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HARMONIC MEASURES SUPPORTED ON CURVES

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NOTATION

We follow standard notation whenever possible. $\mathbb{R}$, $\mathbb{C}$ and $\bar{\mathbb{C}}$ denote the real line, complex plane and Riemann sphere respectively. D is the unit disk, $\mathbb{T}$ its boundary and $D(x,r)$ a disk of radius r centered at x. Ω will always denote an open subset of $\bar{\mathbb{C}}$, Γ a Jordan curve on $\bar{\mathbb{C}}$ and K a compact subset of $\bar{\mathbb{C}}$. Conformal bijections are generally denoted by Φ. As usual, $\omega(z,E,\Omega)$, $E \subset \partial\Omega$, denotes the harmonic measure of E on Ω with respect to $z \in \Omega$. We will use f, g, h, u, v, F, G, H, U, V for various functions and if u is harmonic, u^* denotes its harmonic conjugate. Also, x, y, w, z, ζ will usually be points of $\bar{\mathbb{C}}$. ε, δ, ξ, ν, β will be small positive constants and C, A, B, M, N will be large positive constants. The letters i, j, k, n, m are used as indices, usually integer. $\|f\|_E$ means the sup of f on E. We sometimes drop the subscript if it is clear from context. χ_E is the characteristic function of E.

Results are numbered consecutively within sections. For example, Theorem 1.1.2 refers to the second result of the first section of Chapter One. When referring to a result in the same chapter we omit the first number. Formulas and expressions are handled similarly, but with parentheses; e.g., (2.3.5).

INTRODUCTION

Suppose Γ is a closed Jordan curve dividing the Riemann sphere, $\bar{\mathbb{C}}$, into two domains, Ω_1 and Ω_2. Let ω_1 and ω_2 denote harmonic measures on Γ corresponding to these domains. In this thesis we are mainly concerned with the relation between ω_1 and ω_2 in terms of the geometry of Γ, and with some consequences for the function theory on Ω_1 and Ω_2.

In the first chapter we present joint work with L. Carleson, J.B. Garnett and P.W. Jones ([15]), which characterizes the curves Γ for which ω_1 and ω_2 are mutually absolutely continuous. The condition is that Γ can be approximated by rectifiable curves in a certain way:

Theorem: $\omega_1 \ll \omega_2 \ll \omega_1$ iff for every $\varepsilon > 0$ there are rectifiable curves $R_i \subset \bar{\Omega}_i$, $i = 1,2$, such that $\omega_i(R_1 \cap R_2 \cap \Gamma) > 1 - \varepsilon$, $i = 1,2$.

We also obtain a characterization of the curves for which ω_1 and ω_2 are mutually singular. With the appropriate definition of a tangent point we get:

Theorem: $\omega_1 \perp \omega_2$ iff the set of tangent points of Γ has zero linear measure.

The first chapter also contains results for the special case when Γ is a quasicircle as well as some examples to illustrate our results.

The condition that $\omega_1 \perp \omega_2$ relates to certain function algebras on Γ. By A_Γ we denote the continuous functions on $\bar{\mathbb{C}}$

which are holomorphic off Γ. We say A_T is a Dirichlet algebra if every continuous, real-valued function on Γ can be uniformly approximated on Γ by the real parts of functions in A_T (this says A_T is a "large" subalgebra of $C(\Gamma)$). A. Browder and J. Wermer have shown that A_T is a Dirichlet algebra iff $\omega_1 \perp \omega_2$, so from Chapter One we obtain:

Theorem: A_T is a Dirichlet algebra iff the set of tangent points of Γ has zero linear measure.

In Chapter Two we give a new proof of the Browder-Wermer theorem, replacing the use of the Hahn-Banach theorem in their proof by an explicit construction involving solving a $\bar{\partial}$ equation with L^∞ estimates. One interesting consequence of the construction is:

Theorem: If ψ is a singular homeomorphism of $\mathbb{T} = \{|z| = 1\}$ to itself and $\varepsilon > 0$ is given, then there exists $\varphi \in C(\mathbb{T})$ such that

i) $0 \leq \varphi \leq 1$

ii) $|\{\varphi = 0\}| \geq 1 - \varepsilon$

iii) $|\{\varphi\circ\psi = 1\}| \geq 1 - \varepsilon$

iv) $\|\varphi\|_{BMO} < \varepsilon$

v) $\|\varphi\circ\psi\|_{BMO} < \varepsilon$.

We also obtain new proofs of other conditions equivalent to A_T being Dirichlet due to T. Gamelin and J.B. Garnett.

The definition of A_T still makes sense if Γ is replaced by any compact set K, as does the definition of a Dirichlet algebra. In Chapter Three we give a new proof of a theorem of A. Davie which characterizes the sets K for which A_K is a Dirichlet algebra, in terms of the harmonic measures on the components of $\bar{\mathbb{C}} \setminus K$. The results of Chapter One also give a geometric characterization of these sets.

Part of the construction in Chapter Two is to consider a homeomorphism ψ of $\mathbb{T}$ to itself and functions f on $\mathbb{T}$ such that f and $f \circ \psi$ both extend holomorphically to $D = \{|z| < 1\}$. It is known that if ψ is a small C^∞ perturbation of $z \to \bar{z}$ then f and $f \circ \psi$ both extend holomorphically iff f is constant. In particular, this holds if ψ is biLipschitz with constant near one. In Chapter Four we show this constant can not be taken too large.

Theorem: *There is a biLipschitz homeomorphism* ψ *and a non-constant* $f \in C(\mathbb{T})$ *such that* f *and* $f \circ \psi$ *extend holomorphically to* D.

This is an easy consequence of the following:

Theorem: *For each* $1 < d < 2$ *there is a quasicircle* Γ *and* $C > 0$ *such that* $\dim(\Gamma) = d$ *and for any interval* $I \subset \Gamma$,

$$\frac{1}{C} \le \frac{\omega_1(I)}{\omega_2(I)} \le C.$$

The construction of this curve is the main goal of Chapter Four.

CHAPTER I

HARMONIC MEASURES SUPPORTED ON CURVES

1. Statement of Results

Suppose Γ is a closed Jordan curve on the Riemann sphere, $\bar{\mathbb{C}}$, and let Ω_1 and Ω_2 denote the two components of $\Omega = \bar{\mathbb{C}}\backslash\Gamma$. Choose a point $z_1 \in \Omega_1$ and let $\omega_1 \equiv \omega(z_1,\cdot,\Omega_1)$ denote the harmonic measure on Γ with respect to z_1. Since Ω_1 is simply connected, the Riemann mapping theorem says there is a conformal map Φ_1 from the unit disk $D = \{|z| < 1\}$ to Ω_1 with $\Phi_1(0) = z_1$. Ω_1 is a Jordan domain, so by Carathéodory's theorem Φ_1 extends to a homeomorphism of $\mathbb{T} = \partial D$ to Γ. It is well known that ω_1 is the image of normalized Lebesgue measure on $\mathbb{T}$ under this correspondence. We choose $z_2 \in \Omega_2$ and define ω_2 and Φ_2 similarly.

In this chapter we will characterize those curves Γ for which ω_1 and ω_2 are mutually absolutely continuous, i.e., $\omega_1(E) = 0$ iff $\omega_2(E) = 0$ for all Borel subsets $E \subset \Gamma$. We write this as $\omega_1 \ll \omega_2 \ll \omega_1$. We will also characterize curves for which ω_1 and ω_2 are mutually singular. This means there is a Borel subset $E \subset \Gamma$ such that $\omega_1(E) = \omega_2(\Gamma\backslash E) = 0$ and is written as $\omega_1 \perp \omega_2$. Note that these properties are independent of our choices of z_i and Φ_i, $i = 1,2$, since different choices differ only by composition with a Möbius transformation from D to itself.

If Γ is rectifiable, a theorem of F. and M. Riesz states that ω_1 and ω_2 are both equivalent to the arc length measure on Γ and thus $\omega_1 << \omega_2 << \omega_1$ (see [43], Lemmas 10.7 and 10.12). On the other hand, Beurling and Ahlfors constructed a curve for which $\omega_1 \perp \omega_2$ (see [4]). In [23], Gamelin and Garnett used their work on function algebras together with a result of Browder and Wermer (Theorem 2.1.1) to show $\omega_1 \perp \omega_2$ for a whole class of self-similar curves such as the von Koch snowflake. Thus we might expect the measures to be mutually absolutely continuous if Γ is "almost" rectifiable, and to be singular if Γ oscillates too much. This motivates the following results (relevant definitions are given in the next section):

Theorem 1.1: *With notation as above, the following are equivalent:*

i) $\omega_1 << \omega_2 << \omega_1$

ii) *for every* $\varepsilon > 0$ *there are closed rectifiable curves* $R_i \subset \bar{\Omega}_i$, $i = 1,2$, *such that*

$$\omega_i(R_1 \cap R_2 \cap \Gamma) > 1 - \varepsilon$$

for $i = 1,2$

iii) *the points of* Γ *at which a tangent exists have full measure with respect to both* ω_1 *and* ω_2.

Theorem 1.2: *With notation as above,* $\omega_1 \perp \omega_2$ *iff the set of tangent points of* Γ *has zero linear measure.*

As noted above, the first theorem says the measures are mutually continuous if Γ can be approximated by rectifiable curves in a certain way. The second result says the measures are mutually singular if Γ has large oscillations on arbitrarily small scales, as in the curves considered in [23]. The second result has the advantage of dealing with a purely metric property of Γ, whereas we may need _a priori_ knowledge of harmonic measure on Γ to apply Theorem 1.1. The new directions of Theorem 1.1 are (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). The implication (ii) $\Rightarrow$ (i) is obvious by the Maximum Principle and the F. and M. Riesz theorem and we will see later that (iii) $\Rightarrow$ (i) is an easy consequence of the definition of tangent points, as is the forward implication of Theorem 1.2.

Since all the problems encountered in this chapter are local in nature, in all results we may replace "closed curve" by "arc" and the conclusions remain unchanged. (Instead of having two measures, ω_1 and ω_2, we would only have one harmonic measure. This measure, however, splits naturally into $\omega_1 + \omega_2$ where the two measures correspond to the two sides of the arc.)

2. Some Definitions and Results

Before proving Theorems 1.1 and 1.2 we need to review some definitions and results. We start with the definition of Hausdorff measure and dimension.

Let h be a continuous, increasing function from $[0,\infty)$ to itself such that $h(0) = 0$. Such an h will be called a measure function. For a planar set E and $\delta > 0$ set

$$\Lambda_h^\delta(E) = \inf\{\Sigma h(r_j)\}$$

where the infimum is taken over all coverings of E by disks $\{D_j\}$ satisfying radius$(D_j) \equiv r_j < \delta$. Then

$$\Lambda_h(E) = \lim_{\delta\to 0} \Lambda_h^\delta(E)$$

is called the <u>Hausdorff measure</u> of E with respect to h. If $h(t) = t^\alpha$ we shorten the notation to Λ_α and for $h(t) = t$ we call $\Lambda_h = \Lambda_1$ <u>linear measure</u>. Thus a set E has zero linear measure if we can cover it by disks whose radii sum up to be as small as we wish. We define the <u>Hausdorff dimension</u> of a set E as

$$\dim(E) = \inf\{\alpha : \Lambda_\alpha(E) = 0\}.$$

For further details see [14].

Next we wish to review Plessner's theorem, which basically says that the boundary behavior of a holomorphic function on D is either very good or very bad. More precisely, let $St(\xi)$ denote the Stolz domain in D with vertex ξ on $\mathbb{T}$, i.e., the interior of the convex hull of $\{|z| < 1/2\} \cup \{\xi\}$. Plessner's theorem says that if F is holomorphic in D then for almost every $\xi \in \mathbb{T}$ one of the following two options hold:

(2.1) $$\lim_{z\to\xi,\, z\in St(\xi)} F(z) \equiv F(\xi) \quad \text{exists}$$
and is not 0 or ∞

or

(2.2) $$\overline{F(St(\xi) \cap \{|z| < r\})} = \mathbb{C} \quad \text{for all} \quad 0 < r < 1.$$

The proof is quite simple (e.g. [43], Theorem 10.13); if (2.1) and (2.2) fail on a set E of positive measure we can compose F with a Möbius transformation so it becomes non-tangentially bounded on a positive measure subset of E. The local version of Fatou's theorem on non-tangential limits and Privalov's theorem imply (2.1) holds on this subset of E, a contradiction.

It will also be convenient to consider geometric conditions corresponding to (2.1) and (2.2). Suppose Ω is a simply connected domain. Fix $x \in \partial\Omega$ and define a continuous branch of $\arg(z-x)$ on Ω. We say x is a <u>twist point</u> of Ω if both

$$\liminf_{z\to x, z\in\Omega} \arg(z-x) = -\infty$$

and

$$\limsup_{z\to x, z\in\Omega} \arg(z-x) = +\infty .$$

On the other hand, we say Ω has an <u>inner tangent</u> at x if there is a unique $\theta_0 \in [0,2\pi)$ such that for every $0 < \varepsilon < \pi/2$ there is a $\delta > 0$ such that (see Figure 1):

$$\{x+re^{i\theta} : 0 < r < \delta, |\theta-\theta_0| < \pi/2-\varepsilon\} \subset \Omega.$$

McMillan's twist point theorem states that almost every (with respect to harmonic measure) boundary point of Ω is of one of these two types and that they correspond to the two conditions in Plessner's theorem. For simplicity, we state his theorem only for Jordan domains.

<u>Theorem</u> 2.1 (McMillan, [40]: <u>Suppose Ω is bounded by a closed Jordan curve Γ, $\Phi : D \to \Omega$ is conformal, and $F = \Phi'$.</u>

i) <u>For almost every $\xi \in \Gamma$, (2.1) holds for F iff Ω has an inner tangent at $\Phi(\xi)$ and (2.2) holds iff $\Phi(\xi)$ is a twist point of Ω.</u>

ii) <u>If $E \subset \Gamma$ and Ω has an inner tangent at every point of E then $\omega(E) = 0$ iff $\Lambda_1(E) = 0$ (ω = harmonic measure for Ω).</u>

Actually, to prove Theorems 1.1 and 1.2 all we need is that for almost every ξ (2.1) holds iff Ω has an inner tangent at $\Phi(\xi)$.

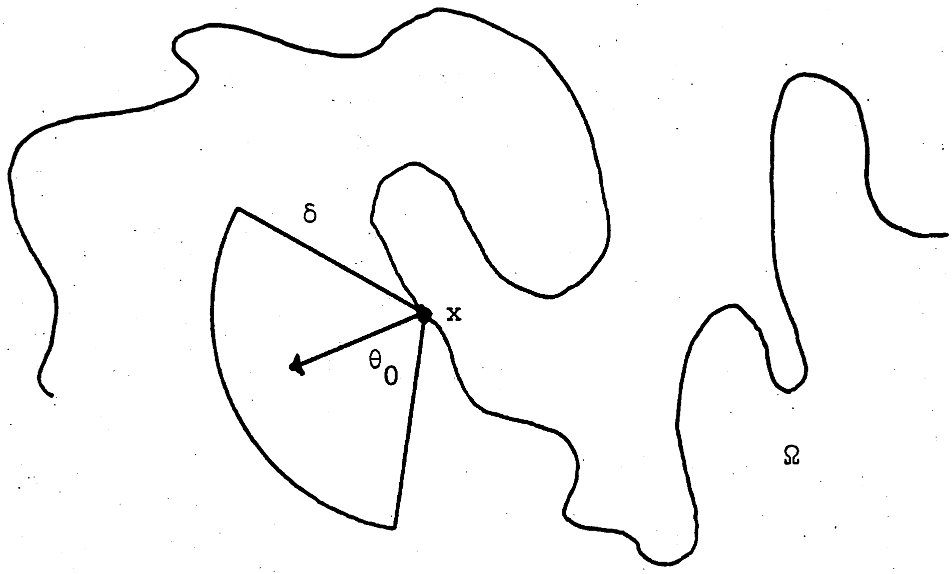

Figure 1: An inner tangent

We will use the result on twist points only for convenience.

We now return to the situation described in Section 1; Γ a closed Jordan curve bounding two domains Ω_1 and Ω_2. For $x \in \Gamma$ it is easy to see that x is a twist point for Ω_1 iff it is a twist point for Ω_2. We let Tw denote this common set of twist points. Also let I_1 and I_2 denote the points of Γ where Ω_1 and Ω_2 have inner tangents. These sets need not be the same, but $T \equiv I_1 \cap I_2$ is the set where Γ has <u>tangents</u>, i.e., the $x \in \Gamma$ such that there is a $\theta_0 \in [0,2\pi]$ so that for any $\varepsilon > 0$ there exists $\delta > 0$ with

$$\{x+re^{i\theta} : 0 < |r| < \delta, |\theta-\theta_0| < \pi/2-\varepsilon\} \cap \Gamma = \emptyset$$

(see Figure 2). Equivalently, x is a tangent point of Γ iff

$$\lim_{z\to x, z\in\Gamma} \arg(z-x)^2$$

exists. By McMillan's theorem we can write

$$\Gamma = \mathrm{Tw} \cup \mathrm{Tn} \cup (I_1\backslash I_2) \cup (I_2\backslash I_1) \cup N$$

where $\omega_1(N) = \omega_2(N) = 0$. By part (ii) of McMillan's theorem ω_1 and ω_2 must be mutually absolutely continuous on Tn (this is proved directly from the definitions in Section 9), and this proves (iii) $\Rightarrow$ (i) of Theorem 1.1. Since $\omega_1(I_2\backslash I_1) = \omega_2(I_1\backslash I_2) = 0$, ω_1 and ω_2 are mutually singular on $I_1\backslash I_2$ and $I_2\backslash I_1$. Thus it only remains to consider what happens on the twist points. We will prove:

<u>Theorem</u> 2.2: <u>For any Jordan curve</u> Γ, ω_1 <u>and</u> ω_2 <u>are mutually singular when restricted to</u> Tw, <u>the twist points of</u> Γ.

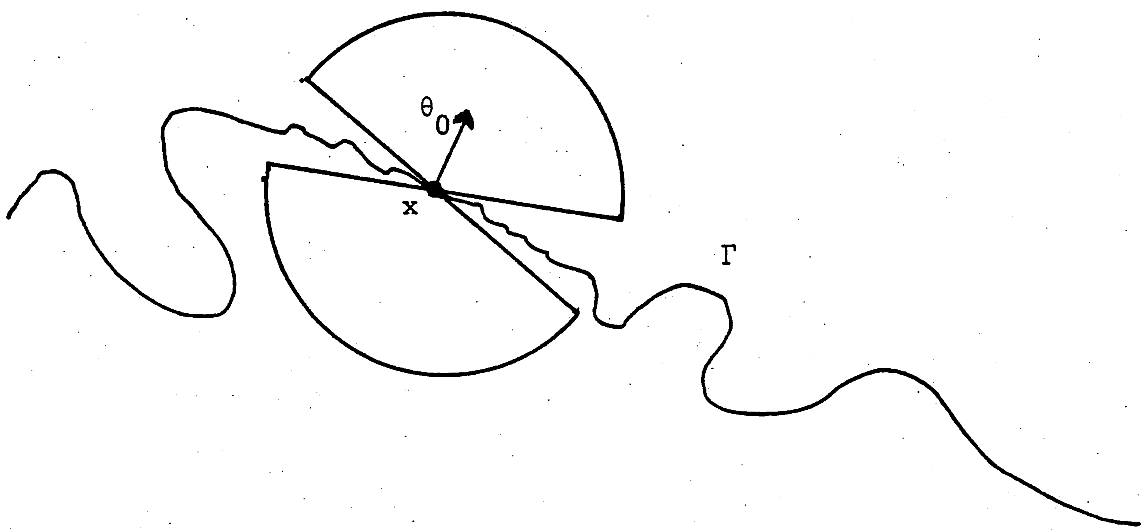

Figure 2: A tangent

3. Proving Theorems 1.1 and 1.2

In this section we deduce Theorems 1.1 and 1.2 from Theorem 2.2. We will then proceed to the proof of Theorem 2.2.

Clearly Theorem 2.2 and our earlier remarks give (i) ⇒ (iii) in Theorem 1.1. The same remarks also show $\omega_1 \perp \omega_2$ unless $\Lambda_1(Tn) > 0$, so we get Theorem 1.2. Thus it only remains to prove (i) ⇒ (ii) in Theorem 1.1.

So suppose (i) holds. By McMillan's theorem and Theorem 2.2, $F = (\Phi_1)'$ has a finite, non-zero non-tangential limit almost everywhere on $\mathbf{T}$. Hence F is bounded in $St(\xi)$ for almost every $\xi \in \mathbf{T}$. It follows that for any $\zeta > 0$ there is a $M > 0$ and a compact $E \subset \mathbf{T}$ such that $|F|$ is bounded by M on

$$\tilde{D} \equiv \bigcup_{\xi \in E} St(\xi)$$

and $|E| > 1 - \eta$ ($|\cdot|$ denotes normalized Lebesgue measure on $\mathbf{T}$). See Figure 3. Note that $\partial\tilde{D}$ is a rectifiable curve in D, so $R_1 \equiv \Phi_1(\partial\tilde{D})$ is a rectifiable curve in $\bar{\Omega}_1$ which satisfies

$$\omega_1(R_1 \cap \Gamma) = |E| > 1 - \eta.$$

Similarly, we construct $R_2 \subset \bar{\Omega}_2$. Since ω_1 and ω_2 are mutually absolutely continuous we have

$$\omega_i(R_1 \cap R_2 \cap \Gamma) > 1 - \varepsilon(\eta) \ , \quad i = 1,2$$

where $\varepsilon(\eta) \to 0$ as $\eta \to 0$. This completes the proof of Theorem 1.1.

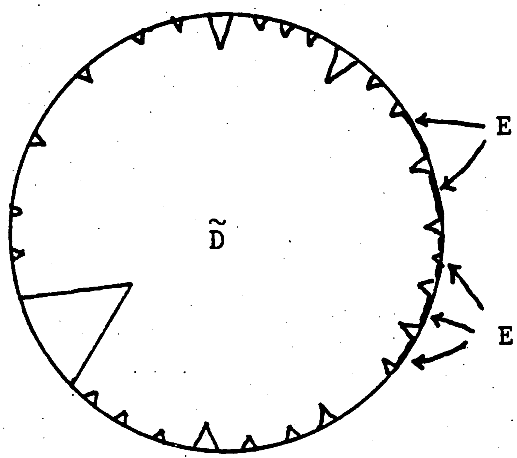

Figure 3: The domain $\tilde{D}$

4. An Estimate on Harmonic Measure

To prove Theorem 2.2 we need two simple lemmas. The first is an estimate on harmonic measure which says that a small piece of Γ cannot simultaneously have large harmonic measure with respect to both sides.

