^n$ (see Figure 1): the $\{\Omega_j\}_0^n$ are precisely the connected components of $p^{-1}(\Omega')$. There is one "large" component Ω_0 that approximates Ω in the Hausdorff metric. The other components $\{\Omega_j\}_1^n$ can be made as small as we wish and to lie in any given neighborhood of ∂K. Moreover, the collection $\{\Omega_j\}_0^n$ forms a tree structure with any two boundaries $\partial\Omega_j$, $\partial\Omega_k$ either disjoint or intersecting at a single point, and with Ω_0 as the "root" of the tree as in Figure 1. Let Ω_∞ denote the unbounded component of $\mathbb{C} \setminus p^{-1}(\Gamma')$, so that

$$\mathbb{C} \setminus p^{-1}(\Gamma') = \Omega_0 \sqcup \left(\sqcup_{j=1}^n \Omega_j\right) \sqcup \Omega_\infty. \tag{1.1}$$

Recalling Notation 1.3, the polynomial p has the following simple structure with respect to the domains in (1.1).

(1) $p(\Omega_0) = \Omega'$ and $\psi_{\Omega'}^{-1} \circ p \circ \psi_{\Omega_0}$ is a finite Blaschke product.
(2) $p(\Omega_j) = \Omega'$ and p is conformal on Ω_j for $1 \leq j \leq n$.
(3) $p(\Omega_\infty) = \mathbb{C} \setminus \overline{\Omega'}$ and $p = \psi_{\mathbb{C}\setminus\overline{\Omega'}} \circ (z \mapsto z^m) \circ \psi_{\Omega_\infty}^{-1}$ on Ω_∞ for $m = \deg(p|_{\Omega_0}) + n$.

In other words, up to conformal changes of coordinates, p is simply a Blaschke product in Ω_0, a conformal map in each Ω_j, $1 \leq j \leq n$, and a power map $z \mapsto z^m$ in Ω_∞. The only finite critical points of p are either in Ω_0, or at a point where two of the curves $(\gamma_j)_{j=1}^n$ intersect, in which case the corresponding critical value lies on $\partial\Omega'$.

Next suppose K is connected, but $\mathbb{C} \setminus K$ has more than one component. In this case, in order to prove Theorem B, we will need to let Ω be a multiply connected analytic domain containing K, and Ω' an analytic Jordan domain containing $f(K)$. We cannot proceed as in the case $\mathbb{C}\setminus K$ is connected, however, without a multiply connected version of Carathéodory's Theorem for the disk (Theorem 2.1). The usual proof of Carathéodory's Theorem is based on power series, and it does not extend to the multiply connected setting. However, there is an alternate proof based on potential theory that does extend. We now briefly sketch the idea.

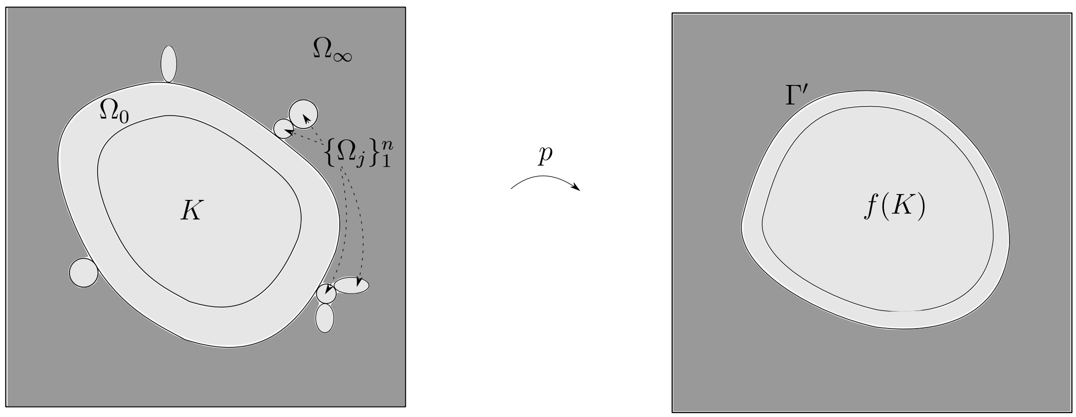

FIGURE 1. This figure illustrates the geometry of the approximant p in Theorem A when K is connected. The notation is explained in the text. Both domain and co-domain are colored so that regions with the same color correspond to one another under p.

Suppose f is holomorphic in a simply connected domain D. To simplify matters, suppose $f(D)$ is compactly contained in $\mathbb{D} \setminus \{0\}$. Then $u := -\log|f|$ is a positive harmonic function on D, and u is also bounded and bounded away from zero. Thus u is the Poisson integral of its boundary values. The Poisson kernel $P_x(z)$ associated to any point $x \in \partial D$ is a limit of normalized Green's functions on D: $P_x(z) \approx G(z, w_n)/G(0, w_n)$ with $w_n \to x$. Approximating the Poisson integral by a Riemann sum gives an approximation of u on any compact set $K \subset D$ by a sum of Green's functions $H(z) := \sum_j G(z, w_j)$ with poles distributed on $\{z : d(z, \partial D) = \delta\}$, and with δ as small as we wish. Figure 2 shows the function $u(x, y) = \frac{1}{2} + xy$ being approximated by a sum of 25 Green's functions on $D = \mathbb{D}$. Since D is simply connected, H has a well defined harmonic conjugate $\widetilde{H}$, and after adding a constant to $\widetilde{H}$ if necessary,

$$B := \exp(-H - i\widetilde{H})$$

approximates f on K. Moreover, B satisfies:

(1) B is holomorphic on D.
(2) $|B|$ extends continuously to a constant function on ∂D, with $||B||_{\partial D} = 1$.

We call such a function B on a (not necessarily simply connected) domain D a (generalized) finite Blaschke product on D (see Definition 2.2). When $D = \mathbb{D}$ this definition coincides with

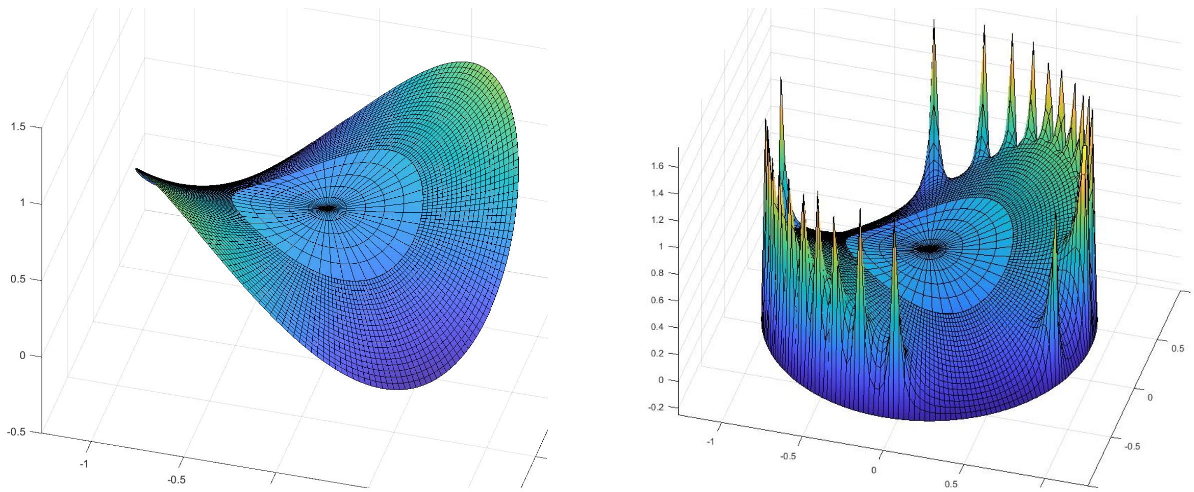

FIGURE 2. On the left is the positive harmonic function $u(x, y) = \frac{1}{2} + xy$ on the unit disk. On the right is a sum of 25 Green's functions with poles on the circle of radius .98. On $D(0, \frac{1}{2})$ the two functions agree to within .03. As expected, the poles are closer together where u is large and farther apart where u is small.

the usual definition of Blaschke product, and the above argument yields Carathéodory's Theorem. In fact, the above argument yields several technical improvements of Carathéodory's Theorem (see Theorem 2.6) which we will need in order to prove Theorem A.

This argument generalizes to finitely connected domains D, except that the sum of Green's functions H may not have a well defined harmonic conjugate (even modulo 2π). We will fix this by adding a small harmonic function h that is constant on each boundary component of D and whose periods match the periods of $-H$ around each boundary component of D. Exponentiating the sum of the modified function $H + h$ with its harmonic conjugate (now well-defined modulo 2π) then gives a (generalized) finite Blaschke product B which approximates the given function f on the desired compact set $K \subset D$. This gives a version of Carathéodory's theorem on finitely connected domains. By choosing H correctly, we can take $\|h\|_D$ as small as we wish, and hence $|B|$ is constant on each connected component of ∂D, with values that can all be taken as close to 1 as desired.

Now we return to the description of our rational approximant in the case that K is connected, but $\mathbb{C} \setminus K$ has more than one component, recalling that Ω is a multiply connected analytic domain containing K, and Ω' is an analytic Jordan domain containing $f(K)$. By the multiply connected version of Caratheodory's Theorem, there exists a (generalized) finite Blaschke product b approximating $\psi_{\Omega'}^{-1} \circ f$ on K, so that $B := \psi_{\Omega'} \circ b$ is a holomorphic map approximating f on K, and B restricts to an analytic, finite-to-1 mapping of each component of $\Gamma := \partial\Omega$ onto $\psi_{\Omega'}(|z| = t)$ for $t = 1$ or $t \approx 1$, where t may depend on the component of Γ.

We will show B can be approximated on Ω by a rational map r so that each component of $\partial\Omega$ can be approximated by a component of $r^{-1} \circ \psi_{\Omega'}(|z| = t)$ for t as above. These components of $r^{-1} \circ \psi_{\Omega'}(|z| = t)$ bound Jordan domains which form a decomposition of the plane as in the previously described polynomial setting, and in the interior of each such domain again r behaves either as a (generalized) finite Blaschke product, a conformal mapping, or a power mapping (up to conformal changes of coordinates).

Lastly, the case when K has more than one connected component is more intricate, and we will leave the precise description to later in the paper. (Briefly, quasiconformal folding is applied not just along the boundary of a neighborhood of K, but also along specially chosen curves that connect different connected components of this neighborhood.)

We remark that while Theorem A strictly improves on Runge's Theorem on polynomial approximation, the relationship between Theorem B and Runge's Theorem on rational approximation is more subtle. Both show existence of rational approximants, and only Theorem B describes the critical point structure of the approximant, however the poles of the approximant in Theorem B are specified only up to a small perturbation, whereas in Runge's Theorem they are specified exactly. We do not know whether it is necessary to consider perturbations of P' in Theorem B, or if the improvement $r^{-1}(\infty) = P'$ is possible (a related problem appears in [BL19], [DKM20], [BLU], where it is known no such improvement is possible).

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2. Interior Approximation

The following classical result of Carathéodory (referenced in the introduction) allows for approximation by Blaschke products in simply-connected domains.

Theorem 2.1. ([Car54]) *Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic and suppose $||f||_{\mathbb{D}} \leq 1$. Then there exists a sequence of finite Blaschke products on $\mathbb{D}$ converging to f uniformly on compact subsets of $\mathbb{D}$.*

The proof of Theorem 2.1 is elementary and may be found, for example, in Theorem I.2.1 of [Gar81] or Theorem 5.1 of the survey [GMR17]. In order to prove Theorem A, we will need to prove a version of Theorem 2.1 in which the Blaschke products satisfy certain boundary regularity conditions, and in order to prove Theorem B, we will need to prove a multiply-connected version of Theorem 2.1. These improvements are stated below in Theorem 2.6. First we need several definitions:

Definition 2.2. Let $D \subset \mathbb{C}$ be a finitely connected domain. We say a non-constant holomorphic function $B : D \to \mathbb{C}$ is a *finite Blaschke product on D* if $|B|$ extends continuously to a non-zero, constant function on each component of ∂D and $||B||_D = 1$.

Remark 2.3. When $D = \mathbb{D}$, the definition above corresponds with the usual definition of finite Blaschke product.

Notation 2.4. For a finite Blaschke product B on a finitely connected domain D, we let $\mathcal{I}_B$ denote the connected components of $\partial D \setminus \{z : B(z) \in \mathbb{R}\}$. In other words, $\mathcal{I}_B$ are the preimages (under B) of the open upper and lower half-circle components of $B(\partial D) \cap \mathbb{H}$, $B(\partial D) \cap (-\mathbb{H})$. We will frequently be dealing with sequences of finite Blaschke products $(B_n)_{n=1}^{\infty}$ on D, in which case we abbreviate $\mathcal{I}_{B_n}$ by $\mathcal{I}_n$.

Definition 2.5. We call a domain $D \subset \mathbb{C}$ an *analytic domain* if D is finitely connected, and each component of ∂D is an analytic Jordan curve.

We remark that a boundary component of an analytic domain D cannot be a single point.

Theorem 2.6. *Let $D \subset \mathbb{C}$ be an analytic domain, suppose $K \subset D$ is compact, and let f be a function analytic in a neighborhood of D satisfying $||f||_{\overline{D}} < 1$ and $\{z : f(z) = 0\} \cap \partial D = \emptyset$. Then there exists $M < \infty$ and a sequence $(B_n)_{n=1}^{\infty}$ of finite Blaschke products on D satisfying:*

$$\inf_{z \in \partial D} |B_n(z)| \xrightarrow{n\to\infty} 1, \tag{2.1}$$

$$||B_n - f||_K \xrightarrow{n\to\infty} 0, \tag{2.2}$$

$$\sup_{I \in \mathcal{I}_n} \operatorname{diam}(I) \xrightarrow{n\to\infty} 0, \text{ and} \tag{2.3}$$

$$\sup_{I,J \in \mathcal{I}_n} \operatorname{diam}(I)/\operatorname{diam}(J) < M \text{ and} \tag{2.4}$$

$$\sup_{I \in \mathcal{I}_n} \frac{\sup_{z\in I} |B_n'(z)|}{\inf_{z\in I} |B_n'(z)|} < M \text{ for all } n. \tag{2.5}$$

Theorem 2.6 will suffice for the proofs of Theorems A and B, although we prove a slightly stronger result in Section 12 (see Theorem 12.2). The proof of Theorem 2.6 will be delayed until Sections 10-12, which are independent of Sections 2-9. We now turn to applying Theorem 2.6 to produce Blaschke approximants as described in the introduction. Given a Jordan curve $\gamma \subset \mathbb{C}$, we denote the bounded component of $\widehat{\mathbb{C}} \setminus \gamma$ by $\operatorname{int}(\gamma)$.

Notation 2.7. We refer to Figure 3 for a summary of the following. For the remainder of this section, we will fix a compact set K, an analytic domain D containing K, and a function f holomorphic in a neighborhood of $\overline{D}$ satisfying $||f||_{\overline{D}} < 1$. Fix $\varepsilon > 0$. We assume that

$$d(z, K) < \varepsilon/2 \text{ and } d(f(z), f(K)) < \varepsilon/2 \text{ for every } z \in \partial D. \tag{2.6}$$

Definition 2.8. We let γ be an analytic Jordan curve surrounding $f(D)$ such that

$$\operatorname{dist}(w, f(K)) < \varepsilon \text{ for every } w \in \gamma, \tag{2.7}$$

and let $\Psi : \mathbb{D} \to \operatorname{int}(\gamma)$ denote a Riemann mapping.

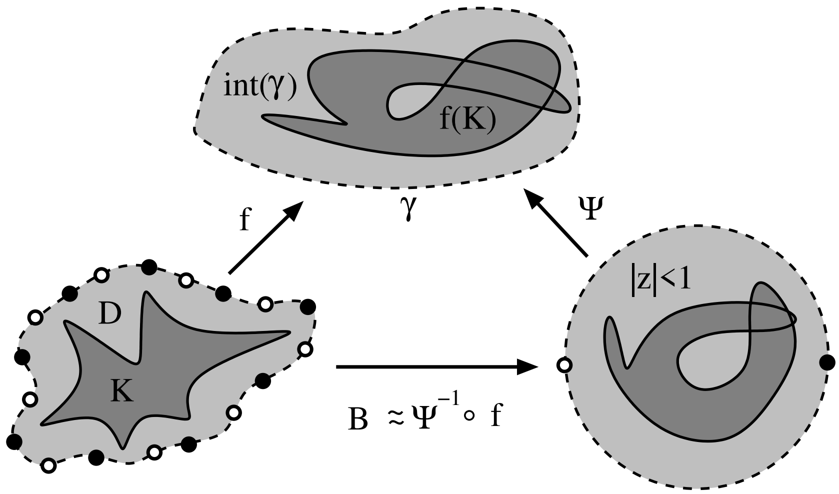


FIGURE 3. This figure illustrates Notations 2.4, 2.7 and Theorem 2.6. The vertices pictured on ∂D are $B^{-1}(\pm 1)$, and the components $\mathcal{I}_B$ of Notation 2.4 are the edges along ∂D connecting these vertices.

Remark 2.9. By enlarging D slightly we may ensure $\{z : \Psi^{-1} \circ f(z) = 0\} \cap \partial D = \emptyset$, so that Theorem 2.6 applies to the triple D, K, $\Psi^{-1} \circ f$ to produce $M < \infty$ and a sequence of finite Blaschke products $(B_n)_{n=1}^{\infty}$ on D satisfying the conclusions of Theorem 2.6.

