MINKOWSKI DIMENSION AND THE POINCARÉ EXPONENT

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ABSTRACT. Let G be a non-elementary, analytically finite Kleinian group, $\Lambda(G)$ its limit set and $\delta(G)$ the critical exponent for the Poincaré series. We give a new proof of the fact that if $\text{area}(\Lambda(G)) = 0$ then $\delta(G)$ equals the upper Minkowski dimension of $\Lambda(G)$. This gives new proofs of the following results:

1. If Λ has zero area then $\delta = \dim(\Lambda)$.
2. The Minkowski dimension of Λ exists equals the Hausdorff dimension.

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1. Statement of results

If $K \subset \mathbb{R}^{\not\models}$ is compact, let $N(K,\epsilon)$ be the minimal number of ϵ balls needed to cover K. We define the *upper* and *lower Minkowski dimension* as

$$\overline{\mathrm{Mdim}}(K) = \limsup_{\epsilon\to 0} \frac{\log N(K,\epsilon)}{\log 1/\epsilon},$$

$$\underline{\mathrm{Mdim}}(K) = \liminf_{\epsilon\to 0} \frac{\log N(K,\epsilon)}{\log 1/\epsilon}.$$

If the two values agree, the common value is simply called the Minkowski dimension of K and is denoted $\mathrm{Mdim}(K)$.

Consider a group G of Möbius transformations acting on the two sphere S^2. Such transformations are identified elements of $\mathrm{PSL}(2,\mathbb{C})$ in a natural way and G is called *Kleinian* if it is discrete in this topology (i.e., the identity is isolated in G). G is called *elementary* if it contains a finite index Abelian subgroup. In this paper we will consider only non-elementary groups. For a non-elementary group, the limit set, $\Lambda(G)$, is the accumulation set (on S^2) of the orbit of any point $z_0 \in S^2$ (and is independent of the point). The complement $\Omega(G) = S^2 \setminus \Lambda$ is called the ordinary set. In this paper we will always assume Ω is non-empty and that the group is conjugated in $\mathrm{PSL}(2,\mathbb{C})$ so $\infty \in \Omega$.

For any Kleinian group, the quotient $R = \Omega/G$ is a union of Riemann surfaces. We say that G is *analytically finite* if $R = R_1 \cup \cdots \cup R_s$ is a finite union of finite type surfaces (i.e., each R_j is compact or compact with a finite number of punctures). The Ahlfors finiteness theorem says that if G is finitely generated then G is analytically finite.

If $z_0 \in \Omega(G)$ then the critical exponent (or Poincaré exponent) is defined as

$$\delta(G) = \inf\{s : \sum_{g \in G} \operatorname{dist}(g(z_0), \Lambda)^s < \infty\},$$

where distance is in the spherical metric. It is easy to show it does not depend on the choice of z_0. Usually $\delta(G)$ is defined by extending the action of G on S^2 to a group of isometries of the hyperbolic 3-ball $\mathbb{B}^{\nvDash} \subset \mathbb{R}^{\nvDash}$, and then considering the series

$$\sum_G \exp(1 - |g(0)|)^s.$$

However, it is easy to see that this definition gives the same number.

Theorem 1.1. *Suppose G is an analytically finite, non-elementary Kleinian group. If area$(\Lambda(G)) = 0$ then $\delta(G) = \overline{Mdim}(\Lambda(G))$.*

The assumption that G is non-elementary is needed in Theorem 1.1, for if G is a rank 1, cyclic, parabolic group then $\delta(G) = 1/2$, but Λ is a single point. Define the Hausdorff content

$$H^\infty_\alpha(K) = \inf\{\sum r_j^\alpha : K \subset \cup_j D(x_j, r_j)\},$$

(the infimum is over all coverings of K by disks) and

$$\dim(K) = \inf\{\alpha : H^\infty_\alpha(K) = 0\}.$$

This is the Hausdorff dimension of K and it is easy to see that $\dim(K) \leq \underline{\mathrm{Mdim}}(K)$. Jones and I proved that $\delta(G) \leq \dim(\Lambda)$ for any non-elementary Kleinian group (Theorem 1.1 of [4]). Combining this result and Theorem 1.1 we easily deduce

Corollary 1.2. *If G is an analytically finite Kleinian group then the Minkowski dimension of Λ exists and equals the Hausdorff dimension.*

Corollary 1.3. *If G is an analytically finite, non-elementary Kleinian group and $\Lambda(G)$ has zero area then $\delta(G) = \dim(\Lambda)$.*

Different proofs of these results are given in [3] and [4] using estimates for the heat kernel on the hyperbolic 3-manifold associated to the Kleinian group G. The proof given here does not require these techniques, i.e., it is a purely "two-dimensional" argument. As such, it may be easier to adapt to other settings, e.g., Julia sets of rational mappings.

G is called *geometrically finite* if it is finitely generated and there is a finite sided fundamental polyhedron for the action of G on $\mathbb{B}$. The limit sets of such groups must have zero area [2], so our results apply to them. For geometrically finite groups, Corollary 1.2 was independently established by Stratmann and Urbański in [11]. Corollary 1.3 is also well known in this case, e.g., [12].

The sections of the paper are organized as follows.

- **Section 2:** We define a related critical exponent δ_{Whit} and show $\delta_{\text{Whit}}(K) \leq \overline{\text{Mdim}}(K)$ for any compact K, with equality if $\text{area}(K) = 0$.
- **Section 3:** We show $\delta \leq \delta_{\text{Whit}}$ for analytically finite groups with equality if $\Omega(G)/G$ is compact.
- **Section 4:** We define good and bad horoballs and prove a lemma giving some of their properties.

Section 5: We prove the main theorem when most horoballs of G are good.

Section 6: We prove the theorem in the case $\dim(\Lambda) = 2$.

Section 7: We state a lemma and finish the proof assuming the lemma and $\dim(\Lambda) < 2$.

Section 8: We prove the lemma.

Notation: In this paper $A \simeq B$ means that A/B is bounded and bounded away from 0. Given a square S in the plane and $\lambda > 0$, λS denotes the concentric square with $\mathrm{diam}(\lambda S) = \lambda \mathrm{diam}(S)$.

