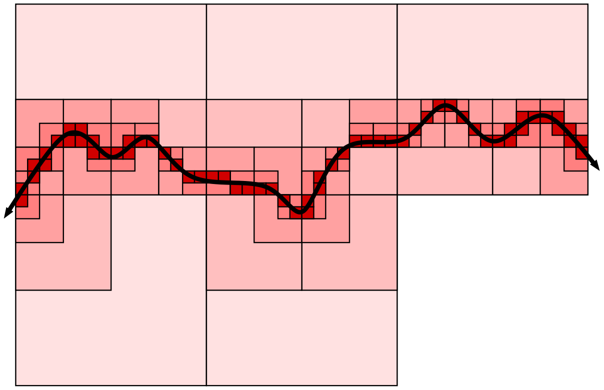
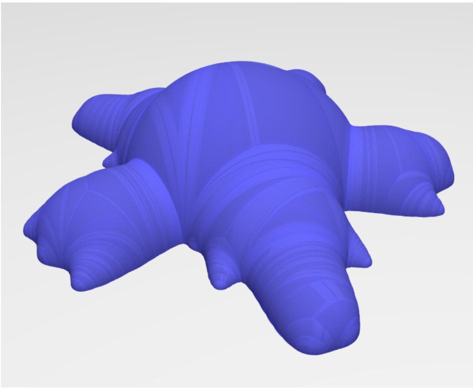
# WEIL-PETERSSON CURVES, TRAVELING SALESMAN THEOREMS, AND MINIMAL SURFACES

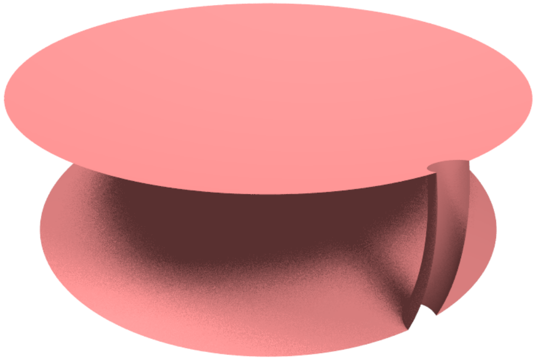
**Christopher Bishop, Stony Brook**

Quasiworld Seminar, Tuesday, 1/12/21

`www.math.sunysb.edu/~bishop/lectures`







Goals for today:

(1) Define Weil-Peterson class of curves.

(2) Give motivation and connections to various areas.

(3) State half a theorem, sketch parts of the proof.

(4) Mention lots of open problems.

First motivation:

(1) String theory studies spaces of loops.

(2) Physicists like Hilbert spaces.

$\Rightarrow$ Physicists want the space of loops to look like a Hilbert space.

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# Weil-Petersson Metric on the Universal Teichmüller Space

Leon A. Takhtajan
Lee-Peng Teo

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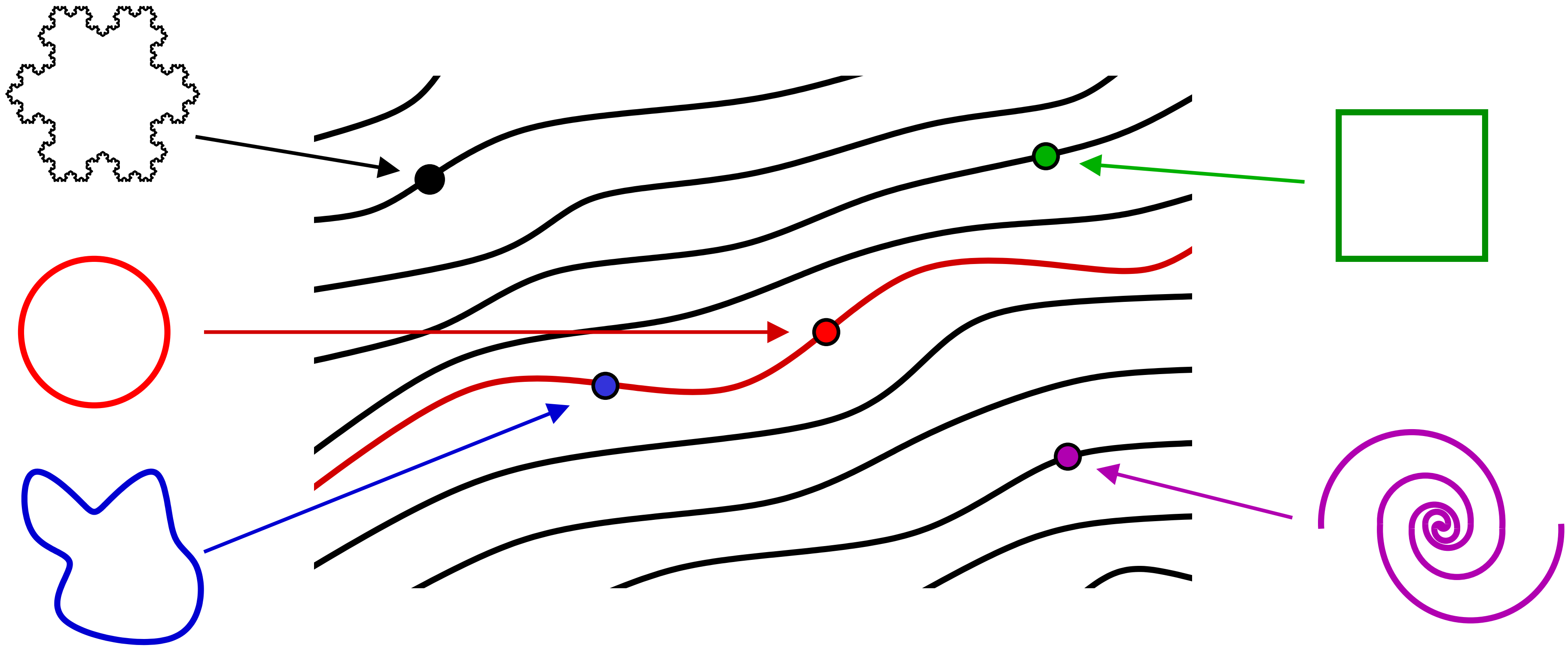
(1) Universal Teichmüller space = $T(1)$ = space of quasicircles.

(2) Teichmüller metric = $L^\infty$ on dilatations.

(3) $L^\infty$ is not a Hilbert space. $T(1)$ is Banach manifold.

$\Rightarrow$ Teichmüller metric not good for string theory.

$\Rightarrow$ We should replace it by an $L^2$ metric.

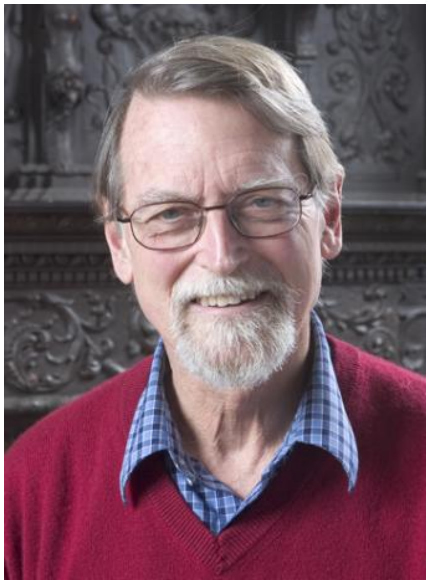


Takhtajan and Teo make $T(1)$ a (disconnected) Hilbert manifold.

$T_0(1)$ = Weil-Petersson class = component containing the circle

= closure of smooth curves = $\infty$-dim Kähler-Einstein manifold.

Question: What non-smooth curves are in $T_0(1)$?



David Mumford asked me how to characterize WP curves in a Dec 1017 email.

"Riemannian geometries on spaces of plane curves, Michor and Mumford, *J. Eur. Math. Soc. (JEMS)*, 2006.

Jan 2019 IPAM workshop: Analysis and Geometry of Random Sets.

Lecture by Yilin Wang:

"Loewner energy via Brownian loop measure and action functional analogs of SLE/GFF couplings"

So the Weil-Petersson class is linked to:

- String theory
- Kähler-Einstein manifolds
- Teichmüller theory
- Pattern recognition
- Brownian motion, SLE, Gaussian free fields, ...

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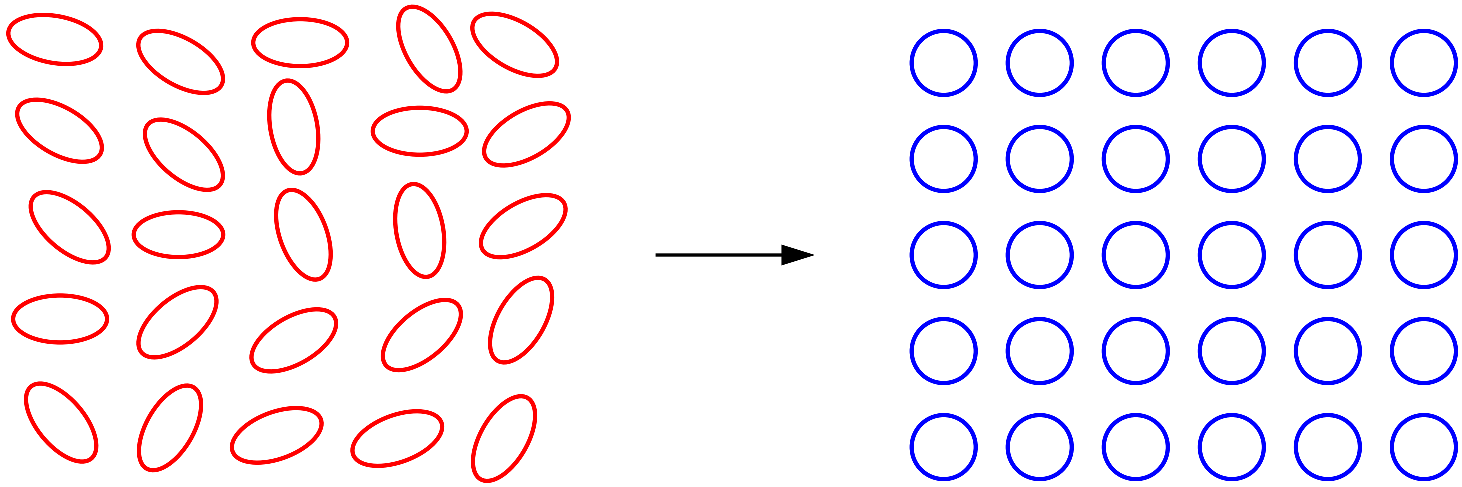
In today's talk I will discuss further connections to:

- Geometric function theory
- Sobolev spaces
- Knot theory
- Geometric measure theorem (Peter Jones's TST)
- Convex hulls in hyperbolic space
- Minimal surfaces
- Isoperimetric inequalities
- Renormalized area

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

For $K$-QC maps, ellipses have eccentricity $\leq K$

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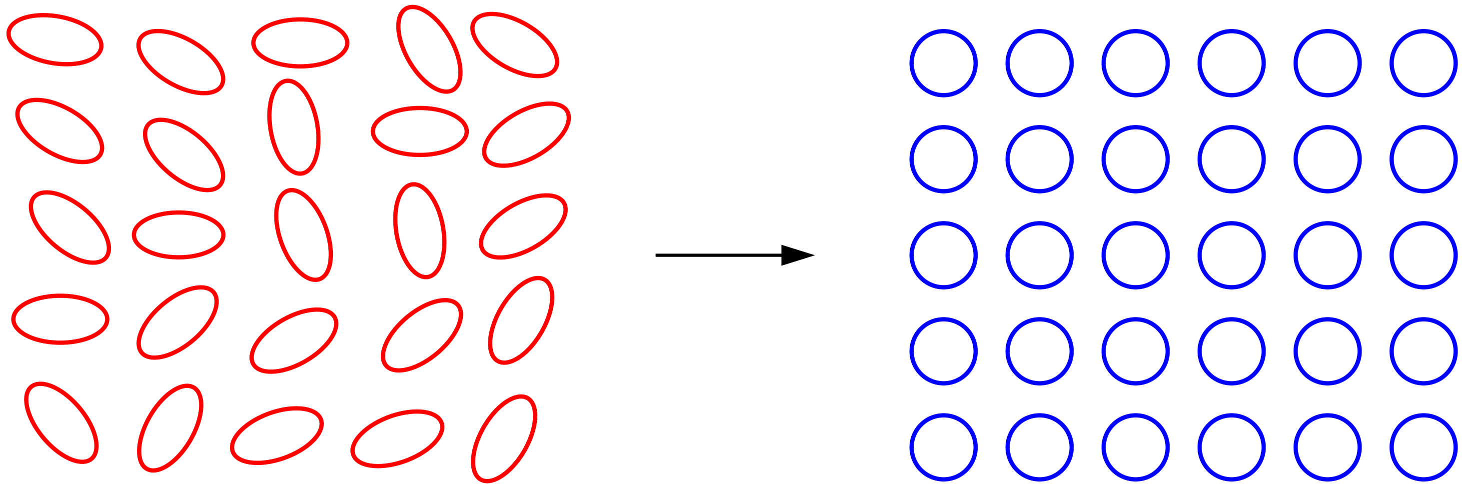

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For $K$-QC maps, ellipses have eccentricity $\leq K$

Ellipses determined a.e. by measurable dilatation $\mu = f_{\overline{z}}/f_z$ with

$$|\mu| \leq \frac{K-1}{K+1} < 1.$$

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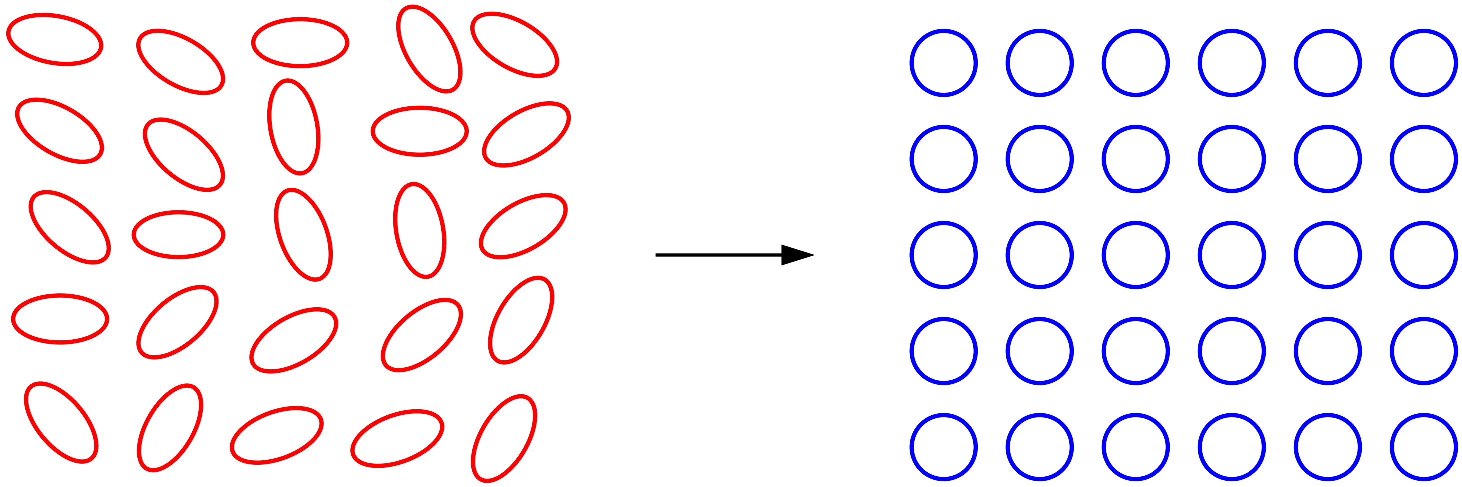

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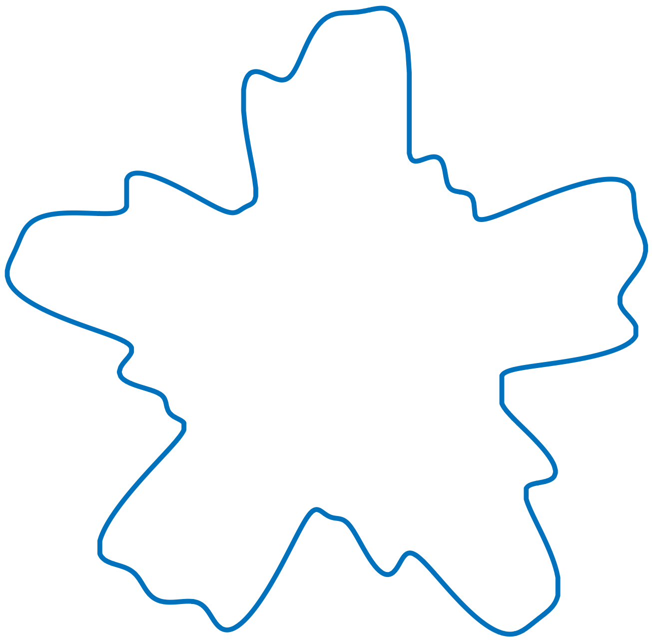
For $K$-QC maps, ellipses have eccentricity $\leq K$

A special case of QC maps are biLipschitz maps (all we need):

$$\frac{1}{C} \leq \frac{|f(x) - f(y)|}{|x - y|} \leq C.$$

A **quasicircle** is the image of circle under a quasiconformal map of $\mathbb{R}^2$.

Any smooth curve is a quasicircle.