Note that in Remark 2.9 we are applying Theorem 2.6 to $\Psi^{-1} \circ f$ (and not f). This will ensure that the critical values of the approximant $\Psi \circ B_n \approx f$ are close to $\text{fill}(f(K))$ as needed for Theorem B. We will now quasiconformally perturb the sequence (B_n) so as to ensure that we can later prove the conclusion $\text{CV}(f|_K) \subset \text{CV}(r)$ of Theorem B:

Theorem 2.10. *We may take the sequence $(B_n)_{n=1}^{\infty}$ of Remark 2.9 so that it also satisfies:*

$$\text{CV}(\Psi^{-1} \circ f|_K) \subset \text{CV}(B_n) \text{ for all large } n. \tag{2.8}$$

Proof. Let $K' \subset D$ be a compact set satisfying $K \subset \text{int}(K') \subset D$. Apply Theorem 2.6 to the triple D, K', $\Psi^{-1} \circ f$ to obtain a sequence of Blaschke products $(\mathcal{B}_n)_{n=1}^{\infty}$ on D. Let $z \in \text{CP}(f|_K)$. Since

$$||\mathcal{B}_n - \Psi^{-1} \circ f||_{K'} \xrightarrow{n\to\infty} 0, \tag{2.9}$$

by Hurwitz's Theorem there exists a sequence $(w_z^n)_{n=1}^{\infty}$ such that

$$\mathcal{B}_n'(w_z^n) = 0 \text{ and } w_z^n \xrightarrow{n\to\infty} z.$$

Let $r < 1$ be so that $\psi^{-1} \circ f(K) \subset r\mathbb{D}$. Define a homeomorphism $h_n : \mathbb{D} \to \mathbb{D}$ to be an interpolation of

$$h_n(\mathcal{B}_n(w_z^n)) := \Psi^{-1} \circ f(z) \text{ for all } z \in \text{CP}(f|_K), \text{ and} \tag{2.10}$$

$$h_n(z) = z \text{ for } r \leq |z| \leq 1. \tag{2.11}$$

By (2.9), we have that

$$|\mathcal{B}_n(w_z^n) - \Psi^{-1} \circ f(z)| \xrightarrow{n\to\infty} 0 \text{ for all } z \in \mathrm{CP}(f|_K),$$

and hence h_n may be taken to satisfy:

$$||(h_n)_{\overline{z}}/(h_n)_z||_{\mathbb{D}} \xrightarrow{n\to\infty} 0. \tag{2.12}$$

By the Measurable Riemann Mapping Theorem (see [Ahl06]), there is a quasiconformal $\phi_n : D \to D$ so that:

$$B_n := h_n \circ \mathcal{B}_n \circ \phi_n^{-1}$$

is holomorphic. We normalize each ϕ_n by specifying $\phi_n(p) = p$ and $\phi_n'(p) > 0$ for some $r < |p| < 1$. Note that

$$\mathcal{B}_n(\partial D) \subset \{z : r \leq |z| \leq 1\} \text{ for large } n \tag{2.13}$$

by (2.1), so it follows from (2.11) that B_n is a Blaschke product on D for large n. We claim the sequence (B_n) satisfies the conclusions of Theorem 2.6, as well as (2.8). Indeed, we have $\phi_n(w_z^n) \in \mathrm{CP}(B_n)$ and

$$B_n(\phi_n(w_z^n)) = \Psi^{-1} \circ f(z).$$

Thus (2.8) is satisfied. By (2.12), we have that

$$||\phi_n(z) - z||_D \xrightarrow{n\to\infty} 0, \tag{2.14}$$

and hence (2.2) follows. The relation (2.1) follows from (2.11), and (2.3) follows from (2.14).

To prove (2.4) and (2.5), we first note that since $(h_n)_{\overline{z}} = 0$ in $r < |z| < 1$, it follows from (2.13) that ϕ_n extends to a holomorphic function in a neighborhood U of ∂D, where U does not depend on n. By (2.14) and the Cauchy integral formula, it follows that $\phi_n'(z) \xrightarrow{n\to\infty} 1$ uniformly for $z \in \partial D$ and hence we deduce (2.4). Similarly, $\phi_n''(z) \xrightarrow{n\to\infty} 0$ uniformly for $z \in \partial D$ and so (2.5) follows. □

Recall from the introduction that we plan to extend the definition of the approximant $\Psi \circ B_n \approx f$ from D to all of $\mathbb{C}$, where we recall Ψ was defined in Definition 2.8. To this end, it will be useful to define the following graph structure on ∂D.

Definition 2.11. For any $n \in \mathbb{N}$, we define a set of vertices on ∂D by $\mathcal{V}_n := (B_n|_{\partial D})^{-1}(\mathbb{R})$, where each vertex v is labeled black or white according to whether $B_n(v) > 0$ or $B_n(v) < 0$, respectively. The curve ∂D will be considered as a graph with edges defined by $\mathcal{I}_n$ (recall from Notation 2.4 that $\mathcal{I}_n$ is precisely the collection of components of $\partial D \setminus \mathcal{V}_n$). We will sometimes write D_n in place of D when we wish to emphasize the dependence of the graph ∂D on n.

Definition 2.12. We define a holomorphic mapping g_n in D by the formula

$$g_n(z) := \Psi \circ B_n(z). \tag{2.15}$$

In Sections 3-7 we will quasiregularly extend the definition of g_n to $\mathbb{C}$, and then in Section 9 we apply the MRMT to produce the rational approximant of Theorem B as described in the introduction.

Remark 2.13. Recall that in Notation 2.7, we fixed $\varepsilon > 0$, a compact set K contained in an analytic domain D, and a function f holomorphic in D (we note ε, K, D, f also satisfied extra conditions specified in Notation 2.7). The objects γ, Ψ, B_n, $\mathcal{V}_n$, g_n we then defined in this section were determined by our initial choice of ε, K, D, f. In future sections, it will be useful to think of γ, Ψ, B_n, $\mathcal{V}_n$, g_n as defining functions which take as input some quadruple (ε, K, D, f) (for any ε, K, D, f as in Notation 2.7), and output whatever object we defined in this section. For instance, $\mathcal{V}_n$ defines a function which takes as input any (ε, K, D, f) as in Notation 2.7 and outputs (via Definition 2.11) a set of vertices $\mathcal{V}_n(\varepsilon, K, D, f)$ on ∂D. Similarly, B_n takes as input any (ε, K, D, f) as in Notation 2.7 and outputs (via Theorem 2.10) a Blaschke product $B_n(\varepsilon, K, D, f)$ on D. Likewise for γ, Ψ, g_n.

3. Quasiconformal Folding

Given a compact set $K \subset \mathbb{C}$ and a function f holomorphic in a domain D containing K, we showed in Section 2 how to approximate f by a holomorphic function g_n defined in D (see Definition 2.12). Moreover g_n is just a Blaschke product in D. If f is a function holomorphic in an arbitrary analytic neighborhood U (where U need not be connected) of a compact set K, then one can apply the results of Section 2 to each component of U which intersects K (this is done precisely in Definition 4.1): this yields a holomorphic approximant of f defined in a finite union of domains, so that the approximant is just a finite Blaschke product on each domain (recall Definition 2.2). In Sections 3-7, we will build the apparatus necessary to systematically extend this holomorphic approximant to a quasiregular function of $\mathbb{C}$ which is holomorphic outside a small set.

It was convenient to assume in Notation 2.7 that the compact set K was covered by a single domain D, however we now begin to work more generally:

Remark 3.1. We refer to Figure 6 for a summary of the following. Throughout Sections 3-7, we will fix $\varepsilon > 0$, a compact set $K \subset \mathbb{C}$, a domain $\mathcal{D}$ containing K, a disjoint collection of analytic domains $(D_i)_{i=1}^k$ such that $K \subset U := \cup_i D_i \subset \mathcal{D}$, and a function f holomorphic in a neighborhood of $\overline{U}$ satisfying $||f||_{\overline{U}} < 1$. We assume that the following analog of Equation (2.6) holds

$$d(z, K \cap D_i) < \varepsilon/2 \text{ and } d(f(z), f(K \cap D_i)) < \varepsilon/2 \text{ for all } z \in \partial D_i \text{ and } 1 \leq i \leq k. \tag{3.1}$$

Applying the methods of the previous section to each component D_i of U, we can define a sequence of finite Blaschke products $(B_n)_{n=1}^{\infty}$ on each D_i (see Remark 2.13). We will let B_n denote the corresponding function defined on U. In particular, $(B_n)_{n=1}^{\infty}$ gives the following definition of vertices on the boundary of $U := \cup_i D_i$ (see Definition 2.11 and Remark 2.13).

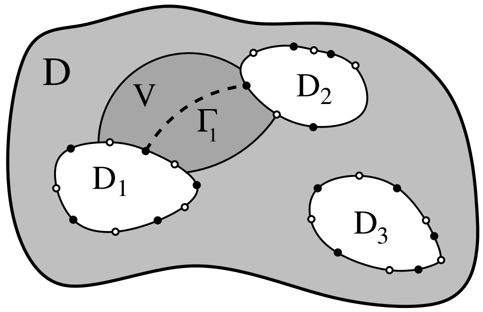


FIGURE 4. Illustrated is the Definition of the domain V in the proof of Proposition 3.3

Definition 3.2. For every $n \in \mathbb{N}$, we define a set of vertices $\mathcal{V}_n$ on ∂U by

$$\mathcal{V}_n := \bigcup_{i=1}^{k} \mathcal{V}_n(\varepsilon, K \cap D_i, D_i, f|_{D_i}) = \bigcup_{i=1}^{k} (B_n|_{\partial D_i})^{-1}(\mathbb{R}).$$

We now extend the graph structure on ∂U by connecting the different components of U by curves $\{\Gamma_i\}_{i=1}^{k-1}$ in Proposition 3.3 below, and defining vertices along these curves in Definition 3.4. We will need to prove a certain level of regularity for these curves and vertices in order to ensure that the dilatations of quasiconformal adjustments we will make later do not degenerate as $n \to \infty$. We will denote the curves by $\{\Gamma_i\}_{i=1}^{k-1}$, and we remark that the curves depend on n, although we suppress this from the notation.

Proposition 3.3. *For each $n \in \mathbb{N}$, there exists a collection of disjoint, closed, analytic Jordan arcs $\{\Gamma_i\}_{i=1}^{k-1}$ in $(\widehat{\mathbb{C}} \setminus U) \cap \mathcal{D}$ satisfying the following properties:*

1. *Each endpoint of Γ_i is a vertex in $\mathcal{V}_n$,*
2. *Each Γ_i meets ∂U at right angles,*
3. *$U \cup (\cup_i \Gamma_i)$ is connected, and*
4. *For each $1 \leq i \leq k-1$, the sequence (in n) of curves Γ_i has an analytic limit.*

Proof. The set $(\widehat{\mathbb{C}} \setminus U) \cap \mathcal{D}$ must contain at least one simply-connected region V with the property that there are distinct i, j with both $\partial V \cap \partial D_i$ and $\partial V \cap \partial D_j$ containing non-trivial arcs (see Figure 4). By (2.3), for all sufficiently large n both $\partial V \cap \partial D_i$, $\partial V \cap \partial D_j$ contain vertices of $\mathcal{V}_n$ which we denote by $v_i \in \partial D_i$, $v_j \in \partial D_j$, respectively. Consider a conformal map $\phi : \mathbb{D} \to V$, and define Γ_1 to be the image under ϕ of the hyperbolic geodesic connecting $\phi^{-1}(v_i)$, $\phi_i^{-1}(v_j)$ in $\mathbb{D}$.

We now proceed recursively, making sure at step l we pick a V which connects two components of U not already connected by a Γ_1, ..., Γ_{l-1}, and so that V is disjoint from Γ_1, ..., Γ_{l-1}. The curves Γ_i satisfy conclusions (1)-(3) of the proposition. We may ensure that for each $1 \leq i \leq k-1$, the sequence (in n) of curves Γ_i has an analytic limit by choosing v_i, v_j above to converge as $n \to \infty$. □

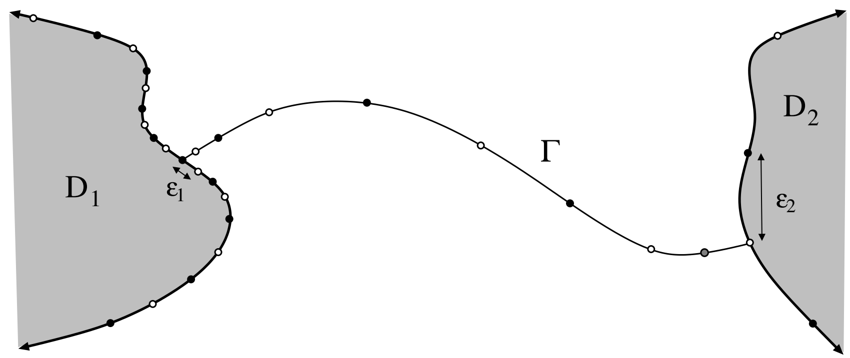


FIGURE 5. Illustrated is Definition 3.4.

Definition 3.4. Consider the vertices $\mathcal{V}_n \subset \partial U$ of Definition 3.2. We will augment $\mathcal{V}_n$ to include vertices on the curves $(\Gamma_i)_{i=1}^{k-1}$ as follows (see Figure 5). Let $\Gamma \in (\Gamma_i)_{i=1}^{k-1}$ denote both the curve as a subset of $\mathbb{C}$ and the arclength parameterization of the curve, and suppose Γ connects vertices

$$\Gamma(0) = v_i \in \partial D_i, \quad \Gamma(\text{length}(\Gamma)) = v_j \in \partial D_j.$$

For $k = i,\ j$, let ε_k denote the minimum length of the two edges with endpoint v_k in ∂D_k, and suppose without loss of generality $\varepsilon_j < \varepsilon_i$. Let l be so that

$$\varepsilon_j/2 \leq \varepsilon_i/2^l \leq 2\varepsilon_j.$$

We place vertices at $\Gamma(\varepsilon_i/2), ..., \Gamma(\varepsilon_i/2^l)$, and we place vertices along $\Gamma([\varepsilon_i/2^l, \text{length}(\Gamma)])$ at equidistributed points. We can label the vertices black/white along Γ so that vertices connect only to vertices of the opposite color by adding one extra vertex at the midpoint of the segment having v_j as an endpoint, if need be.

We introduce the following notation.

Notation 3.5. Throughout Sections 3-7, we will let Ω denote a fixed (arbitrary) component of

$$\widehat{\mathbb{C}} \setminus \left(\overline{U} \cup \bigcup_{i=1}^{k-1} \Gamma_i \right), \tag{3.2}$$

and $p \in \Omega$. Note that Ω is simply connected by Proposition 3.3(3). Denote $\mathbb{D}^* := \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and let σ denote any conformal mapping

$$\sigma : \mathbb{D}^* \to \Omega \tag{3.3}$$

satisfying $\sigma(\infty) = p$. For $z \in \Omega$, we define $\tau(z) := \sigma^{-1}(z)$. The map τ induces a partition of $\mathbb{T}$ which we denote by $V_n := \tau(\mathcal{V}_n)$.

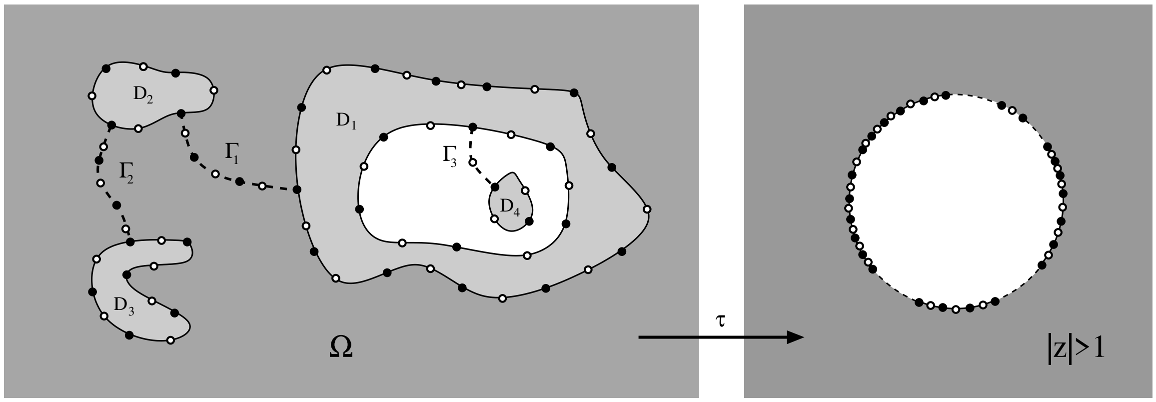


FIGURE 6. This figure illustrates Remark 3.1 and Notation 3.5. As pictured, U has four components $(D_i)_{i=1}^4$ which are connected by curves $(\Gamma_i)_{i=1}^3$. Recall $K \subset U$ (the compact set K is not shown in the figure). The unbounded component Ω of (3.2) is pictured in dark grey. The map $\tau : \Omega \to \mathbb{D}^*$ is a conformal mapping, and sends the vertices on $\partial\Omega$ to (possibly unevenly spaced) vertices on the unit circle.

Remark 3.6. We will sometimes write Ω_n, $\mathbb{D}_n^*$ in place of Ω, $\mathbb{D}^*$, respectively, when we wish to emphasize the dependence of the vertices $\mathcal{V}_n \subset \partial\Omega$, $V_n \subset \partial\mathbb{D}^*$ on the parameter n.