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2. Whitney squares and Minkowski dimension

A Whitney decomposition of a domain $\Omega \subset \mathbb{R}^{\not\models}$ is a collection of disjoint (except for boundaries) squares $\{Q_j\}$ such that $\Omega = \cup_j Q_j$ and

$$\mathrm{diam}(Q) \leq \mathrm{dist}(Q_j, \partial\Omega) \leq 4\mathrm{diam}(Q_j).$$

The existence of a Whitney decomposition for any open set is a standard fact in real analysis (e.g., Theorem VI.1 of [10]). One can simply take a maximal collection of dyadic squares in Ω such that $\mathrm{dist}(Q, \partial\Omega) \leq \mathrm{diam}(Q)$.

For any compact set $K \subset \mathbb{R}^{\not\models}$ we can define an exponent of convergence

$$\delta_{\mathrm{Whit}} = \delta_{\mathrm{Whit}}(K) = \inf\{s : \sum_{Q_j : \mathrm{dist}(Q_j, K) \leq 1} \mathrm{diam}(Q_j)^s < \infty\}.$$

The sum is taken over all squares in a Whitney decomposition of $\Omega = K^c$ which are within distance 1 of K (we have to drop the "far away" squares or the series will not converge). It is easy to check that this does not depend on the particular choice of Whitney decomposition.

Lemma 2.1. *For any compact set K, $\delta_{Whit} \leq \overline{Mdim}(K)$. If, in addition, area$(K) = 0$ then $\delta_{Whit} = \overline{Mdim}(K)$.*

Proof. Suppose $\{Q_j\}$ is a Whitney decomposition of $\Omega = \mathbb{R}^{\mathbb{K}} \setminus \mathbb{K}$. For each Q_j with $\operatorname{diam}(Q_j) \leq \operatorname{diam}(K)$, there is a dyadic cube Q_j' of the same size which hits K and satisfies $\operatorname{dist}(Q_j, Q_j') \leq C\operatorname{diam}(Q_j)$. Clearly each Q_j' is associated to only a bounded number of Whitney cubes. Therefore the number of dyadic cubes of size 2^{-n} which hit K is at least $C2^{n(\delta_{\text{Whit}}-\epsilon)}$ (for n large enough, depending on ϵ). Thus $\delta_{\text{Whit}}(K) \leq \overline{\text{Mdim}}(K)$.

Conversely, if K has zero area, Q is a dyadic square hitting K and $\{Q_k\}$ is the collection of Whitney squares for Ω contained in Q, then

$$\sum_k \operatorname{diam}(Q_k)^2 = \operatorname{diam}(Q)^2.$$

Hence for any $s \leq 2$, (since $\operatorname{diam}(Q) \leq 1$),

$$\sum_k \operatorname{diam}(Q_k)^s \geq \operatorname{diam}(Q)^s.$$

Since there are more than $C2^{n(\overline{\text{Mdim}}(K)-\epsilon)}$ such squares Q, the sum over the whole Whitney collection is greater than

$$C2^{-ns}2^{n(\overline{\text{Mdim}}(K)-\epsilon)},$$

which diverges if $s < \overline{\text{Mdim}}(K) - \epsilon$. Thus $\delta_{\text{Whit}}(K) \geq \overline{\text{Mdim}}(K)$, as desired. $\square$

We can have strict inequality if K has positive area. For example, one can choose a set of disjoint disks $D(x_j, r_j) \subset D(0,1)$, so that $K = \overline{D(0,1)} \setminus \cup_j D_j$, is nowhere dense, has positive area and $r_j \to 0$ as fast as we wish. If we sum the Whitney decomposition of a single disk we get

$$\sum_{Q_k \subset D_j} \mathrm{diam}(Q_j)^s \simeq r_j^s,$$

if $s > 1$ and equals ∞ if $s \leq 1$. By taking $r_j \to 0$ very fast, we can get

$$\delta_{\mathrm{Whit}}(K) = 1 < 2 = \overline{\mathrm{Mdim}}(K).$$

3. Whitney squares and the Poincaré series

In this section we explain the elementary relations between δ and δ_{Whit}.

Suppose Ω is a domain in S^2 with more than two boundary points. Then Ω has a hyperbolic metric ρ defined by the covering map from the disk to Ω. Let $d(z) = \mathrm{dist}(z, \partial\Omega)$. For a general domain (e.g., [6], Theorem 4.3),

$$\frac{(1+o(1))|dz|}{d(z)\log 1/d(z)} \leq |d\rho(z)| \leq 2\frac{|dz|}{d(z)}. \tag{3.1}$$

A set $K \subset \mathbb{R}^{\nvDash}$ is called *uniformly perfect* if there is a constant $C < \infty$ so that

$$\frac{1}{C}\frac{|dz|}{d(z)} \leq |d\rho(z)| \leq 2\frac{|dz|}{d(z)},$$

on each component Ω of $S^2 \setminus K$. (This is one of many equivalent definitions; see [7] and [8].)

The limit set of any finitely generated group is uniformly perfect, [9], [5]. In fact, the proof in Canary's paper [5] shows this is true under the weaker assumption that there is an $\epsilon_0 > 0$ so that any closed geodesic on Ω/G has length $\geq \epsilon_0$. This is certainly true Ω/G is a finite union of finite type surfaces, so the result is still true for analytically finite groups.

Lemma 3.1. *If G is any non-elementary Kleinian group with $\Lambda \neq S^2$ then $\delta \leq \delta_{Whit}$. If $\Omega(G)/G$ is compact then $\delta = \delta_{Whit}$.*

Proof. Fix a point $z_0 \in \Omega(G)$ (not an elliptic fixed point). There is a small hyperbolic disk around z_0 (with radius r_0 depending on z_0)

which projects injectively to $R = \Omega/G$ under the quotient map. Thus points in $G(z_0)$, the orbit of z_0 under G, are separated by at least r_0 in the hyperbolic metric. By (3.1) each Whitney square has a uniformly bounded hyperbolic diameter and area. Thus each Whitney square for $\Omega = S^2 \setminus \Lambda$ contains at most a bounded number M (depending on z_0 and G) of points in $G(z_0)$. Therefore,

$$\sum_{g \in G} \mathrm{dist}(g(z_0), \Lambda)^s \leq M \sum_j \mathrm{diam}(Q_j)^s$$

and hence $\delta(G) \leq \delta_{\mathrm{Whit}}(\Lambda(G))$.