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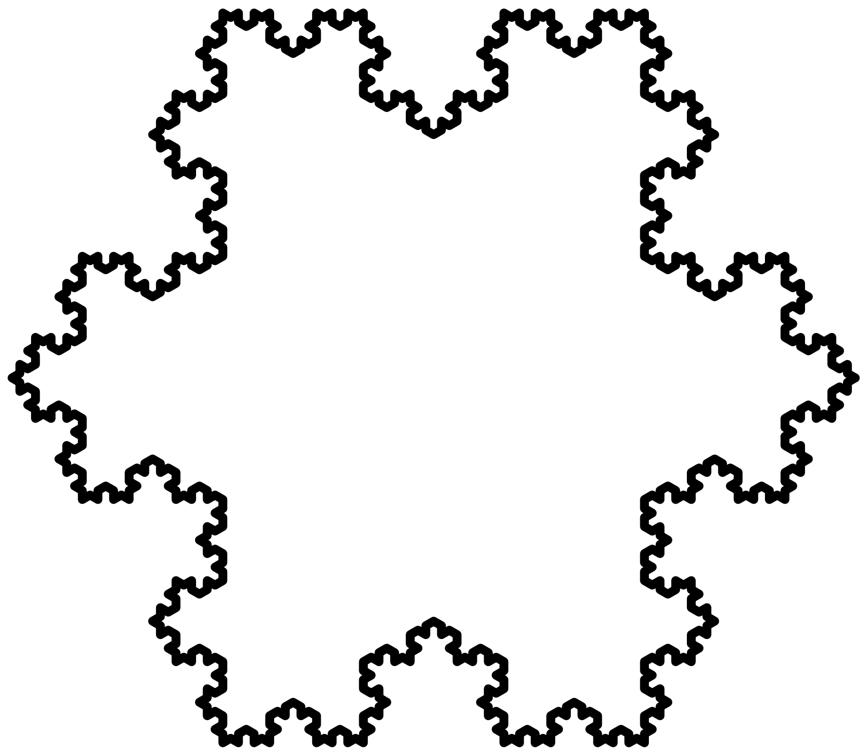
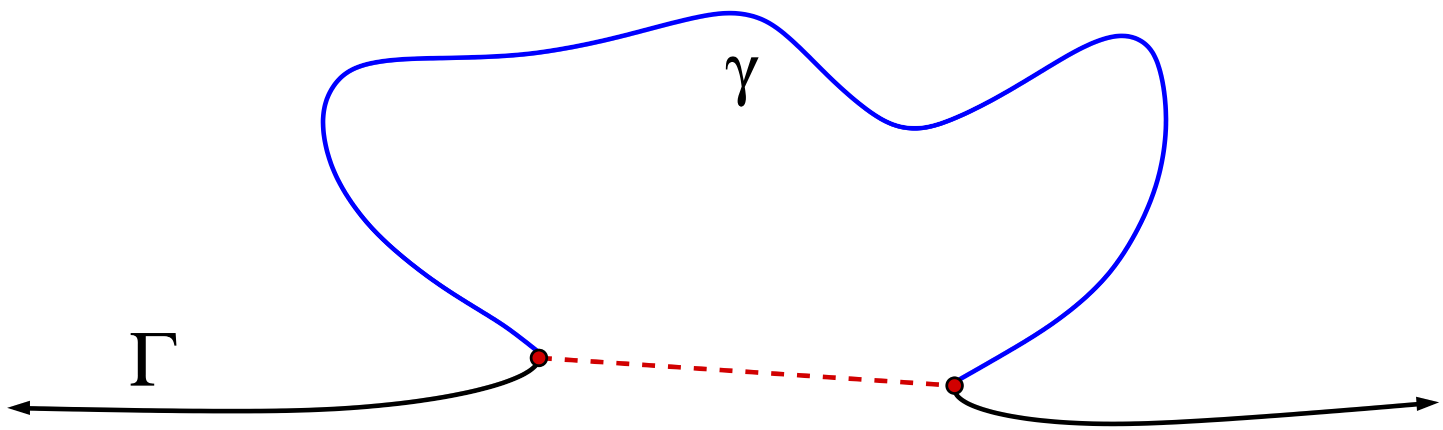
Some fractals are quasicircles.



A **quasicircle** is the image of circle under a quasiconformal map of $\mathbb{R}^2$.

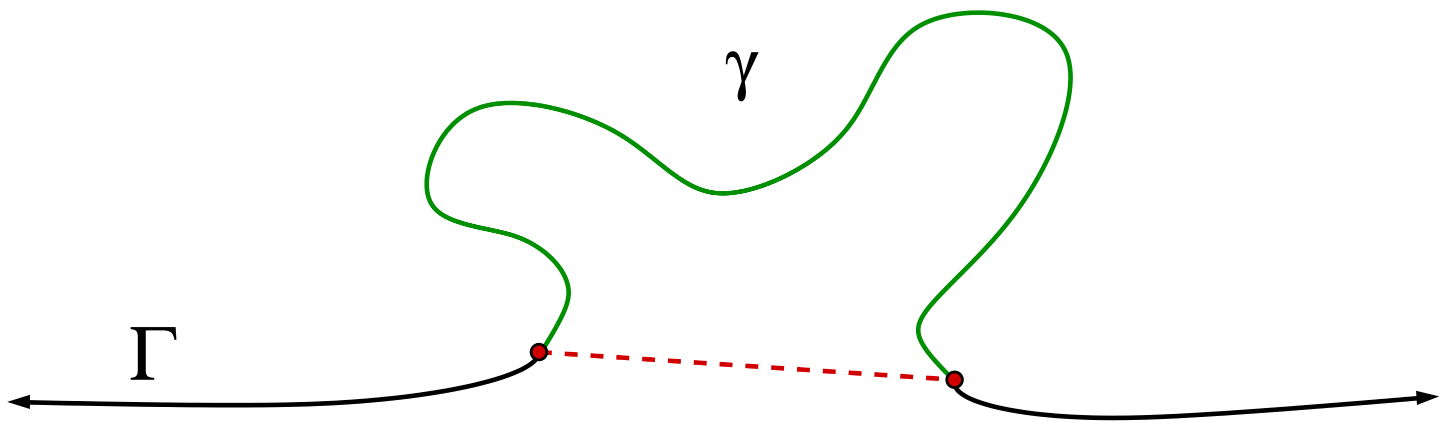
$\Gamma$ is a quasicircle iff $\mathrm{diam}(\gamma) = O(\mathrm{crd}(\gamma))$ for all $\gamma \subset \Gamma$.




$\mathrm{crd}(\gamma) = |z - w|$, $z, w$, endpoints of $\gamma$.

A **quasicircle** is the image of circle under a quasiconformal map of $\mathbb{R}^2$.

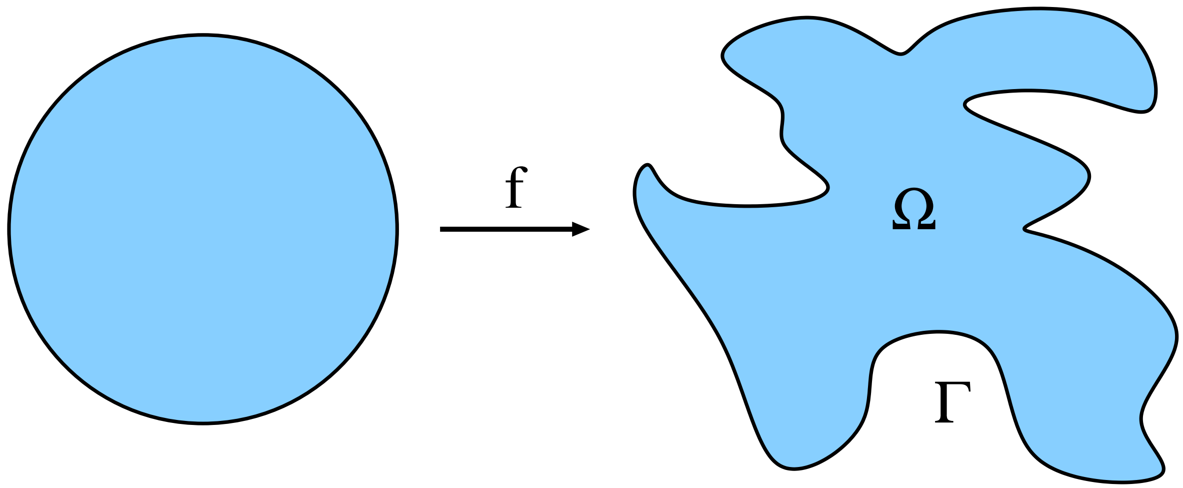
$\Gamma$ is a **chord-arc** iff $\ell(\gamma) = O(\mathrm{crd}(\gamma))$ for all $\gamma \subset \Gamma$.




Chord-arc curves = biLipschitz images of circle.

A **Weil-Petersson curve** is $\Gamma = f(\mathbb{T})$ where $f$ is quasiconformal on the plane, conformal outside $\mathbb{D}$, and $\mu \in L^2(dA_\rho)$.

$dA_\rho = \frac{dxdy}{(1-|z|^2)^2}$ = hyperbolic area. Clearly Möbius invariant.




Quasicircles $= \{\|\mu\|_\infty < 1\}$. WP $= \{\|\mu\|_2 < \infty\} \cap \{\|\mu\|_\infty < 1\}$.

Informally: $T_0(1)$ is to $L^2$, as $T(1)$ is to $L^\infty$.

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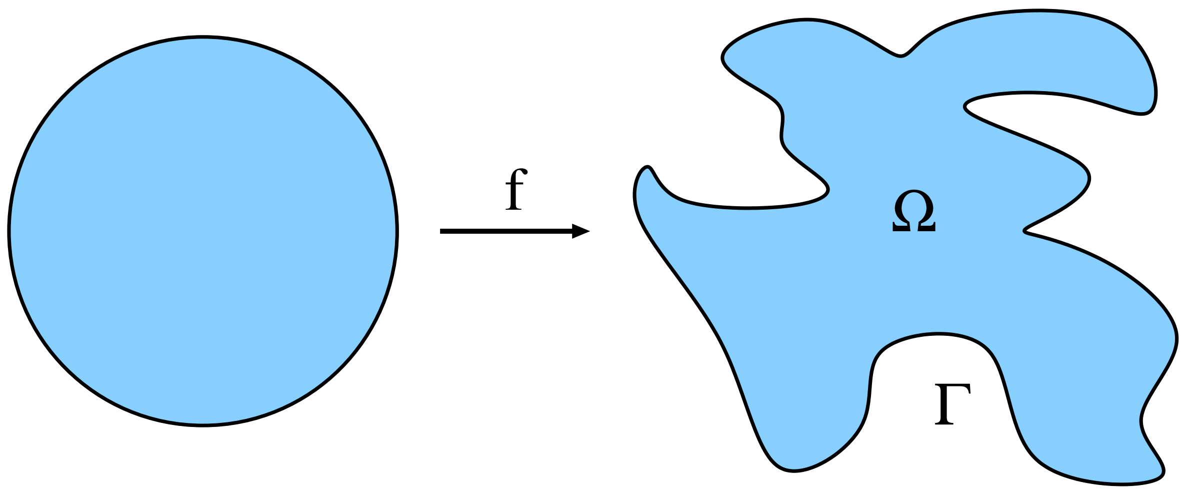
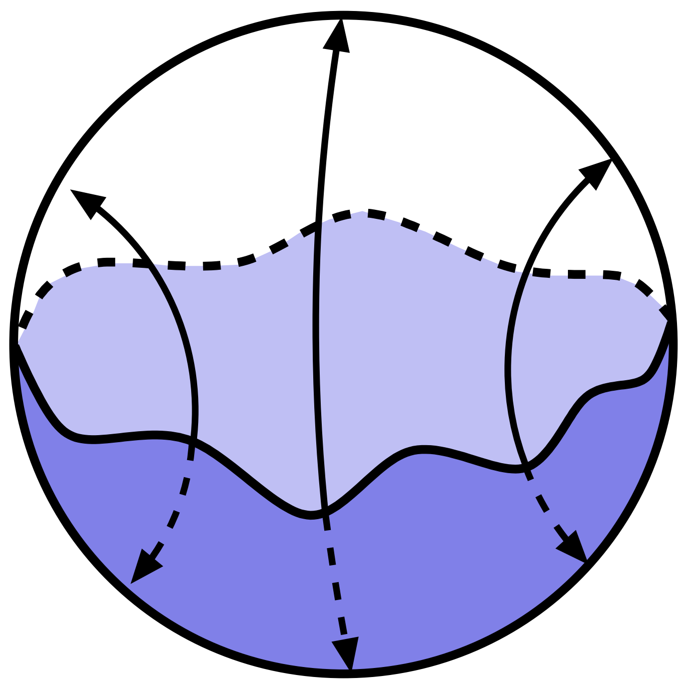



Define $F(z) = f(1/\overline{f^{-1}(z)})$.

$F$ is biLipschitz involution fixing $\Gamma$, and swaps the two sides of $\Gamma$.

A **Weil-Petersson curve** is $\Gamma = f(\mathbb{T})$ where $f$ is quasiconformal on the plane, conformal outside $\mathbb{D}$, and $|\mu| \in L^2(dA_\rho)$.

$\Leftrightarrow$ $\quad\Gamma$ is fixed by QC involution with $\mu \in L^2$ for hyperbolic area on $\mathbb{S}^2 \setminus \Gamma$.



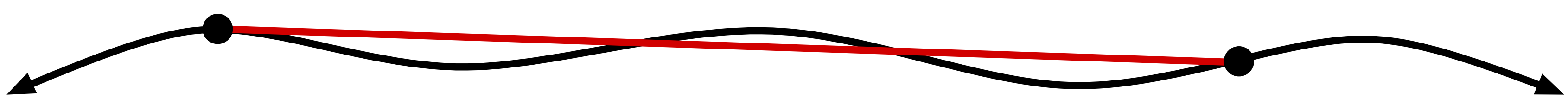
This extends to higher dimensions using biLipschitz involutions.

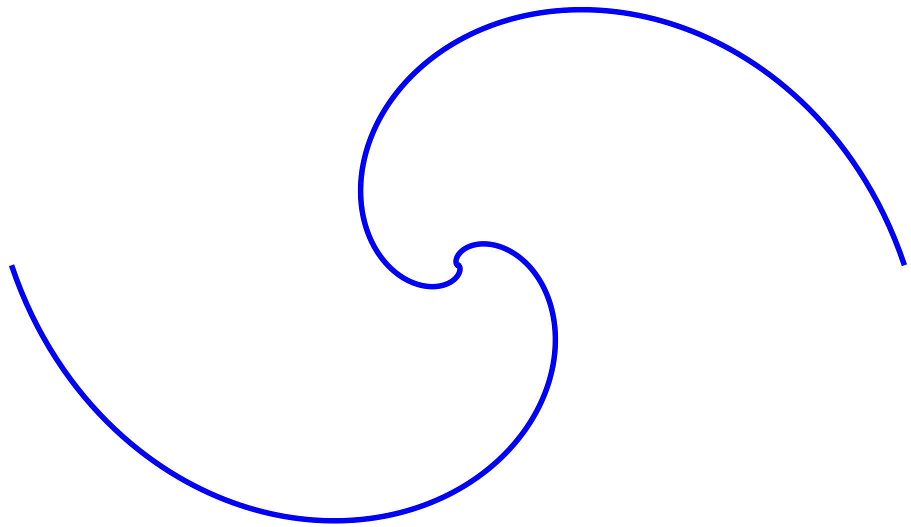
A **Weil-Petersson curve** is $\Gamma = f(\mathbb{T})$ where $f$ is quasiconformal on the plane, conformal outside $\mathbb{D}$, and $|\mu| \in L^2(dA_\rho)$.

Every smooth curve is Weil-Petersson. Every WP curve is chord-arc.

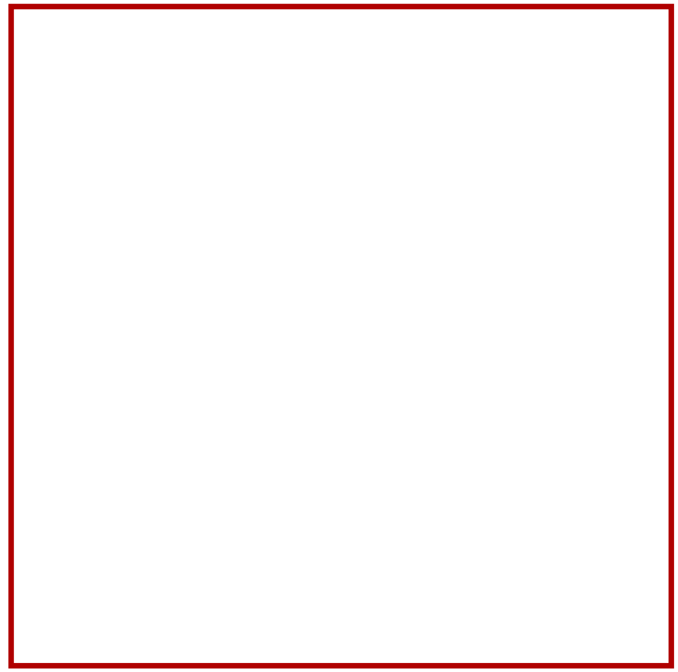
Weil-Petersson curves are almost $C^1$ (but not quite).

WP $\Rightarrow$ Asymptotically smooth $= \gamma \subset \Gamma$, $\ell(\gamma) \to 0$ implies $\frac{\ell(\gamma)}{\mathrm{crd}(\gamma)} \to 1$.





Weil-Petersson curves need not be $C^1$.
$z(t) = \exp(-t + i \log t)$, infinite spiral.