Proposition 3.7. *For the graph $\partial\Omega_n$, we have:*

$$\max\{\mathrm{diam}(e) : e \text{ is an edge of } \partial\Omega_n\} \xrightarrow{n\to\infty} 0.$$

Proof. This follows from (2.3) and Definition 3.4. □

As explained in the introduction, in order to prove uniform approximation in Theorem B, we will need to prove that our quasiregular extension is holomorphic outside a region of small area. This will usually mean proving the following condition holds.

Definition 3.8. Suppose $V \subset \mathbb{C}$ is an analytic domain, and ∂V is a graph. Let $C > 0$. We say a quasiregular mapping $\phi : V \to \phi(V)$ is *C-vertex-supported* if

$$\mathrm{supp}(\phi_{\overline{z}}) \subset \bigcup_{e\in\partial V} \{z : \mathrm{dist}(z, e) < C \cdot \mathrm{diam}(e)\} \tag{3.4}$$

(see Figure 7), where the union in (3.4) is taken over all edges e on ∂V.

It will also be useful to have the following definition.

Definition 3.9. Suppose e, f are rectifiable Jordan arcs, and $h : e \to f$ is a homeomorphism. We say that h is *length-multiplying* on e if the push-forward (under h) of arc-length measure on e coincides with the arc-length measure on f multiplied by $\mathrm{length}(f)/\mathrm{length}(e)$.

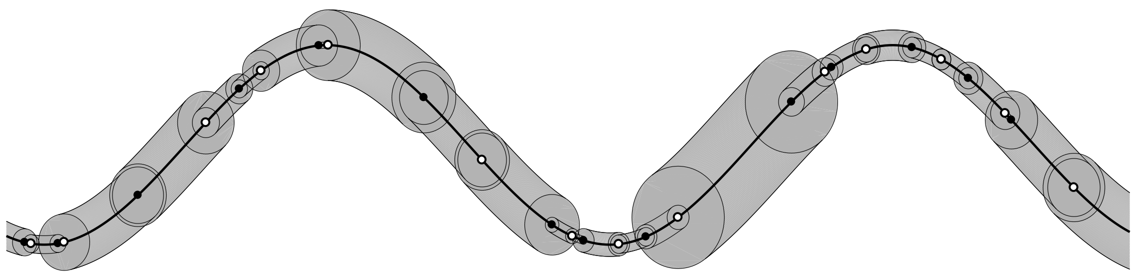


FIGURE 7. Shown as a black curve is part of a graph G, and in light gray the neighborhood $\cup_{e\in G}\{z : \mathrm{dist}(z,e) < C \cdot \mathrm{diam}(e)\}$ of G.

First we will adjust the conformal map τ so as to be length-multiplying along edges of $\partial\Omega$. Recall the vertices $V_n \subset \mathbb{T}$ defined in Notation 3.5.

Proposition 3.10. *For every n, there is a K-quasiconformal mapping $\lambda : \mathbb{D}_n^* \to \mathbb{D}_n^*$ so that:*

(1) *λ is C-vertex-supported for some $C > 0$,*
(2) *$\lambda(z) = z$ on V_n and off of supp$(\lambda_{\overline{z}})$,*
(3) *$\lambda \circ \tau$ is length-multiplying on every component of $\partial\Omega \setminus \mathcal{V}_n$,*
(4) *C, K do not depend on n.*

Proof. This is a consequence of Theorem 4.3 of [Bis15]. Indeed, recall $\tau := \sigma^{-1}$ and consider the $2\pi i$-periodic covering map

$$\phi := \sigma \circ \exp : \mathbb{H}_r \mapsto \Omega. \tag{3.5}$$

The map ϕ induces a periodic partition $\phi^{-1}(\mathcal{V}_n)$ of $\partial\mathbb{H}_r$ which has *bounded geometry* (see the introduction of [Bis15], or Section 2 of [BL19]) with constants independent of n by Proposition 3.3(2) and Definition 3.4. Thus Theorem 4.3 of [Bis15] applies to produce a $2\pi i$-periodic, C vertex-supported, and K-quasiconformal map $\beta : \mathbb{H}_r \to \mathbb{H}_r$ so that $\phi \circ \beta$ is length-multiplying on edges of $\mathbb{H}_r$, and C, K are independent of n. Thus, the inverse

$$\beta^{-1} \circ \log \circ \tau$$

is length-multiplying, and since exp is length-multiplying on vertical edges, the well-defined map

$$\lambda := \exp \circ \beta^{-1} \circ \log : \mathbb{D}^* \to \mathbb{D}^*$$

satisfies the conclusions of the Proposition.

□

The main idea in defining the quasiregular extension in Ω is to send each edge of ∂D_i to the upper or lower half of the unit circle by following $\lambda \circ \tau$ with a power map $z \mapsto z^n$

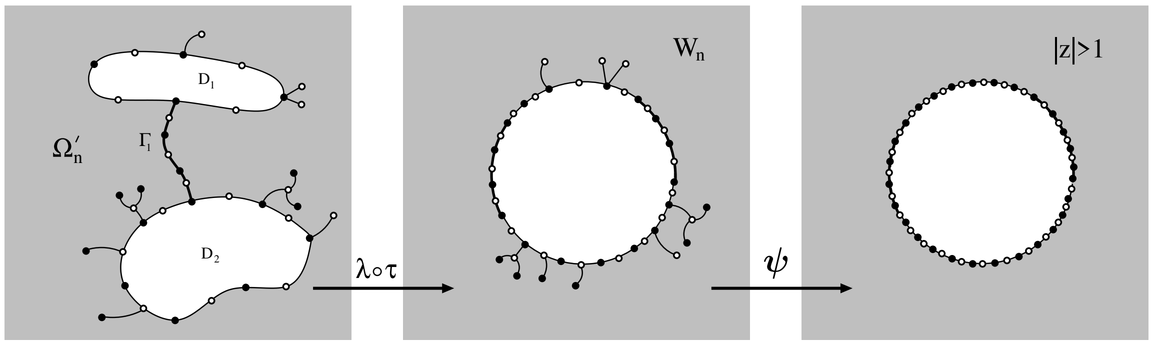


FIGURE 8. This figure illustrates the Folding Theorem 3.13 and Notation 3.14. The simply connected domain Ω_n' is obtained by removing from Ω certain trees based at the vertices along $\partial\Omega$.

of appropriate degree. The main difficulty in this approach, however, is that the images of different edges of ∂D_i under $\lambda \circ \tau$ may differ significantly in size, so that there is no single n with $z \mapsto z^n$ achieving the desired behavior. The solution is to modify the domain Ω by removing certain "decorations" from the domain Ω, so that each edge of ∂D_i is sent to an arc of roughly the same size under $\lambda \circ \tau$. This is formalized below in Theorem 3.13 (see also Figures 8, 9), and is an application of the main technical result of [Bis15] (see Lemma 5.1). The "decorations" are the trees in the following definition.

Definition 3.11. Let $V \subset \mathbb{T}$ be a discrete set. We call a domain $W \subset \mathbb{D}^*$ a *tree domain rooted at* V if W consists of the complement in $\mathbb{D}^*$ of a collection of disjoint trees, one rooted at each vertex of V (see the center of Figure 8).

Notation 3.12. For $m \in \mathbb{N}$, we let

$$\mathcal{Z}_m^{\pm} := \{z \in \mathbb{T} : z^m = \pm 1\},$$
$$\mathcal{Z}_m := \mathcal{Z}_m^+ \cup \mathcal{Z}_m^-.$$

In other words, $\mathcal{Z}_m^+$ denotes the m^{th} roots of unity, and $\mathcal{Z}_m^-$ the m^{th} roots of -1.

Theorem 3.13. *For every n, there exists a tree domain W_n rooted at V_n, an integer $m = m(n)$, and a K-quasiconformal mapping $\psi : W_n \to \mathbb{D}^*$ so that:*

1. *ψ is C-vertex-supported for some $C > 0$, and $\psi(z) = z$ off of supp$(\psi_{\overline{z}})$,*
2. *on any edge e of $\partial W_n \cap \mathbb{T}$, ψ is length-multiplying and $\psi(e)$ is an edge in $\mathbb{T} \setminus \mathcal{Z}_m$,*
3. *for any edge e of $\partial W_n \cap \mathbb{D}^*$, $\psi(e)$ consists of two edges in $\mathbb{T} \setminus \mathcal{Z}_m$. Moreover, if $x \in e$, the two limits $\lim_{W_n \ni z \to x} \psi(z) \in \mathbb{T}$ are equidistant from $\mathcal{Z}_m^+$, and from $\mathcal{Z}_m^-$, and*
4. *C, K do not depend on n.*

Proof. We consider the 2π-periodic covering map

$$\phi := \sigma \circ \lambda \circ \exp \circ (z \mapsto -iz) : \mathbb{H} \mapsto \Omega. \tag{3.6}$$

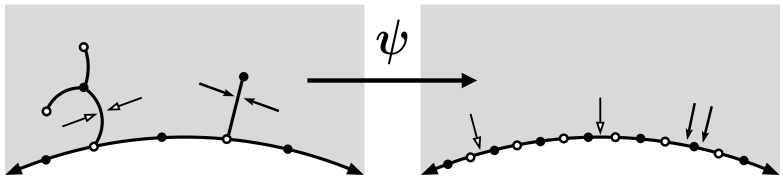


Figure 9. For any $x \in \partial W_n \cap \mathbb{D}^*$, there are two limits $\lim_{W_n \ni z \to x} \psi(z) \in \mathbb{T}$ as illustrated in this figure. Theorem 3.13(3) says that these two limits are equidistant from the nearest black vertex, and are equidistant from the nearest white vertex.

inducing a periodic partition $\phi^{-1}(\mathcal{V}_n)$ of $\partial\mathbb{H}$. By (2.4), Definition 3.4, and Proposition 3.10(2), any two edges of $\mathbb{H}$ have comparable lengths with constant independent of n. Therefore, Lemma 5.1 of [Bis15] applies to yield a 2π-periodic K-quasiconformal map Ψ_n of $\mathbb{H}$ onto a subdomain $\Psi_n(\mathbb{H}) \subsetneq \mathbb{H}$, with K independent of n. We let

$$W_n := \exp(-i\Psi_n(\mathbb{H}))$$

and

$$\psi := \exp \circ - i\Psi_n^{-1} \circ i \log : W_n \to \mathbb{D}^*. \tag{3.7}$$

The map (3.7) is well-defined, and the conclusions of the theorem follow from Lemma 5.1 of [Bis15]. □

Notation 3.14. We will use the notation $\Omega_n' := (\lambda \circ \tau)^{-1}(W_n)$.

4. Annular Interpolation Between the Identity and a Conformal Mapping

Recall from Notation 3.1 that we have fixed $\varepsilon > 0$, a compact set K, disjoint analytic domains $(D_i)_{i=1}^k$ so that $U := \cup_i D_i$ contains K, and f holomorphic in a neighborhood of $\overline{U}$ with $||f||_{\overline{U}} < 1$. In this section, we briefly define two useful interpolations in Lemmas 4.3 and 4.4.

Since the domain D_i contains the compact set $K \cap D_i$, the definitions and results of Section 2 apply to $(\varepsilon, K \cap D_i, D_i, f|_{D_i})$ for each $1 \leq i \leq k$ (see Notation 2.7). Thus Remark 2.13 applies to define (4.1), (4.2) and (4.3) in the following.

Definition 4.1. Let $1 \leq i \leq k$. We define the Jordan curve

$$\gamma_i := \gamma(\varepsilon, K \cap D_i, D_i, f|_{D_i}). \tag{4.1}$$

Recalling that $\mathrm{int}(\gamma_i)$ denotes the bounded component of $\widehat{\mathbb{C}} \setminus \gamma_i$, we define

$$\Psi_i := \Psi(\varepsilon, K \cap D_i, D_i, f|_{D_i}) \tag{4.2}$$

to be a Riemann mapping $\Psi_i : \mathbb{D} \to \mathrm{int}(\gamma_i)$. Lastly, we define the finite Blaschke products

$$B_n := B_n(\varepsilon, K \cap D_i, D_i, f|_{D_i}) \text{ on } D_i, \tag{4.3}$$

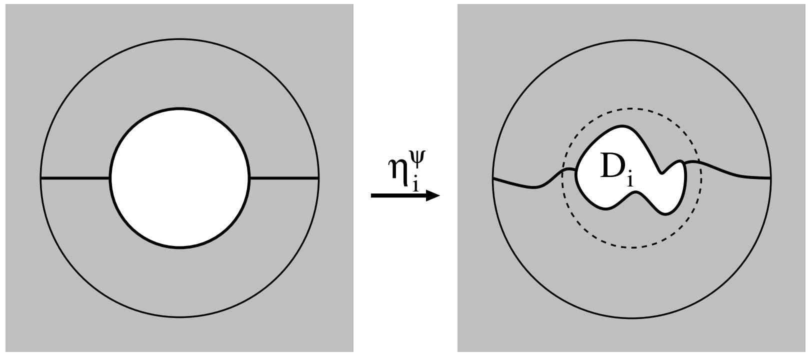


FIGURE 10. Illustrated is the map $\eta_i^\Psi : \mathbb{D}^* \to \widehat{\mathbb{C}} \setminus \Psi_i(r_i\mathbb{D})$ of Lemma 4.3 in the case $r_i = 1$. The dotted circle on the right depicts the unit circle.

where we suppress the dependence of $(B_n)_{n=1}^\infty$ on i from the notation.

Recall that in Section 3, we defined curves $\{\Gamma_i\}_{i=1}^{k-1}$ connecting the domains D_i, and in Notation 3.5 we fixed a component Ω of the complement of $\overline{U} \cup \cup_{i=1}^{k-1}\Gamma_i$.

Notation 4.2. After relabeling the $(D_i)_{i=1}^k$ if necessary, there exists $1 \leq \ell \leq k$ so that $\partial D_i \cap \partial\Omega \neq \emptyset$ if and only if $i \leq \ell$ (see Figure 6 for example). For each $1 \leq i \leq \ell$, note that the intersection $\partial D_i \cap \partial\Omega$ consists of a single Jordan curve. We let $r_i := |B_n(\partial D_i \cap \partial\Omega)|$, so that $1 - \varepsilon \leq r_i \leq 1$ by (2.1).

The two interpolations we will need are given in Lemmas 4.3 and 4.4 below. In Lemma 4.3, we define an interpolation η_i^Ψ between $z \mapsto z$ on $|z| = 2$ with $z \mapsto \Psi_i(r_iz)$ on $|z| = 1$ (see Figure 10), and in Lemma 4.4 we modify η_i^Ψ to define a map η_i so that $\eta_i(z) = \eta_i(\overline{z})$ for $|z| = 1$.

Lemma 4.3. *For each $1 \leq i \leq \ell$, there is a quasiconformal mapping $\eta_i^\Psi : \mathbb{D}^* \to \widehat{\mathbb{C}} \setminus \Psi_i(r_i\mathbb{D})$ satisfying the relations:*

$$\eta_i^\Psi(z) = z \text{ for } |z| \geq 2 \text{ and} \tag{4.4}$$

$$\eta_i^\Psi(z) = \Psi_i(r_iz) \text{ for all } |z| = 1. \tag{4.5}$$

Moreover, if D_i, D_j for $1 \leq i, j \leq \ell$ are connected by one of the curves $(\Gamma_i)_{i=1}^{k-1}$, then

$$\eta_i^\Psi([-2,-1]) \cap \eta_j^\Psi([1,2]) = \eta_i^\Psi([1,2]) \cap \eta_j^\Psi([-2,-1]) = \emptyset. \tag{4.6}$$

Proof. The existence of η_i^Ψ satisfying (4.4) and (4.5) follows from a standard lemma on the extension of quasisymmetric maps between boundaries of quasiannuli (see, for instance, Proposition 2.30(b) of [BF14]). If (4.6) fails for the collection $(\eta_i^\Psi)_{i\in I}$ thus defined, we can renormalize the conformal mappings $(\Psi_i)_{i=1}^\ell$ appropriately (to rotate the points $\Psi_i(\pm r_i)$ along the curve $\Psi_i(r_i\mathbb{T})$), and post-compose a subcollection of the η_i^Ψ by diffeomorphisms of

$2\mathbb{D} \setminus \Psi_i(r_i\mathbb{D})$ so that (4.6) is satisfied for the composition when $i, j \in I$, and (4.4) and (4.5) still hold. $\square$

Lemma 4.4. *For each $1 \leq i \leq \ell$, there is a quasiconformal mapping*

$$\eta_i : \mathbb{D}^* \to \mathbb{C} \setminus \Psi_i([-r_i, r_i])$$

satisfying the relations

$$\eta_i(z) = z \text{ for } |z| \geq 2, \tag{4.7}$$

$$\eta_i(z) = \eta_i(\overline{z}) \text{ for } |z| = 1, \text{ and} \tag{4.8}$$

$$\eta_i(z) = \eta_i^{\Psi}(z) \text{ for } z \in \mathbb{R} \cap \mathbb{D}^*. \tag{4.9}$$

Proof. Define

$$\gamma_i^+ := \eta_i^{\Psi}(\partial(A(1,2) \cap \mathbb{H})).$$

Let η be a quasisymmetric mapping of $\mathbb{T} \cap \mathbb{H}$ onto $[-1, 1]$ fixing ± 1 (one can take $\eta := M|_{\mathbb{T}\cap\mathbb{H}}$ where M is a Mobius transformation mapping -1, 1, i to -1, 1, 0, respectively). Define a mapping g on γ_i^+ by:

$$g(z) := \begin{cases} \Psi_i \circ \eta \circ \Psi_i^{-1}(z) & z \in \Psi_i(\mathbb{T} \cap \mathbb{H}) \\ z & \text{otherwise} \end{cases} \tag{4.10}$$

Since g is a quasisymmetric mapping, a standard lemma on extension of quasisymmetric maps between boundaries of quasidisks (see, for instance, Proposition 2.30(a) of [BF14]) implies that g may be extended to a quasiconformal mapping of $\eta_i^{\Psi}(A(1,2) \cap \mathbb{H})$. Define g similarly in $\eta_i^{\Psi}(A(1,2) \cap (-\mathbb{H}))$. We let $\eta_i := g \circ \eta_i^{\Psi}$. It is then straightforward to check that η_i satisfies (4.7)-(4.9). $\square$

Remark 4.5. Lemmas 4.3 and 4.4 define 2ℓ many quasiconformal mappings: $\{\eta_i^{\Psi}\}_{i=1}^{\ell}$ and $\{\eta_i\}_{i=1}^{\ell}$. The definition of the mappings η_i^{Ψ}, η_i depend on the objects ε, K, $(D_i)_{i=1}^k$, f as fixed in Notation 3.1, but not on the parameter n in (4.3). Thus we record the trivial but important observation that the mappings $\{\eta_i^{\Psi}\}_{i=1}^{\ell}$ and $\{\eta_i\}_{i=1}^{\ell}$ are quasiconformal with a constant independent of n.