Lemma 4.1: Suppose Γ, Ω_i, z_i are as above with the additional normalization that $\mathrm{dist}(z_i,\Gamma) \geq 1$, $i = 1,2$. Then there is a constant $C > 0$ such that for any $x \in \Gamma$ and $0 < r < 1$

$$\omega_1(\Gamma \cap D(x,r)) \cdot \omega_2(\Gamma \cap D(x,r)) \leq C \cdot r^2 .$$

The lemma follows easily from a variation of the Ahlfors distortion theorem (e.g. see [21], Proposition 7.2). For $r < t < 1$ let $\theta_i(t)$ denote the angle measure of $\partial D(x,t) \cap \Omega_i$, $i = 1,2$. Then we have

$$(4.1) \qquad \omega_i(\Gamma \cap D(x,r)) \leq C \cdot \exp\{-\pi \int_r^1 (t\theta_i(t))^{-1} dt\}$$

(the Ahlfors distortion theorem is usually stated for subintervals of Γ, but it is known even if $\Gamma \cap D(x,r)$ is not connected.)

We have $0 \leq \theta_i(t) \leq 2\pi$, $i = 1,2$ and $\theta_1(t) + \theta_2(t) \leq 2\pi$, so we easily see

$$(\theta_1(t))^{-1} + (\theta_2(t))^{-1} \geq \frac{2}{\pi} .$$

So if we multiply the two versions of (4.1) together, we get:

$$\begin{aligned} (4.2) \qquad & \omega_1(\Gamma \cap D(x,r)) \cdot \omega_2(\Gamma \cap D(x,r)) \\ & \leq C \cdot \exp\{-\pi \int_r^1 (\theta_1(t)^{-1} + \theta_2(t)^{-1}) \frac{dt}{t}\} \\ & \leq C \cdot \exp\{-2 \int_r^1 \frac{dt}{t}\} \\ & \leq C \cdot r^2 \end{aligned}$$

as required.

5. Makarov's Theorem

In Section 2 we saw that harmonic measure looks roughly like linear measure when restricted to tangent points. In this section we will prove a lemma which says this fails on the twist points.

The main idea is to use Plessner's theorem to study harmonic measure. This was used by N.G. Makarov in [39] to prove the Øksendal conjecture for simply connected domains, i.e., if Ω is simply connected then there is an $E \subset \partial\Omega$ with full harmonic measure and $\dim(E) \leq 1$. In fact, Makarov proves much more, essentially characterizing those measure functions h for which $\omega << \Lambda_h$. The Øksendal conjecture has recently been proven for all planar domains by P.W. Jones and T. Wolff (see [38]).

The following lemma contains the version of Makarov's result we shall need. It is essentially contained in Pommerenke's paper [44], but we shall include a proof here for completeness.

<u>Lemma 5.1: If Ω is bounded by a Jordan curve Γ, there is a subset T of Tw, the twist points of Γ, such that $\omega(T) = \omega(Tw)$ and for every $k = 1,2,\ldots$ there is a covering of T by disks $\{D_j^k\}$ of radius $\{r_j^k\}$ such that</u>

i) $\sum_j r_j^k \leq 2^{-k}$

ii) $\omega(D_j^k \cap \Gamma) \geq 2^k \cdot r_j^k$.

The first conclusion implies $\Lambda_1(T) = 0$, so that ω is not mutually continuous with linear measure on Tw, as noted above.

To prove the lemma let $\Phi : D \to \Omega$ be the Riemann map and $F = (\Phi)'$. Let $A \subset \mathbf{T}$ be the set where (2.2) holds for F. In particular we have

$$(5.1) \qquad \liminf_{z\to\xi,\, z\in St(\xi)} |F(z)| = 0$$

for all $\xi \in A$. Now fix k and suppose $z \in D$ satisfies

$$|F(z)| \le 2^{-k}\cdot\varepsilon$$

where ε is a small constant we shall choose later. Now let $I = I(z)$ be the interval on $\mathbb{T}$ of length $1 - |z|$ and center $z/|z|$. The collection of all such intervals is a Vitali covering of A (by (5.1)), so we can choose a disjoint subcollection $\{I_j^k\}$ with

$$|A \setminus \bigcup_j I_j^k| = 0 .$$

If $\{z_j^k\}$ denote the corresponding z's, we define

$$w_j^k = \Phi(z_j^k)$$

$$r_j^k = \mathrm{dist}(w_j^k, \Gamma)$$

$$D_j^k = D(w_j^k, 2r_j^k)$$

$$T_n = \bigcup_{k\ge n} (\bigcup_j D_j^k) \cap \Gamma$$

$$T = \bigcap_n T_n$$

It is a well known result that if Ω is simply connected and $w \in \Omega$ with $r = \mathrm{dist}(w, \partial\Omega)$ then

$$\omega(w, \partial\Omega \cap D(w,2r), \Omega) > \eta$$

for some $\eta > 0$ independent of w and Ω. This is just a weak version of Beurling's solution of then Carleman-Milloux problem which says

$$\omega(w, \partial\Omega \cap D(w, \lambda\cdot r), \Omega) \ge \frac{2}{\pi}\arcsin\left(\frac{\lambda-1}{\lambda+1}\right)$$

for $\lambda > 1$ (see [41], Section IV.5).

To apply this to the current situation we set $A_n = \Phi^{-1}(T_n)$, take $k \geq n$ and observe:

$$\begin{aligned}\omega(z_j^k, A_n, D) &= \omega(w_j^k, T_n, \Omega) \\ &\geq \omega(w_j^k, D_j^k, \Omega) \\ &\geq \eta \ .\end{aligned}$$

By our choice of $\{I_j^k\}$ we know that for almost every $\xi \in A$, $St(\xi)$ contains infinitely many of the points $\{z_j^k\}$, $k \geq n$. Combining this with Fatou's theorem on non-tangential limits (applied to the harmonic function $\omega(z, A_n, D)$) we get:

$$\lim_{\substack{z \to \xi \\ z \in St(\xi)}} \omega(z, A_n, D) = 1$$

for almost every $\xi \in A$. Thus

$$\omega(T_n) = |A_n| \geq |A| \ .$$

Hence

$$\omega(T) = \omega(\bigcap_n T_n) = |A| = \omega(Tw)$$

as required. Furthermore, by Koebe's 1/4-theorem ([43], Corollary 1.4)

$$\begin{aligned}\sum_j r_j^k &\leq 4 \cdot \Sigma |\Phi'(z_j^k)| \cdot (1 - |z_j^k|) \\ &\leq 4 \cdot 2^{-k} \cdot \varepsilon \cdot \Sigma\, |I_j^k| \\ &\leq 4 \cdot 2^{-k} \cdot \varepsilon \cdot 2\pi \\ &\leq 2^{-k}\end{aligned}$$

if $\varepsilon \leq \frac{1}{8\pi}$. Also, since

$$\omega(z_j^k, \Phi^{-1}(D_j^k \cap \Gamma), D) > \eta$$

we have

$$\omega(0, \Phi^{-1}(D_j^k \cap \Gamma), D) > C \cdot \eta \cdot (1 - |z_j^k|)$$

by Harnack's inequality, so again by Koebe's theorem

$$\begin{aligned}\omega(D_j^k \cap \Gamma) &> C\cdot(1 - |z_j^k|) \\ &\geq C\cdot r_j^k\cdot(\Phi'(z_j^k))^{-1} \\ &\geq 2^k\cdot r_j^k\end{aligned}$$

if ε is small enough.

6. Proof of Theorem 2.2

It is now easy to prove Theorem 2.2. Apply Lemma 5.1 to Ω_1 and ω_1 to produce a set T_1 satisfying $\omega_1(T_1) = \omega_1(Tw)$ and coverings $\{D_j^k\}$ of T_1 satisfying (i) and (ii) of Lemma 5.1. Then by Lemma 4.1,

$$\begin{aligned}\omega_2(\Gamma \cap D_j^k) &\leq C\cdot(r_j^k)^2\cdot(2^k r_j^k)^{-1} \\ &\leq C\cdot 2^{-k}\cdot r_j^k .\end{aligned}$$

Hence

$$\begin{aligned}\omega_2(T_1) &\leq \inf_k\{\omega_2(\bigcup_j D_j^k \cap \Gamma) \\ &\leq \inf_k\{C\cdot 2^{-k}\cdot\sum_j r_j^k\} \\ &\leq \inf_k\{C\cdot 2^{-2k}\} \\ &= 0 .\end{aligned}$$

Thus $\omega_1(Tw\setminus T_1) = \omega_2(T_1) = 0$ and so ω_1 and ω_2 are mutually singular when restricted to Tw. This proves Theorem 2.2.

7. Quasicircles

Next we would like to point out what happens if we assume Γ has a certain "smoothness". In particular, recall that Γ is called a quasicircle if $\Gamma = g(\mathbb{R})$ where g is a quasiconformal mapping of $\mathbb{C}$ to $\mathbb{C}$. More geometrically, Γ is a quasicircle iff Γ satisfies the "three point condition"

$$|z_3 - z_1| < C \cdot |z_2 - z_1|$$

for some $C = C(\Gamma) > 0$ and any three points on Γ with z_3 on the arc of smaller diameter between z_1 and z_2 (see [2], Theorem IV.5). We shall need only the following property of quasicircles:

Lemma 7.1: *If Ω is bounded by a quasicircle Γ and R is a rectifiable curve, then there is a closed, rectifiable Jordan curve $\tilde{R} \subset \bar{\Omega}$ such that $\tilde{R} \cap \Gamma = R \cap \Gamma$.*

To prove the lemma, let g be the quasiconformal map defining Γ, which we may assume is conformal from the upper half-plane, $H = \{\mathrm{Im}(z) > 0\}$, to Ω. Let $E = g^{-1}(\Gamma \cap R) \subset \mathbb{R}$, which we may assume is compact. Define a closed curve $\hat{R}$ in $\bar{H}$ as follows: let $\{I_n\}_{n\geq 1}$ be the bounded intervals in $\mathbb{R} \backslash E$, and I_0 the hull of E. Then $\hat{R}$ consists of E together with the "tents" (see Figure 4)

$$T_n = \{(x+i\mathrm{dist}(x,E)) : x \in I_n\} \qquad n \geq 1$$

$$T_0 = \{(x+i2\cdot\mathrm{dist}(x,I_0^c) : x \in I_0\}.$$

Now let $\tilde{R} = g(\hat{R})$. Clearly $\tilde{R} \cap \Gamma = R \cap \Gamma = g(E)$, and $\Lambda_1(\tilde{R} \cap \Gamma) < \infty$. Furthermore, for each n, $g(T_n)$ has length comparable to $|g(a)-g(b)|$ where $I_n = [a,b]$ (e.g. see Section One of [34]). Since R was rectifiable these lengths have finite sum and so $\tilde{R}$ is rectifiable, as required.

Thus the existence of R_1 in Theorem 1.1 automatically implies the existence of R_2. (Indeed, this follows from Ahlfors' paper [1], because the quasiconformal reflection defined there is Lipschitz.) By

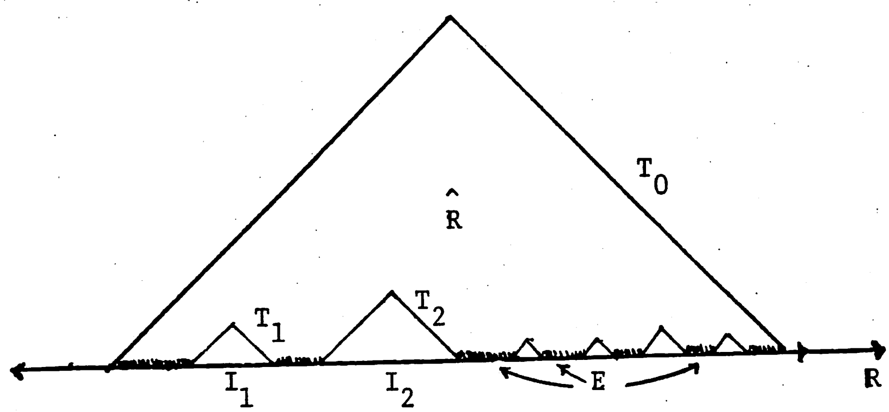

Figure 4: The curve $\hat{R}$

examining the proof in Section 3 we see that (modulo harmonic null sets) the existence of an inner tangent at $x \in \Gamma$ implies the existence of an ordinary tangent. We thus obtain:

Corollary 7.2: *If* Γ *is a quasicircle then the following are equivalent*:

i) $\omega_1 << \omega_2 << \omega_1$

ii) *for all* $\varepsilon > 0$ *there is a rectifiable curve* R *such that*

$$\omega_i(R \cap \Gamma) > 1 - \varepsilon$$

for $i = 1,2.$

iii) $\omega_1(Tw) = \omega_2(Tw) = 0$

iv) $\omega_1 << \Lambda_1$ *and* $\omega_2 << \Lambda_1$.

Corollary 7.3: *If* Γ *is a quasicircle then* $\omega_1 \perp \omega_2$ *iff either* $\omega_1(Tw) = 1$ *or* $\omega_2(Tw) = 1$ *(in which case both are equal to one)*.

8. Some Examples

It is interesting to note that the results of the last section fail for arbitrary Jordan curves. For example, in this section we will construct a curve such that $\omega_1(Tw) = 0$ and $\omega_2(Tw) = 1,$ which is impossible for quasicircles by Corollary 7.3. A slight modification of the construction gives a curve satisfying (ii), (iii) and (iv) of Corollary 7.2 but with $\omega_1 \perp \omega_2$.

The basic building block of the construction is a Cantor set $E \subset [0,1]$ of positive length obtained as follows. Remove the open interval of length 1/8 and center {1/2} from [0,1]. Then remove the intervals of length 1/32 from the centers of the two remaining intervals. At the n^{th} stage we remove 2^{n-1} open intervals, $\{I_j^n\}$, of length $\alpha_n = 2^{-2n-1}$. After removing the n^{th} generation intervals we are left with 2^n closed intervals, $\{J_j^n\}$, of length $\beta_n = (3\cdot 2^n+1)\cdot 2^{-2n-2}$, from which we remove the $(n+1)^{st}$ generation intervals. Clearly

$$E = \bigcap_n \left(\bigcup_j J_j^n\right)$$

is a Cantor set with $|E| = 3/4$.

We now construct a Jordan curve Γ_0 by adjoining "towers" to E. For each interval $I_j^n = (a,b)$ in the complement of E we define a "tower" T_j^n consisting of the three line segments

$$[a, a + i\gamma_n]$$

$$[a + i\gamma_n, b + i\gamma_n]$$

$$[b + i\gamma_n, b]$$

where $\gamma_n = 2^{-n-2}$. Then

$$\Gamma_0 = \bigcup_n \left(\bigcup_j T_j^n\right) \cup E$$

is a Jordan arc connecting {0} to {1} and such that $\Gamma_0 \cap \mathbb{R} = E$ (see Figure 5).

Note that $1/4 \le \gamma_n/\beta_n \le 1/3$. The right hand inequality implies that all the towers are contained in the isosceles triangle with base [0,1] and height 1/6. Similarly, the isosceles triangle with base J_j^n and height $1/6\ \beta_n = 1/6|J_j^n|$ contains all the towers with base in J_j^n (see dotted lines in Figure 5). We will use this later when we iterate the construction to show that the curve does not intersect itself.

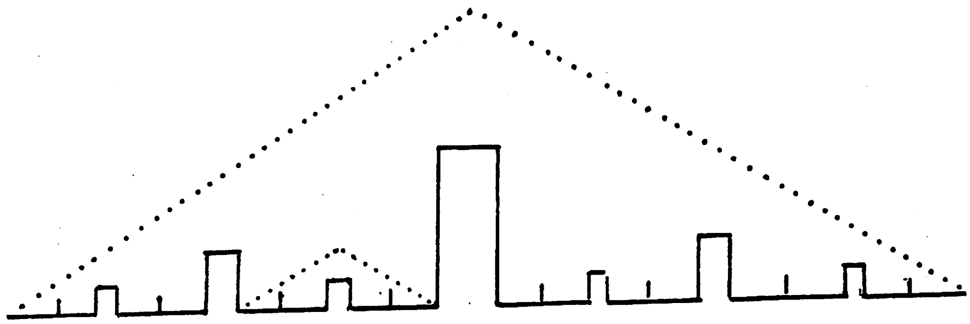

Figure 5: First generation towers

First we wish to make Γ_0 into a closed curve. For example, we can place a copy Γ_0 on each side of a unit square, Q, with the towers pointing out and call this new curve Γ_1. Let Ω_1^1 and Ω_1^2 denote the inside and outside of Γ_1 respectively and let $E_1 = \partial Q \cap \Gamma_1$ be the four copies of E.

By the construction of E and the maximum principle one can show

$$\omega(z, E_1, \Omega_1^1) > \varepsilon$$

for some $\varepsilon > 0$ and all $z \in Q$. We also have

$$\omega(z, E_1 \Omega_1^2) = 0$$

for all $z \in \Omega_1^2$. One way to see this is to observe that no point of E is either a twist point or inner tangent point of Ω_1^2 so by McMillan's theorem, E is a harmonic null set for Ω_1^2. However, this is a silly proof since it uses a non-trivial theorem to prove an easy fact. A better proof is to observe that every $z \in \Omega_1^2$ near Γ_1 lies in a rectangle R as pictured in Figure 6.

Since

$$3/8 \le \frac{\gamma_{n-2} - \gamma_n}{\beta_{n-1}} \le 1/2$$

any such rectangle has bounded eccentricity. So if $z \in R$, the harmonic measure of the adjacent tents with respect to z is bounded below uniformly. Thus

$$\omega(z, E_1, \Omega_1^2) < 1 - \varepsilon$$

for all $z \in \Omega_1^2$. This is well known to imply

$$\omega(z, E_1, \Omega_1^2) = 0$$

for all z as required.

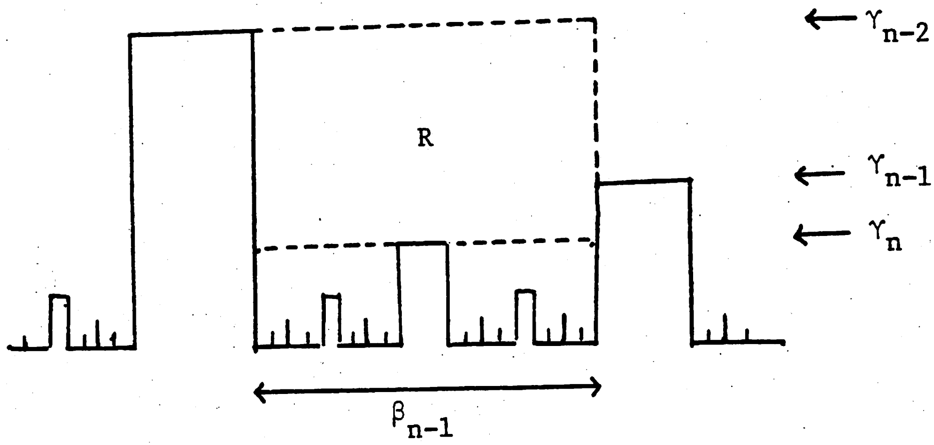

Figure 6: The rectangle R

We have just used the simple fact that if Ω is a domain and $E \subset \partial\Omega$ satisfies

$$\omega(z,E,\Omega) < 1 - \varepsilon$$

for some $\varepsilon > 0$ and all $z \in \Omega$, then $\omega(z,E,\Omega) = 0$ for all $z \in \Omega$. I do not know a reference for this so I will sketch a proof. Let S_E be the Peron family associated to χ_E, i.e., the family of bounded subharmonic functions v on Ω such that

$$\limsup_{z \to \zeta} v(z) \le \chi_E(\zeta)$$

for all $\zeta \in \partial\Omega$. Then one can show,

$$\omega(z,E,\Omega) = \sup_{S_E} \{v(z)\}.$$

So if $v \in S_E$, $v \le 1 - \varepsilon$ on Ω, and so we easily check $\frac{v}{1-\varepsilon} \in S_E$. Thus "supping" over S_E gives

$$\frac{1}{1-\varepsilon}\omega(z,E,\Omega) = \sup_{S_E} \frac{v}{1-\varepsilon} \le \omega(z,E,\Omega)$$

which is possible only if $\omega(z,E,\Omega) \equiv 0$, as required.

We now return to the construction. We obtain a curve Γ_2 from Γ_1 by placing small copies of Γ_0 on the sides of the towers of Γ_1. To be more precise, consider a single tower T_j^n of Γ_1 and let Δ denote an isosceles triangle with base α_n and height $1/6 \cdot \alpha_n$. Since T_j^n has width α_n and height $\gamma_n = 2^{n-1} \cdot \alpha_n$ we can place $2^n + 1$ copies of Δ along the top and sides of T_j^n with the triangle pointing "outward" (into Ω_1^2) as in Figure 7. For each such triangle we replace its base by a scaled copy of Γ_0. Doing this for every tower in Γ_1 gives Γ_2.

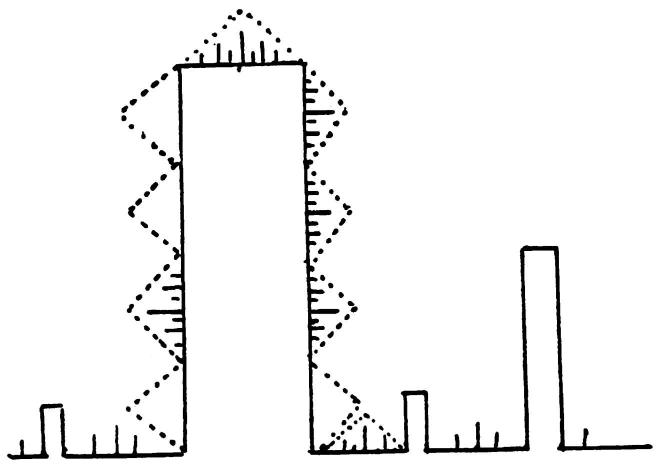

Figure 7: The "outward" towers

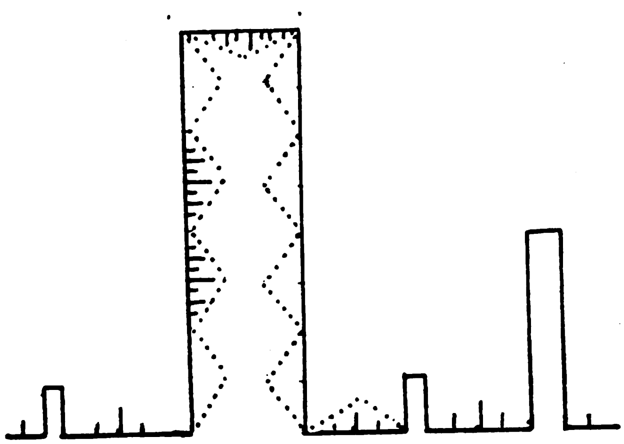

Figure 8: The "inward" towers

It is now clear how to proceed. We apply the construction above to each tower in Γ_2 to obtain Γ_3 and in general apply it to Γ_{n-1} to get Γ_n. The sequence $\{\Gamma_n\}$ converges to a Jordan curve which we will show has the desired properties.