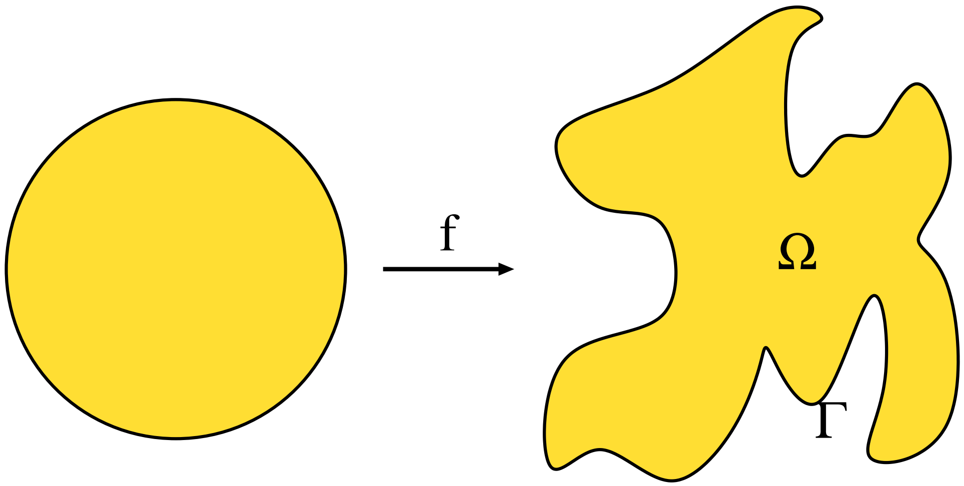


Not Weil-Petersson

In their memoir, Takhtajan and Teo prove:

**Theorem:** $\Gamma$ is Weil-Petersson iff $\Gamma = f(\mathbb{T})$ where $f$ is conformal map on $\mathbb{D}$ so that $u = \log f'$ is in the Dirichlet class.

Dirichlet class $= \{u : |\nabla u| \in L^2(dxdy)\} = \{u \in W^{1,2}(\mathbb{D})\}$.



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Yilin Wang proved $u \in W^{1,2}$ iff **Loewner energy** of $\Gamma$ is finite (as defined by her and Steffen Rohde). Her talk gave connections to large deviation theory of SLE and the Brownian loop soup of Lawler and Werner.






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In 1990's Astala, Zinsmeister invented "BMO-Teichmüller theory" where $\log f' \in$ BMO.

Jones and I characterized such $\Gamma$.

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Dirichlet class $= \{u : |\nabla u| \in L^2(dxdy)\} = \{u \in W^{1,2}(\mathbb{D})\}$.

**Corollary:** $\Gamma$ is WP iff $u = \log f' \in H^{1/2}(\mathbb{T})$ (Sobolev trace thm).

$H^{1/2}$ = Sobolev space = half a derivative in $L^2$.

$$u(e^{i\theta}) = \sum a_n e^{in\theta}, \quad \text{where } \sum n|a_n|^2 < \infty.$$

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|u(z) - u(w)|^2}{|z - w|^2} |dz||dw| < \infty,$$

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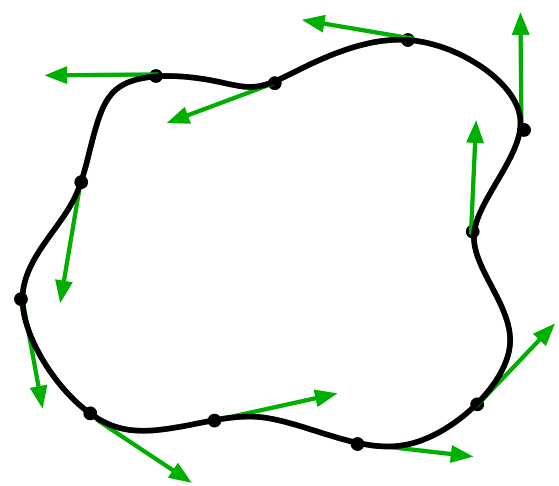
**Corollary:** $\Gamma$ is WP iff $u = \log f' \in H^{1/2}(\mathbb{T})$ (Sobolev trace thm).

Since $\arg z = \mathrm{Im}(\log \mathrm{z})$, we have:

$$\begin{aligned}\log f' \in H^{1/2} &\Rightarrow \arg f' \in H^{1/2} \\ &\Rightarrow \exp(i \arg f') \in H^{1/2} \\ &\Rightarrow f'/|f'| \in H^{1/2}\end{aligned}$$



**Theorem:** $\Gamma$ is WP iff the arc-length parameterization is in $H^{3/2}(\mathbb{T})$.

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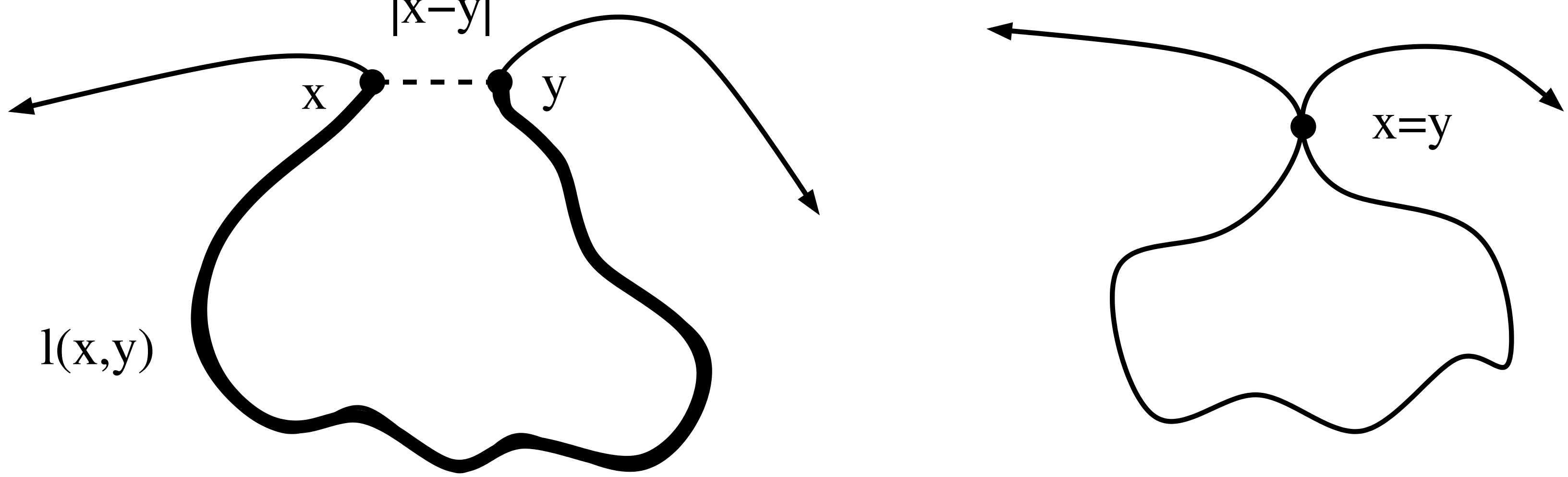
This was proven implicitly in early version of my paper, but I didn't notice. David Mumford pointed out it followed from other characterizations.

Takhtajan and Teo proved tangent space of $T_0(1)$ is $H^{3/2}$.

$H^{3/2}$ curves arise naturally in other areas, e.g., knot theory.




The Möbius energy of a curve $\Gamma \in \mathbb{R}^n$ is

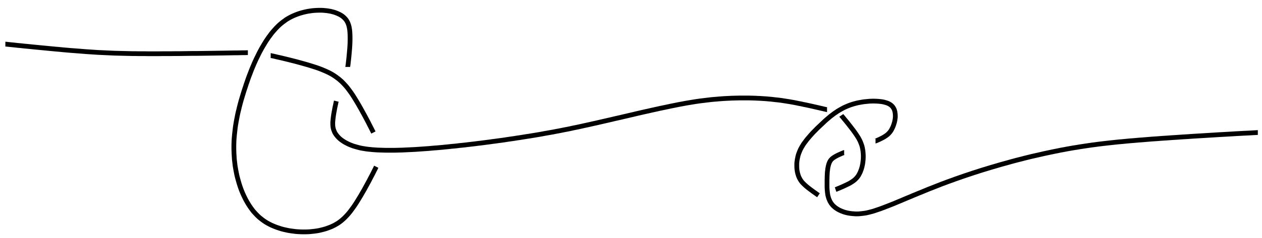
$$\text{Möb}(\Gamma) = \int_\Gamma \int_\Gamma \left( \frac{1}{|x-y|^2} - \frac{1}{\ell(x,y)^2} \right) dxdy.$$

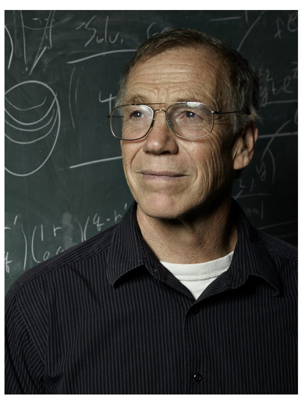
Blows up if curve self-intersects. Is "renormalized" inverse-cube force.

Hadamard renormalization of divergent integral. Connects to SLE?

Möbius energy is one of several “knot energies” due to Jun O’Hara.

Studied by Freedman, He and Wang. They showed:
- Möb($\Gamma$) is Möbius invariant (hence the name),
- that finite energy curves are chord-arc,
- and in $\mathbb{R}^3$ they are topologically tame.



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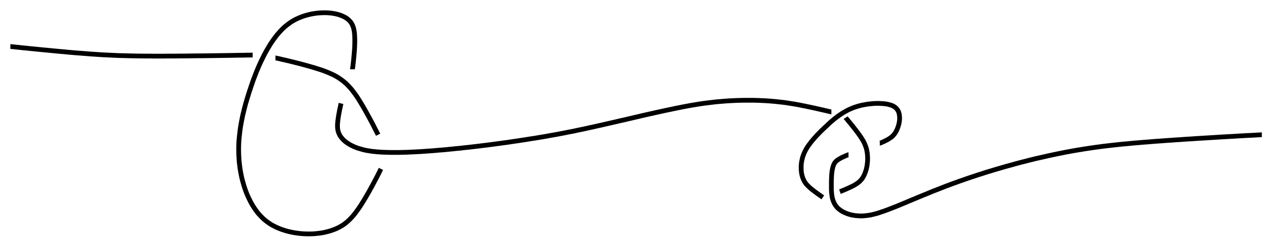


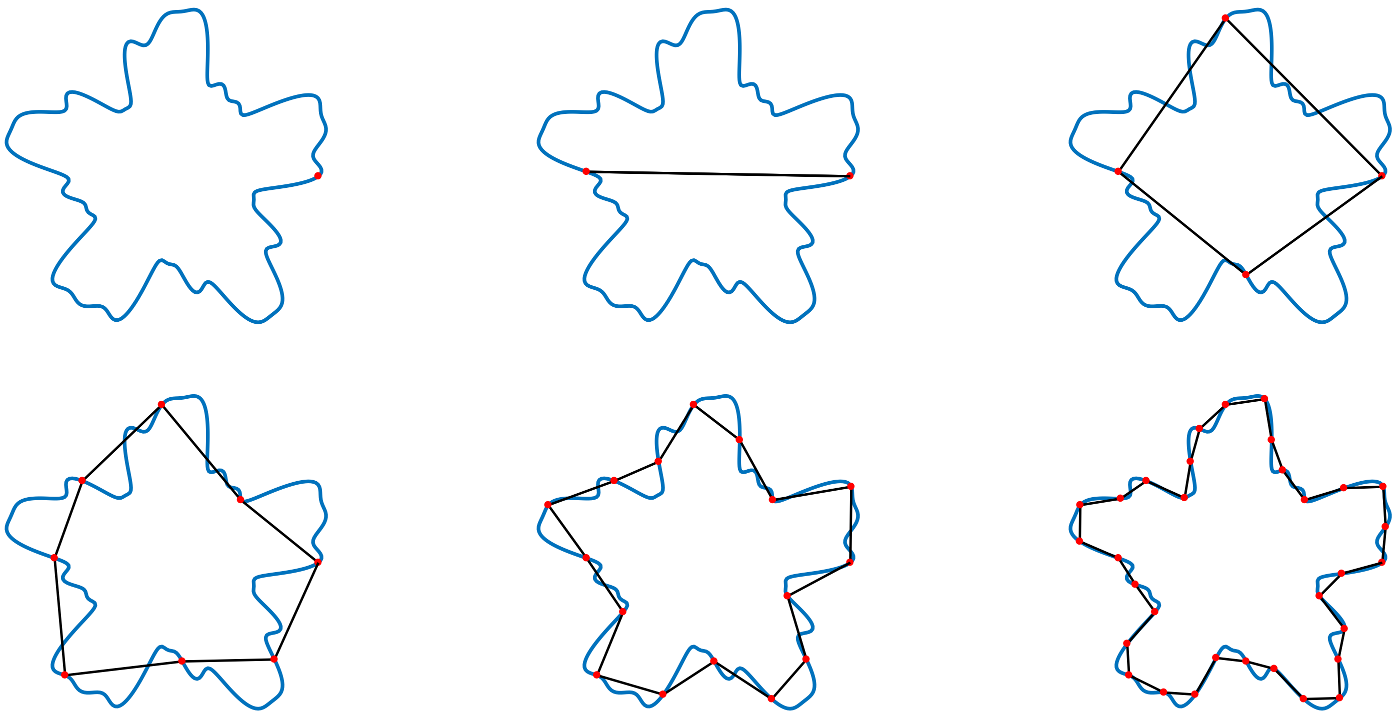
**Theorem (Blatt):** $\text{Möb}(\Gamma) < \infty$ iff arclength parameterization is $H^{3/2}$.

Thus WP curve = finite Möbius energy.

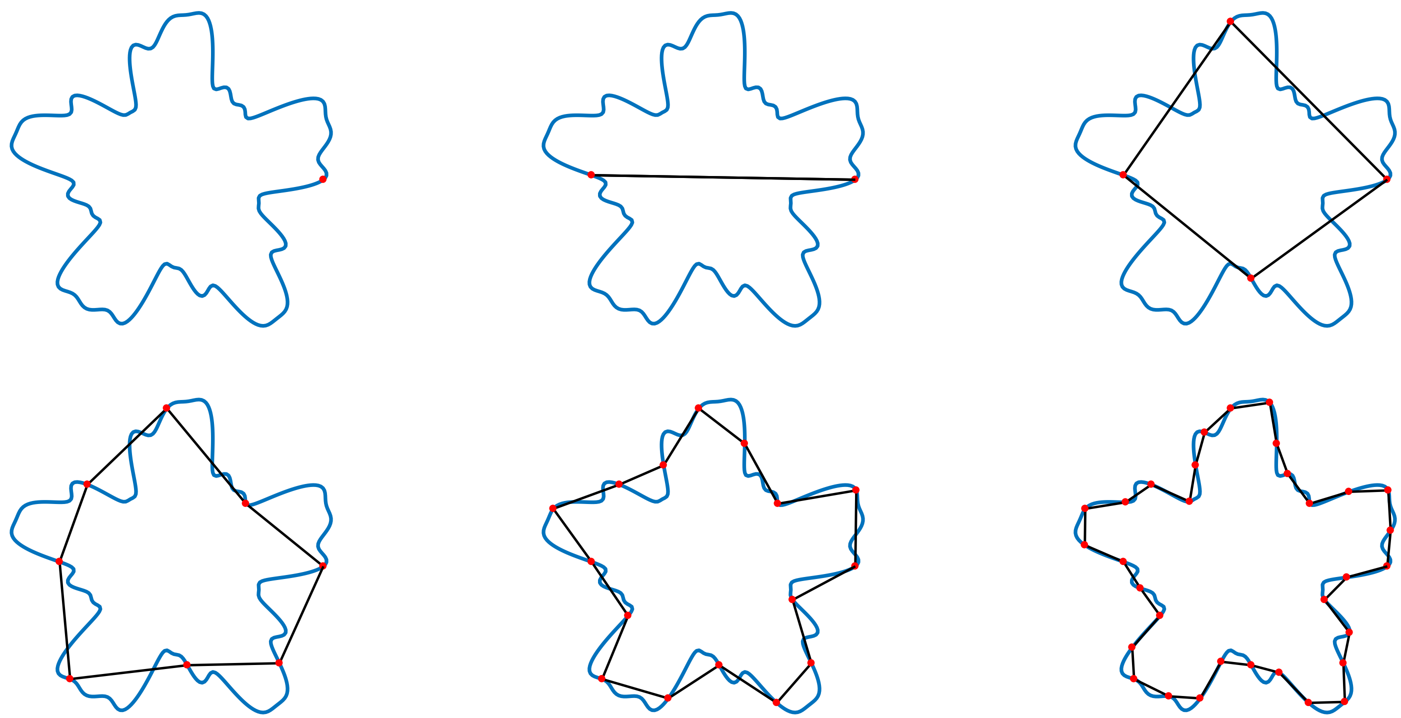
Teichmüller spaces, $H^{3/2}$, and knot theory are pretty sophisticated.

How can you describe WP curves to a calculus student?

**Dyadic decomposition.** Choose a base point $z_1^0 \in \Gamma$ and for each $n \geq 1$, let $\{z_j^n\}$, $j = 1, \ldots, 2^n$ be the unique set of ordered points with $z_1^n = z_1^0$ that divides $\Gamma$ into $2^n$ equal length intervals (called the $n$th generation dyadic subintervals of $\Gamma$).



Let $\Gamma_n$ be the inscribed $2^n$-gon with these vertices. Clearly $\ell(\Gamma_n) \nearrow \ell(\Gamma)$.