5. Annular Interpolation Between a Blaschke Product and a Power Map

Recall that we have fixed $\varepsilon > 0$, a compact set K, disjoint analytic domains $(D_i)_{i=1}^k$ so that $U := \cup_i D_i$ contains K, and f holomorphic in a neighborhood of $\overline{U}$ with $||f||_{\overline{U}} < 1$. The curves $\{\Gamma_i\}_{i=1}^{k-1}$ connect the domains $(D_i)_{i=1}^k$, and Ω is a component of the complement of $\overline{U} \cup \cup_{i=1}^{k-1}\Gamma_i$ with $\tau : \Omega \to \mathbb{D}^*$ conformal. Recall that the domain Ω_n' was defined in Theorem 3.13 and Notation 3.14 by removing from Ω a collection of trees rooted at the vertices along $\partial\Omega$, and the map $\psi \circ \lambda \circ \tau$ maps Ω_n' onto $\mathbb{D}^*$ (see Proposition 3.10 and Theorem 3.13).

Notation 5.1. Recall from Notation 4.2 that $\partial D_i \cap \partial\Omega \neq \emptyset$ if and only if $1 \leq i \leq \ell$. Hence exactly $\ell - 1$ of the curves $(\Gamma_i)_{i=1}^{k-1}$ intersect $\partial\Omega$. By relabelling the $(\Gamma_i)_{i=1}^{k-1}$ if necessary, we may assume Γ_j intersects $\partial\Omega$ if and only if $1 \leq j \leq \ell - 1$.

Let $m = m(n)$ be as in Theorem 3.13. To prove our main results, we will need to modify $z \mapsto z^m$ in $\mathbb{D}^*$ so that, roughly speaking, $(z \mapsto z^m) \circ \psi \circ \lambda \circ \tau(z)$ agrees with the Blaschke products B_n (see Definition 4.1) along ∂D_i. This is done in Theorem 5.3 below. Its proof uses the following.

Proposition 5.2. *Suppose ϕ_1, ϕ_2 are C^1 homeomorphisms of a C^1 Jordan arc e such that:*
 (1) *$\phi_1(e) = \phi_2(e)$,*
 (2) *ϕ_1, ϕ_2 agree on the two endpoints of e, and*
 (3) *$|\phi_1'(z)| = |\phi_2'(z)|$ for all $z \in e$.*

Then $\phi_1 = \phi_2$ on e.

The proof of Proposition 5.2 is a consequence of the Fundamental Theorem of Calculus and is left to the reader. Recall the constant $r_i := |B_n(\partial D_i \cap \partial\Omega)|$ of Notation 4.2.

Theorem 5.3. *For every n, there exists a locally univalent K-quasiregular mapping $h_n : \mathbb{D}^* \to \mathbb{D}^*$ so that:*
 (1) *$h_n(z) = z^m$ for $|z| \geq \sqrt[m]{2}$ where $m := m(n)$ is as in Theorem 3.13,*
 (2) *$h_n \circ \psi \circ \lambda \circ \tau(z) = B_n(z)/r_i$ for every $z \in \partial D_i$ and $1 \leq i \leq l$, and*
 (3) *K is independent of n.*

Proof. Fix the standard branch of log. Given an edge $e \in \partial D_i$, we have by Theorem 3.13 that

$$\log \circ \psi \circ \lambda \circ \tau(e) = \{0\} \times \left[\frac{j\pi}{m}, \frac{(j+1)\pi}{m}\right] \text{ for some } 0 \leq j \leq 2m-1. \tag{5.1}$$

Denote the vertical line segment in (5.1) by v_e. Let $f : v_e \mapsto e$ be a length-multiplying, C^1 homeomorphism so that f^{-1} agrees with $\log \circ \psi \circ \lambda \circ \tau$ on the two endpoints of e. Consider the maps:

$$z \mapsto mz \text{ for } z \in \left\{\frac{\log 2}{m}\right\} \times \left[\frac{j\pi}{m}, \frac{(j+1)\pi}{m}\right], \tag{5.2}$$

$$z \mapsto \log \circ r_i^{-1} B_n \circ f \text{ for } z \in \{0\} \times \left[\frac{j\pi}{m}, \frac{(j+1)\pi}{m}\right]. \tag{5.3}$$

For each $1 \leq i \leq l$, the Blaschke products B_n are orientation-preserving on the unique outer boundary component of D_i, and orientation-reserving on all other boundary components of D_i. This implies that we may choose the branch of log in (5.3) so that the images of (5.2) and (5.3) are horizontal translates of one another (recall $B_n(e)$ is a circular arc of angle π), and the derivative of (5.3) is strictly positive for all $z \in v_e$. Since the derivative of

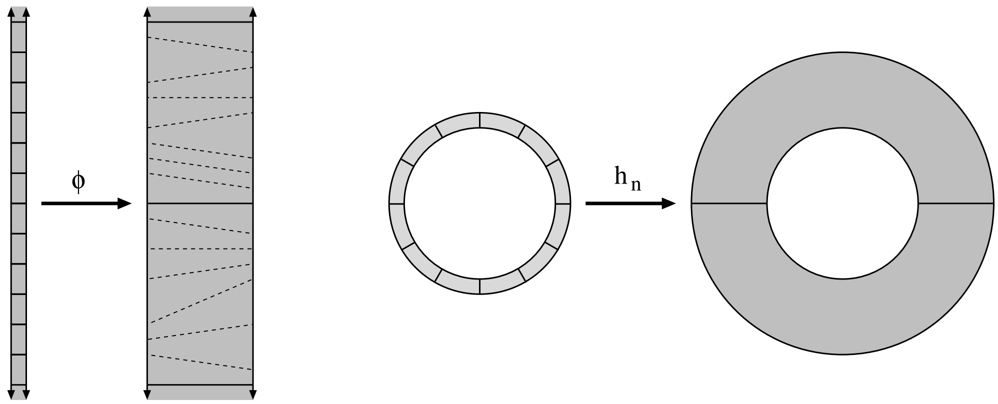


FIGURE 11. Illustrated is the proof of Theorem 5.3. In logarithmic coordinates the desired interpolation is denoted ϕ, and h_n is then defined by (5.5) and (5.6).

(5.2) is also strictly positive, this means the linear interpolation between (5.2) and (5.3) is a homeomorphism.

By (2.5), we have that $|B_n'|$ is comparable at all points of e with constant independent of e and n. Thus, since f is length-multiplying and log is length-multiplying on Euclidean circles centered at 0, we conclude that the derivative of (5.3) is comparable to m at all points of v_e with constant independent of e and n. Thus, we conclude that the linear interpolation between (5.2) and (5.3) in the rectangle

$$\left[0, \frac{\log 2}{m}\right] \times \left[\frac{j\pi}{m}, \frac{(j+1)\pi}{m}\right] \tag{5.4}$$

is K-quasiconformal with K independent of e and n (see, for instance, Theorem A.1 of [MPS20]). Denote the linear interpolation by ϕ (see Figure 11).

We define

$$h_n := \exp \circ \phi \circ \log \ \text{ in } \{z \in \mathbb{D}^* : z/|z| \in \psi \circ \lambda \circ \tau(e)\} \cap \{|z| \leq \sqrt[m]{2}\}. \tag{5.5}$$

The equation (5.5) defines $h_n(z)$ for z in $\{z : 1 \leq |z| \leq \sqrt[m]{2}\}$ and sharing a common angle with the image under $\psi \circ \lambda \circ \tau$ of an edge on some ∂D_i. We finish the definition of h_n by simply setting:

$$h_n(z) := z^m \text{ in } \{z \in \mathbb{D}^* : z/|z| \in \psi \circ \lambda \circ \tau(\partial\Omega_n' \setminus (\cup_i \partial D_i))\}. \tag{5.6}$$

The conclusion (1) now follows by definition of h_n, and (3) follows since h_n is a composition of holomorphic mappings and a K-quasiconformal interpolation where we have already noted that K is independent of n.

We now show that conclusion (2) follows from Proposition 5.2. Fix an edge e on ∂D_i. Recall $v_e := \log \circ \psi \circ \lambda \circ \tau(e)$. Thus, by (5.3) and (5.5) we have that:

$$h_n \circ \psi \circ \lambda \circ \tau = r_i^{-1} B_n \circ f \circ \log \circ \psi \circ \lambda \circ \tau \text{ on } e. \tag{5.7}$$

First note that (5.7) agrees set-wise with $r_i^{-1}B_n$ on e and at the endpoints of e. The map $\psi \circ \lambda \circ \tau$ is length-multiplying (by Proposition 3.10(3) and Theorem 3.13(2)), log is length-multiplying on the circular segment $\psi \circ \lambda \circ \tau(e)$, and f is length-multiplying by definition. Thus the modulus of the derivative of $f \circ \log \circ \psi \circ \lambda \circ \tau$ is constant on e, and so the derivatives of (5.7) and $r_i^{-1}B_n$ have the same modulus at each point of e. Conclusion (2) now follows from Proposition 5.2. □

6. Joining Different Types of Boundary Arcs: the Map E_n

Recall that in Section 4 we defined the maps η_i^Ψ, η_i where $1 \leq i \leq \ell$, and in Section 5 we defined the map h_n for all $n \in \mathbb{N}$. In this section we define a map E_n in $\mathbb{D}^*$ which is roughly given by either $z \mapsto \eta_i^\Psi \circ h_n(z)$ or $z \mapsto \eta_i(z) \circ h_n(z)$, where i is allowed to depend on $\arg(z)$ and which of η_i^Ψ, η_i we post-compose h_n with is also allowed to depend on $\arg(z)$. Thus, we will need a way to interpolate between the definitions of η_i^Ψ, η_i, for different i. The interpolation regions are defined in Definition 6.1 below, and the map E_n in Proposition 6.2. It will be useful to keep Figure 12 in mind for the remainder of this section.

Definition 6.1. Mark one edge e_i on Γ_i for each $1 \leq i \leq \ell - 1$. Label the ℓ components of $\partial\Omega_n' \setminus \cup_i e_i$ as $(G_i)_{i=1}^{\ell}$, where $\partial D_i \subset G_i$. Let
1. $\mathcal{J}_i^D$ denote those edges in ∂D_i,
2. $\mathcal{J}_i^G$ denote those edges in $G_i \setminus \mathcal{J}_i^D$,
3. $\mathcal{J}^e$ denote the edges $(e_i)_{i=1}^{\ell-1}$.

In other words, $\mathcal{J}_i^D$ are the edges shared by $\partial\Omega_n'$ and ∂D_i, $\mathcal{J}^e$ consists of $\ell-1$ edges: one on each of the curves $(\Gamma_i)_{i=1}^{\ell}$, and $\mathcal{J}_i^G$ are the remaining edges on G_i. Thus we have:

$$\partial\Omega_n' = \mathcal{J}^e \cup \bigcup_i \left(\mathcal{J}_i^D \cup \mathcal{J}_i^G\right).$$

For $z \in \mathbb{D}^*$, we define:

$$E_n(z) := \begin{cases} \eta_i^\psi \circ h_n(z) & \text{if} \quad z/|z| \in \psi \circ \lambda \circ \tau(\mathcal{J}_i^D) \\ \eta_i \circ h_n(z) & \text{if} \quad z/|z| \in \psi \circ \lambda \circ \tau(\mathcal{J}_i^G) \end{cases} \tag{6.1}$$

It remains to define $E_n(z)$ for $z \in \mathbb{D}^*$ satisfying $z/|z| \in \psi \circ \lambda \circ \tau(\mathcal{J}^e)$. We do so in the following Proposition.

Proposition 6.2. *The map E_n extends to a locally univalent K-quasiregular mapping $E_n : \mathbb{D}^* \to \mathbb{C}$ satisfying $E_n(z) = z^m$ for $|z| \geq \sqrt[m]{2}$, where $m = m(n)$ is as in Theorem 3.13. Moreover, K does not depend on n.*

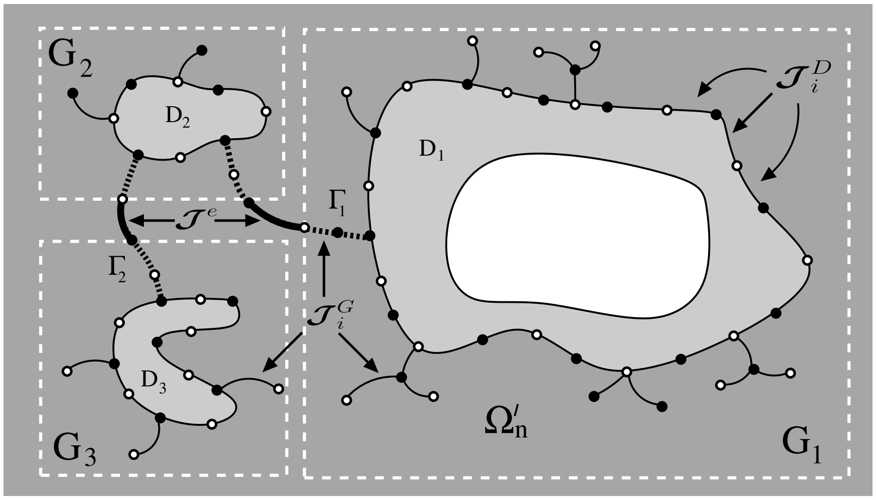


FIGURE 12. Illustrated is Definition 6.1. The curves Γ_1, Γ_2 are depicted as black dotted lines, except for the edges $e_1 \subset \Gamma_1$, $e_2 \subset \Gamma_2$ which are in thick black.

Proof. Consider (6.1). Note that if E_n is defined at z and $|z| \geq \sqrt[m]{2}$, then $E_n(z) = z^m$ by Theorem 5.3(1) and (4.4), (4.7). Thus, setting $E_n(z) := z^m$ for $|z| \geq \sqrt[m]{2}$ extends the definition of E_n.

It remains to extend the definition of E_n to:

$$\{z : 1 \leq |z| \leq \sqrt[m]{2} \text{ and } z/|z| \in \psi \circ \lambda \circ \tau(e_i)\}, \text{ for } 1 \leq i \leq \ell - 1. \tag{6.2}$$

Each of the $\ell - 1$ sets in (6.2) consists of 2 quadrilaterals which we denote by $\mathcal{Q}_i^{\pm}$. The curve Γ_i connects two distinct elements of $(D_i)_{i=1}^{\ell-1}$. In order to avoid complicating notation significantly, we will assume without loss of generality that Γ_i connects D_i to D_{i+1}. Let $\gamma_i \subset 2\mathbb{D}$ be a smooth Jordan arc connecting $\eta_i^{\Psi}(1)$ to $\eta_{i+1}^{\Psi}(-1)$. Moreover, by (4.6), we can choose γ_i so that the union of the arcs

$$\eta_i^{\Psi}([1,2]),\ 2\mathbb{T} \cap \mathbb{H},\ \eta_{i+1}^{\Psi}([-2,-1]),\ \gamma_i \tag{6.3}$$

forms a topological quadrilateral we denote by Q_i^+ (in particular none of the arcs in (6.3) intersect except at common endpoints).