We let Ω_1 and Ω_2 denote the "inside" and "outside" of Γ respectively. It is easy to show Ω_2 has no inner tangents, so $\omega_2(Tw) = 1$, as desired. On the other hand if we set

$$F_n = \Gamma_n \cap \Gamma_{n+1}$$

$$F = \bigcup_n F_n$$

then no point of F is a twist point of Ω_1. However, for z belonging to a tower of Γ_n,

$$\omega(z,F,\Omega_1) \geq \omega(z,F_n,\Omega_1)$$
$$\geq \varepsilon$$

independent of z and n. Thus

$$\omega(z,F,\Omega_1) = 1$$

for all $z \in \Omega_1$, and so $\omega_1(Tw) = 0$. Thus Γ is the desired curve.

If we alter the construction by replacing Figure 7 by Figure 8, i.e., at each stage the new towers point into the old tower instead of out of it, then we obtain a curve Γ for which $\omega_1(Tw) = \omega_2(Tw) = 0$ but $\omega_1 \perp \omega_2$. Rather than give a detailed proof of this, let us consider another example with this property.

Perhaps surprisingly, we can take this example to be the graph of a continuous real valued function on $\mathbb{R}$. For example, consider Γ the graph of the Weierstrass nowhere differentiable function

$$f(x) = \sum_{n=1}^{\infty} b^{-n\alpha} \cos b^n x$$

with $0 < \alpha < 1$ and b an integer. Since Γ is a graph it has no twist points and if b is large enough (depending on α) one can show Γ has no tangents. Thus $\omega_1 \perp \omega_2$ by Theorem 1.2. More simply, one can prove directly that $\omega_1 \perp \omega_2$ if $0 < \alpha < 1$ and $b = b(\alpha)$ is large enough. However, if $\alpha = 1$ then f is in the Zygmund class Λ_* (e.g. [50], Theorem I.4.9), so its graph is a quasicircle ([35], Section 2). Since Γ has no twist points, Corollary 7.2 implies $\omega_1 \ll \omega_2 \ll \omega_1$ even though f is nowhere differentiable.

9. The Double Cone Condition

We will end this chapter by stating a conjecture, but first we need to make a geometrical observation which we have already used implicitly.

If $K \subset \mathbb{C}$ is compact and $x \in K$ we say x satisfies a <u>double cone condition</u> with respect to K if there exists $\theta_0 \in [0,2\pi)$, $0 < \varepsilon < \pi/2$ and $\delta > 0$ such that

$$\{x+re^{i\theta} : 0 < |r| < \delta, |\theta-\theta_0| < \pi/2-\varepsilon\} \cap K = \emptyset$$

i.e., there are two symmetric cones with vertex x which do not hit K. Suppose this condition holds for a set $E \subset K$ of positive linear measure. Then it must hold for some fixed triple $(\theta_0,\varepsilon,\delta)$ on a set of positive measure, F. Rotate the set so $\theta_0 = \pi/2$ and consider the union of cones with vertex in F, as in Figure 9. By considering connected components of this union we see that we have "trapped" part of K

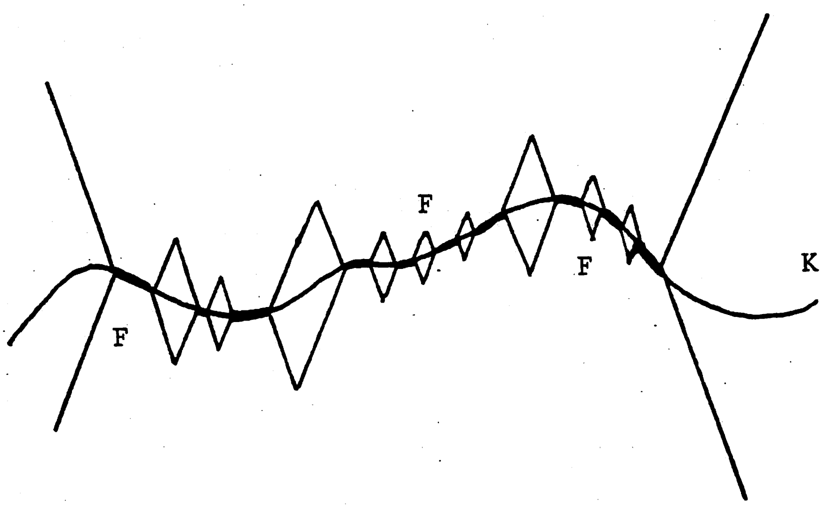

Figure 9: The double cone condition

between the graphs of two Lipschitz functions which agree on a set of positive length. In particular, K has tangents almost everywhere (Λ_1) on F since these two graphs do. Thus, up to sets of zero linear measure, the double cone condition holds at x iff K has a tangent at x.

Thus almost every point of F satisfies the double cone condition with ε as near 0 as we wish (if δ is small enough and for different θ_0's). So choose ε small and fix δ and θ_0 so that the condition is satisfied for $(\theta_0,\varepsilon,\delta)$ on a new set F of positive length. Again rotate and take unions of cones and we get Figure 10: F is trapped between two Lipschitz graphs which agree on F but now the Lipschitz constant is very small (depending on ε). By localizing around a point of density of F (with respect to Λ_1) we may assume the graphs agree on a large fraction of their total length. More formally:

Lemma 9.1: The double cone condition holds on a positive linear measure subset of K iff (after a rigid motion) for all $\varepsilon > 0$ we can find a box $Q = I \times I$ and Lipschitz functions f_1 and f_2 on I such that

i) $\|f_i'\|_\infty \le \varepsilon$

ii) $|\{f_1 = f_2\}| > (1-\varepsilon)\cdot|I|$

iii) $\{(x,f_1(x)):f_1(x) = f_2(x)\} \subset K \cap Q \subset \{(x,y):f_1(x) \le y \le f_2(x)\}$.

As an immediate consequence we have:

Corollary 9.2: For a closed Jordan curve Γ, ω_1 and ω_2 fail to be mutually singular iff the conclusion of Lemma 9.1 holds for Γ.

This is a bit awkward, but is a formulation we will frequently use, as in the next section.

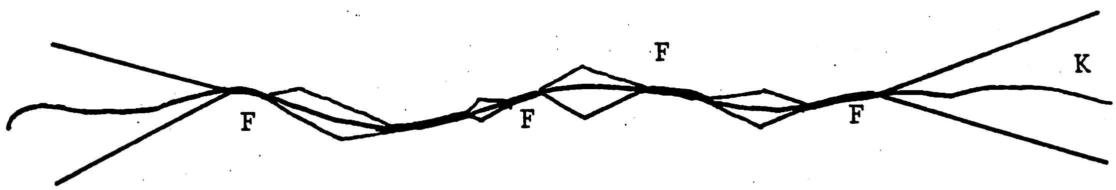

Figure 10: The condition with small ε

10. The ε^2 Conjecture

Suppose Γ is a Jordan curve and that $\omega_1 << \omega_2 << \omega_1$ when restricted to some subset E of Γ. By the arguments of this chapter one can show

$$(10.1) \qquad \lim_{r\to 0} \frac{\omega_1(\Gamma\cap D(x,r))\omega_2(\Gamma\cap D(x,r))}{r^2} > 0$$

for almost every $x \in E$ (with respect to ω_1). With $\theta_1(t)$ and $\theta_2(t)$ as in Section 4 we define

$$\varepsilon(x,r) = \varepsilon(r) = \max_{i=1,2} \{|\pi-\theta_i(r)|\} .$$

By calculus we get

$$\theta_1(r)^{-1} + \theta_2(r)^{-1} \geq \frac{2}{\pi} + \frac{2}{\pi}\left(\frac{\varepsilon(r)}{\pi}\right)^2$$

so by inequalities (10.1) and (4.2)

$$\begin{aligned} 0 &< \lim_{r\to 0}\left(\frac{C}{r^2} \exp\{-\pi \int_r^1 (\theta_1(t)^{-1} + \theta_2(t)^{-1})\frac{dt}{t}\}\right) \\ &\leq C \cdot \exp\{-\frac{2}{\pi^2} \int_0^1 \varepsilon(t)^2 \frac{dt}{t}\} \end{aligned}$$

and so the integral must converge for almost every $x \in E$. We would like to know that the converse holds, i.e., if the integral is finite on E, the harmonic measures are mutually absolutely continuous on E. However, this is false. The problem is that in the definition of $\theta_i(t)$, $D(x,t) \cap \Omega_i$ may consist of many components and in such a case the Ahlfors distortion theorem gives a poor estimate. One can construct curves, similar to those in Section 8, for which the integral converges on a set of positive linear measure but with the harmonic measures mutually singular.

To avoid this difficulty, we define $\tilde{\theta}_i(t)$ as the angle measure of the largest connected arc in $D(x,t) \cap \Omega_i$, $i = 1,2$, and

$$\tilde{\varepsilon}(r) = \max_{i=1,2} \{|\pi - \tilde{\theta}_i(r)|\}$$

Then if ω_1 and ω_2 are not singular, the last section tells us we can find $E \subset \Gamma$, $\Lambda_1(E) > 0$ where the conclusion of Lemma 9.1 holds. Applying the preceeding argument to the domains bounded by the Lipschitz graphs we see

$$\int_0^1 \tilde{\varepsilon}(t)^2 \frac{dt}{t} < \infty$$

for almost every (Λ_1) x in E.

<u>Conjecture</u> 10.1: <u>For a Jordan curve</u> Γ, $\omega_1 \perp \omega_2$ <u>iff</u>

$$\int_0^1 \tilde{\varepsilon}(t,x)^2 \frac{dt}{t} = \infty$$

<u>except for a set of zero linear measure</u>.

This is reminiscent of a result by L. Carleson on the boundary behavior of quasiconformal mappings ([13]) and also of the theorem of Stein and Zygmund relating a functions differentiability to the square integrability of certain second order difference quotients (see [46]), Section VII.5). One can show $\omega_1 \perp \omega_2$ fails iff there is a rectifiable curve R and a positive length subset of $R \cap \partial\Omega$ on which the integral above is finite (and this proves 10.1 if Γ has no twist points). Thus part of the problem is to better understand the twist points in terms of linear measure and rectifiability.

<u>Conjecture</u> 10.2: <u>If</u> R <u>is rectifiable then</u> $\omega_1(R \cap Tw) = \omega_2(R \cap Tw) = 0$.

CHAPTER II

A CONSTRUCTION OF CONTINUOUS FUNCTIONS HOLOMORPHIC OFF A CURVE

1. Statement of Results

If Ω is an open subset of the Riemann sphere, $\bar{\mathbb{C}}$, we let $H^\infty(\Omega)$ denote the space of bounded holomorphic functions on Ω and let $A(\Omega)$ denote the subspace of functions in $H^\infty(\Omega)$ which extend continuously to $\bar{\Omega}$, the closure of Ω. If $K \subset \bar{\mathbb{C}}$ is compact we let $A_K \equiv A(\bar{\mathbb{C}}\backslash K)$. In this chapter we are primarily concerned with A_Γ where Γ is a closed Jordan curve. In particular, we consider the problem of constructing non-constant elements of A_Γ.

Of course, A_Γ need not contain any non-constant functions. For example, if Γ is a straight line, then Morera's theorem implies any continuous function holomorphic off Γ is actually entire, and thus is constant by Liouville's theorem. It is an observation of Riemann that the same is true of any smooth Γ, and Painlevé showed $A_\Gamma = \mathbb{C}$ whenever Γ is rectifiable.

On the other hand, there do exist curves Γ such that A_Γ is non-trivial. For example, suppose $K \subset D(0,1)$ is a compact, totally disconnected subset of $\mathbb{C}$ with positive area. Then

$$F(z) = 1/z * \chi_K(z) = \int \frac{\chi_K(w)}{w-z}\,dxdy$$

is a convolution of a locally integrable function with a bounded, compactly supported function so it is bounded and continuous. Moreover, if $z \notin K$

$$F'(z) = \int \frac{\chi_K(w)}{(w-z)^2}\, dxdy$$

so F is holomorphic off K. If Γ is any curve containing K then $F \in A_\Gamma$ and is non-constant because

$$\begin{aligned} \mathrm{Re}(F'(10)) &= \int \frac{\chi_K(w)}{\mathrm{Re}(w-10)^2}\, dxdy \\ &> 0 . \end{aligned}$$

This construction is given in a 1909 paper of Denjoy ([19]) and is based on an example of Pompéiu (see also [20] where Denjoy constructs a Γ and a $f \in A_\Gamma$ with singularities everywhere on Γ). The example is interesting because it had previously been thought that any continuous function holomorphic off a totally disconnected set must be entire. Moreover, the construction used the (then) recent work of Cantor on the existence of perfect, totally disconnected sets and Lebesgue's work on integration (note that χ_K is not Riemann integrable). In a commentary on Denjoy's paper, Painleve says ([42]):

> "Il convient de signaler le rôle joué, dans ce résultat, par l'extension due à M. Lebesgue, de l'intégrale définie. Grace à cette opération, que nombre de géomètres jugeaient artificielle et trop abstraite, une question naturelle, une question fondamentale qui restait indécise à l'entrée de la théorie des fonctions uniformes, est aujourd'hui tranchée, et tranchée précisément dans le sens qui semblait le moins vraisemblable à la plupart des analystes.... L'intégration de M. Lebesgue pourra contribuer là encore à la formation d'exemples décisifs."

Another interesting observation about A_Γ is due to John Wermer ([9]), and essentially says that any non-constant function in A_Γ is necessarily badly behaved on Γ. More precisely, for any $f \in A_\Gamma$,

$f(\bar{\mathbb{C}}) \subset f(\Gamma)$, so that if f is non-constant $f(\Gamma)$ must cover a disk. To prove this, suppose $f(z) = 0$ for some z, but f is non-zero on Γ. Then f has only a finite number of zeros and the winding number of the curve $f(\Gamma)$ is well defined for both possible orientations of Γ. The sum of these two is obviously zero, but by the argument principle it is also equal to the number of zeros of f, which is positive. This proves Wermer's result.

It would be nice to be able to characterize the curves Γ such that A_Γ is non-trivial, but this problem seems to be very difficult. It is possible, however, to characterize the curves for which A_Γ is "large", assuming of course, we know how to define "large". Clearly $A_\Gamma \subset C(\Gamma)$, but since elements of A_Γ are holomorphic off Γ the imaginary parts are determined (up to a constant) by the real parts. Moreover, our earlier remark shows that not every continuous, real-valued function on Γ can be the real part of an element of A_Γ. Thus the most we can hope for is that such functions can be approximated by the real parts of functions in A_Γ. This motivates the following definition.

We say a function algebra A on a set K is a <u>Dirichlet algebra</u> on K if for every continuous, real-valued function g on K and every $\varepsilon > 0$ there is an $f \in A$ such that

$$\|g - \mathrm{Re}(f)\|_K < \varepsilon$$

where the norm is the "sup norm" on K.

Then A. Browder and J. Wermer proved in 1963 that:

<u>Theorem</u> 1.1 (Browder and Wermer, [9]): A_Γ <u>is a Dirichlet algebra on</u> Γ <u>iff</u> $\omega_1 \perp \omega_2$.

Because of Theorem 1.2 we obtain:

Corollary 1.2: *A_T is a Dirichlet algebra on Γ iff the set of tangent points of Γ has zero linear measure.*

An interesting, but easy, consequence is that for any measure function h such that $h(t) = o(t)$ as $t \to 0$, we can find a curve Γ satisfying $\Lambda_h(\Gamma) = 0$ but such that A_T is a Dirichlet algebra (see Section 6). However, if $\Lambda_h(\Gamma) = 0$ for all such measure functions, then Γ has sigma finite length ([3], Theorem 4) so A_T is trivial ([25], Corollary 2.4).

A_T being a Dirichlet algebra is also related to other types of approximation. We say A_T is *pointwise boundedly dense* (p.b.d.) in $H^\infty(\Omega)$, $\Omega = \bar{\mathbb{C}}\setminus\Gamma$, if there is a $C > 0$ such that for any $f \in H^\infty(\Omega)$ there is a sequence $\{f_n\} \subset A_T$ such that $\|f_n\| \le C\cdot\|f\|$ and $\{f_n\}$ converges pointwise to f on Ω. A_T is called *strongly pointwise boundedly dense* (s.p.b.d.) in $H^\infty(\Omega)$ if we can take $C = 1$. Then the following is known:

Theorem 1.3: *The following are equivalent:*

i) *A_T is a Dirichlet algebra on Γ.*

ii) *A_T is pointwise boundedly dense in $H^\infty(\Omega)$.*

iii) *A_T is strongly pointwise boundedly dense in $H^\infty(\Omega)$.*

The implication (i) $\Rightarrow$ (ii) is due to Hoffman (see [23], [49]), (ii) $\Rightarrow$ (iii) to Davie in [17], and (iii) $\Rightarrow$ (i) to Gamelin and Garnett in [23]. In the remainder of this chapter we shall give new proofs of Theorems 1.1 and 1.3.

2. Proof of Necessity

We start by showing that the mutual singularity of the harmonic measures is a necessary condition for A_T to be either a Dirichlet algebra or pointwise boundedly dense in $H^\infty(\Omega)$.

Suppose ω_1 and ω_2 are not mutually singular. Then by Corollary 1.9.2, the conclusion of Lemma 1.9.1 holds. After rescaling we may assume we are in the situation pictured in Figure 11: a subarc of Γ is trapped between two Lipschitz graphs, Γ_2 and Γ_3, of length about $R >> 1$ and agreeing except for sets of length less than ε. Choose points z_1 and z_2 at distance one from Γ on either side, and let Γ_1 and Γ_4 be circular arcs of radius R connecting the ends of Γ_2 and Γ_3 respectively.

Now suppose $f \in A_T$. We will estimate $|f(z_1) - f(z_2)|$. By the Cauchy integral formula:

$$2\pi i \cdot f(z_1) = \int_{\Gamma_1} \frac{f(w)dw}{z_1-w} + \int_{\Gamma_2} \frac{f(w)dw}{z_1-w}$$

and by our assumptions

$$\left|\int_{\Gamma_2} \frac{f(w)dw}{z_1-w} - \int_{\Gamma_3} \frac{f(w)dw}{z_1-w}\right| \le \|f\| \cdot \varepsilon.$$

So using the Cauchy integral formula again

$$\left|2\pi i\, f(z_1) = \int_{\Gamma_1+\Gamma_4} \frac{f(w)dw}{z_1-w}\right| \le \|f\| \cdot \varepsilon .$$

If we now subtract the corresponding inequality for z_2 we obtain

$$\begin{aligned} 2\pi|f(z_1)-f(z_2)| &\le 2\|f\|\varepsilon + \left|\int_{\Gamma_1+\Gamma_4}\left(\frac{1}{z_1-w} - \frac{1}{z_2-w}\right)f(w)dw\right| \\ &\le \|f\|(2\varepsilon + \frac{C}{R}) \\ &\le 1/4 \end{aligned}$$

if ε is small enough and R is large enough. But this means we

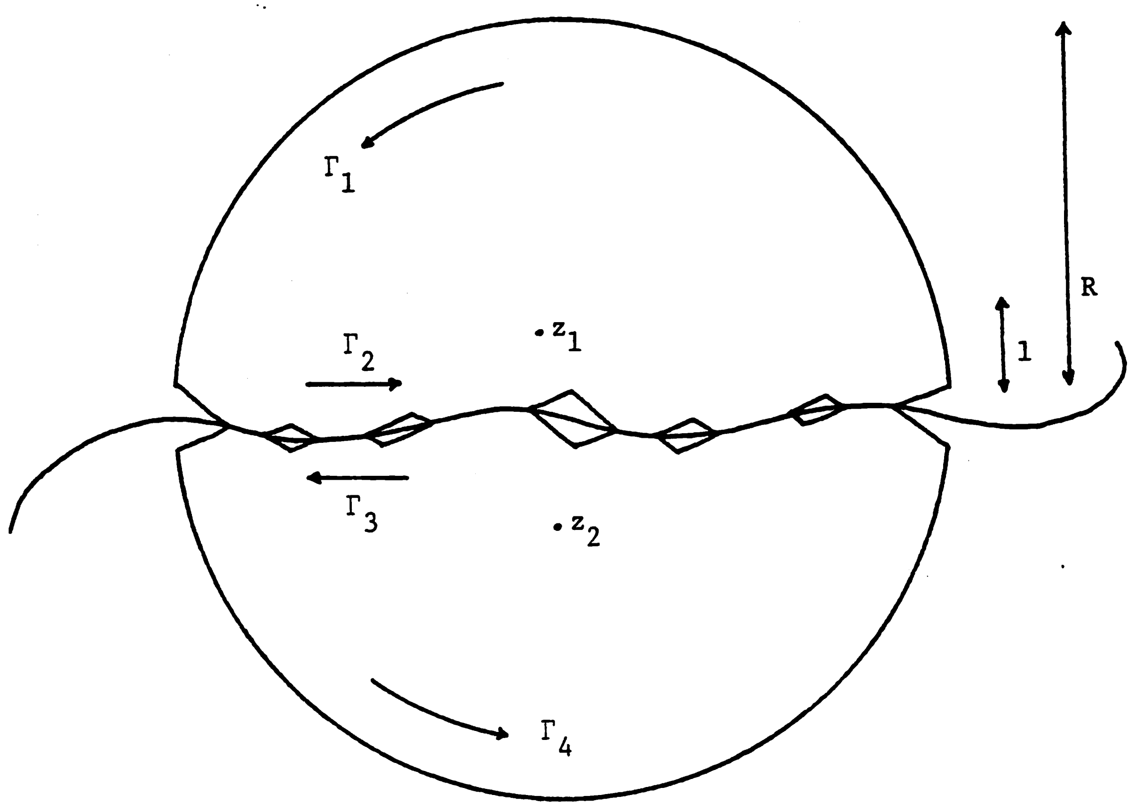

Figure 11: The contours Γ_i

cannot pointwise approximate the holomorphic function which equals zero on one side of Γ and equals one on the other, by any bounded sequence in A_T. Thus A_T is not pointwise boundedly dense in $H^\infty(\Omega)$.

To show A_T cannot be a Dirichlet algebra on Γ we still consider Figure 11, but now estimate $|f'(z_1)|$. Using the Cauchy integral formula and the preceding argument, we get:

$$2\pi|f'(z_1)| \le \left|\int_{\Gamma_1+\Gamma_4} \frac{f(w)\,dw}{(z_1-w)^2}\right| + \|f\| \cdot \varepsilon$$
$$\le \|f\| \left(\frac{C}{R} + \varepsilon\right)$$

so that $|f'(z_1)|$ is small if ε is small and R is large (depending on $\|f\|$).

But if A_T were a Dirichlet algebra, we could take a continuous function g which was zero on the "left" half of Γ and one on the "right" half (except for a small arc in the middle) and approximate it to within 1/10 by $\mathrm{Re}(f)$, $f \in A_T$. Then simple estimates and the mean value theorem say there is a point z_1 satisfying the estimates in the preceding argument and such that

$$|f'(z_1)| \ge |\nabla \mathrm{Re}(f)(z_1)| \ge \frac{1}{100} .$$

We are not quite done since $\|f\|$ may be very large. But since $|\mathrm{Re}(f)| \le 1 + 1/10$

$$F(z) \equiv \exp(f(z))$$

is in A_T, satisfies $\|F\| \le 2$ and

$$|F'(z_1)| = |f'(z_1)| \cdot |\exp(f(z_1))|$$
$$\ge \frac{1}{100} \cdot 1$$

which is a contradiction. Thus A_T cannot be a Dirichlet algebra on Γ.