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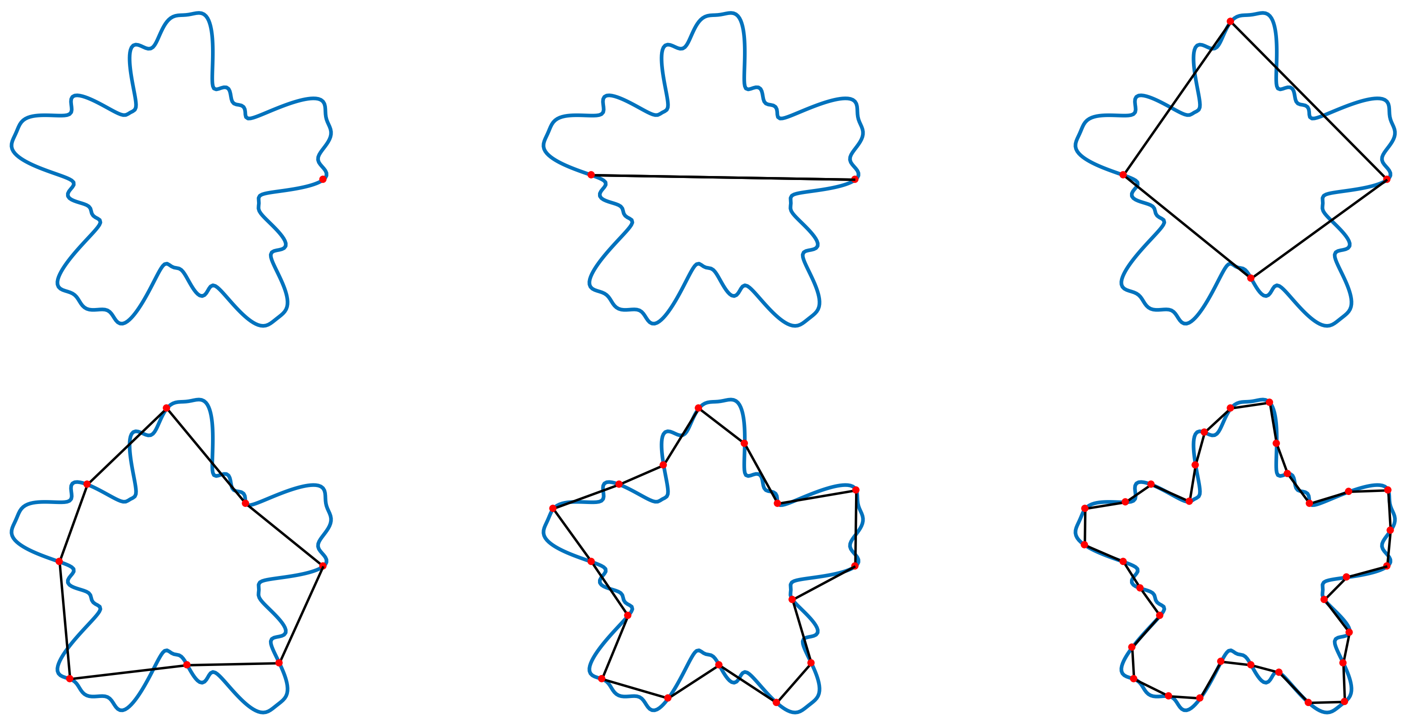


**Theorem:** $\Gamma$ is Weil-Petersson if and only if

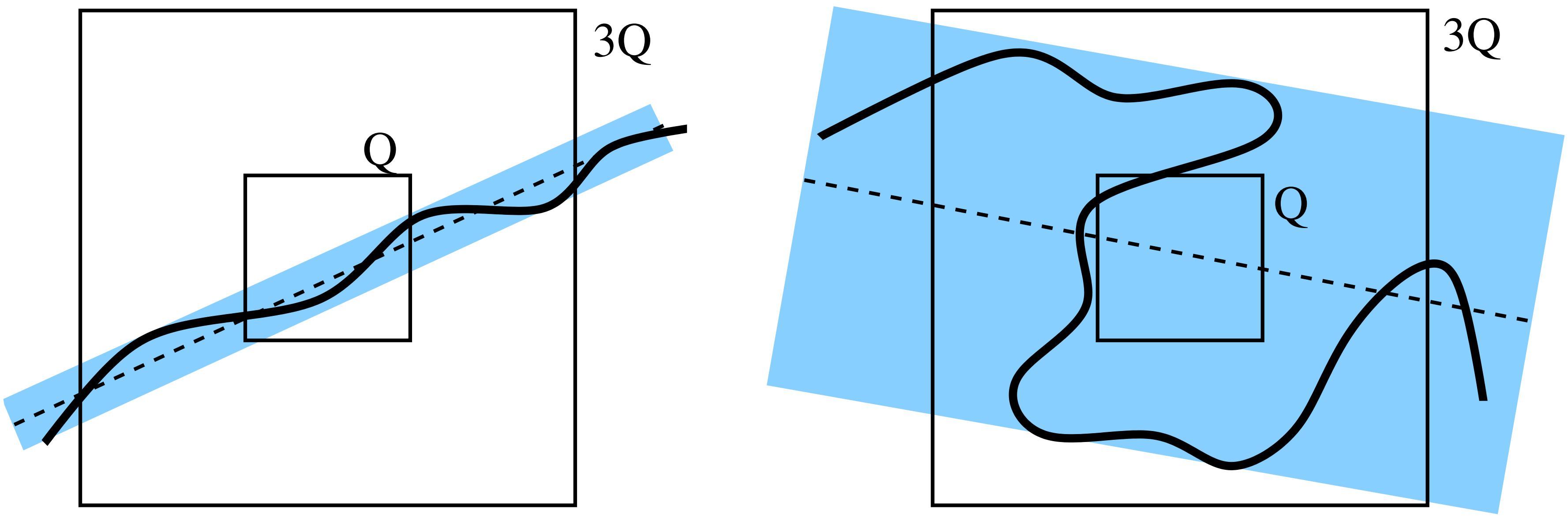
$$\sum_{n=1}^{\infty} 2^n \left[\ell(\Gamma) - \ell(\Gamma_n)\right] < \infty$$

with a bound that is independent of the choice of the base point.

**Peter Jones's $\beta$-numbers:**

$$\beta_\Gamma(Q) = \inf_L \sup\{\frac{\mathrm{dist}(z, L)}{\mathrm{diam}(Q)} : z \in 3Q \cap \Gamma\},$$

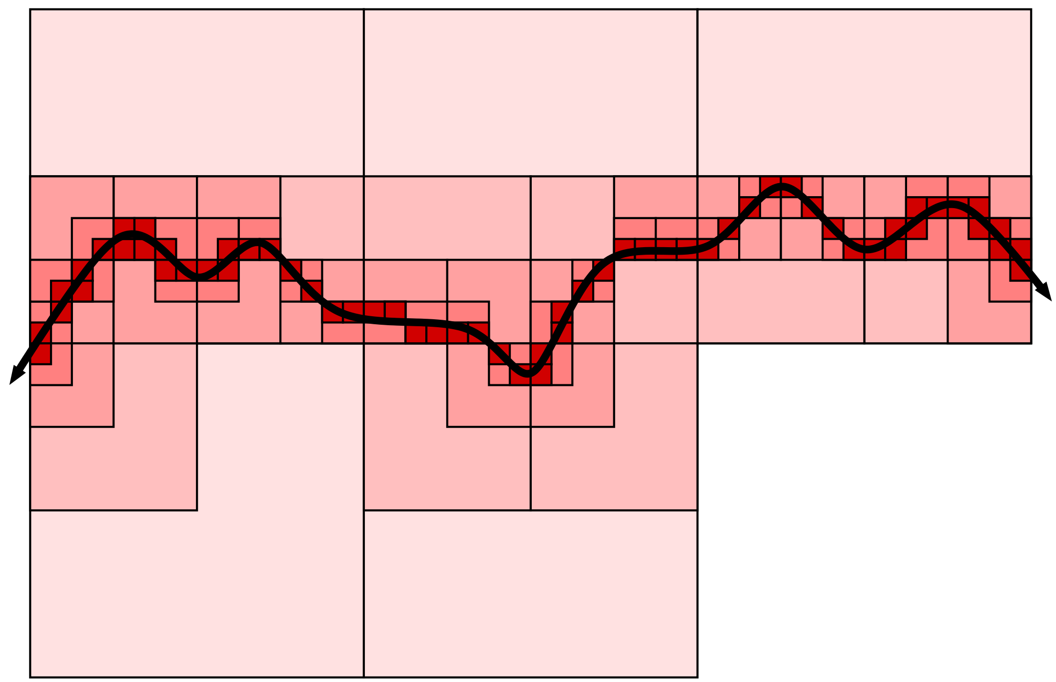
where the infimum is over all lines $L$ that hit $3Q$.



Jones invented the $\beta$-numbers as part of his traveling salesman theorem:

$$\ell(\Gamma) \simeq \mathrm{diam}(\Gamma) + \sum_Q \beta_\Gamma(Q)^2 \mathrm{diam}(Q),$$

where the sum is over all dyadic cubes $Q$ in $\mathbb{R}^n$.



Previous theorem on length should have translation into $\beta$-numbers.

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where the sum is over all dyadic cubes $Q$ in $\mathbb{R}^n$.

**Theorem:** $\Gamma$ is Weil-Petersson iff

$$\sum_Q \beta_\Gamma(Q)^2 < \infty,$$

where the sum is over all dyadic cubes.

WP curves have "curvature in $L^2$, integrated over all positions and scales".

Jones invented the $\beta$-numbers as part of his traveling salesman theorem:

$$\ell(\Gamma) \simeq \mathrm{diam}(\Gamma) + \sum_Q \beta_\Gamma(Q)^2 \mathrm{diam}(Q),$$

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**Theorem:** $\Gamma$ is Weil-Petersson iff

$$\sum_Q \beta_\Gamma(Q)^2 < \infty,$$

where the sum is over all dyadic cubes.

Proof requires improvement of TST for curves:

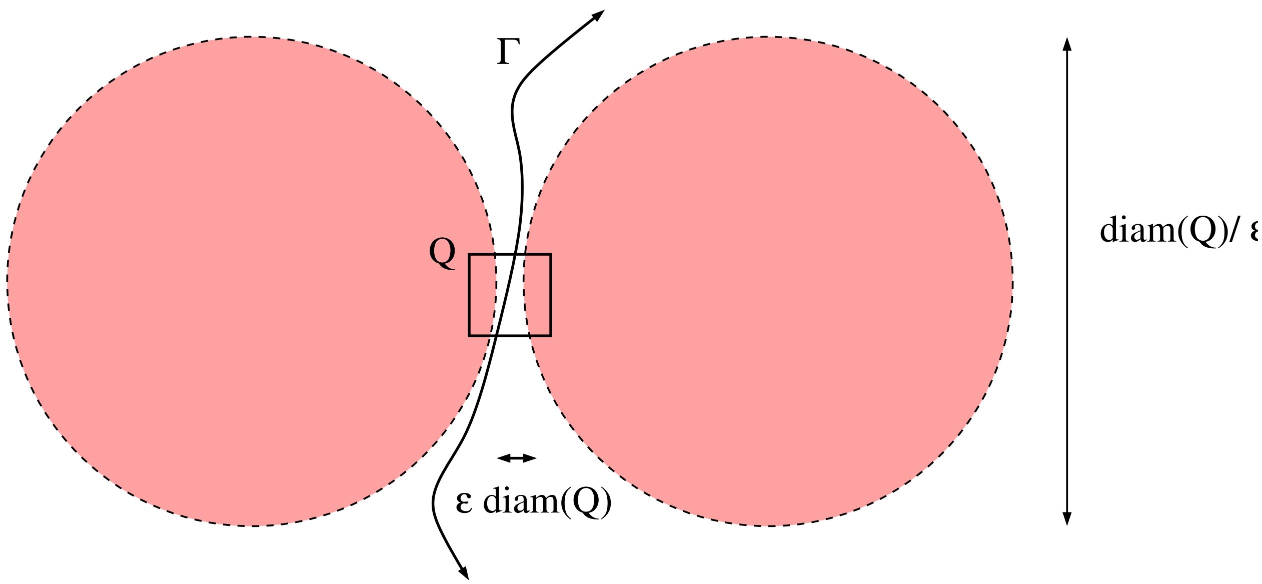
$$\ell(\Gamma) - \mathrm{crd}(\Gamma) \simeq \sum_Q \beta_\Gamma(Q)^2 \mathrm{diam}(Q),$$

The Weil-Petersson class is Möbius invariant.

The $\beta$-numbers are not: lines ($\beta = 0$) can map to circles ($\beta > 0$).

What is a Möbius invariant version of the $\beta$-numbers?

First idea: let $\varepsilon_\Gamma(Q)$ be smallest $\epsilon$ so that there are disks $D, D'$ of radius $\ell(Q)/\epsilon$ separated by $\Gamma$ with $\mathrm{dist}(Q \cap D, Q \cap D') \leq \epsilon \ell(Q)$.

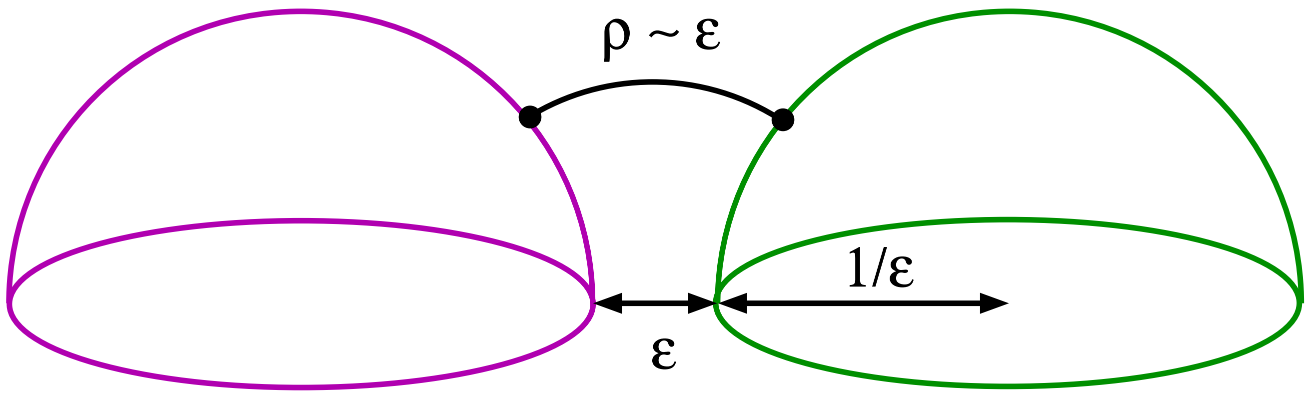



Easy to check $\beta(Q) \lesssim \varepsilon(Q)$. Converse can fail, but

**Theorem:** $\sum_Q \varepsilon^2(Q) < \infty$ iff $\sum_Q \beta^2(Q) < \infty$ iff $\Gamma$ is WP.

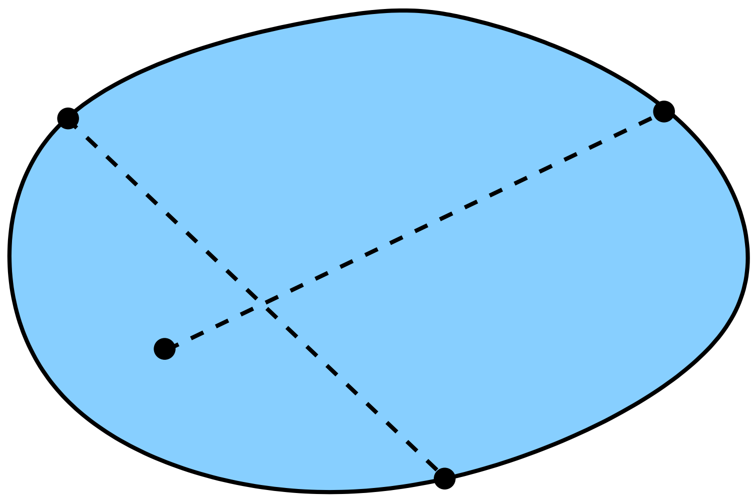
Each disk is the base of a hemisphere in the upper half-space $\mathbb{H}^3 = \mathbb{R}^3_+$.

The hyperbolic distance between these hemispheres is $\lesssim \varepsilon(Q)$.