Define a quasisymmetric homeomorphism $g_i^+ : \partial Q_i^+ \to \partial(A(1,2) \cap \mathbb{H})$ (see Figure 13) by

$$g_i^+(z) = z \text{ for } z \in 2\mathbb{T}$$
$$g_i^+(z) = (\eta_i^{\Psi})^{-1}(z) \text{ for } z \in \eta_i^{\Psi}([1,2])$$
$$g_i^+(z) = (\eta_{i+1}^{\Psi})^{-1}(z) \text{ for } z \in \eta_{i+1}^{\Psi}([-2,-1]),$$

and extending g_i^+ to a quasisymmetric homeomorphism of γ_i to $\mathbb{T} \cap \mathbb{H}$. The mapping g_i^+ extends to a quasiconformal homeomorphism $g_i^+ : Q_i^+ \to A(1,2) \cap \mathbb{H}$ (see Lemma 2.24 of

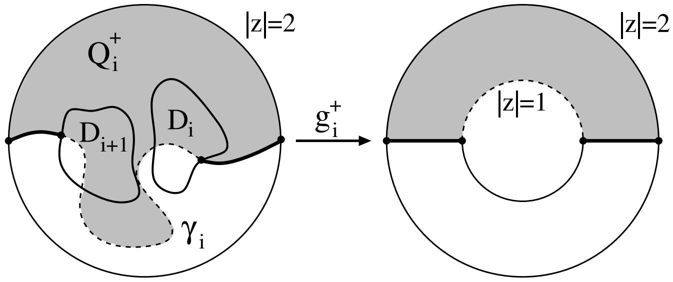


FIGURE 13. Illustrated is the quadrilateral Q_i^+ and the map g_i^+ in the proof of Proposition 6.2.

[BF14]). We define $E_n(z) := (g_i^+)^{-1}(z^m)$ for $z \in \mathcal{Q}_i^+$. A similar definition of $g_i^- : Q_i^- \to A(1,2) \cap -\mathbb{H}$ is given (using the same curve γ_i) so that

$$g_i^+(z) = g_i^-(\overline{z}) \text{ for } z \in \mathbb{T} \cap \mathbb{H}.$$

We let $E_n(z) := (g_i^-)^{-1}(z^m)$ for $z \in \mathcal{Q}_i^-$.

To summarize, we have defined E_n in each of the three regions

$$\{z \in \mathbb{D}^* : z/|z| \in \psi \circ \lambda \circ \tau(\mathcal{J}_i^D)\}, \tag{6.4}$$

$$\{z \in \mathbb{D}^* : z/|z| \in \psi \circ \lambda \circ \tau(\mathcal{J}_i^G)\}, \tag{6.5}$$

$$\{z \in \mathbb{D}^* : z/|z| \in \psi \circ \lambda \circ \tau(\mathcal{J}_i^e)\}, \tag{6.6}$$

Indeed, the definition of E_n in (6.4) and (6.5) was given already in (6.1), and in this proof we have defined E_n in (6.6). The definitions of E_n in each of (6.4), (6.5), (6.6) agree along any common boundary, and thus by removability of analytic arcs for quasiregular mappings, it follows that E_n is quasiregular on $\mathbb{D}^*$. Moreover, E_n has no branched points in $\mathbb{D}^*$, and hence E_n is locally quasiconformal. The dilatation of the map E_n depends only on the dilatation of h_n (which is independent of n by Theorem 5.3(3)) and the dilatations of the the finite collection of quasiconformal maps used in its definition: η_i^Ψ, η_i, g_i^+, g_i^-, and hence we may take K independent of n. □

7. DEFINING g_n IN Ω'_n

First we recall our setup. We have fixed $\varepsilon > 0$, a compact set K, disjoint, analytic domains $(D_i)_{i=1}^k$ so that $K \subset U := \cup_i D_i$, and f holomorphic in a neighborhood of $\overline{U}$ with $||f||_{\overline{U}} < 1$. We defined curves $\{\Gamma_i\}_{i=1}^{k-1}$ connecting the domains $(D_i)_{i=1}^k$, and we denoted by Ω a component of the complement of $\overline{U} \cup \cup_{i=1}^{k-1}\Gamma_i$ with $\tau : \Omega \to \mathbb{D}^*$ conformal. The domain Ω'_n is contained in Ω, and $\psi \circ \lambda \circ \tau$ maps Ω'_n onto $\mathbb{D}^*$. In Section 6 we defined the map E_n.

Definition 7.1. We define the mapping $g_n : \Omega'_n \to \widehat{\mathbb{C}}$ by

$$g_n := E_n \circ \psi \circ \lambda \circ \tau. \tag{7.1}$$

We will now record at which points the function $g_n|_{\Omega'_n}$ is locally $n:1$ for $n>1$.

Definition 7.2. Let g be a quasiregular function, defined in a neighborhood of a point $z \in \mathbb{C}$. We say that z is a *branched point* of g if for any sufficiently small neighborhood U of z, the map $g|_U$ is $n:1$ onto its image for $n>1$. We say $w \in \mathbb{C}$ is a *branched value* of g if $w = g(z)$ for a branched point z of g. We denote the branched points of a quasiregular mapping g by $\mathrm{BP}(g)$, and the branched values by $\mathrm{BV}(g)$.

Remark 7.3. Recall that in Notation 3.5 we fixed a point $p \in \Omega$ satisfying $\tau(p) = \infty$.

Proposition 7.4. *The mapping $g_n : \Omega'_n \to \mathbb{C}$ of Definition 7.1 is K-quasiregular and C-vertex supported for K, C independent of n. Moreover, $g_n^{-1}(\infty) = \{p\}$,*

$$\mathrm{BP}(g_n) \subset \bigcup_{e \in \partial\Omega_n} \{z : \mathrm{dist}(z,e) < C \cdot \mathrm{diam}(e)\}, \text{ and} \tag{7.2}$$

$$\mathrm{BV}(g_n) \subset \bigcup_{i=1}^{k} \Psi_i(r_i\mathbb{T}). \tag{7.3}$$

Proof. Since each of the mappings in the composition (7.1) are K-quasiregular and C-vertex supported for K, C independent of n, the same is true of g_n. The only points where the mapping g_n is locally $l:1$ for $l>1$ are a subset of the vertices of the graph $\partial\Omega'_n$. By Theorem 3.13, the vertices of $\partial\Omega'_n$ all lie in

$$\bigcup_{e \in \partial\Omega_n} \{z : \mathrm{dist}(z,e) < C \cdot \mathrm{diam}(e)\}.$$

Thus, (7.2) is proven. Moreover, any vertex of $\partial\Omega'_n$ is mapped to a point on one of the curves $\Psi_i(r_i\mathbb{T})$ by g_n. Hence, (7.3) follows since $\mathrm{BV}(g_n) = g_n(\mathrm{BP}(g_n))$. It remains to show:

$$g_n^{-1}(\infty) = \{p\}. \tag{7.4}$$

Indeed, note that $E_n \circ \psi \circ \lambda$ fixes ∞ and has no finite poles. The map $\tau : \Omega \to \mathbb{D}^*$ is conformal and hence only one point p is mapped to ∞. The relation (7.4) now follows. □

It will be useful to record the following result.

Proposition 7.5. *Let $r>1$. Then for all sufficiently large n, we have:*

$$g_n(z) = \tau(z)^m \text{ for any } z \in \tau^{-1}(\{z : |z| > r\}). \tag{7.5}$$

Proof. Consider the functional equation (7.1) defining g_n. The maps λ, ψ are vertex-supported, and moreover λ (respectively, ψ) is the identity outside of the support of $\lambda_{\overline{z}}$, (respectively, $\psi_{\overline{z}}$). By Proposition 3.7, we therefore have that $\psi \circ \lambda(z) = z$ if $z \in \tau^{-1}(\{z : |z| > r\})$ and n is sufficiently large. The relation (7.5) now follows from (7.1) and Theorem 5.3(1) since $m \to \infty$ as $n \to \infty$. □

Remark 7.6. As in Remark 2.13, we note that our Definition 7.1 of g_n is determined by a choice of the objects K, U, $\mathcal{D}$, f, ε, Ω, p we fixed in Notations 3.1 and 3.5. When we wish to emphasize this dependence, we will write $g_n(K, U, \mathcal{D}, f, \varepsilon, \Omega, p)$. In particular, it will be useful in the next section to think of g_n as a function taking as input any choice of K, U, $\mathcal{D}$, f, ε, Ω, p satisfying the conditions in Notations 3.1, 3.5, and outputting (via Definition 7.1) a quasiregular function $g_n(K, U, \mathcal{D}, f, \varepsilon, \Omega, p)$ defined on Ω'_n.

8. Verifying g_n is Quasiregular on $\widehat{\mathbb{C}}$

In this section we combine our efforts in Sections 2-7 to define an approximant $g_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of a given f. The approximant g_n will not be holomorphic as required in Theorems A and B, but we will solve this problem in the next section by applying the Measurable Riemann Mapping Theorem. We fix the following for Sections 8-9.

Notation 8.1. Fix K, f, $\mathcal{D}$, ε, P as in the statement of Theorem B. Denote by U the neighborhood of K in which f is holomorphic. Define

$$P' := \{p \in P : p \text{ is contained in a component } V \text{ of } \widehat{\mathbb{C}} \setminus K \text{ such that } V \not\subseteq U\}. \tag{8.1}$$

Compactness of K implies that U contains all but finitely many components of $\widehat{\mathbb{C}} \setminus K$, and so the set P' is finite. Moreover, P' does not depend on ε. By shrinking U if necessary, we may assume that:

(1) $U \cap P' = \emptyset$,
(2) P' contains exactly one point in each component of $\widehat{\mathbb{C}} \setminus U$,
(3) f is holomorphic in a neighborhood of $U \subset \mathcal{D}$, and
(4) the components of U are a finite collection of analytic Jordan domains $(D_i)_{i=1}^k$ so that (2.6) holds for each D_i.

Let K' be a compact set such that $K \subset \text{int}(K') \subset K' \subset U$. We will assume for now that $||f||_{\overline{U}} < 1$.

We now define a quasiregular approximation g_n of f by applying the Blaschke product construction of Section 2 in each D_i, and by applying the folding construction of Sections 3-7 in each complementary component of $\cup_i(\overline{D_i} \cup \Gamma_i)$:

Definition 8.2. For every n, we define a quasiregular mapping g_n as follows. Recalling Remark 2.13, we first set

$$g_n := g_n(\varepsilon, K' \cap D_i, D_i, f|_{D_i}) \text{ in } D_i \text{ for } 1 \le i \le k \tag{8.2}$$

The equation (8.2) defines the curves $(\Gamma_i)_{i=1}^k$ by way of Proposition 3.3, and we enumerate the components of

$$\widehat{\mathbb{C}} \setminus \left(\overline{U} \cup \bigcup_{i=1}^{k-1} \Gamma_i \right)$$

by $(\Omega(i))_{i=1}^{\ell}$. Recalling Remark 7.6 and Notation 3.14, we extend the definition of g_n to the open set

$$\Omega := \widehat{\mathbb{C}} \setminus \left(\bigcup_{i=1}^{\ell} \partial\Omega_n'(i) \right) \tag{8.3}$$

by the formula

$$g_n := g_n(K', U, \mathcal{D}, f, \varepsilon, \Omega(i), P' \cap \Omega(i)) \text{ in } \Omega_n'(i) \text{ for } 1 \leq i \leq \ell. \tag{8.4}$$

Proposition 8.3. *The quasiregular function g_n is C-vertex supported and K-quasiregular for C, K independent of n.*

Proof. For $g_n|_{\Omega_n'(i)}$ this is exactly Proposition 7.4, and so the conclusion follows since g_n is holomorphic in U. □

The function g_n is now defined on all of $\widehat{\mathbb{C}}$ except for the edges of each $\partial\Omega_n'(i)$. We show in Propositions 8.4, 8.5 below that g_n in fact extends continuously across each edge of $\partial\Omega_n'(i)$, and deduce in Corollary 8.6 that g_n extends quasiregularly across $\partial\Omega$.

Proposition 8.4. *The K-quasiregular function $g_n : \Omega \to \widehat{\mathbb{C}}$ extends to a continuous function $g_n : \Omega \cup e \to \widehat{\mathbb{C}}$ for any edge $e \subset \partial\Omega \cap \partial U$.*

Proof. Let i be so that $e \subset \partial D_i$ and denote the unique element of $(\Omega(i))_{i=1}^{\ell}$ that contains e on its boundary by $\Omega(j)$. Recall by Definitions 2.12 and 7.1 that

$$g_n|_{D_i} = \Psi_i \circ B_n, \tag{8.5}$$

$$g_n|_{\Omega_n'(j)} = E_n \circ \psi \circ \lambda \circ \tau. \tag{8.6}$$

Assume that $i \in I$ (the reasoning in the case $i \notin I$ will be the same), so that by Theorem 5.3(2) we have

$$h_n \circ \psi \circ \lambda \circ \tau = r_i^{-1} B_n \text{ on } e.$$

By (4.5) and the definition (6.1) of E_n, it follows from (8.6) that

$$g_n|_{\Omega_n'(j)}(z) = \Psi_i \circ B_n(z) \text{ for } z \in e,$$

in other words $g_n|_{\Omega_n'(j)}$ and $g_n|_{D_i}$ agree pointwise on e. □

Proposition 8.5. *The K-quasiregular function $g_n : \Omega \to \widehat{\mathbb{C}}$ extends to a continuous function $g_n : \Omega \cup e \to \widehat{\mathbb{C}}$ for any edge $e \subset \partial\Omega \cap (\cup_{i=1}^{\ell} \Omega(i))$.*

Proof. Let j be so that $e \subset \partial\Omega_n'(j)$, and as in the proof of Proposition 8.4, recall that

$$g_n|_{\Omega_n'(j)} = E_n \circ \psi \circ \lambda \circ \tau. \tag{8.7}$$

Let $x \in e$. There are two limits

$$\lim_{\Omega_n'(j) \ni z \to x} \psi \circ \lambda \circ \tau(z),$$

each lying on the unit circle. Denote them by $\zeta_\pm$. By Theorem 3.13(3),

$$\zeta_+^m = \overline{\zeta_-^m}.$$

Thus, by (4.8) and (6.1), we conclude that there is a unique limit

$$\lim_{\Omega'_n(j)\ni z\to x} E_n \circ \psi \circ \lambda \circ \tau(z).$$

Hence, setting

$$g_n(x) := \lim_{\Omega'_n(j)\ni z\to x} E_n \circ \psi \circ \lambda \circ \tau(z)$$

defines a continuous extension of g_n across the edge e. □

Corollary 8.6. *The K-quasiregular function $g_n : \Omega \to \widehat{\mathbb{C}}$ extends to a K-quasiregular function $g_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.*

Proof. The set $\widehat{\mathbb{C}} \setminus \Omega = \partial\Omega$ consists of a finite collection of analytic arcs: the edges of the graphs $\partial\Omega'_n(i)$ over $1 \leq i \leq \ell$. Thus, by removability of analytic arcs for quasiregular mappings, it suffices to show that $g_n : \Omega \to \widehat{\mathbb{C}}$ extends continuously across each such edge. There are two types of edges to check: those that lie on the boundary of a domain D_i, and those that lie in the interior of a domain $\Omega(i)$. We have already checked continuity across both types of edges in Propositions 8.4, 8.5, and so the proof is complete. □

9. Proof of the Main Theorems

In Section 9 we prove Theorems A and B, modulo the proof of Theorem 2.6 which is left to Sections 10-12.

Recall that in Section 8 we fixed the objects K, f, $\mathcal{D}$, ε, P as in Theorem B (see Notation 8.1), and we defined a quasiregular approximation g_n to f in Definition 8.2. We also showed in Section 8 that g_n in fact extends to a quasiregular function $g_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. Now we apply the MRMT below in Definition 9.1 to obtain the rational maps $r_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which we will prove satisfy the conclusions of Theorems A and B for large n.

Definition 9.1. The mapping g_n induces a Beltrami coefficient $\mu_n := (g_n)_{\overline{z}}/(g_n)_z$, which, by way of the MRMT, defines a quasiconformal mapping $\phi_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $r_n := g_n \circ \phi_n^{-1}$ is holomorphic. We normalize ϕ_n so that $\phi_n(\infty) = \infty$ and $\phi_n(z) = z + O(1/|z|)$ as $z \to \infty$.

We now begin deducing that for large n, the maps r_n satisfy the various conclusions in Theorems A and B.

Proposition 9.2. *The function r_n of Definition 9.1 is rational, and $r_n^{-1}(\infty) = \phi_n(P')$. In particular, if K is full and $P = \{\infty\}$, then r_n is a polynomial.*

Proof. The function r_n is holomorphic on $\widehat{\mathbb{C}}$ and takes values in $\widehat{\mathbb{C}}$: the only such functions are rational. Note that $g_n^{-1}(\infty) \cap \overline{U} = \emptyset$ since g_n is bounded on $\overline{U}$. Thus, by Proposition 7.4 and (8.4), we have that $g_n^{-1}(\infty) = P'$. Since $r_n := g_n \circ \phi_n^{-1}$, we conclude that $r_n^{-1}(\infty) = \phi_n(P')$.

The last statement of the proposition follows since we normalized $\phi_n(\infty) = \infty$, and the only rational functions with a unique pole at ∞ are polynomials. $\square$

Proposition 9.3. *For all $R < \infty$, the mapping ϕ_n satisfies:*

$$||\phi_n(z) - z||_{R\cdot\mathbb{D}} \xrightarrow{n\to\infty} 0. \tag{9.1}$$

Proof. Since g_n is C-vertex supported by Proposition 8.3, we conclude from Proposition 3.7 that

$$\text{Area}(\text{supp}(\mu_n)) \xrightarrow{n\to\infty} 0. \tag{9.2}$$

The relation (9.1) now follows from (9.2) since $||\mu_n||_{L^\infty} \leq K$ for all n by Proposition 8.3. $\square$

Theorem 9.4. *For all sufficiently large n, the mapping r_n satisfies* $\text{CP}(r_n) \subset \mathcal{D}$.