3. Proof of Sufficiency

We now turn to the sufficiency of the condition "$\omega_1 \perp \omega_2$". The original proofs were based on the Hahn-Banach theorem. For example, to prove that A_T is a Dirichlet algebra on Γ, one shows that if μ is a real measure on Γ such that

$$\int f d\mu = 0 \quad , \quad \forall f \in A_T$$

then μ is the zero measure. Unfortunately, this type of proof provides no description of how to approximate a given function. In this section we will give a simple iterative procedure for approximating a function $f \in H^\infty(\Omega)$ by functions in $H^\infty(\Omega)$ which are "closer" to A_T. In the next section we will show how this proves Theorems 1.1 and 1.3 and in Section 5 we will complete the proof of our main technical lemma.

So suppose $f \in H^\infty(\Omega)$. We can naturally consider f as a pair of functions (f_1, f_2) with f_i defined on Ω_i, $i = 1,2$. Assume, for the moment, that each f_i is actually bounded and holomorphic on a neighborhood of $\bar{\Omega}_i$, so that both functions are defined on a neighborhood A of Γ. Now suppose φ is a continuous function on $\mathbb{C}$ with

$$0 \le \varphi \le 1$$

$$\varphi \equiv 1 \quad \text{on} \quad \Omega_1 \backslash A$$

$$\varphi \equiv 0 \quad \text{on} \quad \Omega_2 \backslash A.$$

Then $g = \varphi \cdot f_1 + (1-\varphi) f_2$ is well defined, continuous and equals f away from Γ. Unfortunately, g is not holomorphic. Recall that if

$$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$$

then g is holomorphic iff $\bar{\partial} g = 0$. However,

$$\bar{\partial}g = \bar{\partial}(\varphi f_1) + \bar{\partial}((1-\varphi)f_2)$$
$$= \bar{\partial}\varphi \cdot (f_1-f_2)$$

will not generally be zero. Therefore we fix $\varepsilon > 0$ and try to find an $h \in C(\Omega)$ such that

$$\bar{\partial}h = -\bar{\partial}g$$
$$\|h\|_\infty \le \min(\varepsilon, \tfrac{1}{4}\|f_1-f_2\|_A).$$

Then $F = h + g$ is in $H^\infty(\Omega)$, approximates f to within ε away from Γ, and has a jump of at most $2\|h\| \le 1/2\|f_1-f_2\|_A$ across Γ. By iterating this procedure, we would obtain in the limit a holomorphic function on Ω approximating f and with no jump across Γ, i.e., an element of A_T.

So the main problem is to construct φ so that we can solve for h. We will do the construction on the unit disk, so let $\Phi_i : D \to \Omega_i$, $i = 1,2$, denote fixed choices of the Riemann maps. By Carathéodory's theorem these maps extend to be homeomorphisms from $\mathbb{T}$ to Γ, so $\psi \equiv (\Phi_2)^{-1} \circ \Phi_1$ defines an orientation reversing homeomorphism of $\mathbb{T}$ to itself. Note that a function $f \in A_T$ corresponds to a pair of functions $f_1, f_2 \in A(D)$ which satisfy $f_1 = f_2 \circ \psi$ on $\mathbb{T}$. Our assumption that $\omega_1 \perp \omega_2$ is equivalent to ψ being singular, i.e., there is set $E \subset \mathbb{T}$ such that $|\mathbb{T}\backslash E| = |\psi(E)| = 0$. We will obtain φ using the following lemma:

<u>Lemma</u> 3.1: <u>For any singular homeomorphism</u> ψ <u>of</u> $\mathbb{T}$ <u>to itself and any</u> $\varepsilon > 0$, $\delta > 0$, $\beta > 0$ <u>there exist functions</u> $\varphi_i \in C(\bar{D}) \cap C^\infty(D)$ <u>and contours</u> $\gamma_i \subset D$, $i = 1,2$, <u>such that</u>

i) $0 \le \varphi_i \le 1$.

ii) $\varphi_1 = 1$ on $\{|z| < 1 - \beta\}$.

iii) $\varphi_2 = 0$ on $\{|z| < 1 - \beta\}$.

iv) $\varphi_1 = \varphi_2 \circ \psi$ on $\mathbb{T}$.

v) arclength on γ_i is a Carleson measure with norm bounded independent of $\psi, \varepsilon, \delta$ and β.

vi) $\mathrm{supp}(\nabla\varphi_i) \subset \{z \in D : \mathrm{dist}(z,\gamma_i) < \delta(1-|z|)\}$.

vii) $|\nabla\varphi_i(z)| \le \varepsilon \cdot \delta^{-1} \cdot (1-|z|)^{-1}$.

We will prove this in Section 5. The first four conditions say that the pair (φ_1,φ_2) corresponds to a continuous function φ as described above. The last three conditions imply we can solve the required $\bar\partial$ problem with the desired estimate. Before proving this we need to review a few facts.

For an arc $I \subset \mathbb{T}$ we let $Q(I)$ denote the "cube" over I,

$$Q(I) = \{r\cdot z: z \in I,\ 1 - |I| < r < 1\}.$$

Then we say a positive measure μ on D is a Carleson measure if

$$\sup_{I\subset \mathbb{T}} \frac{\mu(Q(I))}{|I|} \equiv \|\mu\|_C < \infty$$

and $\|\mu\|_C$ is the Carleson norm of μ. Such measures were introduced by Carleson in his solution of the corona problem on D ([12]). They are related to the $\bar\partial$ problem roughly as follows: if μ is a Carleson measure we can find a function B which solves

$$\bar\partial B = \mu$$

(in the sense of distributions) and satisfies

$$\|B\|_{L^\infty(\mathbb{T})} \le C \cdot \|\mu\|_C$$

for some universal $C > 0$ (see [26], Chapter VIII or [37]). Unfortunately, this estimate only holds on $\mathbb{T}$, not on all of D. For example, if μ is a Dirac mass, then any solution must be unbounded on D. However, with additional assumptions one can get estimates on all of D. For example, if γ is a collection of arcs in D and μ is the arclength measure on γ, then we have:

Lemma 3.2: *If γ and φ satisfy conditions (v)-(vii) in Lemma 3.1, δ is small enough (depending only on $\|\mu\|_C$) and b is a bounded, continuous function on D, then there exists a $F \in L^\infty(D)$ such that:*

i) $\bar\partial F = b\bar\partial\varphi$

ii) $\|F\|_D \le C \cdot \|b\|_D \cdot \varepsilon$

where C depends only on $\|\mu\|_C$.

This is essentially proven in Chapter VIII of [26] (see also [29]), so we will only sketch the proof. Suppose $\{z_n\}$ is a sequence of points on γ which satisfy

$$|z_n - z_m| \ge \frac{1}{10}(1 - |z_n|) \quad , \quad n \ne m.$$

Then it is known ([26], page 341) that such a sequence is interpolating for $H^\infty(D)$ with constant depending only on $\|\mu\|_C$. Thus there exist Pehr Beurling functions $\{h_n\}$ ([26], page 294 or [37]) such that

$$h_n(z_n) = 1$$

$$\Sigma|h_n(z)| \le C_1 = C(\|\mu\|_C).$$

By Schwarz's lemma there is a $\delta > 0$ such that

$$|h_n(w)| > 1/2$$

if $|z_n - w| < \delta(1-|z_n|)$. Let

$$\mathcal{D} = \{z \in D : \text{dist}(z,\gamma) \le \frac{\delta}{2}(1 - |z|)\}$$

and choose a finite collection of sequences $\{z_n^j\}$ $j = 1,\dots,N$ such that

$$|z_n^j - z_m^j| \ge \frac{1}{10}(1 - |z_n|) \quad , \quad n \ne m$$

and so that for every $z \in \mathcal{D}$, there exists z_n^j so that

$$|z - z_n^j| \le \delta(1 - |z_n^j|) .$$

For fixed j, let $\{h_n^j\}$ be the Pehr Beurling functions for $\{z_n^j\}$. Also write $\mathcal{D}$ as a disjoint union of sets

$$\mathcal{D}_n^j \subset \{z : |z_n^j - z| < \delta(1 - |z_n^j|)\} .$$

Now define

$$F(z) \equiv \sum_{n,j} \frac{1}{\pi} \int_{\mathcal{D}_n^j} \frac{h_n(z)}{h_n(w)} \frac{b(w)\bar\partial\varphi(w)}{z-w} dxdy .$$

Then we can check that $\bar\partial F = b\bar\partial\varphi$ formally, so we only have to check the convergence of the series. If we write $F = \Sigma H_n^j$, then

$$|H_n^j(z)| \le \frac{2}{\pi}|h_n^j(z)| \int_{\mathcal{D}_n^j} \frac{\|b\|_\infty \cdot \varepsilon \cdot \delta^{-1}(1-|w|)}{z-w} dxdy$$

$$\le C \cdot \|b\|_\infty \cdot \frac{\varepsilon}{\delta} \cdot |h_n^j(z)| \cdot \delta .$$

Thus

$$|F(z)| \le C \cdot \|b\|_\infty \, \varepsilon \sum_{n,j} |h_n^j(z)|$$

$$\le C \cdot \|b\|_\infty \cdot \varepsilon \cdot C_1 \cdot N$$

$$\le C(\|\mu\|_C) \cdot \|b\|_\infty \cdot \varepsilon .$$

This completes the proof of Lemma 3.2.

4. The Construction

So suppose $\omega_1 \perp \omega_2$. We will first show that A_T is strongly pointwise boundedly dense in $H^\infty(\Omega)$ (assuming Lemma 3.1). Clearly it is enough to show the following: given $f \in H^\infty(\Omega)$, $K \subset \Omega$ compact and $\eta > 0$ there exists $F \in A_T$ satisfying

$$\|F\|_\Omega \le \|f\|_\Omega$$
$$\|F-f\|_K \le \eta \; .$$

To obtain F we will construct a sequence of holomorphic functions $\{F_n\}$ on Ω such that

(4.1) $$\|F_1 - f\|_K < \frac{1}{2}\eta$$

(4.2) $$\|F_n\|_\Omega < \|f\|_\Omega \quad , \quad \forall n$$

(4.3) $$\|F_{n+1} - F_n\|_\Omega < 2^{-n}\cdot \eta \; , \forall n$$

(4.4) $$\text{jump}(F_n) < 2^{-n-1}\cdot\eta \; , \; \forall n$$

where "jump" is defined as

$$\text{jump}(F) \equiv \sup_{x\in\Gamma}(\lim_{\delta\to 0}(\sup_{z,w,\in D(x,\delta)\cap\Omega} |F(z)-F(w)|)) \; .$$

Then $F = \lim_n F_n$ exists, is in A_T and approximates f to within η on K.

Without loss of generality, we assume $\|f\| = 1$. As before we consider f as a pair of functions (f_1, f_2) defined on Ω_1 and Ω_2 respectively. We can approximate f_1 on $K \cap \Omega_1$ by a function in $A(\Omega_1)$ (e.g., pull back to the unit disk and dilate slightly) and this function can be uniformly approximated on $\bar{\Omega}_1$ by a function holomorphic on a neighborhood of $\bar{\Omega}_1$ (e.g., use Mergelyan's theorem, or map Ω_1 conformally to a slightly larger domain). Thus we may assume f_1 is holomorphic on a neighborhood of $\bar{\Omega}_1$ and satisfies

$$\|f-f_1\|_{K\cap\Omega_1} < 1/4 \cdot \eta$$

$$\|f_1\| < 1 - \frac{\eta}{8} < 1$$

(We have extended f_1 only for convenience; we will remark in Section 8 on how to avoid this). We take f_2, holomorphic on a neighborhood of $\bar{\Omega}_2$, similarly. We now map the problem to the unit disk and with the functions from Lemma 3.1 write

$$g_1 \equiv (f_1 \circ \Phi_1) \cdot \varphi_1 + (f_2 \circ \Phi_1)(1-\varphi_1)$$

$$g_2 \equiv (f_1 \circ \Phi_2) \cdot \varphi_2 + (f_2 \circ \Phi_2)(1-\varphi_2)$$

If β is small enough these are well defined and satisfy

$$g_1(x) = g_2(\psi(x)) \quad , \quad x \in \mathbf{T}.$$

Also $\bar{\partial} g_i$, $i = 1,2$, satisfy Lemma 3.2 so we can find h_1 and h_2 so that

$$\bar{\partial} h_i = -\bar{\partial} g_i \qquad i = 1,2$$

$$\|h_i\|_D \le C \cdot \varepsilon \qquad i = 1,2$$

Now set

$$F_1(z) = \begin{cases} (g_1+h_1) \circ \Phi_1^{-1}(z) & , \quad z \in \Omega_1 \\ (g_2+h_2) \circ \Phi_2^{-1}(z) & , \quad z \in \Omega_2 \end{cases}$$

Then F_1 is holomorphic on Ω and

$$\begin{aligned} \|F_1\|_\Omega &\le \max(\|f_1\|_{\Omega_1}, \|f_2\|_{\Omega_2}) + \max(\|h_1\|_D, \|h_2\|_D) \\ &\le 1 - \frac{\eta}{8} + C \cdot \varepsilon \\ &< 1 - \frac{\eta}{16} \end{aligned}$$

if ε is small enough. Also, if β and ε are both small enough we get

$$\|F_1-f\|_K \le \frac{1}{4}\eta + \max(\|h_1\|_D,\|h_2\|_D)$$
$$\le \frac{1}{4}\eta + C\cdot\varepsilon$$
$$\le \frac{1}{2}\eta .$$

Finally,

$$\text{jump}(F_1) \le \|h_1\|_D + \|h_2\|_D \le 2\cdot C\cdot\varepsilon .$$

Thus F_1 satisfies (4.1)-(4.4) if we take β and ε small enough. In general, given F_{n-1} we define F_n by applying the above procedure to F_{n-1}, choosing ε so small that

$$\|h_i\|_D \le \eta \cdot 2^{-n-4} \qquad i = 1,2$$

Then clearly F_n satisfies (4.2) and (4.4) and

$$\|F_n-F_{n-1}\|_\Omega \le \text{jump}(F_{n-1}) + \|h_1\|_D + \|h_2\|_D$$
$$\le 2^{-n-2}\cdot\eta + 2^{-n-3}\cdot\eta + 2^{-n-3}\cdot\eta$$
$$\le 2^{-n}\cdot\eta .$$

Thus Lemma 3.1 implies A_T is strongly pointwise boundedly dense in $H^\infty(\Omega)$.

The proof that A_T is a Dirichlet algebra on Γ is very similar. Suppose g is a continuous, real-valued function on Γ and let u be its harmonic extension to Ω. Since both components of Ω are simply connected, u has a harmonic conjugate u^* on Ω. Now set $G = u + iu^*$ and suppose $\eta > 0$ is fixed. Since $\text{Re}(G)$ is continuous, we can find functions f_1 and f_2, holomorphic on open neighborhoods of $\bar{\Omega}_1$ and $\bar{\Omega}_2$, such that

$$\|u - \text{Re}(f_i)\|_{\Omega_i} < \frac{\eta}{4} \qquad i = 1,2$$

Applying the preceding construction with this pair (f_1, f_2) we can obtain a $F_1 \in H^\infty(\Omega)$ with $\text{jump}(F_1) < \eta/4$. Also, if $z \in \Omega$, say $z = \Phi_1(w)$, $w \in D$, then

$$
\begin{aligned}
|\text{Re}(F_1(z)) - u(z)| &\le \frac{\eta}{4} + |\text{Re}(f_1(w) - g_1(w))| + |h_1(w)| \\
&\le \frac{\eta}{2} + |\text{Re}(\varphi_1(w) f_1(w) + (1-\varphi_1(w)) f_2(w))| \\
&\le \frac{\eta}{2} + |\text{Re}(f_1(w) - f_2(w))| \\
&\le \eta
\end{aligned}
$$

This merely corresponds to the fact that the convex combination of two complex numbers with the same real part also has that same real part. Thus

$$\|\text{Re}(F_1) - u\|_\Omega \le \eta \quad .$$

We now define $F_2, F_3, \ldots$ as before and obtain a $F \in A_\Gamma$ with

$$\|\text{Re}(F) - g\|_\Gamma \le 2 \cdot \eta$$

which proves A_Γ is a Dirichlet algebra on Γ.

5. Proof of Lemma 3.1

We now turn to the proof of Lemma 3.1. We will first construct the contours γ_1 and γ_2 as finite unions of "tents" with endpoints on $\mathbb{T}$. We then define φ_i to be constant on each component of $D \setminus \gamma_i$ and with very small jump across each arc in γ_i. It is then easy to "smooth out" φ_i so that conditions (iv)-(vii) of Lemma 3.1 are satisfied.

We start by dividing $\mathbb{T}$ into two disjoint, finite families of "first generation" intervals, F_1^1 and F_2^1, which satisfy:

(5.1) Intervals in F_1^1 and F_2^1 alternate and cover all of $\mathbb{T}$.

(5.2) $$|I| < \frac{\beta}{2} \quad \forall I \in F_1^1, \quad |\psi(I)| < \frac{\beta}{2} \quad \forall I \in F_2^1.$$

(5.3) $$\sum_{I \in F_1^1} |I| \le \frac{1}{4}, \quad \sum_{I \in F_2^1} |\psi(I)| \le \frac{1}{4}.$$

We can do this because ψ singular implies

$$\lim_{\substack{|I| \to 0 \\ x \in I}} \frac{|I|}{|\psi(I)|} = \infty$$

for almost every $x \in \mathbb{T}$. So if we cover a set of measure larger than 3/4 by disjoint intervals satisfying

$$|I| \ge 4|\psi(I)|$$
$$|I| \le \frac{\beta}{2}$$

we obtain (5.3) and half of (5.2). We can then guarantee (5.1) and the other half of (5.2) by trivial alterations, if necessary.

We define a first generation contour γ_1^1, by drawing "tents" over each interval in F_1^1, i.e.,

$$\gamma_1^1 = \bigcup_{I \in F_1^1} \{r\xi : r = 1 - \mathrm{dist}(\xi, I^c), \ \xi \in I\}$$

and similarly,

$$\gamma_2^1 = \bigcup_{I \in F_1^2} \{r\xi : r = 1 - \mathrm{dist}(\xi, \psi(I)^c), \xi \in \psi(I)\}$$

(see Figure 12). Now suppose we have constructed γ_1^k and γ_2^k, the k^{th} generation contours, by adding "tents" over the intervals in F_1^k and F_2^k. Suppose $I \in F_1^k \cup F_2^k$. We subdivide I into two disjoint, finite families of intervals, $F_1 \equiv F_1^{k+1}(I)$ and $F_2 \equiv F_2^{k+1}(I)$, which satisfy:

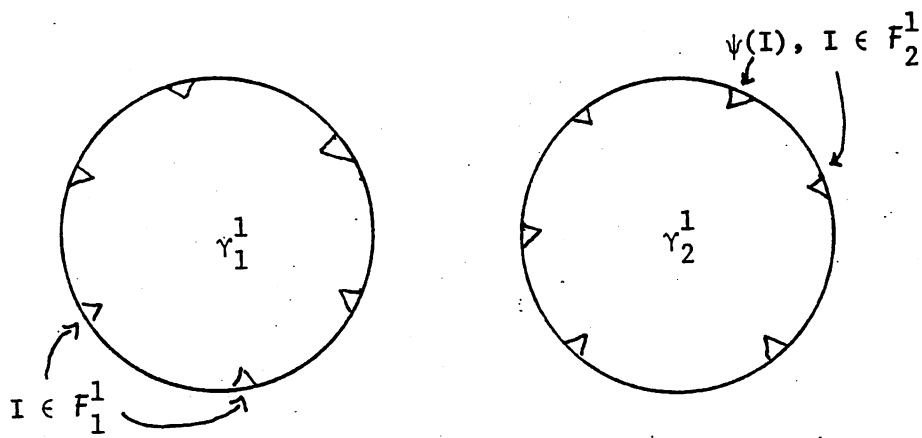

Figure 12: The contours γ_i^1

(5.4) Elements of F_1 and F_2 alternate and if $I \in F_1^k$ then the endpoints of I are in intervals of F_2. If $I \in F_2^k$, the endpoints lie in intervals of F_1.

(5.5) $$\sum_{J \in F_1} |J| \le 4^{-n}|I|, \quad \sum_{J \in F_2} |\psi(J)| \le 4^{-n}|\psi(I)|.$$

(5.6) If $I \in F_1^k$ and $J \in F_1$ then $|J| \le \frac{1}{2}\text{dist}(J, I^c)$ and if $I \in F_2^k$, $J \in F_2$ then $|\psi(J)| \le \frac{1}{2}\text{dist}(\psi(J), \psi(I)^c)$.

(5.7) If $I \in F_1^k$ and $J \in F_2$ contains an endpoint of I then J is adjacent to some interval $\tilde{I} \in F_2^k$. We then require that $|\psi(J)| \le \frac{1}{4}|\psi(\tilde{I})|$. If $I \in F_2^k$, $J \in F_1$, $\tilde{I} \in F_1^k$ then we want $|J| \le \frac{1}{4}|\tilde{I}|$.

As before, (5.5) follows from the singularity of ψ and the other conditions can be obtained by trivial modifications. We now set

$$F_i^{k+1} = \bigcup_{I \in F_1^k \cup F_2^k} F_i^{k+1}(I) \quad , \quad i = 1,2$$

and define γ_i^{k+1} by adding to γ_1^k the tents

$$\gamma_I = \{r\xi : r = 1 - \text{dist}(\xi, I^c), \xi \in I\}$$

for all $I \in F_1^{k+1}$ and adding to γ_2^k the tents $\gamma_{\psi(I)}$ for all $I \in F_2^{k+1}$. However, we make one slight modification. For each tent $\gamma_I \in \gamma_1^k$, I is adjacent to two intervals $J_1, J_2 \in F_1^{k+1}$. Instead of placing new tents over each of these intervals, we remove the tent over I and replace it with a larger tent γ_J, with base $J = J_1 \cup I \cup J_2$ (see Figure 13). This insures that no two tents in γ_1^{k+1} have a common endpoint on $\mathbb{T}$. Also, condition (5.7) implies that no tent more than doubles in size, no matter how many times it is "enlarged". Of course, we do the same for tents in γ_2^k.