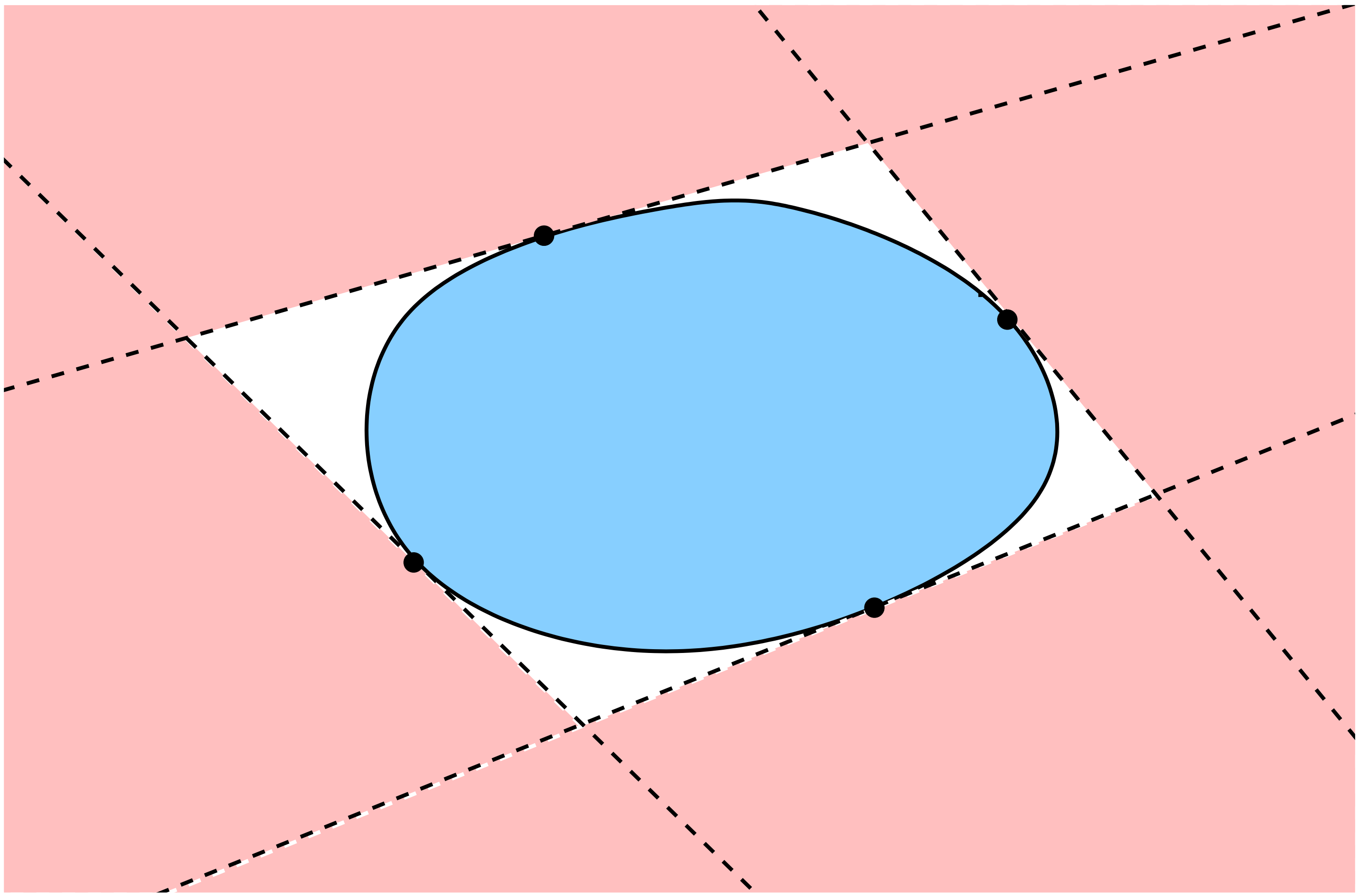



This connects WP class to convex sets in hyperbolic upper half-space.

Usual definition of convex: contains geodesic between any two points.



More useful for us: complement is a union of half-spaces.

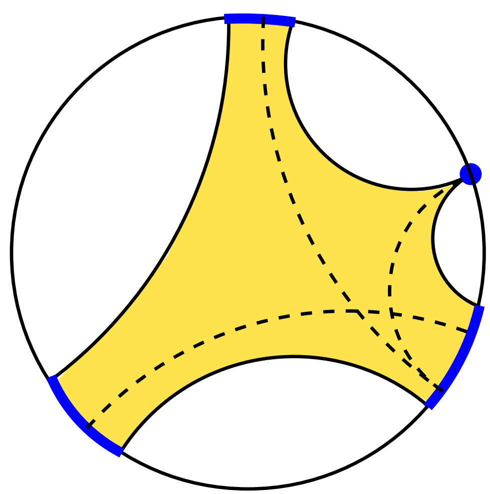


The hyperbolic length of a (Euclidean) rectifiable curve in the unit disk $\mathbb{D}$ or in the n-dimensional ball $\mathbb{B}^n$ is given by integrating

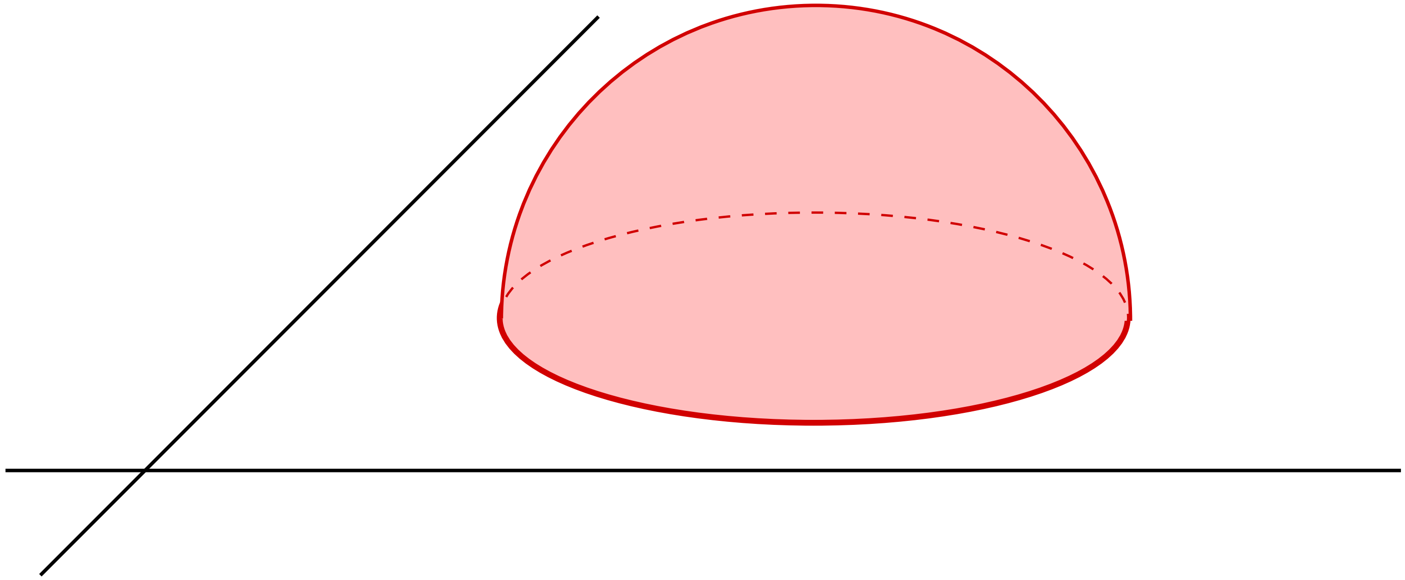
$$d\rho = \frac{ds}{1 - |z|^2},$$

along the curve. In the upper half-space $\mathbb{H}^n$ we integrate $d\rho = ds/2t$.

Geodesics are circles (or lines) perpendicular to boundary.



Hyperbolic convex hull of a boundary set



The hyperbolic convex hull of $\Gamma \subset \mathbb{R}^2$, is the smallest convex set containing all geodesics with endpoints in $\Gamma$.

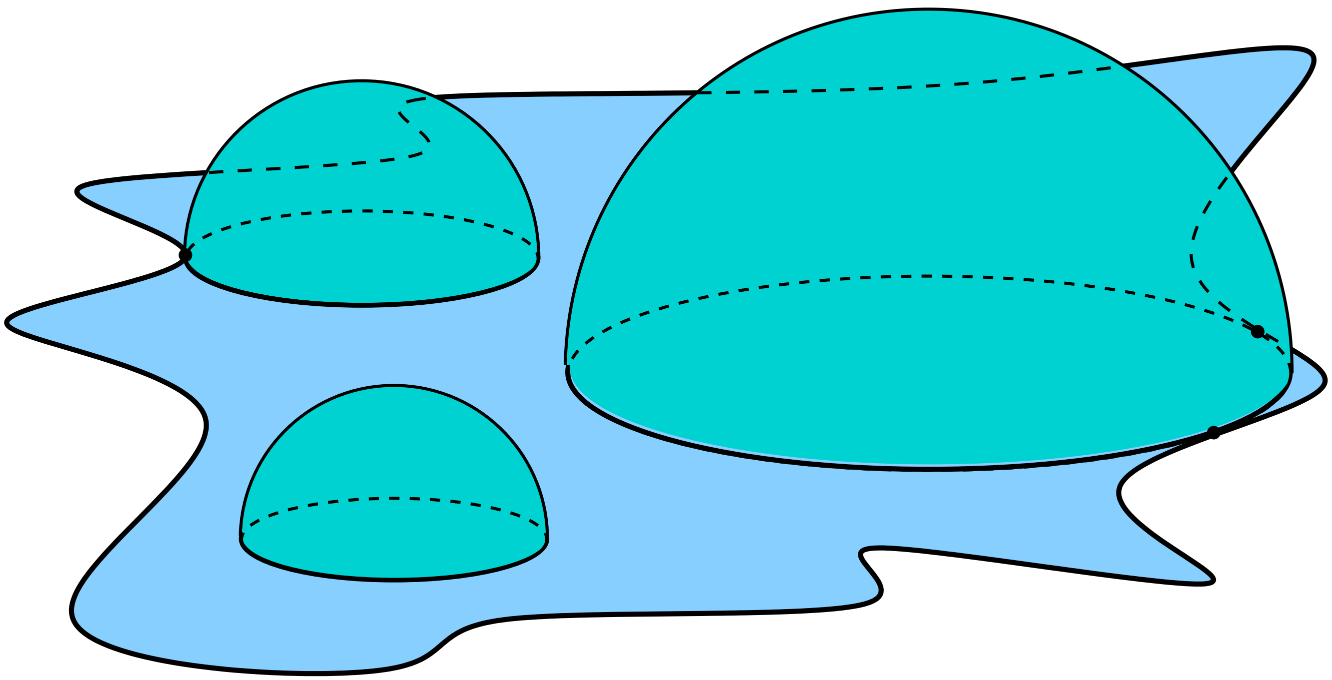
$= \mathrm{CH}(\Gamma) =$ complement of all open half-spaces that miss $\Gamma$.

For a circle in plane, hyperbolic convex hull is a hemisphere.

Otherwise, $\mathrm{CH}(\Gamma)$ has non-empty interior and 2 boundary components.

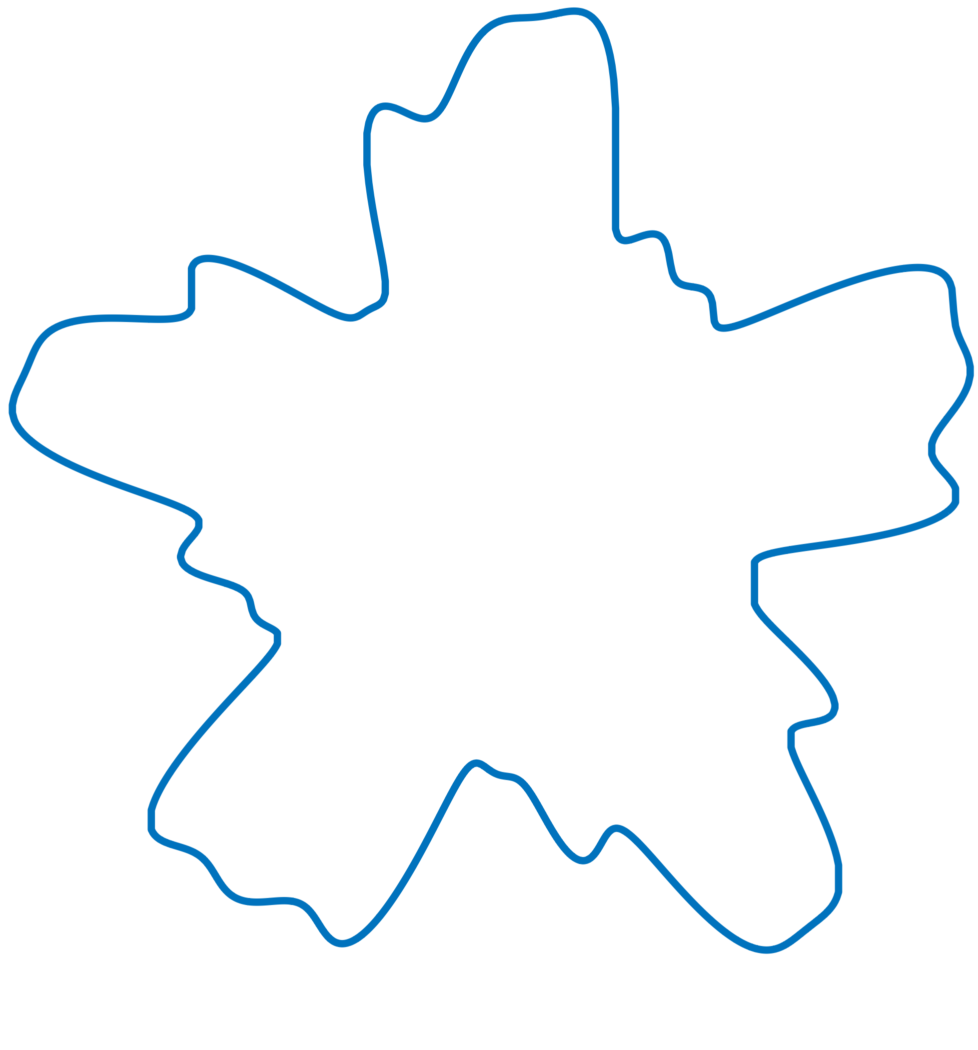
A hyperbolic half-space missing $\mathrm{CH}(\Gamma)$ has boundary disk missing $\Gamma$.

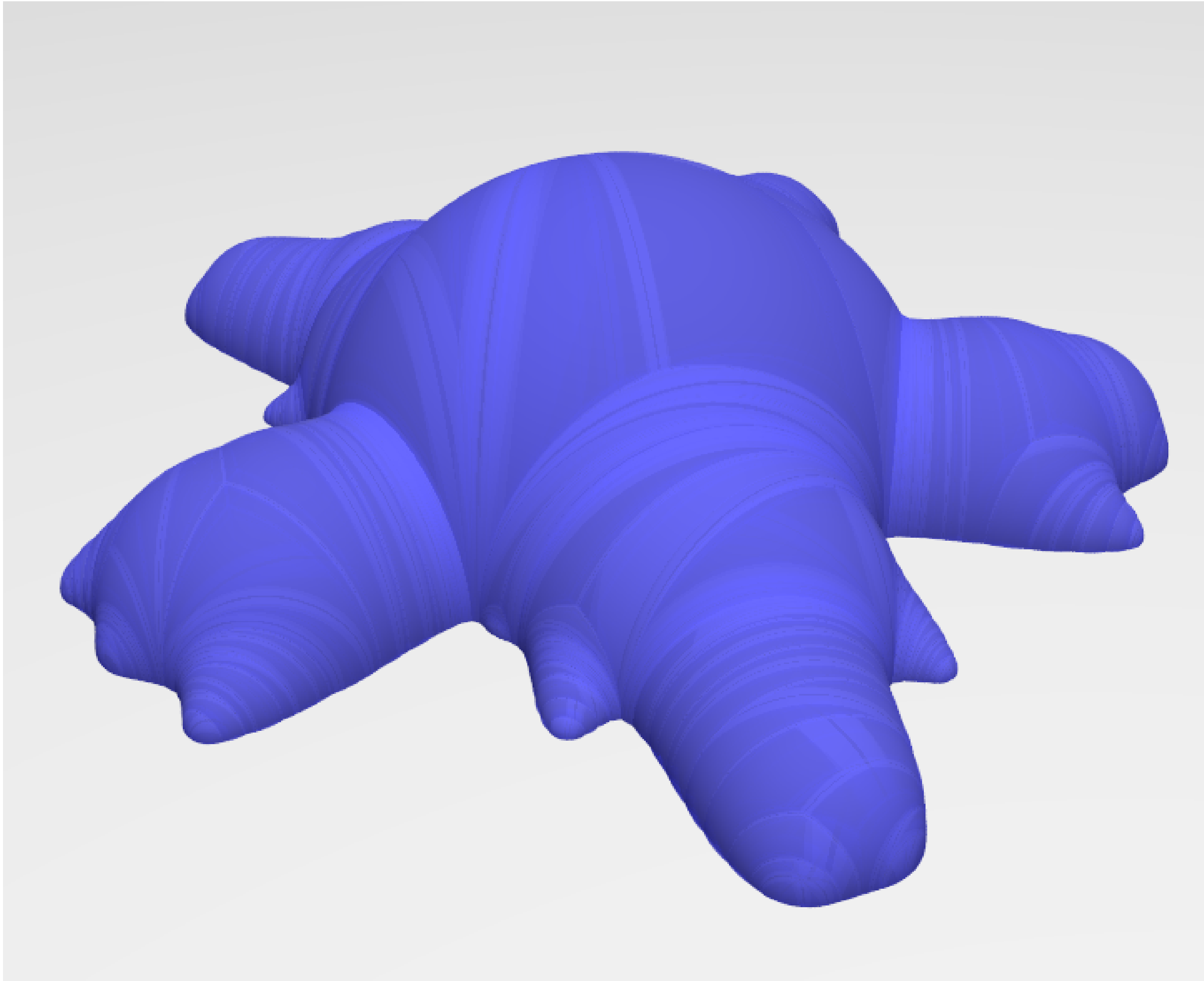
This disk is inside or outside $\Gamma$. Dome($\Omega$) is union over "inside" disks.