Proof. By Proposition 3.7, we have

$$\max\left\{\text{diam}(e) : e \text{ is an edge of } \bigcup_{i=1}^{\ell} \partial\Omega(i)\right\} \xrightarrow{n\to\infty} 0.$$

Thus, since $\mathcal{D}$ is a domain containing $\cup_{i=1}^{\ell}\partial\Omega(i)$, we have by (7.2) that $\text{BP}(g_n) \setminus U \subset \mathcal{D}$ for large n. Since $U \subset \mathcal{D}$ we conclude that $\text{BP}(g_n) \subset \mathcal{D}$ for large n. The result now follows from Proposition 9.3 since $\phi(\text{BP}(g_n)) = \text{CP}(r_n)$. $\square$

Theorem 9.5. *For all sufficiently large n, we have*

$$||r_n - f||_K < 3\varepsilon.$$

Proof. First we note that since f is uniformly continuous on K', there exists $\delta > 0$ so that if z, $w \in K'$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \varepsilon$. By Proposition 9.3, we can conclude that

$$||\phi_n(z) - z||_K < \min(\delta, \text{dist}(K, \partial K')) \tag{9.3}$$

for all sufficiently large n.

Let $z \in K$, $w := \phi_n^{-1}(z)$ and suppose j is such that $z \in D_j$. Then

$$|r_n(z) - g_n(z)| = |g_n(w) - g_n(z)| \leq |g_n(w) - f(w)| + |f(w) - f(z)| + |f(z) - g_n(z)|. \tag{9.4}$$

Let

$$C := \sup_{\substack{z\in\mathbb{D} \\ 1\leq i\leq k}} |\Psi_i'(z)|.$$

Then

$$||g_n - f||_{K'} \leq C \cdot ||\Psi_j^{-1} \circ g_n - \Psi_j^{-1} \circ f||_{K'},$$

and we deduce by (2.15) that:

$$||g_n - f||_{K'} \leq C \cdot ||B_n - \Psi_j^{-1} \circ f||_{K'}.$$

Applying Theorem 2.10 (see also Remark 2.9), we conclude that

$$||g_n - f||_{K'} \xrightarrow{n\to\infty} 0. \tag{9.5}$$

Next, we deduce from (9.3), (9.4) and (9.5) that

$$|r_n(z) - g_n(z)| < 2\varepsilon$$

for sufficiently large n. It follows that for sufficiently large n:

$$||r_n - f||_K \le ||r_n - g_n||_K + ||g_n - f||_K < 3\varepsilon.$$

□

Theorem 9.6. *For all sufficiently large n we have $\mathrm{CV}(f|_K) \subset \mathrm{CV}(r_n)$.*

Proof. Let $z \in \mathrm{CP}(f|_K)$, and let i be so that $z \in D_i$. Then $\Psi_i^{-1} \circ f(z) \in \mathrm{CV}(B_n)$ by Theorem 2.10. Thus $f(z) \in \mathrm{CV}(\Psi_i \circ B_n)$. Thus, by the Definition 2.12 of g_n, we have for large n that $f(z) \in \mathrm{BV}(g_n) = \mathrm{CV}(r_n)$. □

Theorem 9.7. *For all sufficiently large n, we have*

$$\mathrm{CV}(r_n) \subset \mathrm{fill}\{z : d(z, f(K)) < \varepsilon\}.$$

Proof. Since

$$\mathrm{CV}(r_n) = \mathrm{BV}(g_n),$$

it suffices to show that for every $z \in \mathrm{BP}(g_n)$ and sufficiently large n, we have

$$g_n(z) \in \mathrm{fill}\{z : d(z, f(K)) < \varepsilon\}. \tag{9.6}$$

For $z \in \mathrm{BP}(g_n) \setminus U$, (9.6) follows from (7.3). For $z \in \mathrm{BP}(g_n) \cap D_i$, (9.6) follows from Definition 2.12 of $g_n|_{D_i}$. □

Proof of Theorem B: In the special case that $||f||_K < 1$, we have already proven that the mappings r_n satisfy the conclusions of Theorem B for all sufficiently large n. Indeed, Theorem 9.5 says that $||r_n - f||_K < \varepsilon$, conclusion (2) in Theorem B is Theorem 9.4, and conclusion (3) is Theorems 9.6, 9.7. Conclusion (1) follows from Propositions 9.2, 9.3. The general case follows by applying the above special case to an appropriately rescaled f. □

Proof of Theorem A: When K is full, we may take $P = \{\infty\}$ and apply Theorem B, in which case Proposition 9.2 guarantees that the maps r_n are polynomials. □

Theorem 9.8. (Mergelyan+) *Let $K \subset \mathbb{C}$ be full, suppose $f \in C(K)$ is holomorphic in $\mathrm{int}(K)$, and let $\mathcal{D}$ be a domain containing K. For every $\varepsilon > 0$, there exists a polynomial p so that $||p - f||_K < \varepsilon$ and:*

1. *$CP(p) \subset \mathcal{D}$,*
2. *$CV(p) \subset \mathrm{fill}\{z : d(z, f(K)) \le \varepsilon\}$.*

Proof: By the usual version of Mergelyan's Theorem, there exists a polynomial q so that $||q - f||_K < \varepsilon/2$. Apply Theorem A to K, $\mathcal{D}$, q, $\varepsilon/2$ to obtain an approximant of q which we denote by p. The polynomial p satisfies the conclusions of Theorem 9.8. □

Corollary 9.9. (Weierstrass+) *Suppose that $I \subset \mathbb{R}$ is a closed interval, $f : I \to \mathbb{R}$ is continuous, and U, $V \subset \mathbb{C}$ are planar domains containing I, $f(I)$, respectively. Then, for every $\varepsilon > 0$, there exists a polynomial p with real coefficients so that $\|f - p\|_I \leq \varepsilon$, and*

(1) *CP(p) $\subset U$,*
(2) *CV(p) $\subset V$.*

Proof. Let $I = [a, b]$, and f, U, V as in the statement of the corollary. By Theorem 9.8, there exists a complex polynomial q so that $||q - f||_{[a,b]} < \varepsilon/2$. The real polynomial

$$Q(z) := \frac{q(z) + \overline{q(\overline{z})}}{2}$$

satisfies $Q(x) = \mathrm{Re}(q(x))$ for $x \in \mathbb{R}$ and hence $||Q - f||_{[a,b]} < \varepsilon/2$. We will use the symbol $\Subset$ to mean compactly contained. Let $V_1 \Subset V$ be a sufficiently small, $\mathbb{R}$-symmetric domain containing $f(I)$ so that there is a component of $Q^{-1}(V_1)$ (which we denote by U_1) satisfying $U_1 \Subset U$. Let U_2 be a $\mathbb{R}$-symmetric, analytic domain satisfying $I \Subset U_1 \Subset U_2 \Subset U$. Recall Notation 8.1 and consider:

(1) the compact set $\overline{U_1}$,
(2) the analytic function Q,
(3) the analytic domain U_2 containing $\overline{U_1}$,
(4) $\min\{\varepsilon/2, \mathrm{dist}(\partial V_1, \partial V)\}$,
(5) $P = \{\infty\}$.

Applying Definition 8.2 to (1)-(5) yields quasiregular mappings g_n with $\mathbb{R}$-symmetric Beltrami coefficient, so that

$$p_n := g_n \circ \phi_n^{-1} \tag{9.7}$$

is a real polynomial approximant of Q for large n satisfying:

(1) $||p_n - f||_{[a,b]} < \varepsilon$,
(2) $\mathrm{CP}(p_n) \subset U_2$,
(3) $\mathrm{CV}(p_n) \subset V$.

Thus p_n satisfies the conclusion of Corollary 9.9 for large n. □

Remark 9.10. If we make further assumptions on the compact set K, the conclusion

$$\mathrm{CV}(r) \subset \mathrm{fill}\{z : d(z, f(K)) < \varepsilon\}$$

of Theorems A and B can be improved to

$$\mathrm{CV}(r) \subset \mathrm{fill}(f(K)), \tag{9.8}$$

which is equivalent to

$$\mathrm{CV}(r) \subset f(K)$$

if $f(K)$ is full. Indeed, if for instance the interiors of K, $f(K)$ are analytic domains and $f : \mathrm{int}(K) \to \mathrm{int}(f(K))$ is proper, then a similar strategy as in the proofs of Theorems A and B but replacing Ψ in (2.15) with a conformal map $\mathbb{D} \mapsto \mathrm{int}(K)$ can be used to prove (9.8).

Recall the notation $\Omega(i)$ from Definition 8.2, and let $\tau_i : \Omega(i) \to \mathbb{D}^*$ be the conformal mapping satisfying $\tau_i^{-1}(\infty) = P' \cap \Omega(i)$ as in Notation 3.5. The following fact justifies part of our description in the introduction of the behavior of the rational approximants off K.

Proposition 9.11. *Let $1 < r < R < \infty$. Then, for all sufficiently large n, we have*

$$r_n \circ \phi_n(z) = \tau_i(z)^m \text{ and} \tag{9.9}$$

$$|r_n(z)| > R \tag{9.10}$$

for all $z \in \tau_i^{-1}(\{z : |z| > r\})$.

Proof. Fix $R < \infty$ and $r > 1$. From (7.5) and the functional equation (8.4) defining g_n in $\Omega(i)$, it follows that:

$$g_n(z) = \tau_i(z)^m \text{ for all } z \in \tau_i^{-1}(\{z : |z| > (r+1)/2\}) \tag{9.11}$$

for all large n. Since

$$r_n \circ \phi_n = g_n, \tag{9.12}$$

The relation (9.9) follows. Moreover, we have by Proposition 9.3 that:

$$\phi_n \circ \tau_i^{-1}(\{z : |z| > r\}) \subset \tau_i^{-1}(\{z : |z| > (r+1)/2\}) \tag{9.13}$$

for all sufficiently large n. Since $((r+1)/2)^m > R$ for large n, the relation (9.10) also follows. □

10. Some Estimates Involving Harmonic Measure and Green's Functions

In Sections 10-12 we turn to the proof of Theorem 2.6. In fact, we will prove a slightly stronger result (Theorem 12.2 in Section 12) from which Theorem 2.6 follows. We begin by recalling a few standard facts, and sketch the proofs for the convenience of the reader. Let $D(z, r)$ denote the disk of radius r centered at z. Lemma 10.1 is illustrated in Figure 14.

Lemma 10.1. *If $K \subset \mathbb{D}$ is continuum connecting 0 to $\mathbb{T}$, then $\omega(z, K, \mathbb{D} \setminus K) \geq c > 0$ for all $|z| < 1/4$ and some $c > 0$ independent of K.*

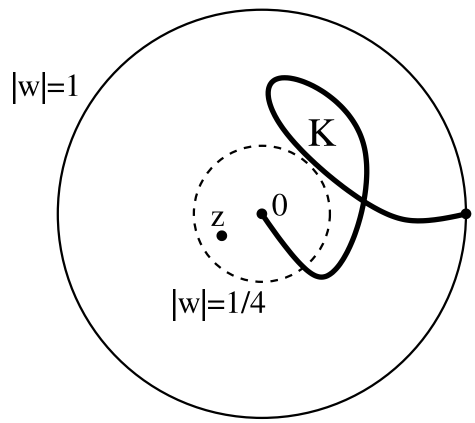


FIGURE 14. Illustrated is the compact set K of Lemma 10.1.

Proof. Exercise III.10 of [GM08] says that if E is a continuum connecting $\{|z| = \frac{1}{2}\}$ to $\mathbb{T}$ in $\mathbb{D}$, then $\omega(0, E, \mathbb{D} \setminus E) \geq c = \frac{2}{\pi} \tan^{-1}(1/\sqrt{8})$. Apply the exercise and the maximum principle to the disk $D = D(z, \frac{1}{2})$ to deduce the lemma. [GM08] gives a simple direct proof of the exercise, but it also follows from Beurling's projection theorem, e.g., Theorem II.9.2 of [GM08]. $\square$

Lemma 10.2. *Suppose u, v are harmonic functions on $\mathbb{D}$ such that $\sup_{\mathbb{D}} |u|, \sup_{\mathbb{D}} |v| \leq M$ and that $|u - v| < \varepsilon$ on some continuum K connecting 0 to $\mathbb{T}$. Then $|u - v| \leq \varepsilon^c M^{1-c}$ on $D(0, 1/4)$, where $c > 0$ is the constant from Lemma 10.1.*

Proof. Consider the subharmonic function $\log |u - v|$. It is less than $\log \varepsilon$ on K and less than $\log M$ on $\partial\mathbb{D}$. Thus for $|z| \leq 1/4$, Lemma 10.1 implies

$$\begin{aligned} \log |u(z) - v(z)| &\leq \omega(z, K, \mathbb{D} \setminus K) \log \varepsilon + \omega(z, \mathbb{T}, \mathbb{D} \setminus K) \log M \\ &\leq c \log \varepsilon + (1 - c) \log M, \end{aligned}$$

so $|u(z) - v(z)| \leq \varepsilon^c M^{1-c}$, as desired. $\square$

Lemma 10.3. *Suppose Ω is a planar domain, $K \subset \Omega$ is compact and connected, and $0 < \varepsilon, M < \infty$. Then there is a $\delta > 0$ so that if $h = u + i\widetilde{u}$ is holomorphic on Ω, $\widetilde{u}$ vanishes at some point of K, $\sup_\Omega |h| \leq M$ and $\sup_K |u| < \delta$, then $\sup_K |\widetilde{u}| < \varepsilon$.*

Proof. If K is single point, this is trivial since $\widetilde{u} = 0$ there by assumption, so assume K is non-trivial. Choose $\eta > 0$ so that $\eta < \operatorname{dist}(K, \partial\Omega)$ and $\eta < \operatorname{diameter}(K)$. Then for any radius η disk D centered at a point of K, u is less than δ on a continuum connecting the center of D to its boundary. This implies $|u| \leq \delta^c M^{1-c}$ (for c as in Lemma 10.1) on a $\eta/4$-neighborhood of K. Thus $|\nabla u| = O(\delta^c M^{1-c}/\eta)$ on the $(\eta/8)$-neighborhood U of K (this uses the Cauchy estimate for $|\nabla u|$, e.g., Theorem 2.4 of [ABR01]). Since $|\nabla \widetilde{u}| = |\nabla u|$ and

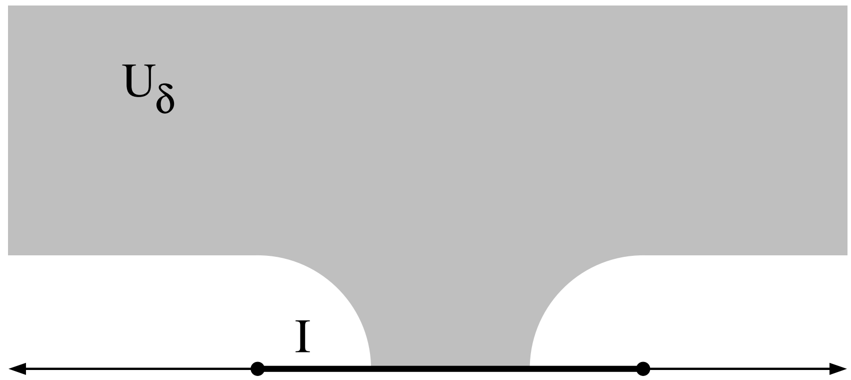


FIGURE 15. Illustrated in gray is the set U_δ of Lemma 10.4.

U is path connected, this implies $\widetilde{u}$ is within ε of zero on K if δ is small enough (depending on ε, η, M and the diameter of U in the path metric). □

We will use this later in the situation that if u and v are harmonic functions on Ω that are close enough on $K \subset \Omega$, then $f = \exp(u + i\widetilde{u})$ and $g = \exp(v + i\widetilde{v})$ are holomorphic functions on Ω that are close on K (if $\widetilde{v}$ is chosen to agree with $\widetilde{u}$ at some point of K).

Next, we recall the well known boundary Harnack inequality (e.g., see Theorem 7.18 of [Mar19] or Exercise I.6 of [GM08]).

Lemma 10.4. *Suppose u and v are positive harmonic functions on $\mathbb{D}$ which extend continuously to the boundary $\mathbb{T}$, and suppose furthermore that u and v are both equal to zero on an arc $I \subset \mathbb{T}$. For $\delta > 0$ let $U_\delta = \{z \in \mathbb{D} : \operatorname{dist}(z, \mathbb{T} \setminus I) > \delta\}$ (see Figure 15). Then for $z \in U_\delta$,*

$$\frac{\delta^2}{4} \cdot \frac{u(0)}{v(0)} \le \frac{u(z)}{v(z)} \le \frac{4}{\delta^2} \cdot \frac{u(0)}{v(0)}.$$

Note that, under these conditions, u and v have well defined inward normal derivatives on I. By letting $z \to \mathbb{T}$ radially, the inequalities above imply that $\frac{\partial u}{\partial n}$ and $\frac{\partial v}{\partial n}$ are comparable (with the same constants as above) at any point of $\overline{U_\delta} \cap \mathbb{T}$.