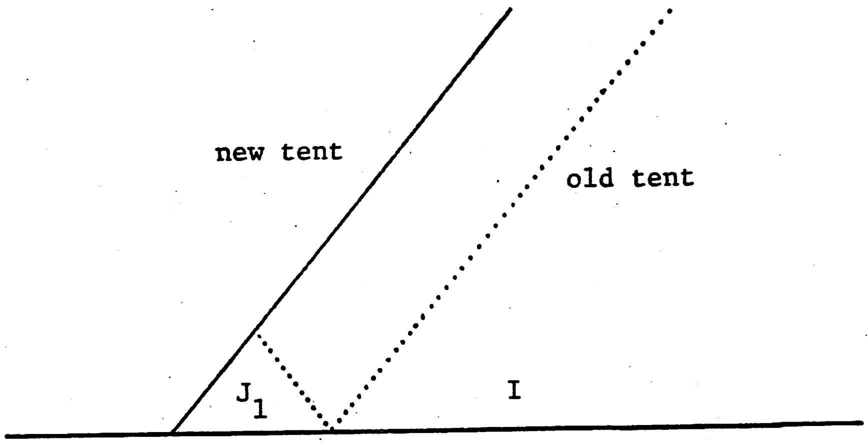

Figure 13: The "new" tent

Now choose N, a large integer, and let $\gamma_i \equiv \gamma_i^N$, $i = 1,2$. We need to verify that arclength on γ_1 and γ_2 are Carleson measures with Carleson norms independent of N. Without loss of generality we consider $\ell = \ell_1$, arclength on γ_1.

Let $\{I_j\}$ be an enumeration of all intervals which occur as the base of some tent in γ_1. Fix $I \subset \mathbf{T}$ and let Q be the associated Carleson cube. To estimate $\ell(Q) = \sum_j \ell(Q \cap \gamma_{I_j})$ we consider three situations.

First, if $3I \subset I_j$, then $\gamma_{I_j} \cap Q = \emptyset$ so these intervals contribute nothing.

Secondly, by (5.5),

$$\sum_{I_j \subset I_k} \ell(\gamma_{I_j} \cap Q) \le 2 \cdot \sum_{I_j \subset I_k} |I_j| \le 2\cdot|I_k|\cdot\sum_0^N 4^{-\ell} \le 3\cdot|I_k|.$$

So if we sum over the (disjoint) collection of maximal I_k's contained in $3I$,

$$\sum_{\substack{I_k \subset 3I \\ \text{maximal}}} \Big(\sum_{I_j \subset I_k} \ell(\gamma_{I_j} \cap Q)\Big) \le 3\cdot\Sigma|I_k| \le 9\cdot|I|$$

Finally, any remaining I_j must contain an endpoint of $3I$. Consider those that contain the left endpoint. We can relabel them so $I_1 \supset I_2 \supset \cdots$ and by condition (5.6),

$$|I_{j+1}| \le \tfrac{1}{2}|I_j \cap 3I| \le 2^{-j}\cdot 3\cdot|I| .$$

Thus

$$\sum_j \ell(\gamma_{I_j} \cap Q) \le 6\cdot|I|\cdot\Sigma 2^{-j} \le 12|I| .$$

Thus ℓ is a Carleson measure of norm less than 33.

For $i = 1,2$, let Ω_i^0 be the component of $D\setminus\gamma_i$ containing zero. Let Ω_i^1 be the components of $D\setminus\gamma_i$ adjacent to Ω_i^0 (i.e., separated by an arc of γ_i). In general, Ω_i^k consists of the components of $D\setminus\gamma_i$ adjacent to Ω_i^{k-1}. Thus we can write $D = \gamma_i \cup \Omega_i^0 \cup \cdots \cup \Omega_i^{N-1}$ (see Figure 14). Set

$$\varphi_1(z) = \sum_{k=0}^{N-1} \left(1 - \frac{k}{N}\right)\chi_{\Omega_1^k}(z)$$

$$\varphi_2(z) = \sum_{k=0}^{N-1} \left(\frac{k}{N}\right)\chi_{\Omega_2^k}(z) .$$

Then $\varphi_1(x) = \varphi_2(\psi(x))$ for all $x \in \mathbb{T}$ except the endpoints of γ_1, γ_2 where these functions have a jump discontinuity of size $1/N \ll \varepsilon$. Also, we can "smooth out" these functions so that they satisfy (vi) and (vii) (e.g., see [26], page 357).

Thus we have proved the lemma except for the finite number of jump discontinuities. In fact, the functions above are quite sufficient for the proofs of Theorems 1.1 and 1.3 (we just pick up a few more small error terms), but to prove the lemma as stated, we need to modify γ_1 and γ_2 near $\mathbb{T}$.

Consider an arc of γ_1 ending at a point ξ of $\mathbb{T}$. Choose an interval I_1^1 around ξ so that $|I_1^1|$ is small compared to the distance to the closest other endpoint of γ_1. We also want $|\psi(I_1^1)|$ to be small compared to the distance between $\psi(\xi)$ and other endpoints of γ_2. We now define a family of intervals $\{I_n^j\}$ $n = 1,2,\ldots,$ $1 \le j \le 2^{n-1}$, by removing a concentric interval from each n^{th} generation intervals. We choose them so that

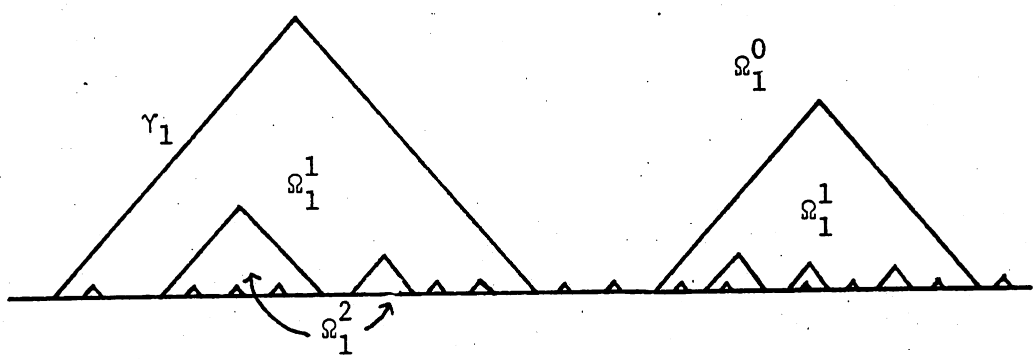

Figure 14: The contour γ_1

$$\sum_{J \subset I} |J| \le \frac{1}{2}|I|$$

$$\sum_{J \subset I} |\psi(J)| \le \frac{1}{2}|\psi(I)|$$

Then arclength on

$$\tilde{\gamma}_1 = \bigcup_{n,j} \gamma_{I_n^j}$$

$$\tilde{\gamma}_2 = \bigcup_{n,j} \gamma_{\psi(I_n^j)}$$

are uniformly bounded Carleson measures. We then replace the "tip" of γ_1 near ξ with $\tilde{\gamma}_1$ (see Figures 15 and 16) and similarly for γ_2. We also redefine φ_1 on $D\backslash\tilde{\gamma}_1$. Suppose φ_1 equaled a and b on the left and right of the replaced arc. Redefine φ_1 as

$$\varphi_1(z) = (\frac{j}{2^n})a + (1 - \frac{j}{2^n})b$$

if $z \in T_n^j$ (the solid tent over I_n^j, minus the solid tents of all small intervals). This is pictured in Figure 16 for $a = 0$, $b = 1$. We redefine φ_2 similarly and do this for every endpoint of the original contours. After smoothing, these functions are continuous on $\bar{D}$, and so satisfy all the conditions of Lemma 3.1.

6. An Example

In the previous section we constructed the "partition of unity" function φ by pulling the problem back to the unit disk. This makes the construction a little simpler and also allows us to deal with homeomorphisms not arising from curves, but the relation between φ and the geometry of Γ becomes less apparent.

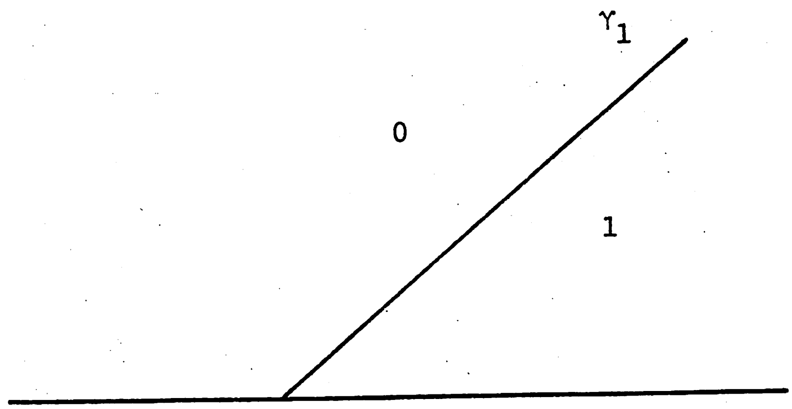

Figure 15: An endpoint of γ_1

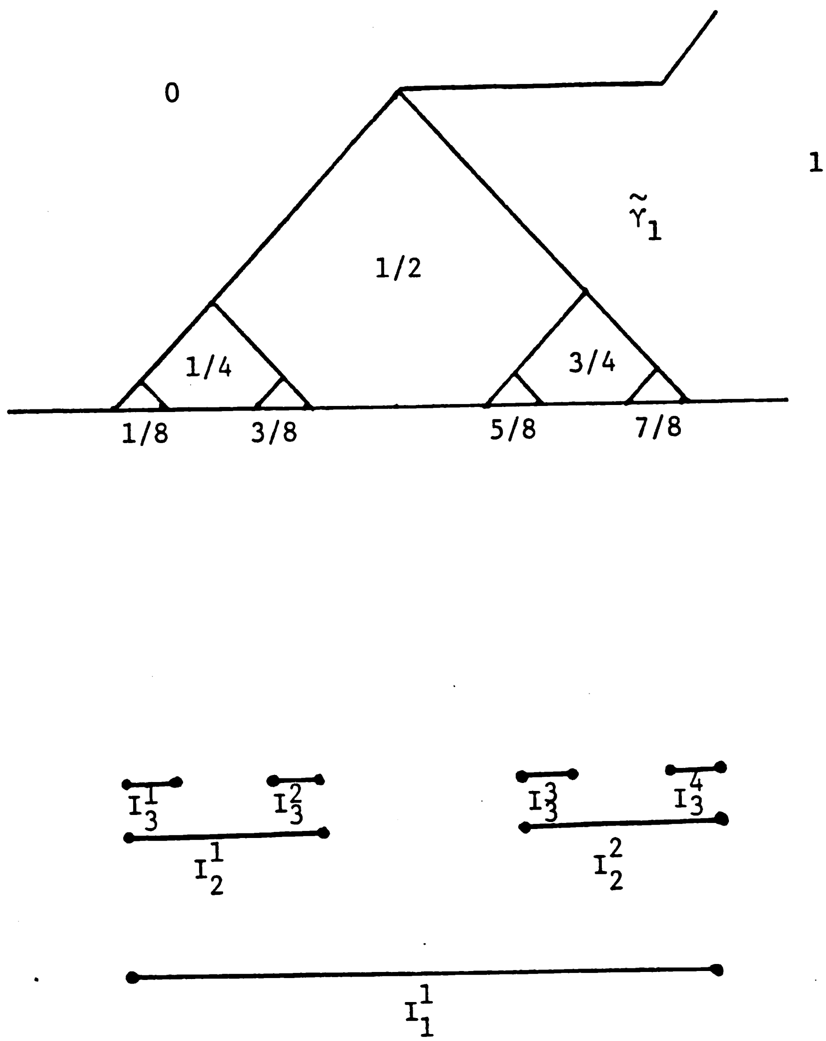

Figure 16: The contour $\tilde{\gamma}_1$

However, it is sometimes quite easy to construct φ without going to the unit disk. For example, consider a curve constructed as follows. Let Γ_0 be the boundary of a square. We obtain Γ_1 from Γ_0 by replacing each of the four line segments in Γ_0 by a polygonal path γ_{n_1} consisting of $3n_1 + 2$ line segments of length $(n_1+2)^{-1}$ arranged in the form of a "square wave" (see Figure 17). In general, we obtain Γ_k from Γ_{k-1} by replacing each line segment in Γ_{k-1} by a scaled copy of γ_{n_k}, $n \in \mathbb{N}^+$. The sequence $\{\Gamma_k\}$ converges to a Jordan curve Γ which has no tangents, so A_Γ is a Dirichlet algebra on Γ.

Each Γ_{k-1} divides Γ_k into intervals (with endpoints $\Gamma_{k-1} \cap \Gamma_k$) and these intervals correspond to the subintervals of $\mathbb{T}$ chosen in the last section. If Ω_1^k, Ω_2^k denote the two sides of Γ_k, we can define φ as follows. Choose integers M and N and define φ on Ω by

$$\varphi(z) = \sum_{j=1}^{N} \frac{1}{N} \chi_{\Omega_1^{M+j}}(z) .$$

This is illustrated for $N = 2$ in Figure 18. After smoothing, φ is the desired function. (Actually, this is not quite true; to solve the $\bar{\partial}$ problem we want the equivalent of condition (5.5). Thus we should replace Ω_1^{M+j} above with $\Omega_1^{M+2^j}$).

We should also note that if h is a measure function with $h(t) = o(t)$ as $t \to 0$, we can take $\Lambda_h(\Gamma) = 0$ by letting the $\{n_k\}$ grow quickly enough. This proves the remark following Corollary 1.2.

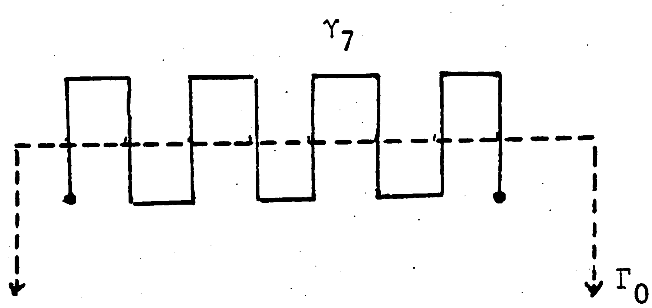

Figure 17: The path γ_7

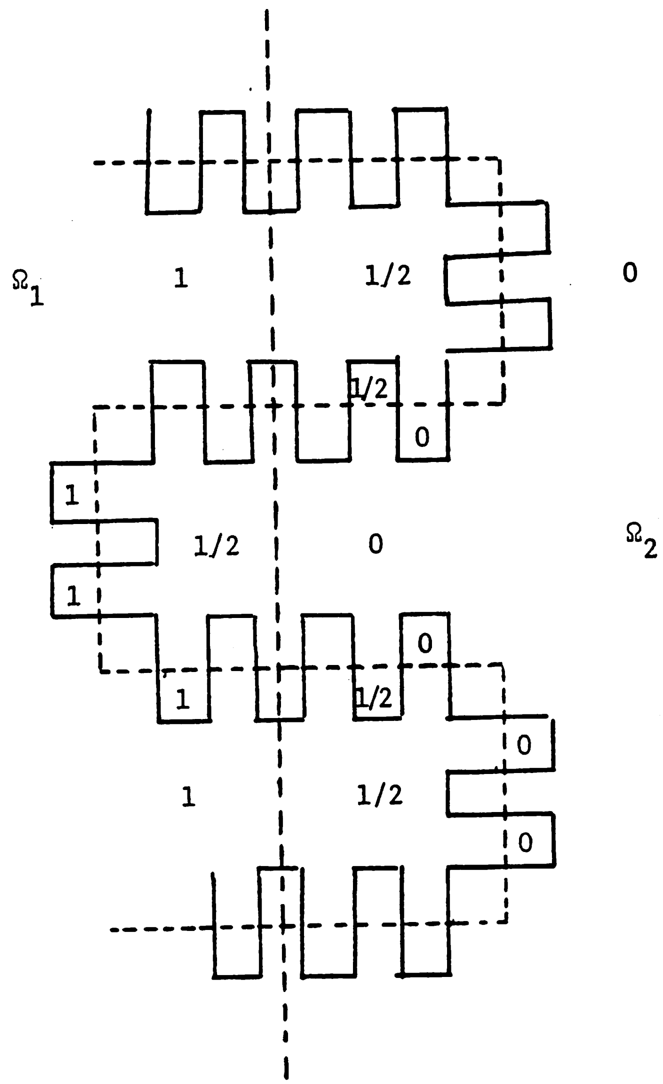

Figure 18: The function φ for N = 2

7. A BMO Corollary

In [47] and [48] Varopoulos shows that a function f on $\mathbb{T}$ is in BMO iff it has an extension φ to D such that $|\nabla\varphi dxdy|$ is a Carleson measure and that

$$\|f\|_{BMO} \sim \inf_{\varphi}\|\nabla\varphi dxdy\|_C$$

Thus the functions in Lemma 3.1 (when restricted to $\mathbb{T}$) have BMO norm about ε. Furthermore, we easily see

$$|\{x : \varphi_1(x) = 1\}| \geq 1/2$$

$$|\{x : \varphi_2(x) = 0\}| \geq 1/2$$

If we replace "4" by a very large constant $(\sim 1/\varepsilon)$ in (5.3) and (5.5) we obtain:

<u>Corollary</u> 7.1: <u>Suppose</u> $\psi : \mathbb{T} \to \mathbb{T}$ <u>is a singular homeomorphism and</u> $\varepsilon > 0$. <u>Then there exists a</u> $\varphi \in C(T)$, $0 \leq \varphi \leq 1$, <u>such that</u>

i) $\|\varphi\|_{BMO} \leq \varepsilon$

ii) $\|\varphi\circ\psi\|_{BMO} \leq \varepsilon$

iii) $|\{\varphi = 1\}| \geq 1 - \varepsilon$

iv) $|\{\varphi\circ\psi = 0\}| \geq 1 - \varepsilon$.

This is reminisant of a result of Garnett and Jones concerning BMO functions taking the values zero and one on small, preassigned subsets of $\mathbb{T}$ (see [28] and [36]).

8. Homeomorphisms of $\mathbb{T}$

In the construction of Section 4, it was unnecessary to assume f_1 and f_2 were holomorphic on a neighborhood of Γ. It is enough to assume $f_i \in A(\Omega_i)$, $i = 1,2$. For example, in the definition of g_1 we can replace "$f_2 \circ \Phi_1$" by a function which is constant on each component of $\text{supp}(1 - \varphi_1)$ and approximates the continuous function $f_2 \circ \Phi_1$ on $\mathbb{T}$. This will work because we see from Section 5 that each component off $\text{supp}(1 - \varphi)$ has diameter less than β, and so the uniform continuity of $f_2 \circ \Phi_1$ provides the necessary estimates.

This observation shows the construction works for any singular homeomorphism ψ of $\mathbb{T}$, regardless of whether it comes from a curve Γ or not. Thus we get:

Corollary 8.1 (Browder and Wermer, [9]): If ψ is a singular homeomorphism of $\mathbb{T}$ to itself then

$$A_\psi = \{f \in A(D) : f \circ \psi \in A(D)\}$$

is a Dirichlet algebra on $\mathbb{T}$.

Similarly, if $\{\psi_1, \ldots, \psi_n\}$ are all homeomorphisms of $\mathbb{T}$ to itself such that $\psi_{jk} \equiv \psi_j \circ \psi_k^{-1}$, $j \neq k$ are all singular, then we can divide $\mathbb{T}$ into n disjoint families of intervals $\mathcal{F}_1, \ldots, \mathcal{F}_n$ such that

$$\sum_{\mathcal{F}_j} |\psi_j(I)| > 1 - \varepsilon \quad , \quad \forall j$$

$$\sum_{\mathcal{F}_j} |\psi_k(I)| < \varepsilon \quad , \quad k \neq j.$$

Using this we can mimic the proof of Lemma 3.1 and the construction of functions in A_T to prove that

$$\{f : f \circ \psi_i \in A(D),\ j = 1, \ldots, n\}$$

is a Dirichlet algebra on $\mathbb{T}$, another result of Browder and Wermer.

We should also mention that everything still works if we replace the closed curve Γ by a Jordan arc Γ. Then $\Omega = \bar{\mathbb{C}}\setminus\Gamma$ has a single simply connected component and the conformal mapping to D induces a homeomorphism ψ of $\mathbb{T}$ to itself such that

$$\psi\circ\psi = \text{identity},$$

i.e., ψ is an involution. We may assume ψ fixes $\{1\}$ and $\{-1\}$. We can now mimic the proof of Lemma 3.1 to construct a φ on D such that

$$\varphi \equiv 1 \quad \text{on} \quad \{|z| < 1 - \beta\}$$

$$\varphi(x) + \varphi(\psi(x)) = 1 \quad \text{on} \quad \mathbb{T}\setminus D(\pm 1,\beta)$$

and $\nabla\varphi$ satisfies the estimates (v)-(vii) of Lemma 3.1. The construction then proceeds as before.

9. Decomposing Continuous Functions

In their paper [10], Browder and Wermer showed that if ψ is any orientation reversing homeomorphism of $\mathbb{T}$ to itself, then

$$A(D) + A_\psi(D) \equiv A(D) + \{f\circ\psi : f \in A(D)\}$$

is a uniformly dense subspace of $C(\mathbb{T})$. A result of Browder implies $A(D) + A_\psi(D)$ is a closed subspace of $C(\mathbb{T})$ if ψ is singular, hence $C(\mathbb{T}) = A(D) + A_\psi(D)$ (see [8], Lemma 7.2.2 and page 235). In fact, this holds for any singular homeomorphism (_ibid._). We can recover this result from our construction, i.e.,

Corollary 9.1: _If_ ψ _is any singular homeomorphism of_ $\mathbb{T}$ _to itself and_ $f \in C(\mathbb{T})$, _then we can find_ $f_1, f_2 \in A(D)$ _such that_

$\|f_1\|,\|f_2\| \le \|f\|$ and $f = f_1 - f_2 \circ \psi$.

Of course, there is a corresponding result in terms of curves. We will not give a complete proof of this result since it is so similar to what we have already done. We merely note that if $f_1, f_2 \in A(D)$ and if φ_1 and φ_2 are as in Lemma 3.1, we can set

$$g_1 = \varphi_1 f_1 + (1-\varphi_1)(f+f_2)$$

$$g_2 = (1-\varphi_2)f_2 + \varphi_2(f_1-f).$$

Then $f = g_1 - g_2 \circ \psi$ and $\bar{\partial} g_i dxdy$, $i = 1,2$, are Carleson measures of the appropriate type. Thus we can find h_1 and h_2 so that

$$F_i = g_i + h_i \in A(D) \qquad i = 1,2$$

$$\|f - (F_1 - F_2 \circ \psi)\| \le \|h_1\| + \|h_2\| < \varepsilon.$$

Iterating this gives Corollary 9.1.

10. Extending Continuous Functions

E. Bishop's generalization of the Rudin-Carleson theorem states that if A_T is a Dirichlet algebra and if $E \subset \Gamma$ has zero harmonic measure from both sides then for any $g \in C(E)$ there exists $f \in A_T$ such that $f|_E = g$ (see [6],[11]). Given g, f can be constructed by the techniques of this chapter. If $|g| \le h$ for some positive continuous function on $\bar{\mathbb{C}}$ we can take $|f| \le h$. This mimics a theorem of Gamelin for $A(D)$.