Now suppose $R = \Omega(G)/G = R_1 \cup \cdots \cup R_s$ is a finite union of compact Riemann surfaces. We can choose points $E = \{z_1, \dots, z_s\} \subset \Omega$, so that z_j projects into R_j, $j = 1, \dots, s$ under the quotient map. By compactness, any point $z \in \Omega$ is a bounded hyperbolic distance from $G(E)$, the orbit of E under G. For each square Q with $\mathrm{dist}(Q, \Lambda) \leq 1$, choose a closest point $z_Q \in G(E)$. Then z_Q is only a bounded hyperbolic distance from Q so the uniform perfectness of Λ implies

$$\mathrm{diam}(Q) \leq C \mathrm{dist}(z_Q, \Lambda).$$

Furthermore, only a bounded number (say M) of the Q_j's are associated to any given point of $G(E)$. Thus

$$\sum_j \mathrm{diam}(Q_j)^s \leq MC^s \sum_{z_j \in E} \sum_{g \in G} \mathrm{dist}(g(z_j), \Lambda)^s,$$

and therefore $\delta_{\mathrm{Whit}}(\Lambda(G)) \leq \delta(G)$. $\square$

One of the main results of [3] is that $\delta_{\mathrm{Whit}} = \delta$ for any non-elementary analytically finite group. This fact and Lemma 2.1 imply Theorem 1.1, but the fact seems harder than the theorem. The purpose of this note is to give a proof of the theorem that does not require proving $\delta_{\text{Whit}} = \delta$.

4. Good and bad horoballs

A horoball in $\Omega(G)$ is a Euclidean ball $B \subset \Omega \subset S^2$ which is invariant under a rank 1 parabolic subgroup of G. The fixed point p of the parabolic element is on the boundary of the horoball and corresponds to a cusp on the surface $\Omega(G)/G$. We say that B is doubly cusped it there is a another (disjoint) ball B_1 fixed by the same subgroup.

Suppose $R = \Omega(G)/G$ is a finite union of finite type Riemann surfaces $R_1, \dots, R_N$, i.e., each is a compact surface with at most a finite number of punctures. Let $\{p_1, \dots, p_m\}$ be the punctures in $R = \cup_{i=1}^N R_i$, and for each p_i let B_i^* be a neighborhood of p_i which lifts to a Euclidean ball B_i in Ω which is invariant under some parabolic element of G (see Lemma 1 of [1]). Then $X = R \setminus \cup_j B_j^*$ is compact, so we can choose a finite set of points $E = \{z_1, \dots, z_P\} \subset \Omega(G)$ which project to an 1-dense subset of X (i.e., every point of X is within hyperbolic distance 1 of a point of E). For the reminder of the paper we fix a finite collection of such horoballs $\mathcal{B} = \{\mathcal{B}_|\}$, one from each equivalence class of horoballs in Ω. It will also be convenient to assume that B_i is contained inside a larger horoball $\hat{B}_i \subset \Omega$ of twice the Euclidean diameter (we can always do this by just taking smaller balls if necessary). If $\infty \in \Omega$ then we may also assume that ∞ is not contained in any of the B_i. Finally, if a parabolic point is doubly cusped, we require that $\mathcal{B}$ contain horoballs of equal size for both "sides" of the parabolic point.

For $z \in \Omega(G)$ we define $d(z) = \mathrm{dist}(z, \Lambda)$ in the Euclidean metric. Normalize the group so that $\infty \in \Omega$ and $\Lambda(G)$ has diameter 1. Now suppose p is a parabolic fixed point of G and that $B \subset \Omega$ is a horoball at p. By our remarks above, B must be the image of one of our finite collection of horoballs $\{B_j\}$ under some element g of G, i.e., $B = g(B_j)$.

We would like to call B good or bad depending on whether the map $g : B_j \to B$ is "close" to being linear. Using a linear Möbius transformation we can map p to 0 and the horoball B to the disk $D(i/2, 1/2)$. The parabolic element of G fixing p is conjugated by the linear map to a transformation of the form

$$\tau(z) = \frac{z}{1+\eta z}.$$

We define $\eta(B) = |\eta|$. Given $\eta > 0$, we say a horoball $B = g(B_i)$ is a "η-bad" horoball if $\eta(B) \leq \eta$, and is η-good otherwise.

A helpful way to think about good and bad horoballs is as follows. Suppose h is a generator of the parabolic subgroup fixing B. Define

$$\eta'(B) = \sup_{z \in \partial B} \frac{|h(z) - z|}{\text{diam}(B)}.$$

Then its easy to see $\eta' \simeq \eta$. In other words, B is η-bad if the parabolic subgroup fixing B has a generator which is close to the identity on B. Alternatively, the group G looks less and less discrete on horoballs with smaller and smaller η.

Lemma 4.1. *Suppose G is analytically finite and normalized so that $\infty \in \Omega(G)$ and diam$(\Lambda) = 1$. Fix a finite collection of horoballs $\mathcal{B}$ as above, and assume ∞ is not in any of these horoballs. Then*

1. *There is a C_1 (depending only on η) so that for any η-good horoball B, and any $w \in \partial B$, there is a point $z \in G(E)$ such that*

$$C_1^{-1} d(w) \leq d(z) \leq C_1 d(w),$$

$$C_1^{-1} d(w) \leq |z - w| \leq C_1 d(w).$$

2. *There is a η_2 so that if B is η_2-bad then it is singly cusped, i.e., there is not a disjoint horoball also tangent to p.*
3. *There is a $\eta_3 > 0$ so that if $\eta \leq \eta_3$ and B is η-bad and $D(x, r) \subset \Omega$ with $\frac{3}{2} diam(B) \leq dist(x, B) \leq diam(B)/(2\eta)$, then $r \leq C_2 \eta \, dist(x, B)^2$, where the constant C_2 depends only on G.*
4. *For any $\delta > 0$ there is a $\eta_4 > 0$ (depending only on δ) such that if B is a η_4-bad horoball then there is a disk $D \subset 3B$ such that $diam(D) \geq \frac{1}{3} diam(B)$, and $D \setminus \Lambda$ contains no balls of radius $\geq \delta \, diam(D)$.*
5. *There is $\eta_5 > 0$ so that if B_1, B_2 are η_5-bad horoballs with $diam(B_1) \leq diam(B_2)$ then $dist(B_1, B_2) \geq 100 \, diam(B_1)$.*
6. *If B_1, B_2 are horoballs with*

$$diam(B_1) \leq diam(B_2) \leq 2 \, diam(B_1),$$

and $dist(B_1, B_2) \leq A diam(B_1)$, then both B_1 and B_2 are A^{-2}-good.