Corollary 10.5. *Suppose u is a positive harmonic function on $\mathbb{D}$ which extends continuously to the boundary $\mathbb{T}$ and that equals zero on an arc $I \subset \mathbb{T}$. Suppose $a, b \in I$ both have distance $> \delta$ from $\mathbb{T} \setminus I$. Then*

$$\frac{\delta^2}{4} \cdot \frac{\partial u}{\partial n}(a) \le \frac{\partial u}{\partial n}(b) \le \frac{4}{\delta^2} \cdot \frac{\partial u}{\partial n}(a).$$

Proof. We simply compare u to a rotation of itself. Let $v(z) = u(\frac{b}{a} z)$. Then v and u both vanish on $J = I \cap \frac{a}{b} \cdot I$, and $a \in J$ is distance $> \delta$ from either endpoint of J. Hence the normal derivatives of u and v at a are comparable by the boundary Harnack principle, and hence so are the normal derivatives of u at a and b. □

Suppose Ω is an analytic domain (see Definition 2.5). For $w \in \Omega$, let $G(z, w)$ be the Green's function on Ω with pole at w, i.e., G is harmonic on $\Omega \setminus \{w\}$, vanishes identically on $\partial\Omega$ and $G(z, w) + \log|z - w|$ is bounded in a neighborhood of w. Our assumptions on Ω imply that Ω is regular for the Dirichlet problem, and hence that the Green's function exists and is unique for any $w \in \Omega$ (e.g., see Sections II.1 and II.2 of [GM08]).

Lemma 10.6. *Suppose Ω is an analytic domain. If $r > 0$ is small enough (depending only on Ω), $x \in \partial\Omega$, and $y \in \Omega \setminus D(x, r)$, then the normal derivative of the Green's function with pole at y has comparable size at all points of $\partial\Omega \cap D(x, r/2)$ with a constant independent of y and Ω.*

Proof. Choose r small enough that $W = D(x, r) \cap \Omega$ is a Jordan domain whose boundary consists of a sub-arc γ of $\partial\Omega$ and an arc of the circle $\partial D(x, r)$. Let $\widetilde{\gamma} = \partial\Omega \cap D(x, r/2)$. Choose a point $z \in W$ that is about distance r from ∂W and choose a conformal map $\varphi : W \to \mathbb{D}$ taking z to 0. If r is small enough, φ extends analytically across γ to all of $D(x, r)$ and by the Koebe distortion theorem it has comparable derivative at all points of $\widetilde{\gamma}$. Also, since γ and each component of $\gamma \setminus \widetilde{\gamma}$ has harmonic measure with respect to z that is bounded away from zero, the image arcs on $\mathbb{T}$ all have lengths bounded uniformly from below. Thus $u(z) = G(\varphi^{-1}(z), y)$ is a positive harmonic function on $\mathbb{D}$ vanishing on $\varphi(\gamma)$. By Corollary 10.5 the normal derivatives of u are comparable at all points of $\varphi(\widetilde{\gamma})$. Since the values of $|\varphi'|$ are comparable at all points of $\widetilde{\gamma}$, we can deduce the lemma from the chain rule. □

Corollary 10.7. *Suppose Ω is an analytic domain. If $r > 0$ is small enough (depending only on Ω), $x \in \partial\Omega$, and $y \in \Omega$, then $\partial G(x, y)/\partial n$ is comparable at all points of $\gamma = \partial\Omega \cap D(x, r)$ with a constant depending only on an upper bound for*

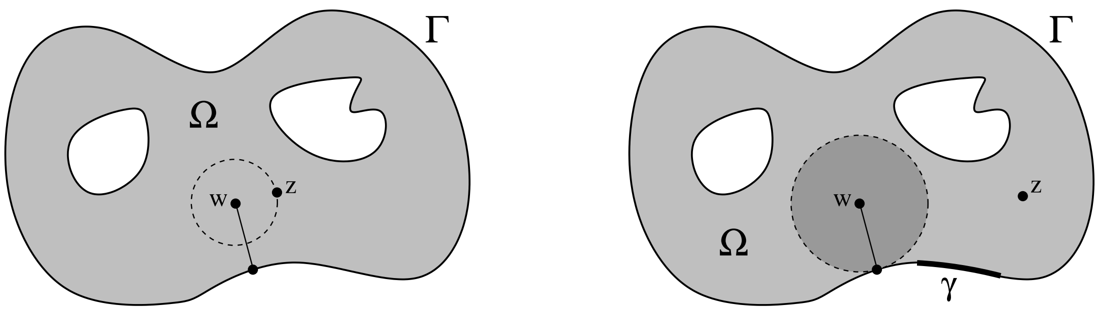
$$M = \max\left(1, \frac{\ell(\gamma)}{\operatorname{dist}(y, \partial\Omega)}\right),$$

where $\ell(\gamma)$ denotes the length of γ). Moreover, $\partial G(x, y)/\partial n = O(1/r)$ on γ.

Proof. We can cover γ by at most $O(M)$ disks $D_j = D(x_j, r_j)$ whose doubles are all disjoint from y. Lemma 10.6 implies the normal derivatives are comparable with some uniform constant C on each corresponding arc, so they are comparable with constant $C^{O(M)}$ on γ. (In fact, if r is so small that $\partial\Omega$ looks "straight" on scale r, then only $O(\log M)$ disks are needed, since they become geometrically larger as we move away from y.)

The final claim follows because the integral of the normal derivative over the whole boundary is 2π, and hence the integral over γ is $\leq 2\pi$. Since the size of the normal derivative is comparable at all points of γ, this implies it is bounded above by $O(1/\ell(\gamma)) = O(1/r)$ (recall ℓ denotes length). □

The following two lemmas relate harmonic measure to Green's function: we refer to Figure 16 for an illustration.



(A) (B)

FIGURE 16. In (A) the setup for Lemma 10.8 is shown, and in (B) the setup for Lemma 10.9.

Lemma 10.8. *There is a constant $C_1 < \infty$ so the following holds. Suppose Ω is an analytic domain, that Γ is a connected component of $\partial\Omega$ and that $w \in \Omega$ satisfies*

$$\operatorname{dist}(w, \Gamma) = \operatorname{dist}(w, \partial\Omega) \leq \operatorname{diameter}(\Gamma).$$

Then $G(z, w) \leq C_1$ on $\sigma = \{z : |z - w| = \frac{1}{2} \operatorname{dist}(w, \Gamma)\}$.

Proof. Let Ω' be the component of $\widehat{\mathbb{C}} \setminus \Gamma$ containing Ω. Then $\Omega \subset \Omega'$, so by the maximum principle the Green's function for Ω is less than the Green's function for Ω'. Thus it suffices to prove the lemma for the simply connected domain Ω'. But by Koebe's distortion theorem, σ contains a ball of fixed hyperbolic radius around w and hence its image contains a ball of fixed radius if we conformally map Ω' to the disk with w going to zero. On the disk, the Green's function is $\log \frac{1}{|z|}$ which is clearly bounded by some C outside a fixed ball around the origin. □

Lemma 10.9. *There is a constant $C_2 < \infty$ so that the following holds. Suppose that Ω is an analytic domain, $z \in \Omega$, that $\gamma \subset \partial\Omega$ is a subarc, and that $w \in \Omega$ satisfies*

(1) $\operatorname{dist}(w, \Gamma) = \operatorname{dist}(w, \partial\Omega) \leq \operatorname{diameter}(\Gamma)$, *where Γ is the component of $\partial\Omega$ containing γ,*
(2) $|z - w| \geq \operatorname{dist}(w, \partial\Omega)$.

Then $G(z, w) \leq C_2 \omega(z, \gamma, \Omega)/\omega(w, \gamma, \Omega)$.

Proof. Let C_1 and $\sigma = \{z : |z - w| = \frac{1}{2} \operatorname{dist}(w, \Gamma)\}$ be as in Lemma 10.8. By assumption z is in the component Ω' of $\Omega \setminus \sigma$ not containing w and by the maximum principle applied to Ω', $\omega(z, \sigma, \Omega') \geq G(z, w)/C_1$. By the maximum principle (again applied to Ω'),

$$\omega(z, \gamma, \Omega) \quad \geq \quad \omega(z, \sigma, \Omega') \cdot \min_{x \in \sigma} \omega(x, \gamma, \Omega).$$

By Harnack's inequality (see Theorem 7.17 of [Mar19]), all the values of $\omega(x, \gamma, \Omega)$ are comparable on σ and hence there is an $\varepsilon > 0$ so that

$$\omega(z, \gamma, \Omega) \quad \geq \quad \omega(z, \sigma, \Omega') \cdot \varepsilon \cdot \omega(w, \gamma, \Omega).$$

Finally, by Lemma 10.8 we have

$$\omega(z, \gamma, \Omega) \quad \geq \quad (\varepsilon / C_1) \cdot G(z, w) \cdot \omega(w, \gamma, \Omega),$$

which is the desired inequality with $C_2 = C_1/\varepsilon$. □

Corollary 10.10. *Suppose Ω is an analytic domain, $z \in \Omega$ and $\varepsilon > 0$. Suppose also that $\{\gamma_k\}_1^n \subset \partial\Omega$ is a collection of disjoint arcs and $\{w_k\}_1^n \subset \Omega$ is a collection of points, so that for all k we have:*

1. $\mathrm{dist}(w_k, \Gamma_k) = \mathrm{dist}(w_k, \partial\Omega) \leq \mathrm{diameter}(\Gamma_k)$, *where Γ_k is the component of $\partial\Omega$ containing γ_k,*
2. $|z - w_k| \geq \mathrm{dist}(w_k, \partial\Omega)$,
3. $\omega(w_k, \gamma_k, \Omega) \geq \varepsilon > 0$.

Then $\sum_{k=1}^n G(z, w_k) \leq C_2/\varepsilon$ where C_2 is the constant from Lemma 10.9.

Proof. By Lemma 10.9, $\sum_k G(z, w_k) \leq (C_2/\varepsilon) \sum_k \omega(z, \gamma_k, \Omega)$ and since the arcs $\{\gamma_k\}$ are disjoint, we have $\sum_k \omega(z, \gamma_k, \Omega) \leq 1$. □

11. Periods of Harmonic Functions

Suppose Ω is an analytic domain with $N+1$ boundary components $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$, so that Ω is regular for the Dirichlet problem. Suppose h is harmonic in Ω. In any sub-disk $D \subset \Omega$, h has a harmonic conjugate $\widetilde{h}$ that is well defined up to an additive constant. If γ is a closed curve in Ω, then we can analytically continue $\widetilde{h}$ along γ until we return to the starting point. The period of h along γ is the difference between the starting and ending values of $\widetilde{h}$. If γ is homologous to a point, the period is zero, but if γ is homologous to a boundary component of Ω, then the period may be non-zero. The sum of the periods corresponding to all $N+1$ boundary components is always zero (the union of all boundary curves is homologous to zero in Ω).

Next we consider the periods of certain special functions. For $j = 0, \ldots, N$ let ω_j be the harmonic function on Ω that has boundary value 1 on Γ_j and is 0 on the other boundary components. Since the boundary components are analytic, each ω_j extends to be analytic across $\partial\Omega$, so the normal and tangential derivatives are well defined, and themselves analytic, at every boundary point. The period of ω_j along Γ_k is the integral of the tangential derivative of $\widetilde{\omega}_j$ around Γ_k, and this equals the integral of the normal derivative of ω_j, i.e., the period is

$$\lambda_{jk} = \int_{\Gamma_k} \frac{\partial \omega_j}{\partial n} ds.$$

Note that since $0 < \omega_j < 1$ in Ω and $\omega_j = 1$ on Γ_j, the inward normal derivative of ω_j on Γ_j is non-positive (and strictly negative by analyticity: in that case, G can't have critical point on the boundary). Similarly, the inward normal derivative of ω_j is strictly positive on Γ_k for $k \neq j$. Thus $\lambda_{jj} < 0$ and $\lambda_{jk} > 0$ for $k \neq j$. Since the periods of ω_j sum to zero we have

$$\sum_{k=0}^{N} \lambda_{jk} = 0 \text{ for every } j,$$

and since $\sum_j \omega_j$ is the constant function 1 on Ω, we have

$$\sum_{j=0}^{N} \lambda_{jk} = 0 \text{ for every } k.$$

Thus for each j we have:

$$\lambda_{jj} = - \sum_{k \geq 0 : k \neq j} \lambda_{jk}.$$

In other words, the row and column sums are all zero for the $(N+1) \times (N+1)$ matrix (λ_{jk}), $0 \leq j, k \leq N$.

If we drop the first row and column of this matrix, we are removing all positive terms (except for λ_{00}), so we have $j = 1, \dots, N$,

$$\lambda_{jj} < - \sum_{k \geq 1 ; k \neq j} \lambda_{jk}.$$

Thus the $N \times N$ matrix $\Lambda = (\lambda_{jk})$, $1 \leq j, k \leq N$ is diagonally dominant, and this implies it is invertible by the Levy-Desplanques theorem: if the kernel of Λ contains a non-zero vector $v = (v_1, \dots, v_N)$, then $\sum_{k \geq 1} v_k \lambda_{jk} = 0$, and if $|v_k|$ is the largest component of v, then

$$|v_k \lambda_{kk}| = | \sum_{j \geq 1 : j \neq k} \lambda_{jk} v_j | \leq |v_k| \cdot \sum_{j \geq 1 : j \neq k} |\lambda_{jk}| < |v_k| \cdot |\lambda_{kk}|,$$

which is a contradiction. See Olga Taussky-Todd's paper [Tau49] for some history of this oft-rediscovered fact. Also note that $\|\Lambda\|$ and $\|\Lambda^{-1}\|$ only depend on Ω.

We have now proven the following result (due in a slightly different form to Heins [Hei50] and in greater generality to Khavinson [Kha84]).

Lemma 11.1. *Suppose Ω, Λ and $\{\omega_j\}_1^N$ are as above and suppose we assign real numbers $v = (v_1, \dots, v_N)$ to the N boundary components $\Gamma_1, \dots, \Gamma_N$. Then there is a linear combination $h = \sum_{j=1}^{n} a_j \omega_j$ so that the period of h around Γ_j is exactly v_j for $j = 1, \dots, N$. The coefficients $a = (a_1, \dots, a_N)$ are solutions of the linear equation $\Lambda a = v$ and hence $\|a\| \leq \|v\| \cdot \|\Lambda^{-1}\|$. Thus if $\|v\| \leq \varepsilon$ then $\|a\| = O(\varepsilon)$ with a constant that depends only on Ω.*

We say that the periods of a harmonic function h on Ω are well defined modulo 2π if every period is some integer multiple of 2π. In this case, $f = \exp(h + i\widetilde{h})$ is a well defined analytic function on Ω. For example, if Ω is the complement of a finite set of points $\{z_k\}_1^n$, then $\sum_{k=1}^n \log|z - z_n|$ has periods that are well defined modulo 2π. The corresponding holomorphic function is a polynomial with zeros at $\{z_k\}$ (and hence extends holomorphically from Ω to the whole plane). Note that $\nabla\widetilde{u}$ is always well defined even if $\widetilde{u}$ is not, since any two different branches of $\widetilde{u}$ differ by a constant.

Corollary 11.2. *Suppose Ω is an analytic domain and $K \subset \Omega$ is a compact set that contains curves $\{\sigma_j\}_0^N$ homologous to each of the boundary components $\{\Gamma_j\}_0^N$. Suppose $K \subset W \subset \Omega$ is open, let $\eta = \operatorname{dist}(K, \partial W)$ and set $U = \{z \in W : \operatorname{dist}(z, \partial W) > \eta/2\}$. Suppose $P \subset \Omega$ is a finite set and suppose u and H are harmonic functions on $\Omega \setminus P$, and each is either bounded or has a logarithmic pole at each point of P. Suppose $|u - H|$ is bounded by M on U, and that $|u - H| < \varepsilon$ on K. If u has a well defined harmonic conjugate modulo 2π on Ω, then there is an harmonic h on Ω so that*

(1) *$h + H$ also has a well defined harmonic conjugate modulo 2π on $\Omega \setminus Z$,*
(2) *$|h| \leq C\varepsilon^c$ on all of Ω (not just K),*
(3) *h is constant on each component of $\partial\Omega$.*

The constant c is the same as in Lemma 10.1 and C depends only on Ω and K.

Proof. Since $v = u - H$ is bounded on U, it extends to be harmonic at each point of $P \cap U$. Since $|v| < \varepsilon$ on K, it is bounded by $O(\varepsilon^c M^{1-c})$ on U and hence the gradient of v is bounded by $O(\varepsilon^c M^{1-c}/\eta)$ on U. Thus the gradient of the (possibly multi-valued) harmonic conjugate of v is bounded by the same quantity. We deduce that the harmonic conjugates of H and u differ by $\delta = O(L\varepsilon^c M^{1-c}/\eta)$ on U, where L is the diameter of U in the path metric. Thus the periods of H on the $\{\sigma_j\}$ differ from multiples of 2π by at most δ. Now apply Lemma 11.1 to define a harmonic function h on Ω that is bounded by $O(\delta)$ and has exactly the periods of $-v$. □

12. The Generalized Caratheodory Theorem on Blaschke Approximation

If B is a finite Blaschke product (see Definition 2.2) on an analytic domain Ω, then it has non-zero, continuous boundary values, and hence can have only finitely many zeros inside Ω. Moreover, the Schwarz reflection principle implies B extends holomorphically across $\partial\Omega$. The following result extends Carathéodory's Theorem (Theorem 2.1), and its statement uses the notation $\mathcal{I}_B$ (see Notation 2.4), and the following notation:

Notation 12.1. If $\Omega \subset \mathbb{C}$ is an analytic domain and $\Gamma := \partial\Omega$, we denote $\Gamma_\delta := \{z \in \Omega : \operatorname{dist}(z, \Gamma) = \delta\}$.