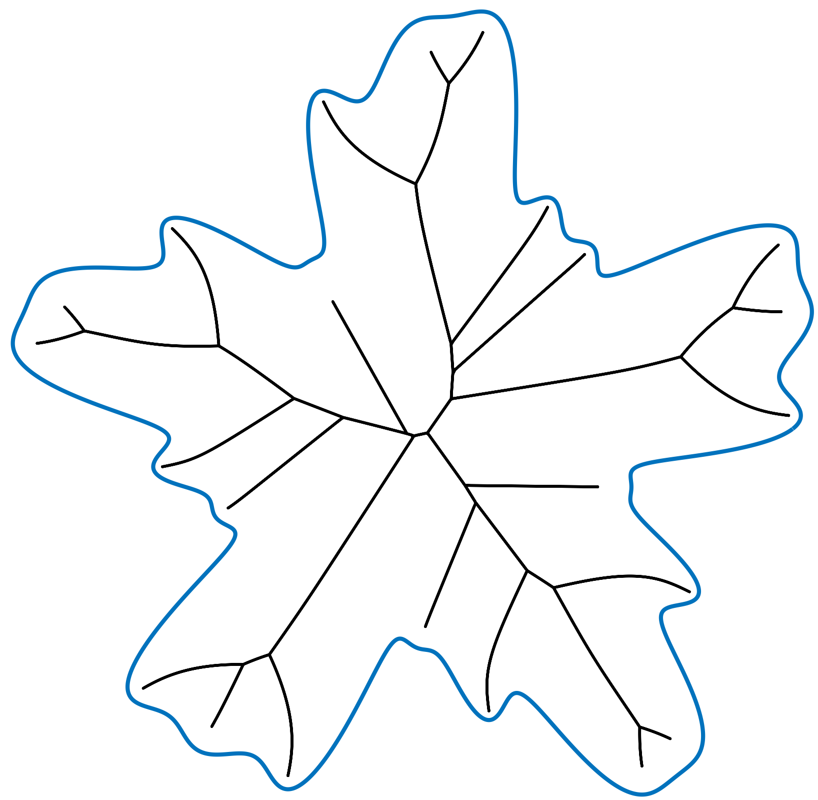


Region above dome is intersection of half-spaces, hence convex.

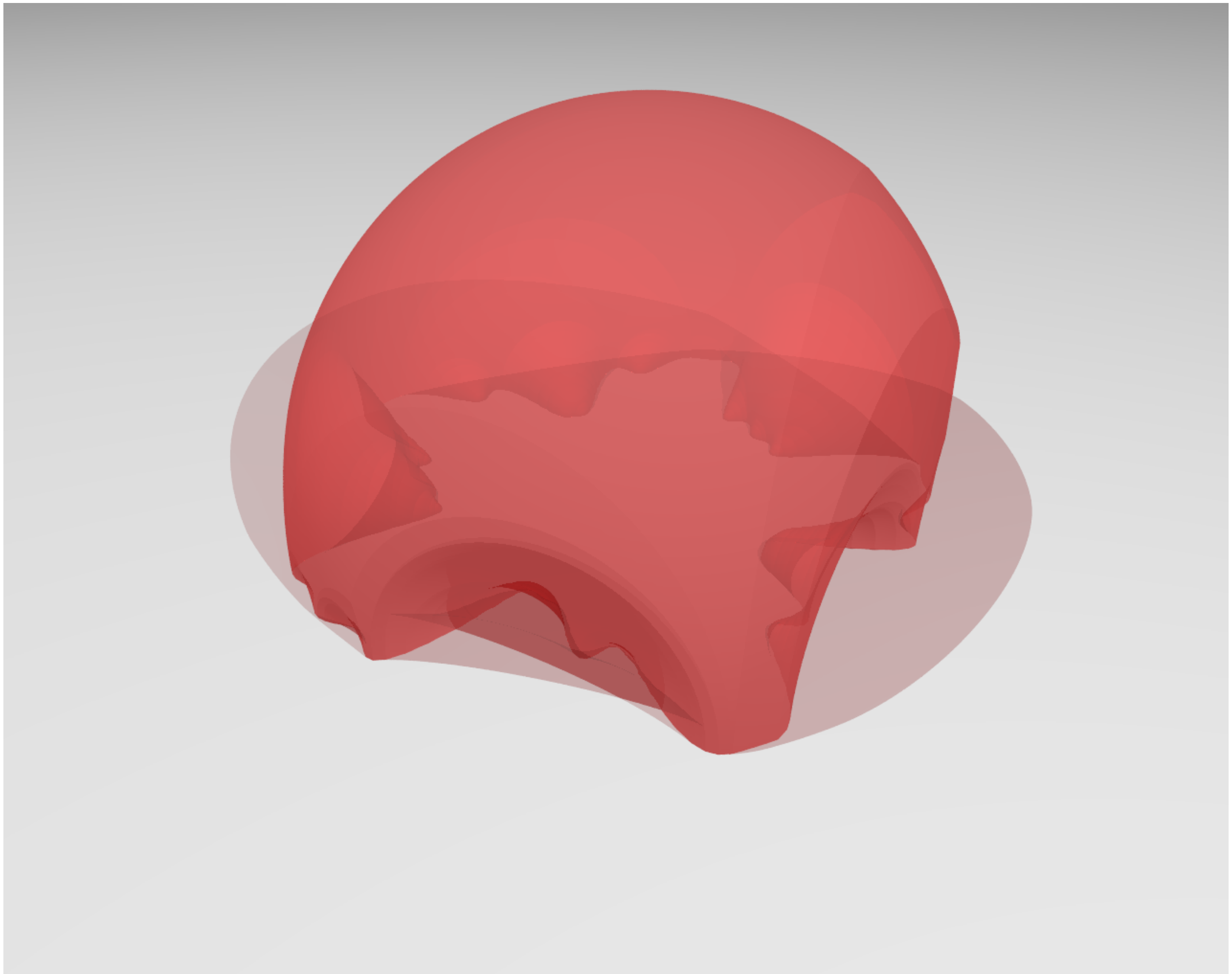
$\mathrm{CH}(\Gamma)$ is region between domes of "inside" and "outside" of $\Gamma$.