Proof. To prove (1), note that it is true for all the balls in $\mathcal{B}$ by finiteness. If B is η-good then B is the image of some B_j in our fixed, finite collection under a map which is the composition of conformal linear map and a map of bounded distortion (depending on η).

To prove (2), suppose $B_1, B_2 \in \mathcal{B}$ are paired horoballs at a doubly cusped parabolic point p with parabolic generator h. For $i = 1, 2$, let z_i be the point on ∂B_i farthest from p and consider the the cross ratio of $z_1, h(z_1), z_2, h(z_2)$. Now suppose $B = g(B_1)$ is some η-bad image of B_1, and suppose we have chosen g so $|g(z_1) - g(p)|$ is maximized among elements mapping B_1 to B. Since cross ratio is preserved by Möbius transformations we can deduce that

$$|g(z_1) - g(h(z_1))| \simeq |g(z_1) - g(z_2)| \simeq |g(z_2) - g(h(z_2))| \simeq \eta \mathrm{diam}(B).$$

Thus all four points are mapped within $C\eta(B)\mathrm{diam}(B)$ of $g(z_1)$. If η is small enough, this is only possible if $\infty \in g(B_2)$, contrary to hypothesis.

To prove (3) note that it is enough to do it in the case when $p = 0$, $B = D(i/2, 1/2)$ and the subgroup fixing p is generated by

$$h(z) = \frac{z}{1 \pm \eta z}.$$

Then

$$|h(x) - x| = |x - \frac{x}{1 \pm \eta x}| \leq 2\eta |x|^2 \leq C_0 \eta \mathrm{dist}(x, B)^2.$$

Suppose $C_2 > 100 C_0$ and that $r \geq C_2 \eta \mathrm{dist}(x, B)^2$. Then both x and $h(x)$ are in the disk $D(x, r) \subset \Omega$ so the line segment connecting them projects to a loop on Ω/G of hyperbolic length $\leq 10 C_0/C_2$. If C_2 is large enough (depending only on G), this implies the loop is contained in a neighborhood of a cusp on Ω/G, thus the lift is in a horoball of Ω which is tangent to 0. This horoball is obviously not B since B does not contain x, so there is a second horoball B' at 0. This contradicts part (2),so we are done.

The final three statements are all easy consequences of part (3). $\square$

We noted in the previous section that if $\Omega(G)/G$ is compact then there is a close connection between the Poincaré and Whitney sums. When there are punctures in $\Omega(G)/G$ we need to take account of the fact that horoballs contain many Whitney squares, but no orbit points. The next observation is very easy and left to the reader.

Lemma 4.2. *Suppose Ω is an open set and $B \subset \Omega$ is a Euclidean ball such that $\partial B \cap \partial\Omega \neq \emptyset$. Let $\{Q_j\}$ be a Whitney decomposition for Ω. Then if $s > 1$,*

$$\sum_{j:Q_j \cap B \neq \emptyset} diam(Q_j)^s \simeq diam(B)^s,$$

where the constants depend only on s.

The following will be useful later.

Lemma 4.3. *Suppose G is analytically finite and $E \subset \Omega$ is a finite set with one point in each equivalence class of components. Assume the group has been normalized so $\infty \in \Omega$. Suppose $\{Q_j\}$ is a Whitney decomposition for $\Omega(G)$, and let $\mathcal{B}$ be a choice of horoballs for G as above. Then if $s > 1$,*

$$\sum_{j:diam(Q_j)\leq 1} diam(Q_j)^s \simeq \sum_{z\in E}\sum_{g\in G} dist(g(z), \Lambda)^s + \sum_{B\in\mathcal{B}} diam(B)^s,$$

where the constants depend on $G, s, E, \mathcal{B}$. If $\eta > 0$ then

$$\sum_{j:diam(Q_j)\leq 1} diam(Q_j)^s \simeq \sum_{z\in E}\sum_{g\in G} dist(g(z), \Lambda)^s + \sum_{B\in\mathcal{B},\eta(\mathcal{B})\leq \eta} diam(B)^s,$$

where the constants depend on $G, s, E, \mathcal{B}, \eta$.

Proof. The first equation is simply the observation that Whitney squares which do not hit any horoballs can be associated (as in the previous section) to orbits of E, whereas the Whitney squares which hit a horoball are controlled by the previous lemma. The second equation is proved using part (1) of Lemma 4.1 to associate to each η-good horoball a nearby orbit point. Thus the part of the horoball sum we are omitting is controlled by the orbit sum. $\square$

5. Theorem 1.1 when G has many good horoballs

We now start the proof of Theorem 1.1. It is enough to prove

Theorem 5.1. *If G is an analytically finite group and area$(\Lambda(G)) = 0$ then $\delta(G) = \delta_{Whit}(\Lambda(G))$.*

Let $D = \overline{\mathrm{Mdim}}(\Lambda) = \delta_{\mathrm{Whit}}(\Lambda(G))$ and $d = \dim(\Lambda)$. If G has no parabolic elements then $\Omega(G)/G$ is compact, and we have already proven this case in Lemma 3.1. Therefore we may assume G has parabolics. This implies $\delta(G) \geq 1/2$. If $D = 1/2$, we have nothing to do, so we may assume that $D > 1/2$.

Suppose $\epsilon > 0$ is so small that $D - \epsilon > 1/2$. Let $E = \{z_1, \dots, z_s\}$ be a finite collection of points in Ω, one projecting to each component of Ω/G. We will show that

$$\sum_{z \in G(E)} d(z)^{D-\epsilon} = \infty,$$

and thus $\delta(g) \geq D$.