Theorem 12.2. *Suppose that $\Omega \subset \mathbb{C}$ is an analytic domain, $K \subset \Omega$ is compact, and f is holomorphic on a neighborhood of $\overline{\Omega}$ with $\sup_\Omega |f| \leq 1$. Let $\Gamma := \partial\Omega$. Then for any $\varepsilon > 0$ and sufficiently small $\delta > 0$, there is a finite Blaschke product B on Ω so that*

(1) $\sup_K |f - B| \leq \varepsilon$,

(2) *$1-\varepsilon < |B| \le 1$ on $\partial\Omega$,*
(3) *Every component of $\Gamma_\delta \setminus \{B=0\}$ has length comparable to δ and adjacent connected components have length comparable to within a factor of $1+\varepsilon$.*
(4) *All components of $\mathcal{I}_B$ have length which is comparable to δ with constants that depend only on ε and Ω.*
(5) *on each component γ of $\mathcal{I}_B$, the ratio $\max_\gamma |B'|/\min_\gamma |B'|$ is bounded depending only on ε and Ω.*

Proof. Without loss of generality, we may assume K is connected. Otherwise, replace K by a compact, connected superset, for instance, its closed convex hull in the hyperbolic metric. As before, let $G(z,w)$ denote the Green's function on Ω with pole at w.

Since f is holomorphic on a neighborhood of $\overline{\Omega}$, it only has finitely many zeros in Ω. We consider $g := (1-a)\cdot f + b$ for some constants $0 < a < \varepsilon$ and $|b| < \varepsilon$ such that $|g| < 1-\varepsilon$ on Ω and g has no zeros on $\partial\Omega$. If we construct a finite Blaschke product B approximating g to within ε on K, then it approximates f to within 3ε. Fix a finite number of smooth curves $\{\sigma_k\}$ that are homologous to each boundary curve of Ω. By enlarging K, if necessary, we may assume it contains all these curves. Let $\{z_k\}_1^N$ be the zeros of g, counted with multiplicity. By enlarging K again, if necessary, we may assume all the zeros of g are in K. Let W be an open domain with $K \subset W \subset \overline{W} \subset \Omega$. By compactness we have $\min_{\partial W} |g| \ge \eta$ for some $\eta > 0$. In what follows, $\delta > 0$ will always be chosen so small that $\overline{W}$ is disjoint from $\{z \in \Omega : \operatorname{dist}(z,\partial\Omega) < 2\delta\}$. In particular, g has no zeros in this neighborhood of the boundary.

Let $u(z) = -\log|g(z)|$. Then $u \ge -\log(1-\varepsilon) \ge \varepsilon$ is positive and harmonic on Ω except for finitely many logarithmic poles at the $\{z_k\}_1^N$, the zeros of g listed with multiplicity. Let $p(z) = \sum_{k=1}^N G(z,z_k)$ be the sum of the Green's functions with these poles. Then $v(z) = u(z) - p(z)$ is harmonic on Ω, and equals u on $\partial\Omega$. Thus v is continuous and non-zero on $\partial\Omega$, and hence it is bounded and bounded away from zero there, say $m \le v \le M$ on $\partial\Omega$. Hence $m \le v \le M$ on all of Ω by the maximum principle.

By Theorem II.2.5 of [GM08] v is the Poisson integral of its boundary values, i.e.,

$$v(z) = \frac{1}{2\pi}\int_{\partial\Omega} v(w)\frac{\partial G(w,z)}{\partial n} ds(w)$$

where $\frac{\partial}{\partial n}$ is the inward normal and ds denotes length measure on the boundary. For $w \in \Gamma_\delta$, denote by $w^* \in \partial\Omega$ the closest point to w on $\partial\Omega$. For $z \in K$ and $w \in \Gamma_\delta$ we have

$$G(w,z) = \delta \cdot \frac{\partial G(w^*,z)}{\partial n} + O(\delta^2),$$

where the constant depends on z, but is uniformly bounded as long as z is in the compact set K (the constant in the "big-Oh" depends on a bound for $|\nabla^2 G|$ between Γ_δ and $\partial\Omega$ and since G is harmonic and extends analytically across $\partial\Omega$, this is bounded as long as the pole of Green's function is not too close to $\partial\Omega$). It follows that

$$v(z) = \frac{1}{2\pi\delta}\int_{\Gamma_\delta} v(w)G(w,z)ds(w) + O(\delta). \tag{12.1}$$

Next use the identity $G(z,w) = G(w,z)$ (e.g., Theorem II.2.8 of [GM08]), to deduce

$$v(z) = \frac{1}{2\pi\delta}\int_{\Gamma_\delta} v(w)G(z,w)ds(w) + O(\delta).$$

We now discretize the integral by cutting Γ_δ into disjoint subarcs $\{\gamma_k\}$ chosen so that

$$1 \le \int_{\gamma_k} \frac{v(w)}{2\pi\delta} ds(w) \le 1 + O(\delta). \tag{12.2}$$

This is possible since the integral over a component Γ_δ^k of Γ_δ is at least $A = m \cdot \ell(\Gamma_\delta^k)/2\pi\delta$ and this tends to infinity as $\delta \searrow 0$. We can therefore cut each boundary curve into sub-arcs where the integral is between 1 and $1+O(1/A) = 1+O(\delta)$, i.e., we can make the sub-integrals all as close to 1 as we wish, by taking δ small enough.

The left side of Equation (12.2) implies that for each k, we have $\ell(\gamma_k)M/2\pi\delta \ge \int_{\gamma_k} v/2\pi \ge 1$ and hence $\ell(\gamma_k) \ge 2\pi\delta/M$. Similarly, the other side implies $\ell(\gamma_k) \le (1 + O(\delta))2\pi\delta/m$. Thus every such arc has length comparable to δ. If δ is small enough, then the continuity implies that on the union of two adjacent arcs with common endpoint x, v is close to $v(x)$, and hence by (12.2), the lengths of these intervals are both close to $2\pi\delta/v(x)$, and hence are close to each other. This fact, together with each γ_k have length comparable to δ, will imply part (3) of the Theorem once we have defined the generalized Blaschke product B.

Adding and subtracting a term from (12.1), have

$$\begin{aligned} v(z) &= \frac{1}{2\pi\delta}\sum_j G(z,w_j)\int_{\gamma_j} v(w)ds(w) \\ &\quad + \frac{1}{2\pi\delta}\sum_j \int_{\gamma_j} v(w)[G(z,w) - G(z,w_j)]ds(w) + O(\delta). \end{aligned} \tag{12.3}$$

The curve Γ_δ is parallel to the boundary, which is the level line $G(z,w) = 0$ of the Green's function with pole at w. Thus, the gradient of G along Γ_δ is nearly perpendicular to Γ_δ. Denoting by w_j the center of γ_j, we conclude:

$$|G(z,w) - G(z,w_j)| = O(\delta^2). \tag{12.4}$$

Using (12.4) in the last term of Equation 12.3 gives

$$v(z) = \frac{1}{2\pi\delta}\sum_j G(z,w_j)\int_{\gamma_j} v(w)ds(w) + \sum_j \frac{1}{2\pi\delta}\int_{\gamma_j} v(w)O(\delta^2)ds(w) + O(\delta).$$

Simplifying, we get

$$\begin{aligned} v(z) &= \frac{1}{2\pi\delta}\sum_j G(z,w_j)\int_{\gamma_j} v(w)ds(w) + O(\delta)\sum_j \int_{\gamma_j} v(w)ds(w) + O(\delta) \\ &= \sum_j G(z,w_j)\int_{\gamma_j} \frac{v(w)}{2\pi\delta} ds(w) + O(\delta) \\ &= \sum_j G(z,w_j)(1+O(\delta)) + O(\delta), \\ &= \sum_j G(z,w_j) + O(\delta), \end{aligned}$$

where in the last line we have used Corollary 10.10 to bound the sum of Green's functions by $O(1)$. Therefore, v is approximated on K by a finite sum of Green's functions on Ω (indeed, we can even take the approximation to hold on the larger compact set $\overline{W}$).

If Ω is simply connected, then we are essentially done. In this case,

$$H(z) \;=\; \sum_k G(z,z_k) + \sum_j G(z,w_j) = p(z) + \sum_j G(z,w_j)$$

is harmonic except for a finite number of logarithmic poles at $P = \{\cup_k z_k\}\cup\{\cup_j w_j\}$. Therefore H has a harmonic conjugate $\widetilde{H}$ that is well defined modulo 2π on $\Omega\setminus P$. Then

$$B(z) = \exp(-H - i\widetilde{H})$$

is holomorphic on $\Omega\setminus P$, but tends to zero at each point of P, so B is holomorphic on all of Ω with zeros exactly at the points of P. Moreover, on $\partial\Omega$ we have $|B| = \exp(0) = 1$, so B is a finite Blaschke product on Ω. Moreover,

$$H(z) = p(z) + \sum_j G(z,w_j) = u(z) - v(z) + \sum_j G(z,w_j) \approx u(z) = -\log|g(z)|,$$

so $\log|B| = -H$ approximates $\log|g|$ on $\overline{W}$ as closely as wish. In particular, we may assume $|B| \ge \eta/2$ on ∂W. In this case, g/B is holomorphic on W (since every zero of B inside W is also a zero of g of the same multiplicity), and so by the maximum principle $|g/B|$ is bounded on K by $\max_{\partial K}|g/B| \le \max_K 2|g|/\eta$. Now apply Lemma 10.3 to $h = \log g/B = u + \widetilde{u}$ on W to deduce that $\arg(B)$ approximates $\arg(g)$ on K, at least if we add an appropriate constant to $\arg(B)$. Therefore B (times an appropriate unit scalar) uniformly approximates g on K. This extends Carathéodory's theorem to simply connected domains.

However, if Ω is multiply connected, then H need not have a well defined harmonic conjugate modulo 2π on $\Omega\setminus P$. Therefore $B = \exp(-H - i\widetilde{H})$ need not be well defined: if we analytically continue B along one of the closed loops σ_k, we return to the same absolute value, but possibly a different value of the argument. The change in the argument is as small as we wish, tending to zero as the difference between $\log|B| = H$ and $\log|g|$ tends to zero on K. This is because g has a well defined harmonic conjugate modulo 2π on $\Omega \setminus P$, and so the discussion following Lemma 10.3 applies. In order to get a well defined (generalized) finite Blaschke product B on Ω, we apply Corollary 11.2 to u and H to construct h so that

(1) h is harmonic on all of Ω, and
(2) $h + H$ has a well defined harmonic conjugate modulo 2π on $\Omega \setminus P$.

(Note the H was constructed exactly so that the corollary can be applied: $u - H$ is harmonic except for poles on Γ_δ which are outside W, and we may make $u - H$ is as close to zero on K as we wish, while keeping it uniformly bounded on W.)

Now set $F = -H - h$. This function has all periods equal to zero modulo 2π, so $B = \exp(F + i\widetilde{F})$ is a well defined holomorphic function on Ω. Since $|h| = O(\delta)$, we can deduce that B still approximates g on the compact set K. Moreover, since $H = 0$ on $\partial\Omega$ and $h = a_j$ on Γ_j, we see that $|B| = \exp(a_j) = 1 + O(\delta)$ on Γ_j, so $|B|$ is constant on each boundary component. Dividing by the largest such value, we get another finite Blaschke product that satisfies (2) and still approximates g to within $O(\delta)$ on K.

To prove (4), it suffices to show that the modulus of the tangential derivative of B along $\partial\Omega$ (which is equal to $|B'|$ since B is holomorphic) is comparable to $1/\delta$ everywhere: then the preimage of a half-circle will have length comparable to δ. Note that on $\partial\Omega$, $|B'|$ is also equal to the normal derivative of $h + H$. The function h is a linear combination of fixed functions ω_j that depend only on Ω, and although the coefficients of the combination may depend on δ, they remain small as $\delta \searrow 0$. Thus the normal derivative of h remains bounded on $\partial\Omega$ as $\delta \searrow 0$, and this is negligible compared to $1/\delta$.

The function H is a sum of two sets of Green's functions, one with poles $\{z_j\}$ corresponding to the zeros of g and the other with poles $\{w_k\}$ along the curve Γ_δ. The first set of poles is fixed independent of δ and their contribution to the normal derivative of H is also bounded independent of δ. Again, these terms are negligible.

The main contribution to the normal derivative of H comes from the poles $\{w_k\}$ lying on Γ_δ. For each such point w_k consider the arc $\sigma_k = D(w_k, 2\delta) \cap \partial\Omega$; the harmonic measure of this arc with respect to w_k is bounded uniformly away from zero, i.e., $\omega(w_k, \sigma_k, \Omega) > c > 0$. This harmonic measure is the integral over σ_k of the normal derivative of the Green's function with pole at w_k, and thus it is less than the integral of the normal derivative of H, since this Green's function is one term of the sum defining H. Thus the integral of $|B'|$ over σ_k is $> c$ and so any single arc l_j in $\mathcal{I}_B$ can contain at most a bounded number of arcs of the form σ_k. By Part (3) of the theorem, adjacent points w_k are at most distance $O(\delta)$ apart and hence l_j can have length at most $O(\delta)$ (otherwise it would cover too many of the arcs σ_k).

Next, we need to prove $\ell(l_j)$ is bounded below by a multiple of δ, and this is equivalent to proving an upper bound $|B'(x)| = O(1/\delta)$ for any $x \in \partial\Omega$. As noted above, the contributions to $|B'|$ from h and from the poles of H coming from the zeros of g are both bounded independent of δ. To deal with the poles $\{w_k\}$ of H on Γ_k, we choose a point $z \in \Gamma_\delta$ that is close to $x \in \partial\Omega$, say $|x - z| \le 2\delta$ and note that

$$\frac{\partial}{\partial n} G(x, w_k) \simeq \frac{1}{\delta} G(z, w_k)$$

Thus by Corollary 10.10, the total contribution of all the poles $\{w_k\}$ to $|B'|$ is at most

$$\sum_k \frac{1}{\delta} G(w_k, z) = O(\frac{1}{\delta}),$$

and hence (4) is proven.

Let γ denote a component of $\mathcal{I}_B$. By Corollary 10.7 and conclusion (4) just proven, the normal derivative of Green's function with a pole at least distance δ from $\partial\Omega$ has comparable values at all points of γ, and so the same holds for any finite sum of such functions (with the same constant). Since on $\partial\Omega$ we have $|B'| = |\sum \frac{\partial G}{\partial n}|$, we deduce (5).

□

Remark 12.3. If $\Omega = \mathbb{D}$, then it suffices to assume f is holomorphic on Ω instead of a neighborhood of $\overline{\Omega}$. In that case if $r < 1$ then $g(z) = f(rz)$ is holomorphic on a neighborhood of $\overline{\mathbb{D}}$ and approximates f on a compact set $K \subset \mathbb{D}$ if r is close enough to 1. So if B is a finite Blaschke product on Ω approximating g to within $\varepsilon/2$, and r is chosen so that g approximates f to within $\varepsilon/2$, then B approximates f to within ε, as desired.

Remark 12.4. The analyticity of f on a neighborhood of $\overline{\Omega}$ is only used to deduce that f has a finite number of zeros inside Ω. The same proof would work if we assumed that f is holomorphic on Ω, extends continuously to $\partial\Omega$, and is non-zero on $\partial\Omega$.

Remark 12.5. If we make the previous assumption on f, then it suffices to assume Ω is bounded and finitely connected. If so, and any component of $\partial\Omega$ is a single point p, then by the Riemann removable singularity theorem, f extends to be holomorphic at p, and it suffices the prove the theorem for the extended function on the domain $\Omega' = \Omega \cup \{p\}$. By removing all the point components of $\partial\Omega$, we may assume every component of Ω is non-trivial. By repeated applications of the Riemann mapping theorem to the complement of each complementary component of Ω, the domain Ω can be mapped to a domain Ω' bounded by a finite number of analytic curves indeed, with more work, the Koebe circle domain theorem says it can be mapped to a domain bounded by circles. Transferring f and K to Ω we can use Theorem 12.2 to construct an approximating finite Blaschke product on Ω', and then transfer this to a finite Blaschke on Ω that approximates f on K.

Remark 12.6. We placed the poles of our Green's function all at the same distance δ from $\partial\Omega$, but this was not necessary. If we fix $z_0 \in \Omega$ and let $w \in \Omega$ approach $x \in \partial\Omega$, then $G(z, w)/G(z_0)$ approaches a positive harmonic function on Ω with zero boundary values on $\partial\Omega$, except at x, where it blows up. Thus the limiting function must be a multiple of the Poisson kernel on Ω with respect to $x \in \partial\Omega$. Thus it is easy to find finite weighted sums of Green's functions (with positive real weights) that approximate the Poisson integral of v.

Then one must cluster the poles to approximate this sum by a sum of Green's functions with integral weights; exponentiating such a sum gives a finite Blaschke product on Ω. Possibly this extra flexibility would be useful in other problems, such trying to minimize the number of poles needed.

Proof of Theorem 2.6: The hypotheses of Theorem 2.6 are stronger than those of Theorem 12.2: namely we assume in Theorem 2.6 that $||f||_\Omega < 1$ (rather than ≤ 1) and that the zeros of f are disjoint from $\partial\Omega$. Under these additional assumptions, there is no need in the second paragraph of the proof of Theorem 12.2 to replace f by $g := (1-a)\cdot f + b$ since we already have $|f| < 1-\varepsilon$ and f has no zeros on $\partial\Omega$. Since the B produced in Theorem 12.2 approximates g to within $O(\delta)$ and we may take $\delta \to 0$, the conclusions of Theorem 2.6 follow from the conclusions of Theorem 12.2. □

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