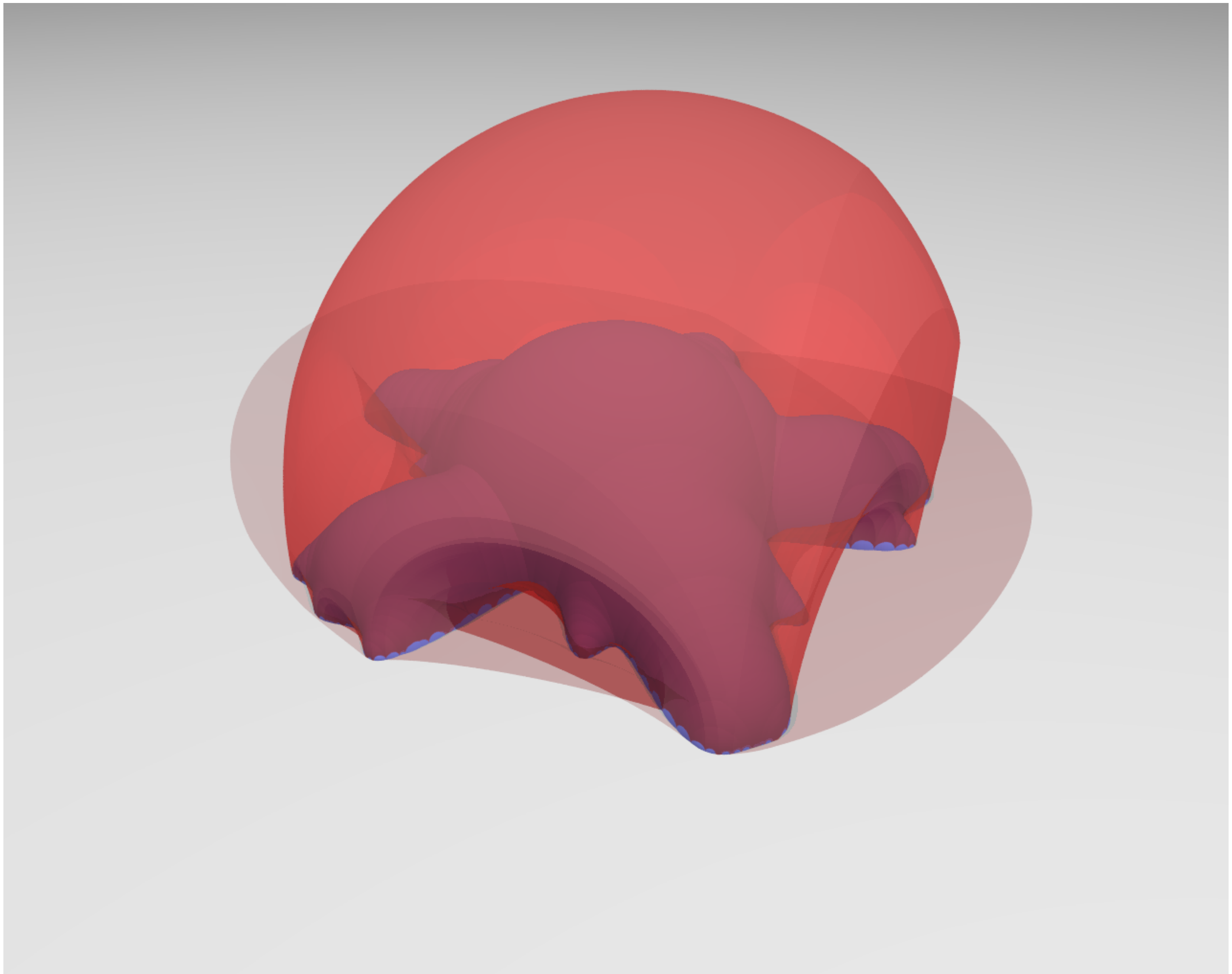


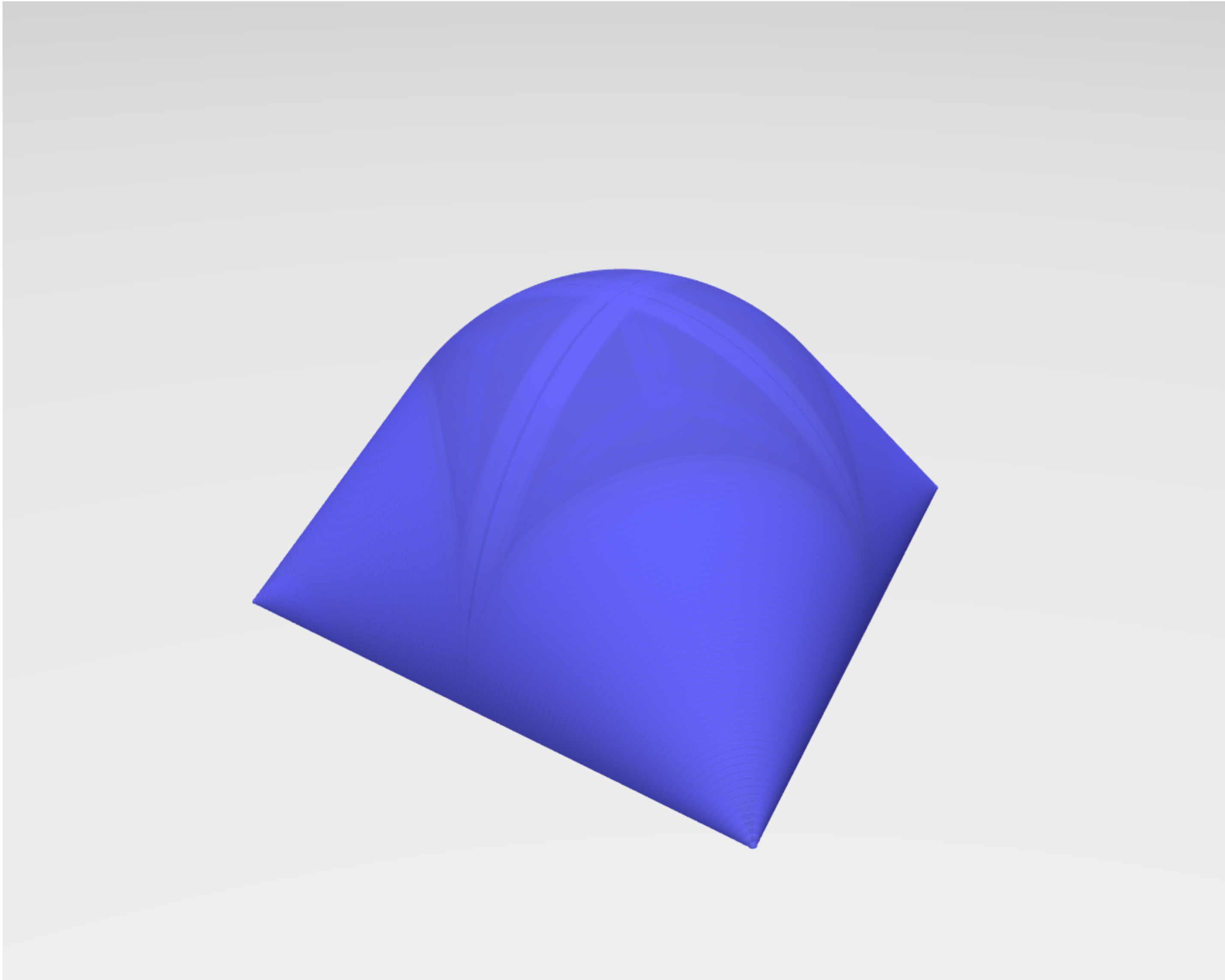


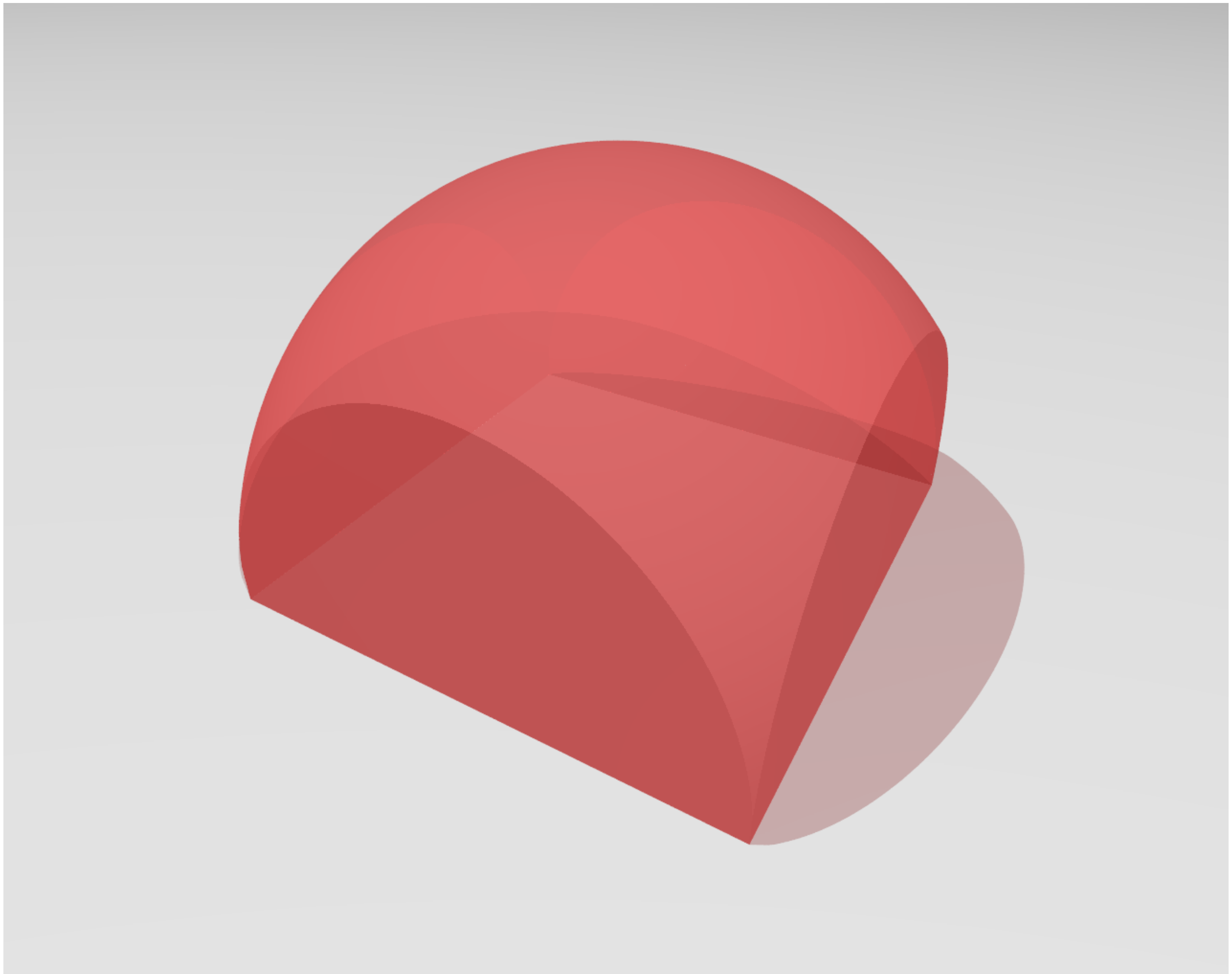


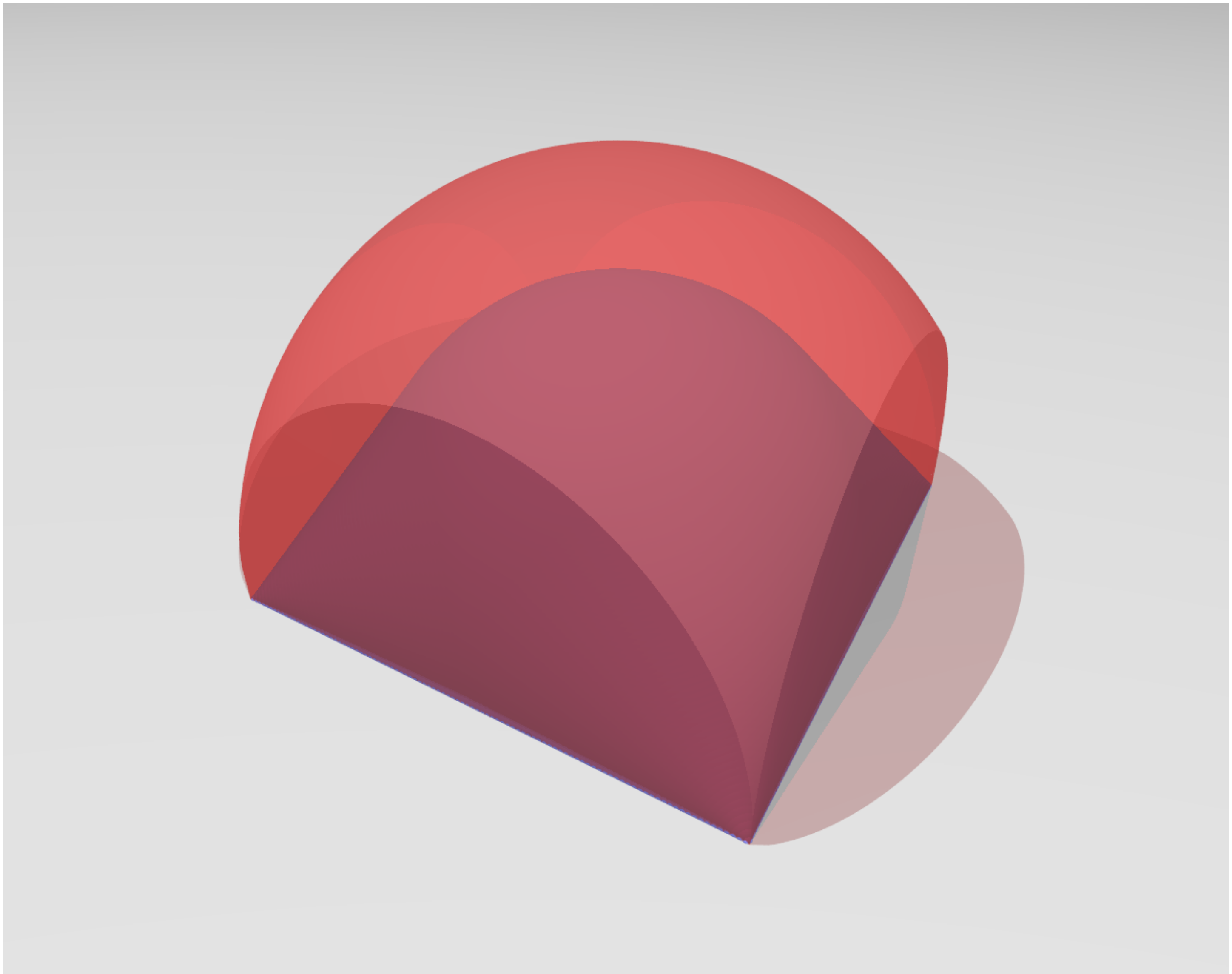
The medial axis. Equidistant from at least two boundary points.
Corresponding hemispheres give the dome.
Medial axis is widely studied in computational geometry.

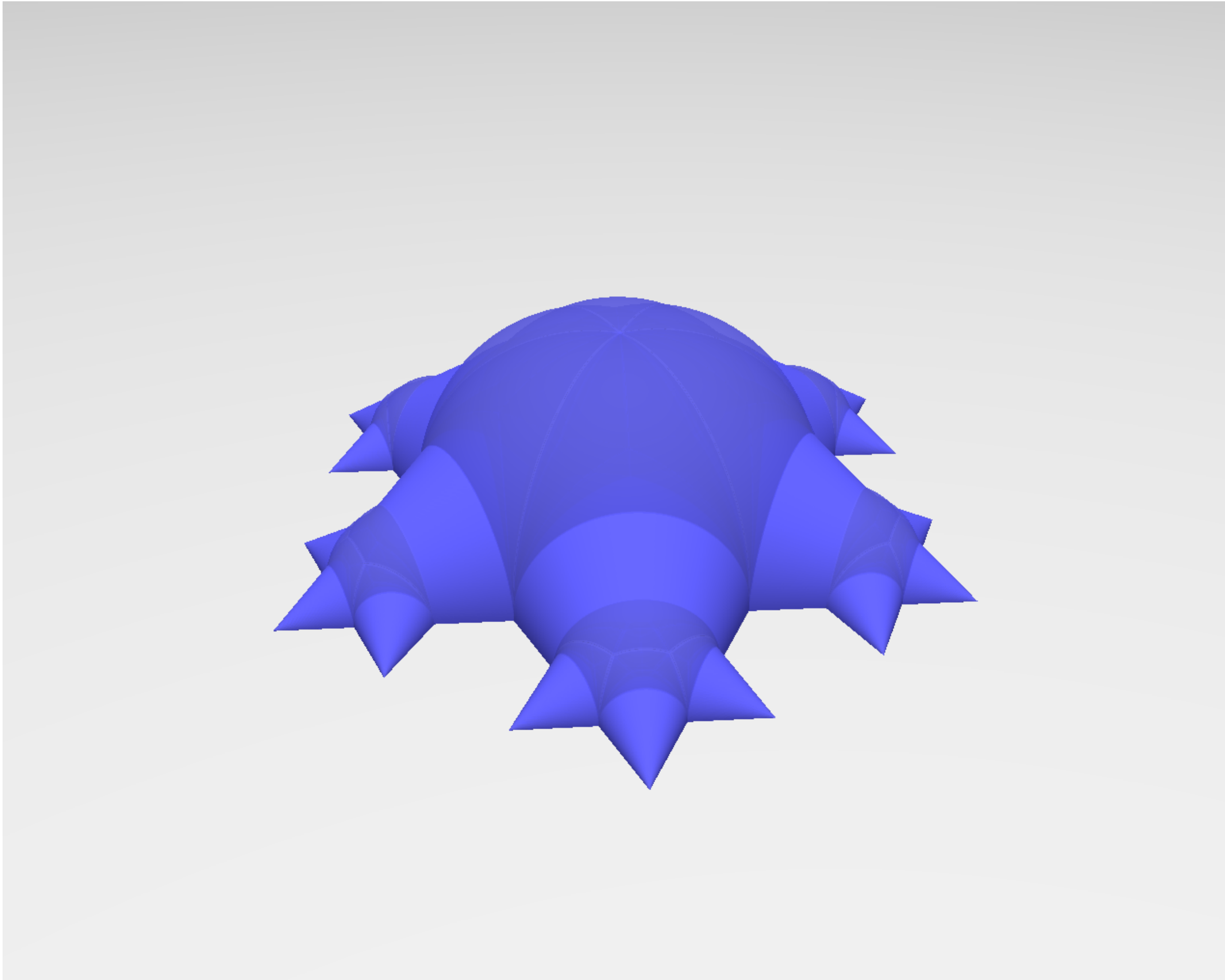


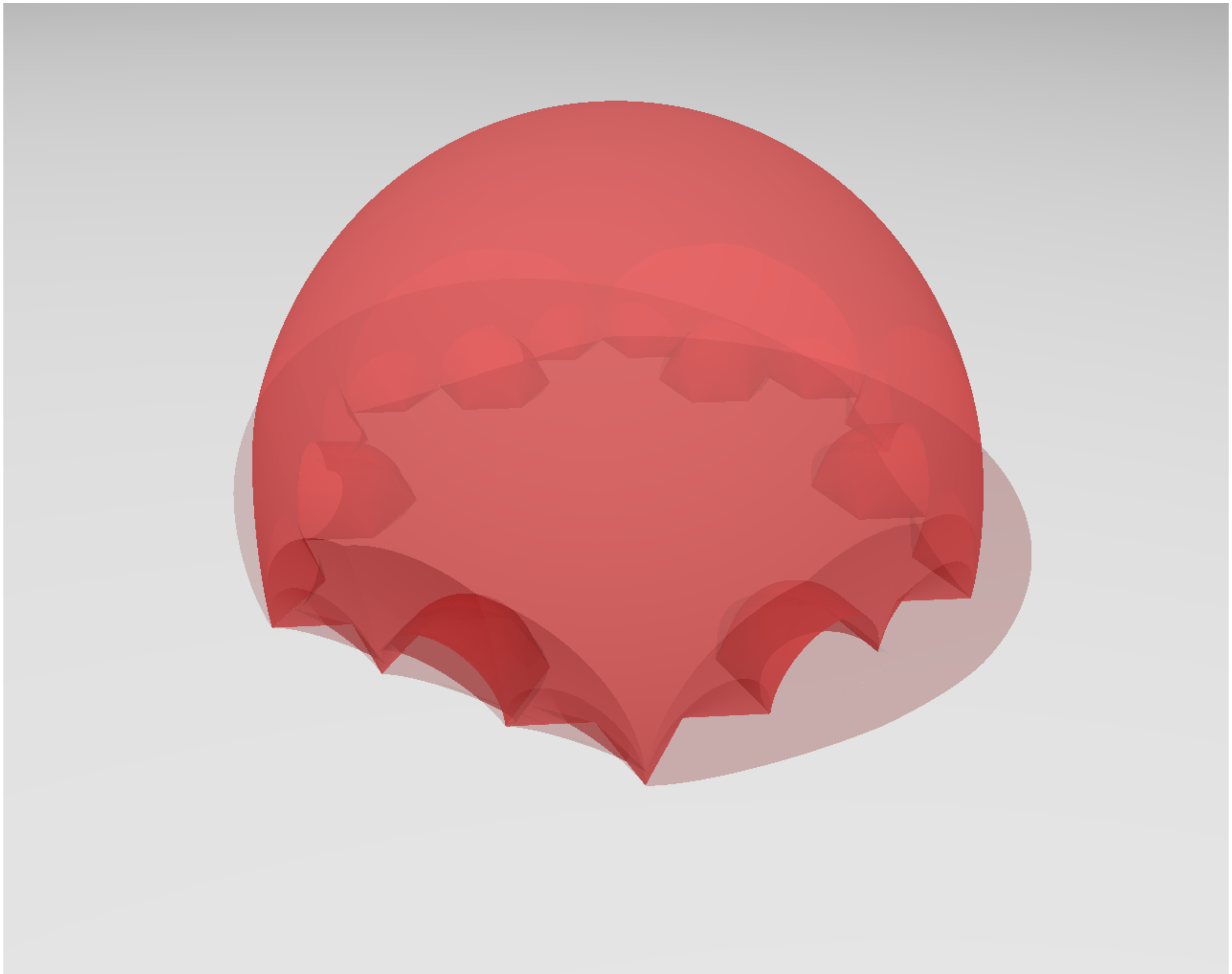


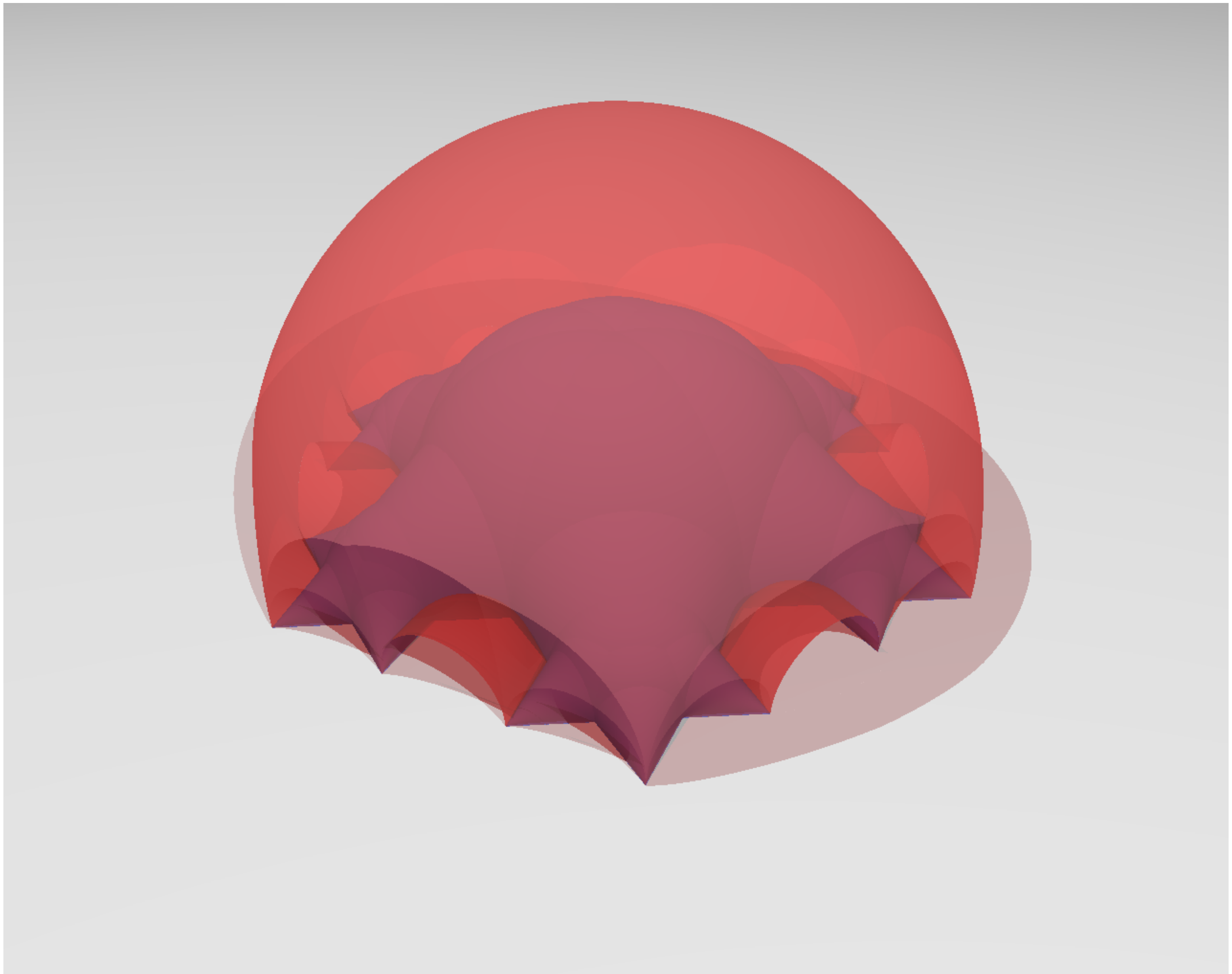




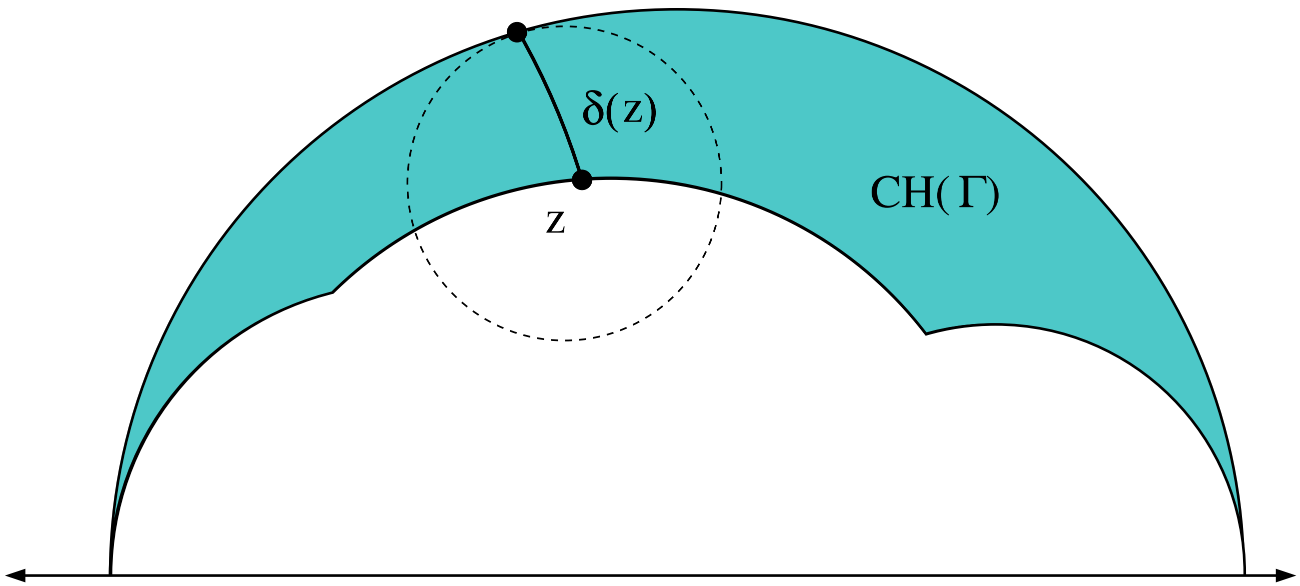








We define $\delta(z)$ to be the maximum distance from $z$ to the components of $\partial\text{CH}(\Gamma)$. Can show $\delta(z) \lesssim \varepsilon_\Gamma(Q)$, for some $Q$.




**Theorem:** $\Gamma$ is Weil-Petersson iff $\int_{\partial\text{CH}(\Gamma)} \delta^2(z)\,dA_\rho < \infty$.

Easy to see: Quasicircle $\quad\Rightarrow\quad \delta \in L^\infty \quad\not\Rightarrow\quad$ Quasicircle.