Choose an integer n_0 so that

$$N(\Lambda, 2^{-n_0}) \geq 1000 \cdot 2^{n_0(D-\epsilon/2)}.$$

By passing to a subcollection with at least $2^{n_0(D-\epsilon/2)}$ elements we claim that we may assume that for any two squares S_j, S_k we have $9S_j \cap 9S_k = \emptyset$. This is easy. Just enumerate the list of squares and inductively remove any square S_k for which there is a $j < k$ with $9S_j \cap 9S_k \neq \emptyset$. Since $9S_j \cap 9S_k \neq \emptyset$ implies $S_k \subset 15S_j$, each S_j can cause at most $30^2 = 900$ later squares to be removed. Thus the final list has at least $2^{n_0(D-\epsilon/2)}$ elements.

Let $r = 3 \cdot 2^{-n_0}$. Let $\mathcal{S} = \{\mathcal{S}_{||}\}$ be a collection of $2^{n_0(D-\epsilon/2)}$ squares of size r so that the triples $3S_k$ are pairwise disjoint and $\frac{1}{3}S_k \cap \Lambda \neq \emptyset$ for each k.

First we deal with the case when most of the horoballs of size r are good. Let $\eta > 0$ (to be fixed later). For each η-good horoball with $\mathrm{diam}(B) \geq r/3$, let $\mathcal{G}_{\mathcal{B}}$ be the collection of squares in $\mathcal{S}$ which are such that $\frac{1}{3}S$ hits B. Let $\mathcal{G}$ be the union of all the $\mathcal{G}_{\mathcal{B}}$.

The proof breaks into three cases:

1. $\#(\mathcal{S} \cap \mathcal{G}) \geq \frac{\infty}{\in}\#(\mathcal{S})$ for all large enough n.
2. $\dim(\Lambda) = 2$.
3. $\#(\mathcal{S} \cap \mathcal{G}) < \frac{\infty}{\in}\#(\mathcal{S})$ for infinitely many n and $\dim(\Lambda) < 2$.

If η is chosen small enough (depending on G) then one can show the last case is impossible (e.g., [4]). However, this is hard result using heat kernel estimates on hyperbolic 3-manifolds and one of our purposes here is to give a self-contained proof that uses only two dimensional techniques.

Proof of case 1: By part (1) of Lemma 4.1 there is an orbit point $z \in G(E) \cap B$ such that $d(z) \simeq \operatorname{diam}(B)$. For this point,

$$d(z)^{D-\epsilon} \geq C \sum_{S \in \mathcal{G}_{\mathcal{B}}} \operatorname{diam}(S)^{D-\epsilon}.$$

If more than half the squares in $\mathcal{S}$ belong to $\mathcal{G}$ then this argument shows

$$\sum_{z \in G(E)} d(z)^{D-\epsilon} \geq \frac{1}{2} C 2^{n_0 \epsilon/2}.$$

If this happens for arbitrarily large n_0 then

$$\sum_{z \in G(E)} d(z)^{D-\epsilon} = \infty,$$

so we have shown that $\delta(G) \geq D - \epsilon$, as desired. This is the end of case (1).

6. Proof of case (2) of Theorem 1.1

Case 2 follows easily from the following.

Lemma 6.1. *Suppose G is a analytically finite Kleinian group, normalized so $\infty \in \Omega(G)$ and diam$(\Lambda(G)) = 1$. If $\delta_{Whit} = 2$ then $\delta = 2$.*

Proof. We know the lemma if $\Omega(G)/G$ is compact (see Section 2), so we may assume that $\Omega(G)$ contains horoballs. Consider the sum over all Whitney squares for Ω,

$$\sum_j \text{diam}(Q_j)^{2-2\epsilon}.$$

By the definition of δ_{Whit}, this diverges. Using Lemma 4.3 we can split the sum into two pieces; one corresponding to all squares Q_j which hit some horoball and the other corresponding to all Whitney squares which miss every horoball, i.e.,

$$\sum_{j:\text{diam}(Q_j)\le 1} \text{diam}(Q_j)^{2-2\epsilon} \simeq \sum_{z\in E}\sum_{g\in G} \text{dist}(g(z),\Lambda)^{2-2\epsilon} + \sum_{B\in\mathcal{B}} \text{diam}(B)^{2-2\epsilon}.$$

If the first sum on the right diverges then $\delta > 2 - 2\epsilon$ and we are done. Thus we may assume the second sum on the right diverges for all $\epsilon > 0$. Thus if

$$\mathcal{B}_\backslash = \{\mathcal{B}_| : \in^{-\backslash-\infty} \le \text{diam}(\mathcal{B}_|) < \in^{-\backslash}\},$$

and $N_n = \#\mathcal{B}_\backslash$, we must have $N_n \ge 2^{n(2-2\epsilon)}$, for infinitely many values of n. Fix a value of n_0 where this holds and note that for at least half the balls B in $\mathcal{B}_\backslash$ there is a second ball $B' \in \mathcal{B}_\backslash$ such that

$$\text{dist}(B, B') \le 2^{-n(1-\epsilon)} \le \text{diam}(B) 2^{n\epsilon},$$

otherwise we would have so many disjoint balls of radius $2^{-n(1-\epsilon)}$ that they could not all be contained in a bounded neighborhood of Λ (recall that $\mathrm{diam}(\Lambda) = 1$).

By part (4) of Lemma 4.1, this implies that for such a pair both B and B' are $2^{-2\epsilon n}$-good horoballs. Let $\mathcal{G}_{\backslash\prime} \subset \mathcal{B}_{\backslash\prime}$ be the subcollection of $2^{-2\epsilon n}$-good horoballs. For any η-good horoball B let z be the point given in part (1) of Lemma 4.1 such that $d(z) \geq C\eta \mathrm{diam}(B)$. Let H be the parabolic subgroup fixing B_j. Then an easy calculation shows that there are at least $C\eta^{-1}$ orbits of z under H with distance $\geq C\eta\mathrm{diam}(B)$ from Λ. Thus

$$\sum_{h\in H} d(h(z))^\alpha \geq C\mathrm{diam}(B)^\alpha \eta^{\alpha-1},$$

for any $1 < \alpha \leq 2$ and some C depending on G and α.