11. P.B.D. and Distance Estimates

In [18], Davie, Gamelin and Garnett show that A_k is pointwise boundedly dense in $H^\infty(\Omega)$ iff

$$\text{dist}(h,A_K) = \text{dist}(h,H^\infty(\Omega))$$

for every $h \in C(\bar{\mathbb{C}})$. The inequality "$\geq$" is trivial, of course. If $K = \Gamma$, a curve, and $\omega_1 \perp \omega_2$ our construction gives "$\leq$". This is because if $f \in H^\infty(\Omega)$ satisfies $\text{dist}(h,f) = d$, then applying the construction of f gives a function $F \in A_\Gamma$ which at a given point is essentially a convex combination of f's values near that point plus a small error. Using the uniform continuity of h we can deduce $\text{dist}(h,F) \leq d + \varepsilon$. On the other hand, using the methods of Section 2, one can show the equality above fails if ω_1 and ω_2 are not singular.

12.. A Capacity Characterization

Another "geometric" characterization of Dirichlet algebras is due to Gamelin and Garnett. In [23] they prove that A_Γ is a Dirichlet algebra on Γ iff for all $x \in \Gamma$ and $0 < \delta < \text{diam}(\Gamma)$,

$$\alpha(D(x,\delta) \cap \Gamma) \geq \delta/4.$$

Here α is the continuous analytic capacity,

$$\alpha(E) \equiv \sup\{|f'(\infty)| : f \in A_E, \|f\|_\infty \leq 1\}.$$

By a Cauchy integral argument (as in Section 2) one can show the inequality above false if ω_1 and ω_2 are not singular. On the other hand, suppose $\omega_1 \perp \omega_2$, and let E be a connected subarc of $D(x,\delta) \cap \Gamma$ of diameter at least δ. It is a well known consequence of the Riemann mapping and Koebe 1/4 theorems that

$$\gamma(E) \equiv \sup\{|f'(\infty)| : f \in H^{\infty}(\bar{\mathbb{C}}\setminus E), \|f\|_{\infty} \leq 1\}$$

$$\geq 1/4 \text{ diam}(E)$$

$$\geq \delta/4$$

(see [25], page 9). Now take $f \in H^{\infty}(\bar{\mathbb{C}}\setminus E)$ with $\|f\|_{\infty} \leq 1$ and $|f'(\infty)| > \delta/4 - \varepsilon$. By the construction we can approximate f uniformly on a compact neighborhood of $\{\infty\}$ and so we can find $F \in A_E$ with $\|F\|_{\infty} \leq 1$ and

$$|F'(\infty)| > |f'(\infty)| - \varepsilon > \delta/4 - 2\varepsilon.$$

This proves the desired inequality.

CHAPTER III

THE CONSTRUCTION FOR COMPACT, CONNECTED SETS

1. Statement of Results

In the previous chapter we considered the question of when A_Γ is a Dirichlet algebra on Γ, Γ a closed curve. However, the question makes sense for any compact set $K \subset \bar{\mathbb{C}}$. In this chapter we shall give a new proof of a result of Davie which characterizes Dirichlet sets, i.e., those sets K for which A_K is a Dirichlet algebra on K.

One necessary condition is that K be connected. For suppose we can write $K = K_1 \cup K_2$, a disjoint union of non-empty, compact sets. If either K_1 or K_2 has zero logarithmic capacity, we cannot approximate $\chi_{K_1} - \chi_{K_2}$ by functions in $\mathrm{Re}(A_K)$ since such sets are removable for bounded harmonic functions (e.g. [14]). On the other hand, if both sets have positive logarithmic capacity one can show the harmonic extension of χ_{K_1} does not have a single valued harmonic conjugate, and neither will any sufficiently close approximating function (however, see Section 5). Thus K must be connected. In particular, each of the complementary components, $\{\Omega_j\}$, of K is simply connected. Thus we can fix conformal mappings $\{\Phi_j\}$ from D to each Ω_j. It is well known that these maps extend non-tangentially to almost every point of $\mathbb{T}$ and we let $\{\Phi_j\}$ also denote these extensions. Then following Glicksberg (in [31]) we say Ω_j is nicely connected if there is a set of full measure $E_j \subset \mathbb{T}$ so that Φ_j is injective on E_j (this is independent of the particular choice of Riemann mapping). The following is Theorem 4.3.1 in [16].

Theorem 1.1 (Davie): *For a compact connected set* K, *the following are equivalent*:

i) A_K *is a Dirichlet algebra on* K.

ii) A_K *is pointwise boundedly dense in* $H^\infty(\bar{\mathbb{C}}\setminus K)$.

iii) A_K *is strongly pointwise boundedly dense in* $H^\infty(\bar{C}\ K)$.

iv) *Each complementary component* Ω_j *of* K *is nicely connected and harmonic measures for different components are mutually singular*.

The equivalence of (i), (ii) and (iii) is a generalization of Theorem 2.1.3 and the references are exactly the same. Next, we should note that condition (iv) has a geometrical interpretation. The following two results are easily deduced from the arguments in Chapter I.

Lemma 1.2: *If* Ω_1 *and* Ω_2 *are disjoint and simply connected, with harmonic measures* ω_1 *and* ω_2, *then* $\omega_1 \perp \omega_2$ *iff the set of points in* $\mathbb{C}\setminus(\Omega_1 \cup \Omega_2)$ *which satisfy a double cone condition (with a cone in each* Ω_1 *and* Ω_2*) has zero linear measure*.

Lemma 1.3: *If* Ω *is simply connected, then* Ω *is nicely connected iff the set of points in* $\mathbb{C}\setminus\Omega$ *satisfying a double cone condition (with respect to* $\mathbb{C}\setminus\Omega$*) has zero linear measure*.

Thus we obtain:

Corollary 1.4: K *is a Dirichlet set iff it is connected and the set of points in* K *satisfying a double cone condition have zero linear measure*. (*Equivalently, iff the set of tangent points of* K

has zero linear measure).

In the remainder of this chapter we will give a new proof of Theorem 1.1. This proof is very similar to that given in Chapter Two, but sufficiently different (I hope) to justify giving both. As before, the main idea is to take an $f \in H^\infty(\Omega)$, $\Omega = \bar{\mathbb{C}}\setminus K$, modify it so it is "closer" to being continuous and then add a very small error term to make it holomorphic again.

2. Two $\bar{\partial}$ Lemmas

Before describing the construction in detail, we will review a few more facts about the $\bar{\partial}$ problem that we shall need later. The first corresponds roughly to the fact that given a closed set E of measure zero on $\mathbb{T}$ we can find a non-zero holomorphic function on D vanishing on E.

Lemma 2.1: *Suppose $\mathcal{O} \subset \mathbb{T}$ is open and satisfies $|\mathcal{O}| < \varepsilon$. Let $\mathcal{O} = \bigcup_j I_j$ be its decomposition into disjoint connected arcs, and let*

$$T_j = \{r\xi : 1 > r > \mathrm{dist}(\xi, I_j^c), \xi \in I_j\}$$

be the solid "tent" above I_j. Then for any $\beta > 0$, $\delta > 0$ there is an $\varepsilon > 0$ and a $\varphi \in C^\infty(D)$ such that:

i) $\varphi \equiv 0$ *on* $\{|z| \le 1 - \beta\}$

ii) $\varphi \equiv 1$ *on* T_j, *all* j.

iii) *For any bounded, continuous function b on D, the equation*

$$\bar{\partial}B = b\bar{\partial}\varphi$$

has a solution with $\|B\|_D < \delta \cdot \|b\|_D$.

The proof is essentially due to Garnett and Jones (see [28]). Consider the Hardy-Littlewood maximal function of $\chi_{\mathcal{O}}$

$$m(\xi) \equiv M(\chi_{\mathcal{O}})(\xi) = \sup_{\xi \in I} \frac{|\mathcal{O} \cap I|}{|I|} .$$

It is clearly equal to one on $\mathcal{O}$ and is lower semi-continuous. Fix $N \in \mathbb{N}$ and suppose $|I_j^n|$ are the intervals of the open set $\{m > 2^{-n}\}$, $1 \le n \le N$. Clearly

$$\sum_{I_j^n \subset I_k^{n+1}} |I_j^n| \le 1/2 |I_k^{n+1}|$$

and if $I_j^n \subset I_k^{n+1}$ then

$$|I_j^n| \le 1/2 \operatorname{dist}(I_j^n, (I_k^{n+1})^c)$$

Then if γ is the contour consisting of all the "tents"

$$\gamma_j^n = \{r\xi : r = \operatorname{dist}(\xi, (I_j^n)^c), \xi \in I_j^n\}$$

one can show arclength on γ is a Carleson measure with norm bounded independent of N. If ε is small enough the longest arc in $\{I_j^n\}$ has length less than $\beta/2$. We define φ, constant on each component of $D\backslash\gamma$, by

$$\varphi(z) = \sum_{n=1}^{N} \frac{1}{N} \chi_{T_j^n}(z)$$

where T_j^n is the solid tent over I_j^n. Thus φ satisfies (i) and (ii) and by "smoothing" it, we can apply Lemma 2.3.2 to get (iii). This proves the lemma.

The next fact we wish to recall involves solving a $\bar{\partial}$ problem, not on the unit disk, but on an arbitrary simply connected domain Ω. Fix an $\eta > 0$ and consider the grid of squares of the form

$$Q = \{z = x+iy : k\eta \le x \le (k+1)\eta, j\eta \le y \le (j+1)\eta\}$$

for integers k and j. Let C be the collection of such squares satisfying

$$Q \cap \Omega \neq \emptyset , \quad \text{dist}(Q,\partial\Omega) \leq 3\cdot\eta .$$

Set

$$\Omega_\eta = \bigcup_C Q \cap \Omega$$
$$\Gamma_\eta = \bigcup_C \partial Q \cap \Omega$$

(see Figure 19). Then we have:

Lemma 2.2: Suppose Ω is simply connected and Γ_η is as above. Also, suppose $\varphi \in C^\infty(\Omega)$ satisfies

i) $\text{supp}(\nabla\varphi) \subset \{z : \text{dist}(z,\Gamma_\eta) \leq \delta\ \text{dist}(z,\partial\Omega)\}$.

ii) $|\nabla\varphi(z)| \leq \varepsilon\cdot\delta^{-1}\cdot\text{dist}(z,\partial\Omega)^{-1}$.

Then if δ is small enough, there is a universal $C > 0$ such that for any bounded, continuous function b on Ω, there is a solution to

$$\bar\partial B = b\bar\partial\varphi$$

which satisfies $\|B\|_\Omega \leq C\cdot\varepsilon\cdot\|b\|_\Omega$. (In particular, C does not depend on Ω or η).

The proof is exactly the same as that of Lemma 2.3.2 once we know that any collection of points, $\{z_j\}$, on $\Gamma_\eta \cap \Omega$ which satisfies

$$|z_m - z_n| > \frac{1}{10}\text{dist}(z_n,\partial\Omega), \quad n \neq m$$

is an interpolating sequence for $H^\infty(\Omega)$. This is essentially proven by Garnett, Gehring and Jones in [27]. Actually, they only consider the case when $\{z_n\} \subset \Omega \cap \mathbb{R}$, but the desired result is an easy consequence. Divide the squares into 64 subcollections C_j, $1 \leq j \leq 64$ so that

$$Q_1,Q_2 \in C_j \Rightarrow \text{dist}(Q_1,Q_2) \geq 8\cdot\eta$$

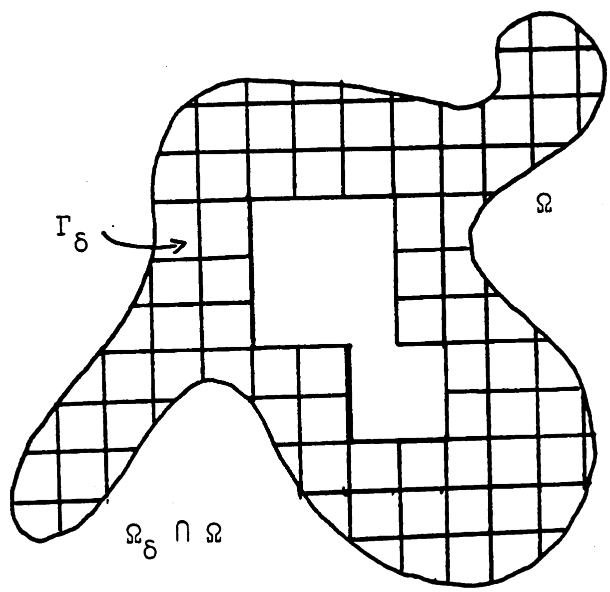

Figure 19: Γ_δ and Ω_δ

and divide Γ into 128 subcontours $\{\Gamma_j\}$, each corresponding to taking one side from each square in a given C_j. The proof in [27], together with easy estimates on harmonic measure, shows that $\{z_n\} \cap \Gamma_j$ is interpolating for $H^\infty(\Omega)$. This is enough to prove the lemma.

We should also mention that this lemma essentially corresponds to the theorem of Hayman and Wu which states that if $\Phi : D \to \Omega$ is conformal, then arclength on $\Phi^{-1}(\mathbb{R})$ is a Carleson measure with norm bounded independently of Ω and Φ (see [33] or [27]).

3. The Construction

We now turn to the proof of Theorem 1.1. If condition (iv) fails, then exactly the same argument as in Section 2.2 shows (i)-(iii) also fail. Therefore we need only show the other direction. So assume (iv) holds. In this section we shall show A_K is strongly pointwise boundedly dense in $H^\infty(\Omega)$, $\Omega = \bar{\mathbb{C}} \setminus K$. We will than prove A_K is Dirichlet on K.

First we introduce some notation concerning the continuity of a function f defined on $\mathbb{C}$. We set

$$J(f,\delta,x) = \text{diameter}(f(D(x,\delta)))$$

$$= \sup\{|f(z)-f(w)| : z,w \in D(x,\delta)\}$$

$$J(f,x) = \lim_{\delta \to 0} J(f,\delta,x)$$

$$J(f) = \sup_x J(f,x).$$

Clearly $J(f,x)$ is upper semi-continuous, and f is continuous at x iff $J(f,x) = 0$.

Now take $f \in H^\infty(\Omega)$, $\|f\|_\Omega = 1$. It is enough to show f can be uniformly approximated on compact sets, so fix a compact set $\tilde{K}$ in Ω and a $\eta > 0$. We will construct a sequence $\{F_n\}$ in $H^\infty(\Omega)$ which satisfies

(3.1) $$\|f-F_n\|_{\tilde{K}} \le \eta$$

(3.2) $$\|F_n\|_\Omega \le 1 - 2^{-n}\cdot\eta < 1$$

(3.3) $$\|F_n-F_{n+1}\|_\Omega \le 95/100\cdot J(F_n)$$

(3.4) $$J(F_{n+1}) \le 95/100\ J(F_n).$$

Such a sequence obviously converges to the desired function F.

We start by setting $F_1 = (1-\frac{\eta}{2})f$. In general, given F_n we will define F_{n+1} in two steps. First we modify it to obtain a function g_n such that

$$g_n = F_n \quad \text{on } \tilde{K}$$

$$\|g_n\|_\Omega < 1$$

$$\|g_n-F_n\|_\Omega < 89/100\cdot J(F_n)$$

$$J(g_n) < 90/100\cdot J(F_n)$$

Unlike Chapter 2 this g_n is not continuous, but merely has smaller jump. Like Chapter 2, g_n is not holomorphic, so we need to find h_n such that

$$\bar\partial h_n = -\bar\partial g_n \quad \text{on } \Omega$$

$$\|h_n\|_{\tilde{K}} \le 2^{-n-2}\cdot\eta$$

$$\|h_n\|_\Omega \le 1/100\ J(F_n)$$

Then $F_{n+1} \equiv g_n + h_n \in H^\infty(\Omega)$ satisfies (3.1)-(3.4), as required.

To ease notation, we shall drop the "n" and write $F = F_n$, $g = g_n$ and $h = h_n$. Our first step in defining g to to construct a smooth function H on a neighborhood of ∂K which approximates F, i.e., satisfies

$$(3.5) \qquad |H(z)-F(z)| \le \frac{88}{100} J(F).$$

To do this, first choose a $\delta = \delta_1 > 0$ so that

$$\delta < \frac{1}{10}\operatorname{dist}(\tilde{K},K)$$

and so small that

$$\sup_x J(F,10\cdot\delta,x) < \frac{100}{99} J(f).$$

The key observation is that if E is a planar set and $z,w \in \bar{E}$ satisfy $|z-w| = \operatorname{diam}(E)$, then

$$E \subset \bar{D}(\tfrac{1}{2}(z+w),(\sqrt{3}/2)\cdot\operatorname{diam}(E))$$
$$\subset D(\tfrac{1}{2}(z+w),(87/100)\cdot\operatorname{diam}(E))$$

since $\sqrt{3}/2 = .86602\cdots < 87/100 < 1$. To define H first consider points of the form $z = n\delta + im\,\delta$, $n,m \in \mathbb{Z}$, and $\operatorname{dist}(z,\partial K) < 10\cdot\delta$ and for such a z choose $H(z)$ so

$$F(B(z,3\delta)) \subset D(H(z),\frac{88}{100} J(F)).$$

(This is possible by the above observation applied to $E = F(D(z,3\delta))$ which satisfies $\sqrt{3}/2 \operatorname{diam}(E) \le 88/100\ J(F)$).

Next, if z is in the square formed by four adjacent points $\{z_1,\ldots,z_4\}$ as above, then

$$F(D(z,\delta)) \subset F(\bigcap_{i=1}^{4} D(z_i,3\delta))$$
$$\subset \bigcap_{i=1}^{4} F(D(z_i,3\delta))$$

$$\subset \bigcap_{i=1}^{4} D(H(z_i), \frac{88}{100} J(F))$$

$$\subset D(w, \frac{88}{100} J(F))$$

for any w in the convex hull of $\{F(z_1),\dots,F(z_4)\}$. Therefore we can extend H smoothly to a neighborhood of ∂K so that it satisfies (3.5) and $\|H\|_\Omega \le \|F\|_\Omega$.

Now suppose $\varepsilon = \varepsilon_1 > 0$ is small (to be fixed later). Since H is uniformly continuous on a neighborhood of ∂K we can choose a $\delta = \delta_2 > 0$ so that $|z_1 - z_2| < \delta$, $\text{dist}(z_1, \partial K) < \delta$ imply $|H(z_1) - H(z_2)| < \varepsilon$. Consider the grid of squares formed by the lattice $\delta\mathbb{Z} + i\delta\mathbb{Z}$ (as in Section 2) and let $C = C_\delta$ be the subcollection of such squares satisfying

$$\text{dist}(Q,K) < 3\cdot\delta$$

and set $\Omega_\delta = \bigcup_C Q$, $\Gamma_\delta = \bigcup_C \partial Q$. Define an approximation to H which is constant on each square (e.g., the value of H at the center), fix $\delta_3 > 0$ and then "smooth out" this function to obtain an $\tilde{H} \in C^\infty(\Omega \cap \Omega_\delta)$ satisfying

$$\|\tilde{H}\|_\Omega \le \|F\|_\Omega$$

$$\|\tilde{H} - F\|_\Omega \le \frac{88}{100} J(F) + \varepsilon \le \frac{89}{100} J(F)$$

$$\text{supp}(\nabla\tilde{H}) \subset \{z : \text{dist}(z,\Gamma_\delta) \le \delta_3 \text{dist}(z,\partial\Omega)\}$$

$$|\nabla\tilde{H}(z)| \le \varepsilon\cdot(\delta_3 \text{dist}(z,\partial\Omega))^{-1} .$$

In particular, $\tilde{H}$ satisfies the hypotheses of Lemma 2.2 for each complementary component Ω_j.

The function g will be written as a convex combination of the values of F and $\tilde{H}$, i.e.,

$$g = (1-\varphi)F + \varphi\cdot\tilde{H}$$

where $0 \le \varphi \le 1$. We define φ by constructing its restriction to each complementary component Ω_j. First of all, note that all but finitely many of the Ω_j lie inside $\Omega_{\delta/4}$ (otherwise infinitely many would contain a disk of radius $\delta/4$, contradicting the boundedness of K). On these we define $\varphi \equiv 1$.

For the remaining components, say $\Omega_1,\ldots,\Omega_N$, we fix Riemann mappings $\Phi_j : D \to \Omega_j$, $1 \le j \le N$. By compactness we can choose a $\beta > 0$ so that

$$\Phi_j^{-1}(\Omega \setminus \Omega_{\delta/4}) \subset \{|z| < 1 - \beta\}$$

for each $j = 1,\ldots,N$. Using Fatou's theorem (applied to both Φ_j and $F\circ\Phi_j$) and hypothesis (iv) of Theorem 1.1 we can find compact subsets $E_j \subset \mathbb{T}$ which satisfy

(3.6) Φ_j and $F\circ\Phi_j$ have non-tangential limits at each point of E_j,

(3.7) Φ_j is injective on E_j,

(3.8) $\Phi_j(E_j)$, $1 \le j \le N$, are N disjoint, compact subsets of K,

(3.9) $|\mathbb{T}\setminus E_j| < \varepsilon = \varepsilon_2$

for any given ε_2. We now apply Lemma 2.1 to each $\mathbb{T}\setminus E_j$ with β as above and $\delta < \min(2^{-n-2}\cdot\eta, \frac{1}{100})$, to get φ_j. We define φ on Ω_j as $\varphi = \varphi_j\circ\Phi_j^{-1}$.

If we write

$$g = (1-\varphi)F + \varphi\cdot\tilde{H}$$

then g clearly satisfies the first three conditions it's supposed to, so we need only estimate $J(g)$. Choose $x \in \partial\Omega$, $\delta > 0$ and suppose

$z, w \in D(x,\delta) \cap \Omega$. If $\varphi(z) = 1$, then

$$\begin{aligned} |g(z)-g(w)| &= |\tilde{H}(z) - (1-\varphi(w))F(w) - \varphi(w)\tilde{H}(w)| \\ &\le |\tilde{H}(z)-\tilde{H}(w)| + (1-\varphi(w))|F(w) - \tilde{H}(w)| \\ &\le \varepsilon_1 + \frac{89}{100} J(F) \\ &\le \frac{90}{100} J(F) \end{aligned}$$

if ε_1 is small enough. Similarly, if $\varphi(w) = 1$. Thus we may assume $\varphi(w), \varphi(z) < 1$. By construction $\operatorname{supp}(1-\varphi)$ lies in the union of N disjoint, compact sets, each corresponding to one of the components $\Omega_1, \ldots, \Omega_N$. So if δ is small enough, z and w must lie in the same component, say Ω_1. Also by our construction, F is continuous on $\operatorname{supp}(1-\varphi)$ so $|F(z)-F(w)|$ is small if δ is small. Thus

$$\begin{aligned} |g(z)-g(w)| &\le |\tilde{H}(z)-\tilde{H}(w)| + |\tilde{H}(w)-F(w)| + |F(w)-F(z)| \\ &\le \varepsilon_1 + \frac{89}{100} J(F) + \varepsilon(\delta) \\ &\le \frac{90}{100} J(F) \end{aligned}$$

if ε_1 and δ are small enough. Hence

$$J(g,\delta,x) \le \frac{90}{100} J(F)$$

as required.