Thus if z_j is the good point in B_j given by (1) of Lemma 4.1,

$$\begin{aligned}
\sum_{z\in G(E)} d(z)^\alpha &\geq C \sum_{B_j\in\mathcal{G}_{\backslash\prime}} \sum_{k\in\mathbb{Z}} d(h^k(z_j))^\alpha \\
&\geq C \sum_{B_j\in\mathcal{G}_{\backslash\prime}} \mathrm{diam}(B)^\alpha 2^{-2\epsilon n_0(\alpha-1)} \\
&\geq C 2^{n_0(2-\epsilon)} 2^{-n_0\alpha} 2^{-2\epsilon n_0(\alpha-1)}.
\end{aligned}$$

Taking $\epsilon = 0$ and solving for α we see this diverges for small enough ϵ if $\alpha < 2$. Thus $\delta \geq 2$, as desired. $\square$

7. Theorem 1.1 when G has few good horoballs

We now do case 3, i.e., we assume that fewer than half the elements of $\mathcal{S}$ are in $\mathcal{G}$ for all large enough n_0 (i.e., we assume most horoballs are bad) and that $\dim(\Lambda) < 2$. We use a stopping time construction which is described by the following lemma. Recall that $d = \dim(\Lambda)$ and $D = \overline{\text{Mdim}}(\Lambda)$.

Lemma 7.1. *Suppose $\epsilon > 0$ and $r > 0$. There is a constant C_0 (depending only on G and ϵ) and constants $\eta_0 > 0$ and ν_0 (depending on G, ϵ and r) such that the following holds: Suppose we have a square S such that $\frac{1}{3}S \cap \Lambda \neq \emptyset$ and S does not intersect any η_0-good horoball with diameter $\geq diam(S)/3$. Then either*

$$\sum_{z \in S \cap G(E)} d(z)^{D-\epsilon} \geq \nu_0 \, diam(S)^{D-\epsilon},$$

or there is a collection of subsquares $\mathcal{C}(\mathcal{S}) = \{\mathcal{S}_{|}\} \subset \mathcal{S}$ with

1. *$diam(S_j) \leq r \, diam(S)$ for all j,*
2. *$\{3S_j\} \subset S$ and are pairwise disjoint,*
3. *$S_j \setminus \Lambda$ does not contain a ball of radius $diam(S_j)/10$,*
4. *$\sum_j diam(S_j)^2 \geq C_0 diam(S)^2$.*

We shall refer to the two possibilities as alternatives one and two respectively.

Let us assume that the lemma holds and see how to finish the proof. We will prove the lemma in the next section.

Suppose $\epsilon > 0$. Let $C_0 = C_0(\epsilon, G)$ be as given by the lemma. Let $r = C_0^{1/\epsilon}$ and let η_0, ν_0 be as given by the lemma.

We define generations of nested squares $\mathcal{S}_{\prime}, \mathcal{S}_{\infty}, \dots$ as follows. Let $\mathcal{S}_{\prime} = \mathcal{S}$, be the collection considered in the previous section; i.e., $\mathcal{S}_{\prime}$ is a collection of $\frac{1}{2} \cdot 2^{n_0(D-\epsilon/2)}$ squares of size $r = 2^{-n_0}$ which have disjoint triples, so that $\frac{1}{3}S \cap \Lambda \neq \emptyset$ and such that S does not hit any η-good horoball of size $\geq \mathrm{diam}(S)/3$.

Suppose $\mathcal{S}_\backslash$ has been defined. Let $\mathcal{A}^\backslash_\infty \subset \mathcal{S}_\backslash$ be the subcollection of squares for which alternative one holds and $\mathcal{A}^\backslash_\in$ be the subcollection for which alternative two holds. Then we define

$$\mathcal{S}_{\backslash+\infty} = \bigcup_{\mathcal{S} \in \mathcal{A}^\backslash_\in} \mathcal{C}(\mathcal{S}).$$

In other words, given $\mathcal{S}_\backslash$, we define $\mathcal{S}_{\backslash+\infty}$ by throwing away all the squares where alternative one of Lemma 7.1 holds and for each square $S \in \mathcal{S}_\backslash$ where the second alternative holds we replace it by the collection $\mathcal{C}(\mathcal{S})$ of subsquares satisfying (1)-(4) in Lemma 7.1.

To each square $S \in \mathcal{S}_\infty = \cup_\backslash \mathcal{S}_\backslash$ we associate a positive number $\mu(S)$ as follows. For $S \in \mathcal{S}_/$ let

$$\mu(S) = \mathrm{diam}(S)^{D-\epsilon}. \tag{7.1}$$

For $S \in \mathcal{S}_\backslash$, $n \geq 1$, there is a unique $S_0 \in \mathcal{S}_{\backslash-\infty}$ containing S (i.e., S_0 is its "parent") and by definition alternative two holds for S_0. Set

$$\mu(S) = \frac{\mathrm{diam}(S)^2}{\sum_{S' \in \mathcal{C}(\mathcal{S}_/)} \mathrm{diam}(S')^2} \mu(S_0). \tag{7.2}$$

Note that $\sum_{S' \in \mathcal{C}(\mathcal{S}_/)} \mu(S') = \mu(S_0)$. Let $\mu(n) = \sum_{S \in \mathcal{S}_\backslash} \mu(S)$. It is clear that $\{\mu(n)\}_{n=0}^\infty$ is non-increasing.

Lemma 7.2. *If* $\mu(n) \not\to 0$ *then* $\dim(\Lambda) \geq 2 - \epsilon$. *(Recall* μ *depends on* ϵ.*)*

Proof. Suppose $\mu(n) \not\to 0$. Then the numbers $\{\mu(S)\}$ define a measure positive measure on the set $Y = \cap_n \cup_{S \in \mathcal{S}_\backslash} S \subset \Lambda$. We claim that

$$\mu(S) \leq C \mathrm{diam}(S)^{2-\epsilon},$$

for every square with $\mathrm{diam}(S) \leq r$ where C is a constant that depends on the choice of n_0, but not on S.

We first verify this by induction for squares in $\mathcal{S}_\infty$. If $S \in \mathcal{S}_\prime$, the claim is true by definition with $C = A_0 = 2^{n_0(2-D)}$.

If $S \in \mathcal{S}_\backslash$ is contained in $S_0 \in \mathcal{S}_{\backslash - \infty}$, then part (4) of Lemma 7.1 implies

$$\mu(S) \quad = \quad \frac{\mathrm{diam}(S)^2}{\sum_{S' \in \mathcal{C}(\mathcal{S}_\prime)} \mathrm{diam}(S')^2} \mu(S_0) \leq A_0 \frac{\mathrm{diam}(S)^2}{C_0 \mathrm{diam}(S_0)^2} \mathrm{diam}(S_0)^{2-\epsilon}$$

and hence

$$\begin{aligned} \mu(S) &\leq A_0 C_0^{-1} \mathrm{diam}(S)^2 \mathrm{diam}(S_0)^{-\epsilon} \\ &\leq A_0 C_0^{-1} r^{-\epsilon} \mathrm{diam}(S)^{2-\epsilon} \\ &\leq A_0 \mathrm{diam}(S)^{2-\epsilon}, \end{aligned}$$

as desired. Also note (for future use) that this proves the weaker estimate

$$\mu(S) \leq \mathrm{diam}(S)^{D-\epsilon}. \tag{7.3}$$

Now consider a general square $S \subset S_0 \in \mathcal{S}_{\prime}$. Let S_1 be the smallest square in $\mathcal{S}_\infty$ containing S. Suppose $S_1 \in \mathcal{S}_{\backslash - \infty}$. Then by (7.2),

$$\begin{aligned}
\mu(S) &= \frac{\sum_{S' \in \mathcal{S}_\backslash, \mathcal{S}' \subset \mathcal{S}} \operatorname{diam}(S')^2}{\sum_{S' \in \mathcal{S}_\backslash, \mathcal{S}' \subset \mathcal{S}_\infty} \operatorname{diam}(S')^2} \mu(S_1) \\
&\leq A_0 \frac{\operatorname{diam}(S)^2}{C_0 \operatorname{diam}(S_1)^2} \operatorname{diam}(S_1)^{2-\epsilon} \\
&\leq A_0 C_0^{-1} \operatorname{diam}(S)^2 \operatorname{diam}(S_1)^{-\epsilon} \\
&\leq A_0 C_0^{-1} \operatorname{diam}(S)^{2-\epsilon} \left(\frac{\operatorname{diam}(S_1)}{\operatorname{diam}(S)}\right)^{-\epsilon} \\
&\leq A_0 C_0^{-1} \operatorname{diam}(S)^{2-\epsilon}.
\end{aligned}$$

Thus the inequality holds for general squares with the constant $C = A_0 C_0^{-1}$.

Thus if $\{S_j\}$ was any covering of Λ we would have

$$0 < \mu(\Lambda) \leq \sum_j \mu(S_j) \leq C \sum_j \operatorname{diam}(S_j)^{2-\epsilon}.$$

Therefore, $\dim(\Lambda) \geq 2 - \epsilon$, as desired. □

By the previous lemma if $\epsilon < 2 - \dim(\Lambda)$, then we must have $\mu(n) \to 0$. Assume this is the case.

Recall that $\mathcal{A}^\backslash_\infty$ is the collection of all squares in $\mathcal{S}_\backslash$ where alternative one held (i.e., the construction above stopped) and $\mathcal{A}^\backslash_\in$ are the remaining squares where alternative two held. Note that $\cup_n \mathcal{A}^\backslash_\infty$ is a union of disjoint squares. For $S \in \mathcal{A}^{\backslash - \infty}_\in$, let $\mathcal{A}_\infty(\mathcal{S})$ be the collection of all squares in $\mathcal{C}(\mathcal{S}) \subset \mathcal{S}_\backslash$ where alternative one holds and let $\mathcal{A}_\in(\mathcal{S})$ be the squares where alternative two holds. For $S' \in \mathcal{A}_\infty(\mathcal{S}) \in \mathcal{A}^\backslash_\infty$, define

$$\nu(S') = \sum_{z \in S' \cap \mathcal{G}(\mathcal{E})} d(z)^{D-\epsilon}.$$

Since alternative one of Lemma 7.1 holds for S', $\nu(S') \geq \nu_0 \mathrm{diam}(S')^{D-\epsilon}$, so by (7.3) $\nu(S') \geq \nu_0 \mu(S')$, ($\mu(S')$ must be defined because alternative two holds for its parent). Thus,

$$\sum_{S' \in \mathcal{A}_\infty(\mathcal{S})} \nu(S') \geq \nu_0 \sum_{S' \in \mathcal{A}_\infty(\mathcal{S})} \mu(S') = \nu_0[\mu(S) - \sum_{S' \in \mathcal{A}_\in(\mathcal{S})} \mu(S')].$$

Therefore,

$$\begin{aligned} \sum_{S' \in \mathcal{A}^{\backslash}_{\infty}} \nu(S') &\geq \nu_0[\sum_{S' \in \mathcal{A}^{\backslash-\infty}_{\in}} \mu(S') - \sum_{S'' \in \mathcal{A}^{\backslash}_{\in}} \mu(S'')] \\ &= \nu_0[\mu(n-1) - \mu(n)]. \end{aligned}$$

Hence, since $\mu(n) \to 0$, a telescoping series argument gives

$$\sum_{n=0}^{\infty} \sum_{S' \in \mathcal{A}^{\backslash}_{\infty}} \nu(S') \geq \nu_0 \sum_{n=0}^{\infty} (\mu(n-1) - \mu(n)) = \nu_0 \mu(0).$$

Thus by (7.1),

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{S' \in \mathcal{A}^{\backslash}_{\infty}} \nu(S') &\geq \nu_0 \sum_{S \in \mathcal{S}_\prime} \mathrm{diam}(S)^{D-\epsilon} \\ &\geq \nu_0 \#(\mathcal{S}_\prime) \in^{-\backslash_\prime(\mathcal{D}-\epsilon)} \\ &= \nu_0 2^{n_0(D-\epsilon/2)} 2^{-n_0(D-\epsilon)} \\ &= \nu_0 2^{n_0 \epsilon/2} \end{aligned}$$

Thus, since the squares in $\cup_n \mathcal{A}^{\backslash}_{\infty}$ are all disjoint

$$\sum_{z \in G(E)} d(z)^{D-\epsilon} \geq \sum_n \sum_{S \in \mathcal{A}^{\backslash}_{\infty}} \nu(S) \geq \nu_0 2^{n_0 \epsilon/2}.$$

Taking $n_0 \to \infty$ proves $\delta(G) \geq D - \epsilon$. Taking $\epsilon \to 0$ shows $\delta(G) \geq D = \overline{\mathrm{Mdim}}(\Lambda)$, as desired.

8. Proof of Lemma 7.1

The idea of the proof is that the two alternatives for S in the lemma simply depend on whether S contains many bad horoballs or not. If it does not then the first alternative holds, but if it does then the second is true.