We now take care of the fact that g is not holomorphic. By our construction and Lemmas 2.1 and 2.2, each term of

$$-\bar{\partial} g = \bar{\partial}\varphi \cdot F - \bar{\partial}\varphi \cdot \tilde{H} - \varphi \cdot \widetilde{\bar{\partial} H}$$

corresponds to a $\bar{\partial}$ problem on each Ω_j that we know how to solve with uniform estimates. Thus we can find h such that

$$\begin{aligned} &\bar{\partial} h = -\bar{\partial} g \\ &\|h\|_\Omega \le \min(2^{-n-2}\eta, \frac{1}{100} J(F)) \end{aligned}$$

as desired. Thus we have proven that A_K is strongly pointwise boundedly dense in $H^\infty(\Omega)$.

4. Proving A_K is Dirichlet

Next we wish to show A_K is a Dirichlet algebra on K. Suppose g is real-valued and continuous on K and let u be its harmonic extension to Ω. Since each component of Ω is simply connected, u has a well defined harmonic conjugate u^* on Ω. We wish to apply the construction of Section 2 to $f = u + iu^*$, but u^* may be unbounded, so first we must find a $F \in H^\infty(\Omega)$ such that

$$\|Re(f) - Re(F)\|_\Omega \leq \eta$$

where η is given. What we need is:

<u>Lemma</u> 4.1: <u>Suppose</u> Ω <u>is simply connected and</u> u <u>is harmonic on</u> Ω, <u>with continuous extension to</u> $\bar{\Omega}$. <u>Then for any</u> $\eta > 0$ <u>there is a</u> $F \in H^\infty(\Omega)$ <u>with</u>

$$\|u - Re(F)\|_\Omega < \eta.$$

To prove this, let $\Phi: D \to \Omega$ be conformal and $\tilde{u} = u \circ \Phi$. First note that $\tilde{u} \in VMO$ (see [26], page 250). For if $z \in D$, $w = \Phi(z)$, $r = dist(w, \partial\Omega)$, then by Beurling's solution of the Carleman-Miloux problem (see Section 1.5),

$$\int_{\mathbb{T}} |\tilde{u}(\xi) - \tilde{u}(z)| P_z(\xi) d\xi = \int_{\partial\Omega} |u(\zeta) - u(w)| d\omega_w(\zeta)$$

$$\leq \sum_{n=0}^{\infty} J(u, 2^n \cdot r, w) \cdot \omega_w(D(w, 2^{n-1} \cdot r))$$

$$\leq C \cdot \sum_{n=0}^{\infty} J(u, 2^n \cdot r, w) 2^{-n/2}.$$

Since u is continuous, this tends to zero uniformly as r does and this proves $\tilde{u} \in VMO$. In particular, it proves that for any $\varepsilon > 0$ we can write

$$\tilde{u} = v + w$$

where v is harmonic and continuous on $\bar{D}$ and $\|w\|_{BMO} < \varepsilon$ (w restricted to $\mathbb{T}$). By Varopoulos' theorem (see Section 2.6), we can find an extension $\tilde{w}$ of $w|_{\mathbb{T}}$ so that $|\nabla\tilde{w}\, dxdy|$ is a Carleson measure of norm $C \cdot \varepsilon$. Thus we can solve the $\bar{\partial}$ problem

$$\bar{\partial} b = -\bar{\partial}\tilde{w}$$

with $\|b\|_{L^\infty(\mathbb{T})} \leq \eta/2$ if ε is small enough. Then if we choose r close enough to one so $|v(z)-v(rz)| < \eta/2$ for $z \in D$,

$$F(z) \equiv v(rz) + iv^*(rz) + \tilde{w}(z) + b(z)$$

(v^* = conjugate of v) defines a bounded, holomorphic function on D and

$$\begin{aligned} |\mathrm{Re}(F(z))-\tilde{u}(z)| &\leq |v(z)-v(rz)| + |w(z)-\tilde{w}(z) - \mathrm{Re}(b(z))| \\ &\leq \eta/2 + \eta/2 \end{aligned}$$

as required. (I would like to thank Peter Jones for pointing out this argument to me).

We now return to the problem of finding a bounded function F approximating f. Let $\varepsilon > 0$ (to be fixed later). Since u is continuous on $\bar{\mathbb{C}}$, there is a $\delta > 0$ such that $|z-w| < \delta$ implies $|u(z)-u(w)| < \varepsilon$. For this δ, consider Ω_δ, Γ_δ as in the previous section. If Ω_j is a component of $\bar{\mathbb{C}}\backslash K$ such that $\Omega_j \subset \Omega_\delta$, then we can approximate u by a function $\tilde{u}$ satisfying the hypotheses of Lemma 2.2. Thus we can solve

$$\bar{\partial} v = -\bar{\partial}\tilde{u}$$

with

$$\|v\|_{\Omega_j} \leq C\cdot\varepsilon$$

and so $F = \tilde{u} + v$ is the desired function on Ω_j (if ε is small enough). On the finitely many remaining components we apply Lemma 4.1. Thus we have the desired F.

We now apply the construction to F to obtain a sequence $\{F_n\}$ which, in addition to (3.1)-(3.4), satisfies

$$J(\mathrm{Re}(F-F_n)) \leq (1-2^{-n})\cdot\eta. \tag{4.1}$$

This can be done by taking $\delta = \delta_1$ in the definition of H so small that

$$J(\mathrm{Re}(F-F_n),10\cdot\delta,x) \leq (1-2^{-n} + \frac{2^{-n-1}}{3})\cdot\eta$$

for all x. From this we get

$$\|\mathrm{Re}(H)-\mathrm{Re}(F_n)\|_{\Omega} \leq (1-2^{-n} + \frac{2^{-n-1}}{3})\cdot\eta$$

Taking ε_1 and ε_2 small enough we get

$$\|h_n\|_{\Omega} \leq 1/3\cdot 2^{-n-1}$$

$$\|H-\tilde{H}\|_{\Omega} \leq 1/3\cdot 2^{-n-1}$$

which gives (since H is continuous),

$$J(\mathrm{Re}(F)-\mathrm{Re}(F_n)) \leq (1-2^{-n-1})\cdot\eta$$

as required. This proves A_K is a Dirichlet algebra on K.

5. Finitely Connected Sets

The conditions described in Sections 2.11 and 2.12 for curves also characterize the sets K such that A_K is pointwise boundedly dense in $H^\infty(\bar{\mathbb{C}} \setminus K)$. If K is connected the construction in this chapter proves these characterizations. It also works if K has only finitely many components (with some minor modifications). I do not know how to use these techniques to prove these characterizations for arbitrary compact sets K.

We should note that if K has finitely many components, $K_1, \ldots, K_N$, then the proof that A_K is Dirichlet on K still works, except for two problems. First, if some component K_j is a single point, then it is a removable singularity for bounded holomorphic functions. Hence the value of a function $f \in \mathrm{Re}(A_K)$ on K_j is determined by its values on the other components. Second, if $K_1, \ldots, K_n$, $n \le N$, are the non-degenerate components of K, then any continuous, real-valued function g on $K_1 \cup \cdots \cup K_n$ has a continuous, harmonic extension u, but u need not have a single valued harmonic conjugate. However, for an appropriate choice of $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$, the harmonic extension of

$$\tilde{g} = g + \sum_{j=1}^{n} \lambda_j \chi_{K_j}$$

does have a well defined conjugate. Thus A_K fails to be a Dirichlet algebra but $\mathrm{Re}(A_K)$ has finite codimension in $C(K, \mathbb{R})$ (equal to $N-1$, the number of components minus 1). This is usually stated by saying A_K is <u>hypo-Dirichlet</u> on K (e.g., see [24]).

6. Three Open Problems

As a final remark in this chapter we mention three problems. Each has a more abstract flavor than the problems we have considered so far, but perhaps they can be attacked by similar ideas.

Does A_K have a Shauder basis (when considered as a Banach space with the sup norm)? The answer is probably yes, but I don't know how to prove it. The corresponding question for the disk algebra was answered affirmatively in 1976 by S.V. Bočkarev (see [7]).

Does A_K have a finitely generated dense subalgebra? A related question is whether there exists a curve Γ in $\mathbb{C}$ such that A_Γ contains a finitely generated subalgebra whose maximal ideal space is $\bar{\mathbb{C}}$. This is related to a conjecture about function algebras on curves in $\mathbb{C}^n$, and is discussed more carefully in [32].

Does there exist a proper, closed subalgebra of the disk algebra, $A(D)$, which is a Dirichlet algebra on $\mathbb{T}$ and whose maximal ideal space is $\bar{D}$ (i.e., the same as $A(D)$'s)? The algebras A_ψ defined in Section 2.8 are proper, closed Dirichlet subalgebras of $A(D)$, but their maximal ideal spaces are homeomorphic to $\bar{\mathbb{C}}$. (I would like to thank John Wermer for mentioning this problem to me.)

CHAPTER IV

A COUNTEREXAMPLE IN CONFORMAL WELDING

1. Statement of Results

As before, if Γ is a closed Jordan curve we let Ω_1 and Ω_2 denote the two complementary components, and for given $z_1 \in \Omega_1$, $z_2 \in \Omega_2$ we let ω_1 and ω_2 denote the harmonic measures with respect to these points. The goal of this chapter is to prove:

Theorem 1.1: *For any* $1 \le d < 2$ *there is a quasicircle* Γ, *a* $C > 0$ *and points* $z_1 \in \Omega_1$, $z_2 \in \Omega_2$ *such that* $\dim(\Gamma) = d$ *and for any Borel* $E \subset \Gamma$,

$$C^{-1} \le \frac{\omega_1(E)}{\omega_2(E)} \le C. \tag{1.1}$$

The inequalities imply that the associated homeomorphism ψ (Section 2.3) is biLipschitz. Furthermore, one can show that A_Γ contains a nonconstant function. Thus we obtain:

Corollary 1.2: *There is a biLipschitz, orientation reversing homeomorphism* ψ *of* $\mathbb{T}$ *to itself and a non-constant* $f \in A(D)$ *such that* $f \circ \psi \in A(D)$.

Before discussing the proof, we should mention some related results, and following the usual conventions we consider ψ as an increasing homeomorphism of $\mathbb{R}$ (instead of $\mathbb{T}$) determined by mapping the upper and lower half-planes to Ω_1 and Ω_2. Then a very well known result (see [2]) says that Γ is a quasicircle (Section 1.7) iff there is a $C > 0$ so that

$$C^{-1} \le \frac{\psi(x+t)-\psi(x)}{\psi(x)-\psi(x-t)} \le C \tag{1.2}$$

for any $x,t \in \mathbb{R}$. In particular, if ψ is biLipschitz it automatically satisfies this condition. Also, if ψ is any increasing homeomorphism satisfying (1.2) it corresponds to some quasicircle Γ. We say Γ is a chord-arc curve if Γ is locally rectifiable and there is a $C > 0$ such that

$$\ell(z,w) \le C|z-w| \quad , \quad z,w \in \Gamma$$

where ℓ denotes the arclength on Γ between two points. By a theorem of David ([15]) this holds with C close to 1 iff ψ is absolutely continuous and $\log(\psi')$ is in BMO with small norm. This happens if ψ is biLipschitz with constant close to 1, so Theorem 1.1 fails if C is close to 1. Since A_T is trivial if Γ is locally rectifiable, Corollary 1.2 also fails in this case.

A related example is given in [45], where Semmes constructs a non-locally rectifiable curve Γ satisfying (1.1). Also, in [30] Garnett and O'Farrell give an example of an absolutely continuous ψ on $\mathbb{T}$ and a non-constant $f \in A(D)$ with $f \circ \psi \in A(D)$.

In the next section we motivate the construction, and in Section 3 we prove some simple estimates on harmonic measure. In Section 4 we prove Theorem 1.1.

2. The Basic Construction

The way we will build a curve with a given dimension, is to pass a curve through a Cantor set with the desired dimension, constructed as follows.

Fix an α, $0 < \alpha < 1/2$, and subdivide the unit square into four squares of side length α by removing a "cross" from the center (see Figure 20). Each of these squares is then divided in four squares of side length α^2, and so on, obtaining at the n^{th} stage 4^n squares of side length α^n. The limiting set is the Cantor set $E = E(\alpha)$. Since E can be covered by 4^n disks of radius α^n one easily shows

$$\dim(E) \leq \log 4/-\log \alpha \equiv d_\alpha$$

To prove equality consider the usual singular measure μ on E which gives mass 4^{-n} to each n^{th} generation square. We wish to show

$$\mu(Q) \leq C\cdot\ell(Q)^{d_\alpha} \tag{2.1}$$

where $\ell(Q)$ is side length of Q. If Q is a n^{th} generation square then

$$\mu(Q) = 4^{-n} = (\alpha^n)^{-\log 4/\log \alpha} = \ell(Q)^{d_\alpha}$$

For a general Q with $\ell(Q) \leq 1$, choose n so that $\alpha^n \leq \ell(Q) < \alpha^{n-1}$ and let $\{Q_j\}$, $1 \leq j \leq N$, be the collection of n^{th} generation squares which intersect Q. Then $E \cap Q \subset \cup Q_j$ and $N < 9\alpha^{-2}$, so

$$\mu(Q) \leq N\ell(Q)^{d_\alpha} \leq C\cdot\ell(Q)^{d_\alpha}$$

Thus (2.1) holds. Now suppose $\{D_j\}$ is any covering of E by disks of radius $\{r_j\}$, $r_j \leq 1$. Then

$$0 < \mu(E) \leq \sum_j \mu(D_j) \leq C \sum_j (r_j)^{d_\alpha}$$

and so $\dim(E(\alpha)) = d_\alpha$.

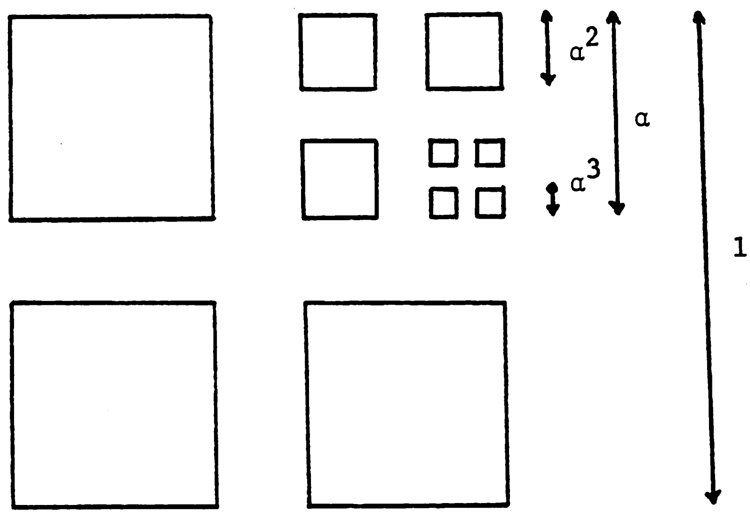

Figure 20: Constructing the Cantor set

Now we want a curve which contains $E(\alpha)$. Consider Figure 21. It shows a square of side length $(1+\delta)$ with four squares of the side length $\alpha(1+\delta)$ removed, one around each 1^{st} generation square of E. In order for these small boxes not to overlap, we need $\delta < (1-2\alpha)/2\alpha$, say $\delta = (1-2\alpha)$. We then draw a curve Γ in the remaining portion of the large square, consisting of 5 connected arcs, connecting the midpoints of the vertical sides of the squares, as shown.

Now dilate this picture by α and place one copy in each of the smaller squares, and extend Γ by adding the corresponding arcs. Iterating this construction and adding $E(\alpha)$ we obtain a curve Γ resembling Figure 22. Since $E(\alpha) \subset \Gamma$ and Γ is smooth away from E,

$$\dim(\Gamma) = \dim(E) = d_\alpha = \log 4/-\log \alpha$$

which varies from 1 to 2 as α goes from 1/4 to 1/2. Moreover, it is easy to see A_Γ is non-trivial since inequality (2.1) implies that

$$F(z) \equiv 1/z * \mu = \int \frac{d\mu(w)}{z-w}$$

defines a non-constant element of A_E.

Furthermore, one can verify that for any $z \notin \Gamma$, $\omega(z,\Gamma\setminus E,\Gamma^c) > \varepsilon > 0$ uniformly, so that $\omega(z,E,\Gamma^c) = 0$, as in Section 1.8. Since Γ is smooth elsewhere, the two harmonic measures on Γ are mutually continuous and so the corresponding homeomorphism ψ of $\mathbb{T}$ is absolutely continuous. This is essentially the construction of Garnett and O'Farrell mentioned in Section One.

Unfortunately, this curve will not satisfy (1.1). For example, if we consider Figure 22 and the portion of Γ in the dotted box, it

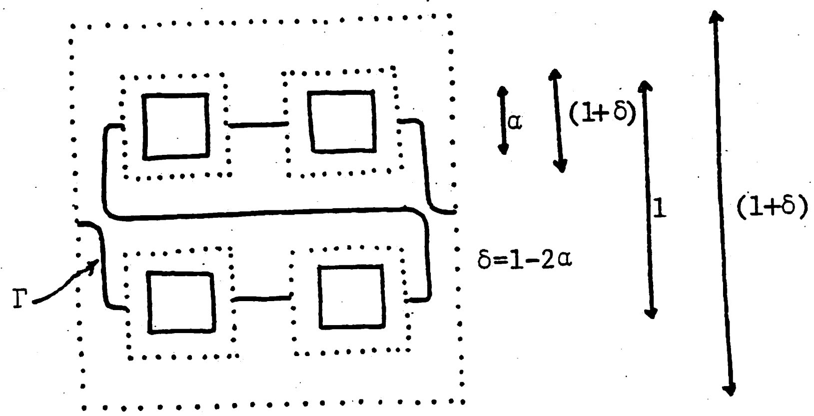

Figure 21: A building block

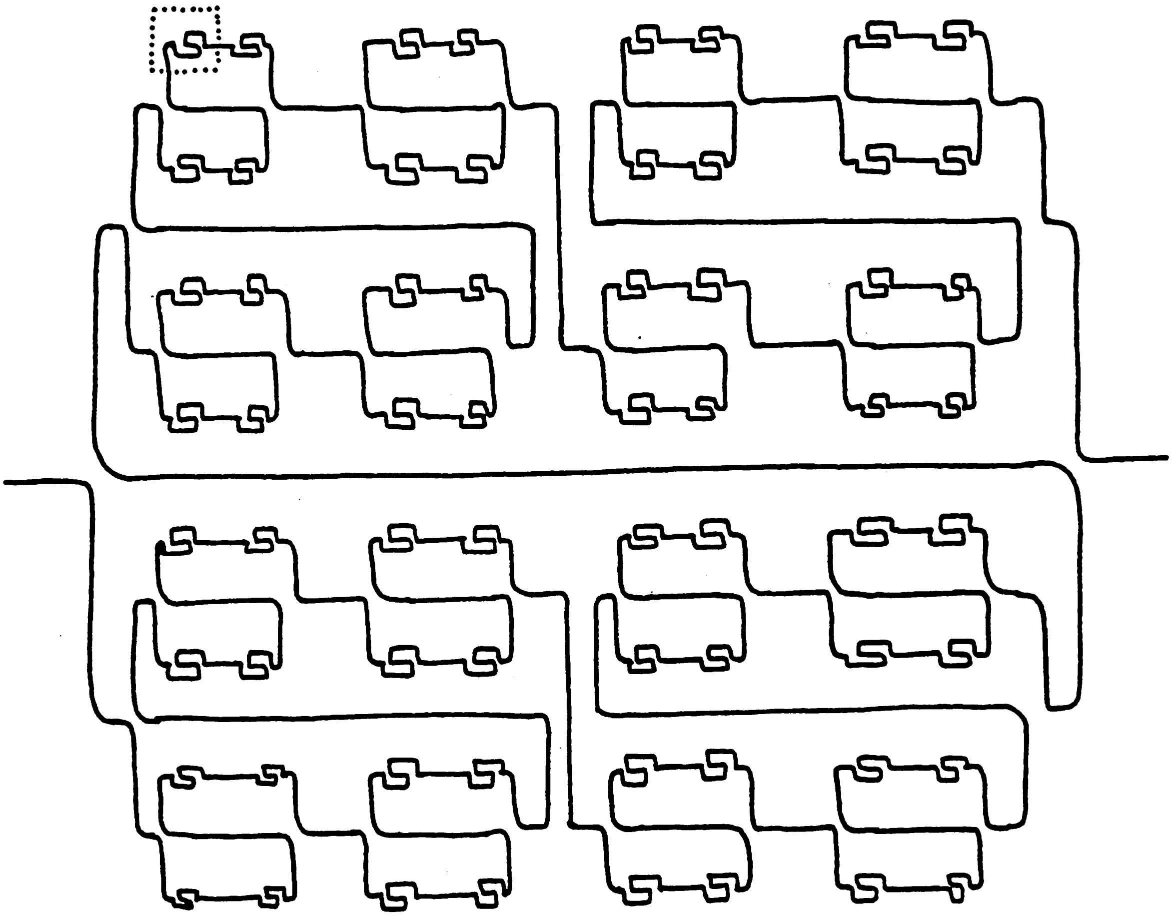

Figure 22: A curve with $\dim(\Gamma) > 1$

has much larger harmonic measure from one side than the other, and this gets worse on smaller scales. Part of the problem is that portions of Γ lying in two different n^{th} generation boxes can have very different harmonic measures (with respect to a point outside the 1^{st} generation box) and this ratio grows with n. We will modify the construction to obtain a Γ so that the harmonic measures of any two n^{th} generation squares is roughly the same, uniformly in n.

We begin by replacing Figure 21 with the more complicated "building block" pictured in Figure 23 and iterating to get Γ. This Γ still has the problem that not all n^{th} generation squares get the same harmonic measure, so we modify it as follows. We think of the n^{th} stage square as being connected to a $(n-1)^{st}$ stage square by a long narrow tube (see arrows in Figure 23). If a particular n^{th} generation box is getting more than its "fair share" of harmonic measure, we "pinch off" the tube leading to it to decrease the measure (see Figure 24). The main problem is to show that perturbing the curve to "fix" harmonic measure in one place does not ruin the estimates elsewhere.

3. Some Estimates on Harmonic Measure

In this section we will prove some very simple lemmas which quantify the idea that a local perturbation of Γ does not greatly effect harmonic measure far away.

We will only need to consider domains constructed as follows. Take two disjoint squares Q_1 and Q_2 (possibly of different sizes) with centers z_1 and z_2. For $i = 1,2$, let D_i be the disk with

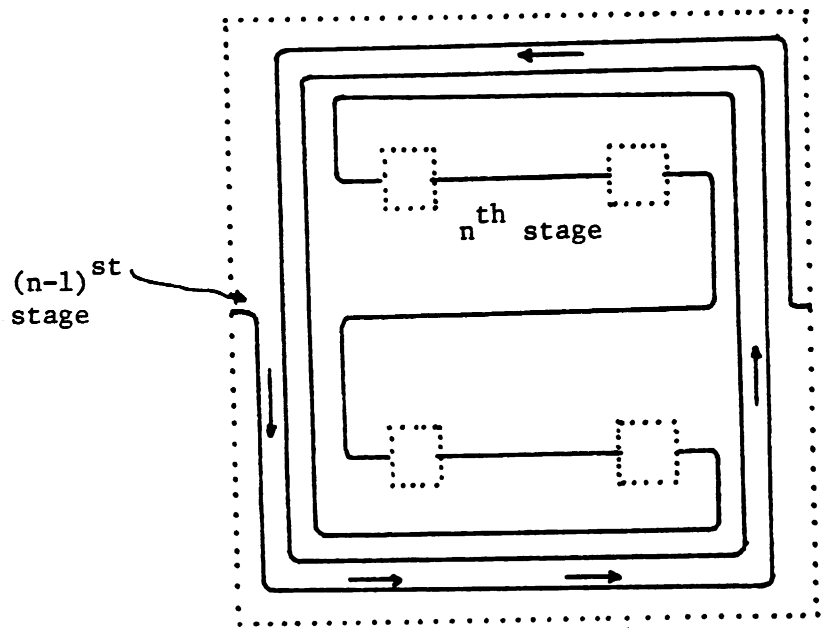

Figure 23: Another building block

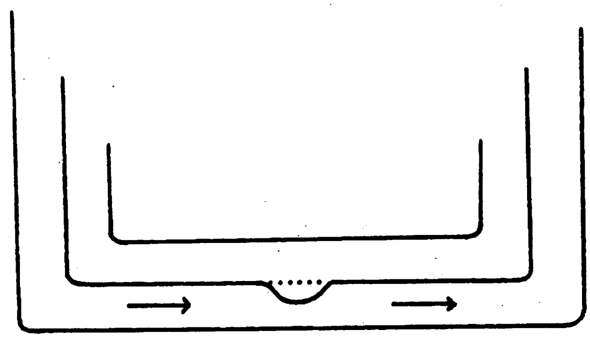

Figure 24: "Pinching" the tube

center z_i and diameter 1/4 that of Q_i (see Figure 25). Replace one side of Q_1 with a tent shaped arc σ_1 in Q_1, as in Figure 25, and let C_1 (for "collar") denote the middle thirds of the two adjacent sides. Similarly for σ_2 and C_2.

Now let Ω_0 be as in Figure 26, a Jordan domain bounded by arcs σ_1, σ_2, Γ_1 and Γ_2 and each Γ_i hits each square exactly along an edge. We obtain another domain Ω from Ω_0 by replacing σ_1 and σ_2 by arcs Σ_1 and Σ_2 which do not hit $Q_1 \cup \Omega_0 \cup \Omega_2$. We let Ω_1 be the subdomain of Ω bounded by σ_1 and Σ_1 and similarly for Ω_2 (see Figure 27). Thus we think of Ω_0 as a "tube" connecting Ω_1 and Ω_2.

Because of the specific geometry, we can make a few immediate observations about harmonic measure on these domains. (In this section C is a positive constant, different at different places, but always independent of the particular choice of $\Omega, \Omega_0, \ldots$). For example, for any $z \in \Omega_1$,

$$C^{-1} \leq \frac{\omega(z, \partial D_1, \Omega \backslash D_1)}{\omega(z, C_1, \Omega)} \leq C$$

and both pieces of C_1 receive equivalent measure. Suppose $\tilde{\sigma}_1$ is the segment through z_1 connecting the midpoints of the two segments of C_1 and suppose $0 \leq u \leq 1$ is harmonic and vanishes on C_1. Then for $z \in \tilde{\sigma}_1$,

$$u(z) \sim u(z_1) \cdot \operatorname{dist}(z, C_1)/\operatorname{dist}(z_1, C_1) .$$

The "$\geq$" is Harnack's inequality and "$\leq$" is the obvious estimate for $\omega(z, C_1^c, \Omega)$. Using these observations we can prove:

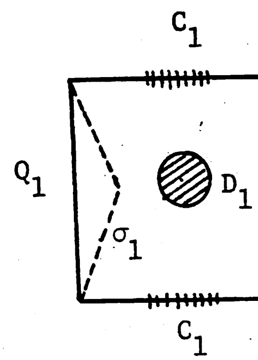

Figure 25: The square Q_1

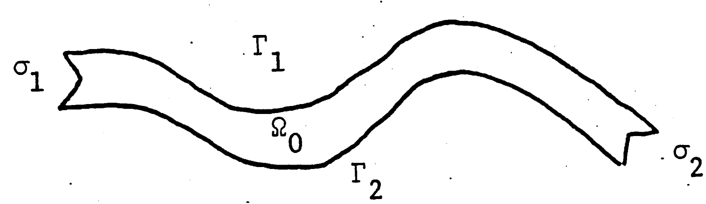

Figure 26: The domain Ω_0

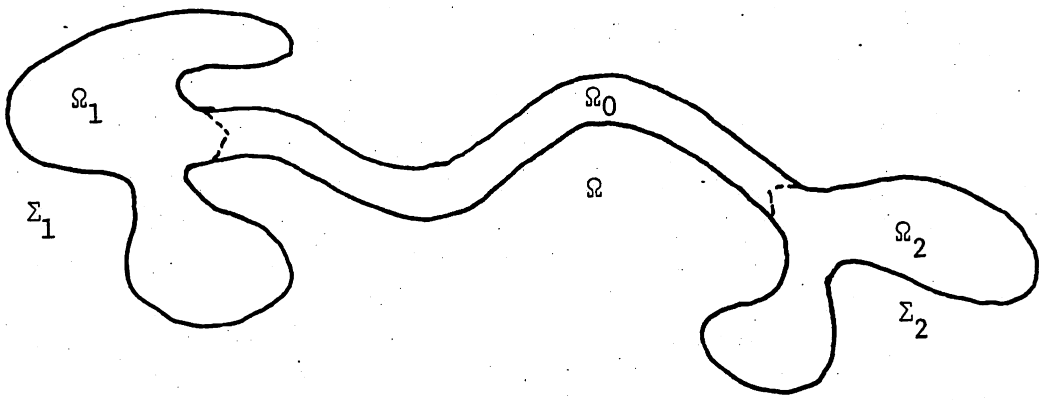

Figure 27: The domain Ω

Lemma 3.1: There is a $C > 0$ such that

$$C^{-1} \le \frac{\omega(z,E,\Omega)}{\omega(z,C_2,\Omega)\omega(z_2,E,\Omega)} \le C$$

for all $z \in \Omega_1$, and any (Borel) $E \subset \Sigma_2$.

Proof: To prove this, first note that by Harnack's inequality,

$$\omega(z,E,\Omega) > C\cdot\omega(z_2,E,\Omega)$$

for all $z \in D_2$. Thus for all $z \in \partial(\Omega \setminus D_2)$,

$$\omega(z,E,\Omega) \ge C\cdot\omega(z_2,E,\Omega)\cdot\omega(z,\partial D_2,\Omega\setminus D_2)$$

and so by the maximum principle this holds for all $z \in \Omega_1 \subset \Omega\setminus D_2$. By our early remarks, the right hand side is equivalent (for $z \in \Omega_1$) to

$$\omega(z_2,E,\Omega)\cdot\omega(z,C_2,\Omega)$$

which gives the right hand side of the lemma. On the other hand, for $z \in \tilde{\sigma}_2$

$$\omega(z,E,\Omega) \sim \omega(z_2,E,\Omega)\cdot \text{dist}(z,C_2)/\text{dist}(z_2,C_2)$$

$$\le C\cdot\omega(z_2,E,\Omega)\omega(z,C_2,\Omega) .$$

This also holds (trivially) for $z \in \Sigma_1 \cup \Gamma_1 \cap \Gamma_2$, so by the maximum principle it holds on Ω_1, proving the lemma. □

By a similar argument we can get:

Corollary 3.2: There is a $C > 0$ such that

$$C^{-1} \le \frac{\omega(z,E,\Omega)}{\omega(z,C_1,\Omega)\omega(z_1,C_2,\Omega)\omega(z_2,E,\Omega)} \le C$$

for all $z \in \Omega_1$, $E \subset \Sigma_2$.

We want to think of Ω_0 as a long, narrow tube connecting Ω_1 and Ω_2. We quantify this by assuming

$$\omega(z_1, C_2, \Omega) \le \delta$$
$$\omega(z_2, C_1, \Omega) \le \delta$$

for some very small δ. The following lemma makes precise the notion that perturbing Ω_2 does not really change harmonic measure in Ω_1.

Lemma 3.3: There is a $C > 0$ so that if δ is sufficiently small and $z \in \Omega_1$, $E \subset \Sigma_1$, then

$$1 \le \frac{\omega(z,E,\Omega)}{\omega(z,E,\tilde{\Omega})} \le 1 + C\cdot\delta$$

where $\tilde{\Omega} = \Omega_0 \cup \Omega_1 \subset \Omega$.

Proof: The left inequality is obvious since $\tilde{\Omega} \subset \Omega$. To get the other direction fix z and let Φ be a conformal mapping of Ω to the upper half-plane with $\Phi(z) = i$ and $\Phi(\Sigma_2)$ an interval of length r centered at zero. Then $\Phi(C_1)$ consists of two intervals each of size roughly r/δ. Thus there is a $A_1 > 0$ such that

$$\Phi(E) \subset \{|x| > A_1 \cdot r/\delta\}.$$

Since $\omega(z,\sigma_2,\tilde{\Omega}) \sim \omega(z,\Sigma_2,\Omega)$ and $\Phi(\sigma_2)$ is an arc with an endpoint on either side of zero, there is a A_2 such that

$$\Phi(\sigma_2) \subset \{|z| < A_2 \cdot r\}.$$

So if $\hat{\Omega} = \{\mathrm{Im}\, z > 0\} \cap \{|z| > A_2 r\}$, it is enough to show (by the maximum principle) that for $F \subset \{|x| > A_1 \cdot r/\delta\}$

$$\omega(i,F,H) \le (1+C\delta)\omega(i,F,\hat{\Omega}).$$

But this is easy since there is an explicit mapping $g : \hat{\Omega} \to H$, $g(i) = i$ given by

$$g(z) = b(az + (az)^{-1})$$

$$a = A_2 \cdot r \quad , \quad b = a/(1-a^2)$$

which satisfies

$$|g'(x)-1| \le C\cdot\delta \quad , \quad |x| > A_1\cdot r/\delta$$

$$|g(x)-x| \le C\cdot\delta \quad , \quad |x| > A_1\cdot r/\delta$$

This proves the lemma. □

We will apply Lemma 3.3 in the following form: if $\tilde{\Omega}$ is obtained from Ω by replacing Σ_2 by some other a $\tilde{\Sigma}_2$, then for $z \in \Omega_1$, $E \subset \Sigma_1$,

$$(1+C\delta)^{-1} \le \frac{\omega(z,E,\Omega)}{\omega(z,E,\tilde{\Omega})} \le (1+C\delta)^2 \le 1 + C\delta .$$

Recall that for a C^∞ Jordan domain harmonic measure is some smooth function times arc length. The following fact is based on this idea.

<u>Corollary</u> 3.4: <u>Suppose</u> $\Gamma_1 \cup \Sigma_2 \cup \Gamma_2$ <u>is a</u> C^∞ <u>arc</u>. <u>Then there is a</u> $C = C(\Gamma_1 \cup \Sigma_2 \cup \Gamma_2)$ <u>such that</u>

$$C^{-1} \le \frac{\omega(z,E,\Omega)}{|E|\cdot\omega(z,C_2,\Omega)} \le C$$

<u>for any</u> $z \in \Omega_1$, $E \subset \Sigma_2$ ($|\cdot|$ <u>denotes arclength</u>).

Proof: Clearly there is a C such that

$$C^{-1} \le \frac{\omega(z_2,E,\Omega_0 \cup \Omega_2)}{|E|} \le C . \tag{3.1}$$

So by Lemma 3.3,

$$C^{-1} \le \frac{\omega(z_2,E,\Omega)}{|E|} \le C$$

and we obtain the result by Lemma 3.1. □

Note that this still holds if Σ_2 is replaced by a parameterized family of curves Σ_t, as long as (3.1) holds uniformly for the family.

Finally, suppose we replace Ω_0 by a family of domains Ω_0^t, $0 < t < 1$, satisfying the same conditions as Ω_0 and

$$\omega(z_1,C_2,\Omega_t) \le t\cdot\delta$$

$$\omega(z_2,C_1,\Omega_t) \le t\cdot\delta$$

where $\Omega_t = \Omega_1 \cup \Omega_0^t \cup \Omega_2$.

Corollary 3.5: For all $\varepsilon > 0$ there is a $\tau = \tau(\varepsilon)$ such that

$$\omega(z,E,\Omega_\tau) \le \varepsilon\cdot\omega(z,E,\Omega)$$

for all $z \in \Omega_1$, $E \subset \Sigma_2$.

The proof is immediate from Lemma 3.1. The point is that τ depends on ε and not on Ω.

4. Proof of Theorem 1.1

We will now use the elementary observations of the previous section to prove Theorem 1.1. The proof is quite simple, but the notation is a bit awkward and I apologize for any discomfort it may cause.

We will index parts of the construction by $S_n = \{1,2,3,4\}^n$, i.e., $j \in S_n$ is a sequence of length n with values in the four element set $\{1,2,3,4\}$. For $j \in S_n$ and $1 \le m \le n$ we let $j^m \in S_m$ be the sequence whose terms are the first m terms of j. For $j \in S_n$ we write $|j| = n$ and define a metric on S_n by

$$|k-j| \equiv n - \sup\{m : k^m = j^m\}.$$

Suppose we have constructed Γ as in Section 2 by iterating Figure 23 inside a unit square centered at the origin, adding E and letting $\Gamma = \mathbb{R}$ away from the square. We fix $z_1 = i$, $z_2 = -i$ and let Ω_1, Ω_2 and ω_1, ω_2 be the corresponding domains and measures. We write

$$\Omega_i = \Omega_0^i \cup \bigcup_{n=1}^{\infty} \bigcup_{|j|=n} (T_j^i \cup \Omega_j^i)$$

for $i = 1,2$, where Ω_0^1, T_j^1, Ω_j^1 are pictured in Figure 28. We think of $\{\Omega_j^i\}$, $j \in S_n$ as the 4^n pieces of the n^{th} generation of Ω^i. We index them so that T_j^i is the "tube" connecting Ω_j^i to Ω_k^i, $k = j^{n-1} \in S_{n-1}$.

For convenience we drop the "i's", except where needed. We let C_j denote the "collar" joining T_j to Ω_j (Figure 29). We also define the "pinched" versions of T_j by taking a C^∞ family of domains $\{T_j(t)\}$ $0 < t \le 1$, with $\tilde{T}_j(1) = T_j$ and so that the "bottleneck" closes completely as $t \to 0$ (Figure 30). Note that perturbing T_j^1 only changes the shape of itself and T_j^2 but no other part of the domains (Figure 31). Once we have chosen values of t for both T_j^1 and T_j^2, we denote the two resulting tubes by $\tilde{T}_j^1$ and $\tilde{T}_j^2$.

Now suppose we have already constructed

$$\Omega_{N-1} = \Omega_0 \cup \bigcup_{n=1}^{N-1} \bigcup_{|j|=n} (\tilde{T}_j \cup \Omega_j).$$

For $|j| = N$, consider all $2 \cdot 4^N$ values of $\omega(z_i, C_j^i, \Omega_{N-1} \cup T_j^i \cup \Omega_j^i)$ and let $a > 0$ be the minimum value. For each j, choose t so that

$$\omega(z_i, C_j^i, \Omega_{N-1} \cup T_j^i(t) \cup \Omega_j^i) = a \quad , \quad i = 1,2 .$$

Now set

$$\Omega_N = \Omega_{N-1} \cup \bigcup_{|j|=N} (\Omega_j \cup \tilde{T}_j).$$

Then $\tilde{\Omega}_i = \bigcup_N \Omega_N^i$ defines two Jordan domains, with common boundary Γ,

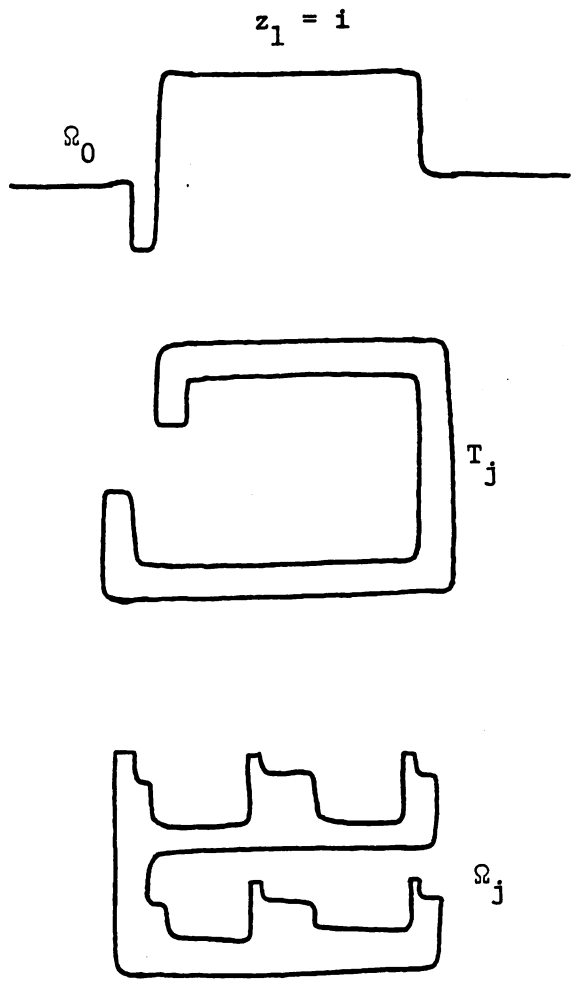

Figure 28: Parts of Ω

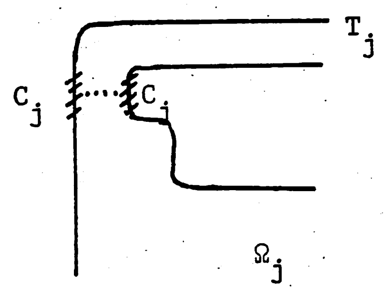

Figure 29: The "collar" C_j

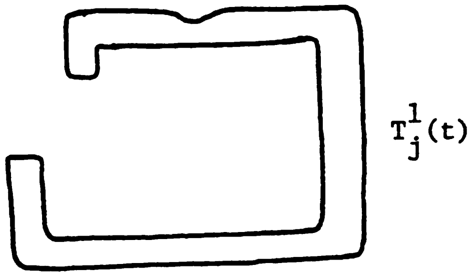

Figure 30: The "bottleneck" $T_j(t)$

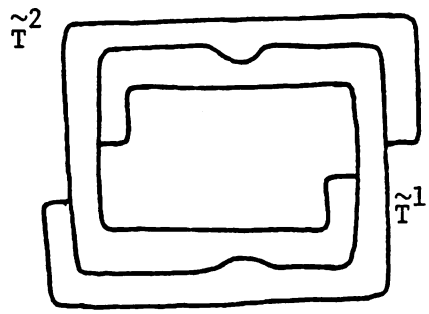

Figure 31: The tubes $\tilde{T}^1$ and $\tilde{T}^2$

which we claim satisfy Theorem 1.1.

First we claim that

$$a = \omega(z_i, C_j^i, \Omega_{N-1} \cup T_j^i(t) \cup \Omega_j^i) \sim \omega(z_i, C_j^i, \Omega_N). \tag{4.1}$$

Note that (dropping the i's)

$$\omega(z, C_j, \Omega_{N-1} \cup T_j(t) \cup \Omega_j) \sim \omega(z, C_j, \Omega_{N-1} \cup \tilde{T}_j \cup \Omega_j)$$

since replacing $T_j(t)$ by $\tilde{T}_j$ (the perturbation due to the "other" tube T_j) increases the harmonic measure by at most an absolute factor (independent of $t, j, \ldots$).

Now let

$$D_j = \bigcup_{n=1}^{N} (T_{j^n} \cup \Omega_{j^n}) \cup \Omega_0$$

so D_j is the subdomain of Ω_N consisting of the $\tilde{T}_k$'s and Ω_k's "over" T_j and Ω_j. Also, for $j, k \in S_n$, let $\Omega_{kj} = \Omega_{N-1} \cup \tilde{T}_k \cup \Omega_k \cup \tilde{T}_j \cup \Omega_j$.

Now apply Lemma 3.3 with $\Omega = \Omega_{kj}$, $\Omega_1 = D_j$, $\Omega_2 = D_k$ and $\delta = \eta^{|k-j|}$ where $\eta > 0$ is very small (of our choice, but independent of N) depending only on the curve in Figure 23. We get:

$$1 \le \frac{\omega(z, C_j, \Omega_{jk})}{\omega(z, C_j, D_j)} \le 1 + C \cdot \eta^{|j-k|}$$

Applying the lemma for all pairs (j,k), $j \neq k$, we have

$$1 \le \frac{\omega(z, C_j, \Omega_N)}{\omega(z, C_j, D_j)} \le \left(\prod_{i=1}^{N} (1 + C \cdot \eta^i)^{4^i} \right)^2$$

If η is small enough, say $\eta \ll 4/C$, the right hand side is bounded independent of N. This proves (4.1).

Thus for any N, we have Ω_N^i, $i = 1,2$ such that all $2 \cdot 4^N$ values of $\omega(z_i, C_j^i, \Omega_N^i)$ are comparable (with constants independent of N).

In particular, since this was true for Ω_{N-1} in the preceding construction, all $2\cdot 4^n$ values of $\omega(z,C_j,\Omega_{N-1} \cup T_j \cup \Omega_j)$ were comparable (with some constant independent of N) by Lemma 3.1. Hence by Corollary 3.5 our choices of $\tilde{T}_j$ did not involve arbitrarily small choices of t but only $t \geq t_0 > 0$ (independent of N). So by Corollary 3.4, applied to the compact family of possible $\tilde{T}_j$'s, we see

$$C^{-1} \leq \frac{\omega(z,E,\Omega_N)}{|E|\omega(z,C_j,\Omega_N)} \leq C$$

for all $E \subset \partial(\tilde{T}_j \cup \Omega_j)$ and $C > 0$ independent of N. In particular,

$$\omega(z_1,E,\Omega_N^1) \sim \omega(z_2,E,\Omega_N^2).$$

By another application of Lemma 3.3, as before, we see that for $E \subset \partial\Omega \cap \partial(\tilde{T}_j \cup \Omega_j)$

$$\omega(z,E,\Omega_N) \sim \omega(z,E,\Omega).$$

By dividing any subset of $\Gamma = \partial\Omega$ into "generations" we see that (1.1) holds in the unit square. Since it holds trivially away from the unit square, we have proven Theorem 1.1.

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