Let $\eta_0 > 0$ (we will show the lemma is correct if η_0 is small enough, depending only on G, ϵ and r). Let H_g be the union of the η_0-good horoballs and let H_b be the union of all the η_0-bad horoballs. Let $U = \Omega \setminus (H_b \cup H_g)$ (this is part of $\Omega(G)$ that projects to the "compact part" of $\Omega(G)/G$). The proof divides into three cases depending on the relative sizes of $S \cap H_b$, $S \cap H_g$ and $S \cap U$. Fix $\epsilon > 0$ and let $E \subset \Omega$ be a finite set so that points in $G(E)$ are at least hyperbolic distance ϵ apart, but so that every point of U is within 10ϵ of some point of $G(E)$.

Case A: First suppose that "a lot" of S corresponds to the compact part of $\Omega(G)/G$. More precisely, let $U_S = \frac{1}{3}S \cap U$ and assume

$$\text{area}(U_S) \geq \frac{1}{2000}\text{area}(S).$$

We will show that the first alternative holds for S. If ϵ in the definition of E is small enough (depending only of the uniformly perfect constant of Λ), then for each point $w \in U$ there is a point $z \in G(E)$ such that $|z - w| \leq d(z)$. Moreover, each $z \in G(E)$ is the center of a disk $D_z = D(z, \frac{\epsilon}{100} d(z))$. These disks are all disjoint, but they cover a fixed fraction of the area of U. This is because given any point of U we can move hyperbolic distance $\leq 10\epsilon$ and reach one of the ϵ-disks. Moreover, since $\Lambda \cap \frac{1}{3}S \neq \emptyset$, moving hyperbolic distance 10ϵ does not allow leaving S from starting inside $\frac{1}{3}S$ (if ϵ is small enough).

Hence,

$$\sum_{z \in G(E) \cap U_S} \text{area}(D_z) \geq A\, \text{area}(U_S) \geq \frac{A}{2000} \text{area}(S).$$

Thus,

$$\sum_{z \in G(E) \cap \frac{1}{3}S} d(z)^2 \geq C \text{diam}(S)^2.$$

Then since $d(z) \leq \text{diam}(S)$ for all $z \in G(E) \cap S$,

$$\sum_{z \in G(E) \cap \frac{1}{3}S} d(z)^{D-\epsilon} \geq C \text{diam}(S)^{D-\epsilon},$$

and we are done.

Case B: Now we assume "a lot" of S lies in H_g. Let $V_S = \frac{1}{3}S \cap H_g$, and suppose

$$\text{area}(V_S) \geq \frac{1}{2000} \text{area}(S),$$

(i.e., the part of S in good horoballs has large area). By hypothesis, the only η_0-good horoballs hitting $\frac{1}{3}S$ have diameter $\leq \frac{1}{3}S$, and so we can associate to each such horoball B a point $z \in G(E) \cap CS$ so that $d(z) \simeq \epsilon \text{diam}(B)$. Thus, as above,

$$\sum_{z \in G(E) \cap CS} d(z)^2 \geq C \sum_{\text{good horoballs in } S} \text{diam}(B)^2 \geq C \text{diam}(S)^{D-\epsilon} \simeq \text{area}(S).$$

Since $D - \epsilon < 2$, this and bounded overlaps of the squares CS imply

$$\sum_{z \in G(E) \cap CS} d(z)^{D-\epsilon} \geq C \text{diam}(S)^{D-\epsilon}.$$

Thus alternative one holds in this case also.

Case C: Now assume

$$\text{area}(U_S) + \text{area}(V_S) \leq \frac{1}{1000} \text{area}(S).$$

Since Λ has zero area, the "bad" part of S must have large area, i.e.,

$$\text{area}(\frac{1}{3}S \cap H_b) \geq \frac{99}{100}\text{area}(S).$$

Next we want to show that we may assume that $\frac{1}{3}S$ does not hit any bad horoball of comparable size.

Suppose that $\frac{1}{3}S$ hits a η_0-bad horoball with $\text{diam}(B) \geq \frac{1}{10}\text{diam}(S)$. If η_0 is small enough then part (3) of Lemma 4.1 implies that B can be the only η_0-bad horoball hitting $\frac{1}{3}S$ with diameter $\geq \frac{1}{50}\text{diam}(S)$. Thus we can find another square $S' \subset S$ with $\text{diam}(S') = \frac{1}{3}(S)$ which only hits small horoballs. More precisely, we can choose $S' \subset S$ so that

$$\text{area}(S' \cap H_b) \geq \frac{1}{100}\text{area}(S'),$$

and such that S' does not hit any η_0-bad horoballs with diameter $\geq \frac{1}{10}\text{diam}(S')$.

So by replacing S by S' if necessary, we may now assume that S is a square such that

$$\text{area}(S \cap W_g) \geq \frac{1}{100}\text{area}(S),$$

and such that S does not hit any η_0-bad horoballs with diameter $\geq \frac{1}{10}\text{diam}(S')$. Let $\{B_j\}$ be the collection of bad horoballs which hit $\frac{1}{3}S$. Since they all have diameter $\leq \text{diam}(S)/10$, they are contained in S and their areas sum to be at least $C\text{area}(\frac{1}{3}S)$.

The horoballs are all disjoint, so by the simple Vitali covering lemma (e.g., Lemma 7.4 of [13]) there is a subcollection of these balls $\{\hat{B}_j\}$ which have disjoint triples and whose areas sums to be more that $C'\text{area}(S)$.

By part (2) of Lemma 4.1, to each of these balls we can associate a square $S_j \subset 3\hat{B}_j$ such that

$$\operatorname{diam}(S_j) = \frac{1}{2}\operatorname{diam}(\hat{B}_j),$$

and so that $S_j \setminus \Lambda$ contains no balls of radius $\delta \operatorname{diam}(S_j)$. Here δ may be as small as we wish, assuming we take η_0 small enough. Choose η_0 so small that $\delta < r/1000$ (where r is the number given in the statement of Lemma 7.1). Inside S_j choose a collection $\{S_{jk}\}$ of r^{-2} squares of diameter $r/100$ which have disjoint triples. Then $\cup_j \cup_k S_{jk}$ obviously has all the desired properties. $\square$

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