Let $S$ be a surface in $\mathbb{H}^3$ that has asymptotic boundary $\Gamma$.

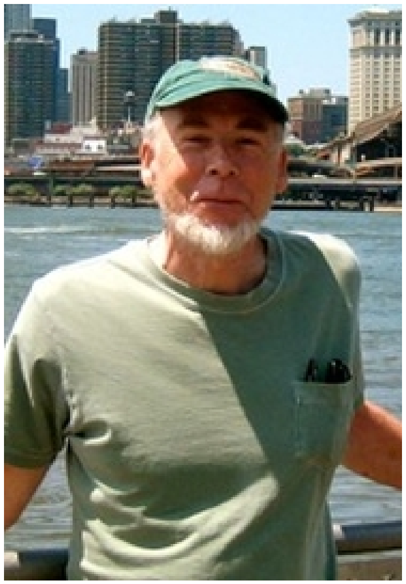
We let $K(z)$ denote the Gauss curvature of $S$ at $z$.

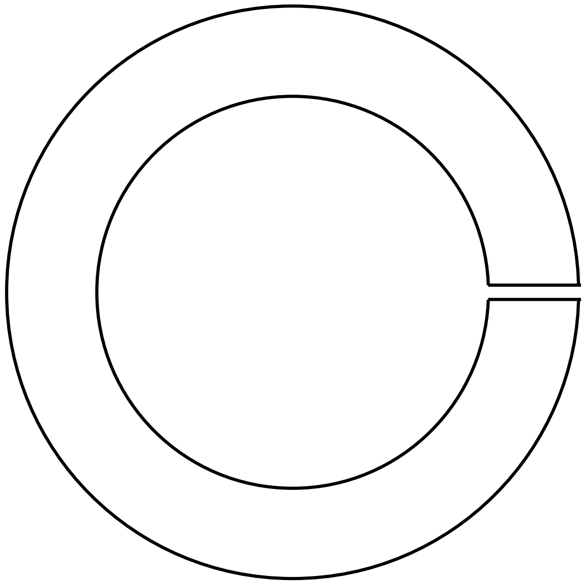
Gauss equation says $K(z) = -1 + \kappa_1(z)\kappa_2(z)$ (principle curvatures).

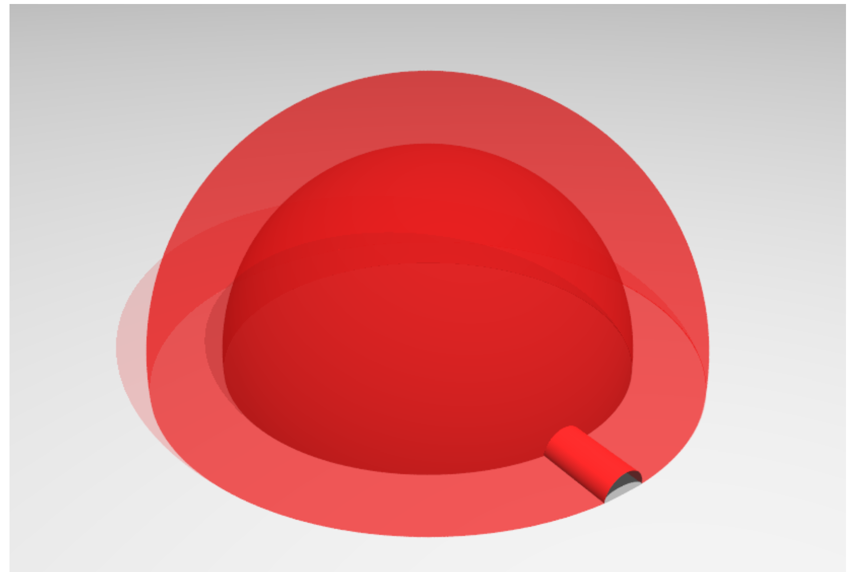
$S$ is a **minimal surface** if $\kappa_1 = -\kappa_2$ (the mean curvature is zero).

In that case, $K(z) = -1 - \kappa^2(z) \leq -1$.

**Theorem (Anderson, 1983):** Every closed Jordan curve $\Gamma \subset \mathbb{R}^2$ bounds a minimal disk $S \subset \mathrm{CH}(\Gamma) \subset \mathbb{H}^3$.







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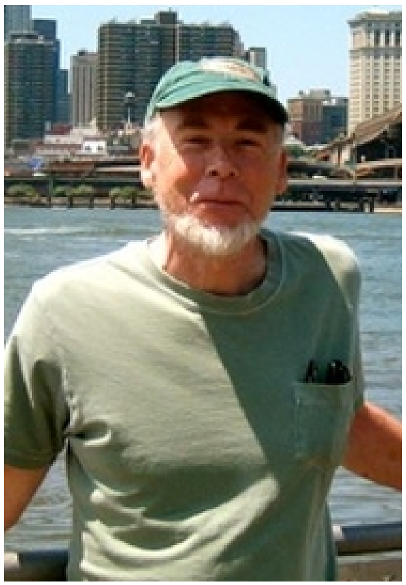
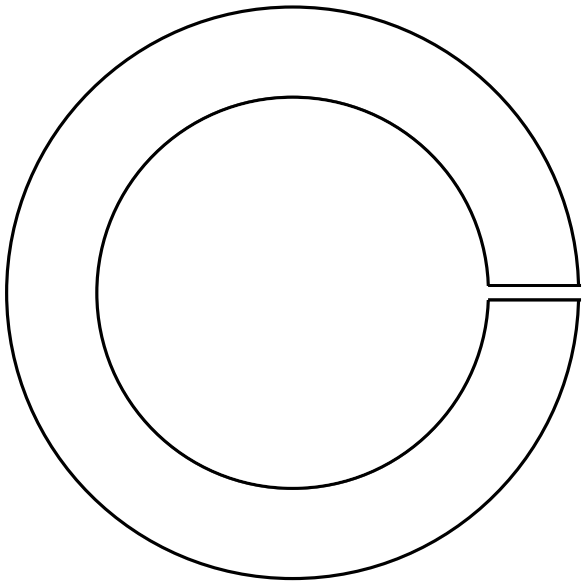
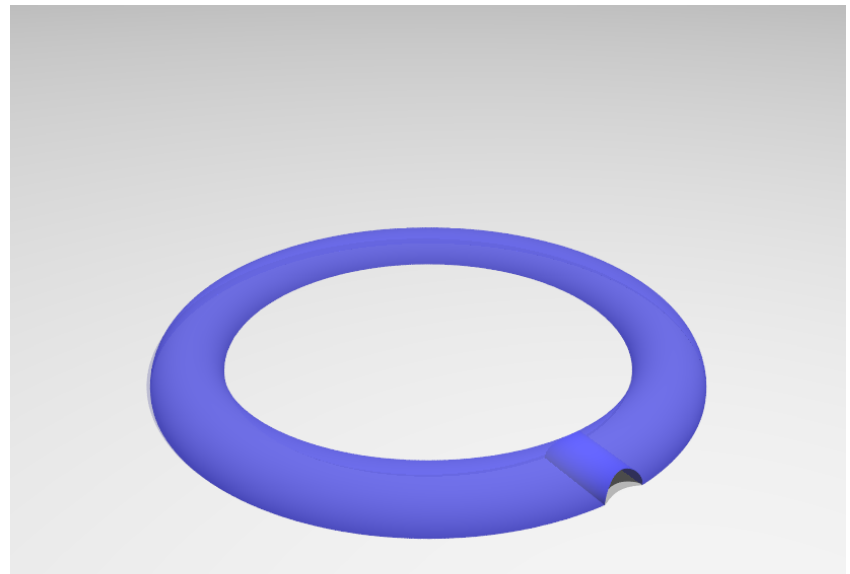
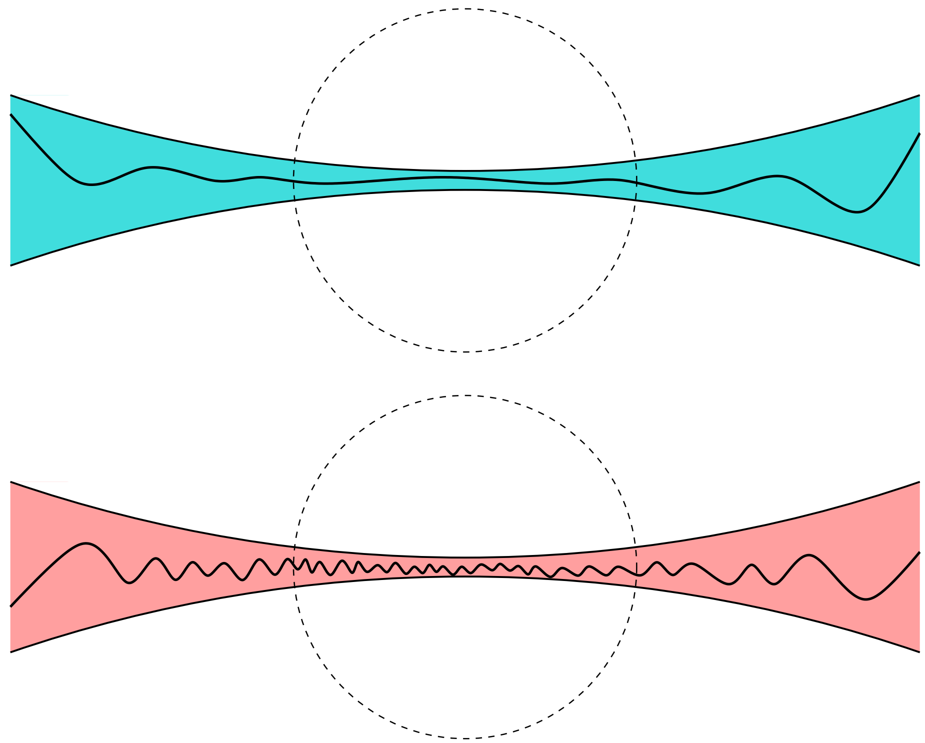






**Theorem (Anderson, 1983):** Every closed Jordan curve $\Gamma \subset \mathbb{R}^2$ bounds a minimal disk $S \subset \mathrm{CH}(\Gamma) \subset \mathbb{H}^3$.

**Theorem (Seppi, 2016):** Principle curvatures satisfies $\kappa(z) = O(\delta(z))$.



$\sinh(\mathrm{dist}(z, P))$ satisfies $\Delta_S u - 2u = 0$. Use Schauder estimate $\|\nabla^2 u\|_\infty \le C\|u\|_\infty$.

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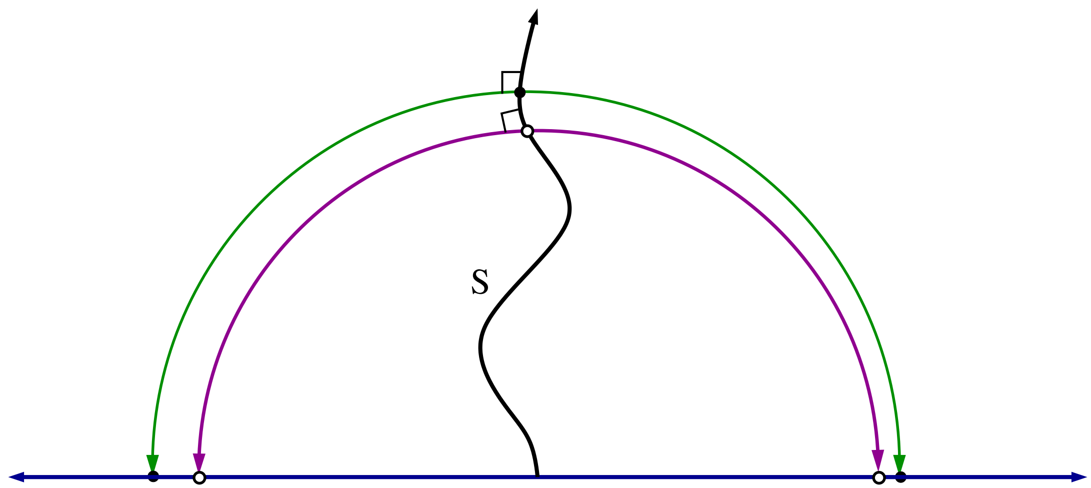
**Theorem:** $\Gamma$ is WP iff it bounds a minimal disk with finite total curvature

$$\int_S |K+1| dA_\rho = \int_S \kappa^2(z) dA_\rho < \infty.$$

Gauss map: follow normal geodesic from surface $S$ to $\mathbb{R}^2 = \partial \mathbb{H}^3$.

Two directions. Defines reflection across $\Gamma$.

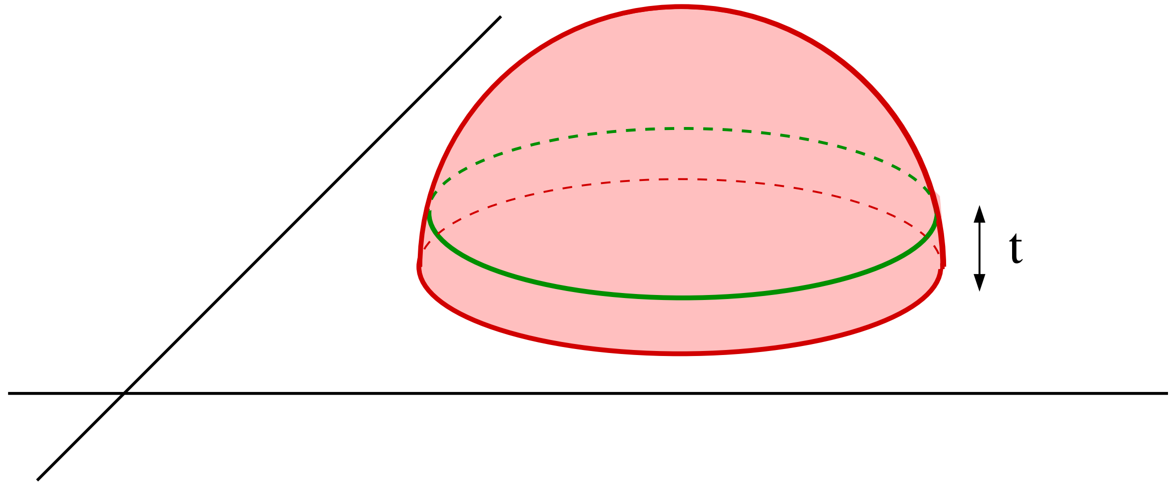
**Theorem (C. Epstein, 1986):** If $S$ is a surface with $|K| < 1$, then the Gauss maps $G_j$, $j = 1, 2$ define a quasiconformal reflection across $\Gamma$ with dilatation $|\mu(G(z))| = O(|\kappa_1|(z) + |\kappa_2|(z))$.




If $S$ is has finite total curvature, then $\int_{\mathbb{C}\setminus\Gamma} |\mu|^2 d\mathrm{A}_\rho < \infty$.

$\Rightarrow$ $\Gamma$ is fixed by a QC involution with $\mu \in L^2(dA_\rho) \Rightarrow$ Weil-Petersson.

For surface in upper half-space with boundary on $\mathbb{R}^2$, we can form sub-domains by cutting at a certain height.




Truncate $S \subset \mathbb{R}^3_+$ at a fixed height above the boundary, i.e.,

$$S_t = S \cap \{(x, y, s) \in \mathbb{R}^3_+ : s > t\}, \quad \partial S_t = S \cap \{(x, y, s) \in \mathbb{R}^3_+ : s = t\}$$

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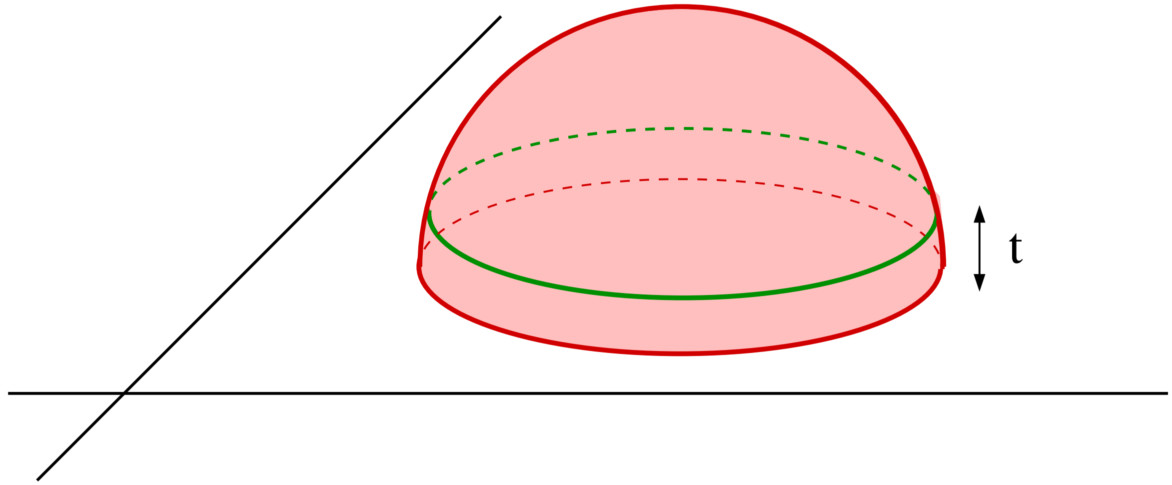


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Surfaces with curvature $K \leq -1$, such as minimal $S \subset \mathbb{H}^3$, satisfy

$$L(\partial S_t) \geq A(S_t) + 4\pi\chi(S).$$

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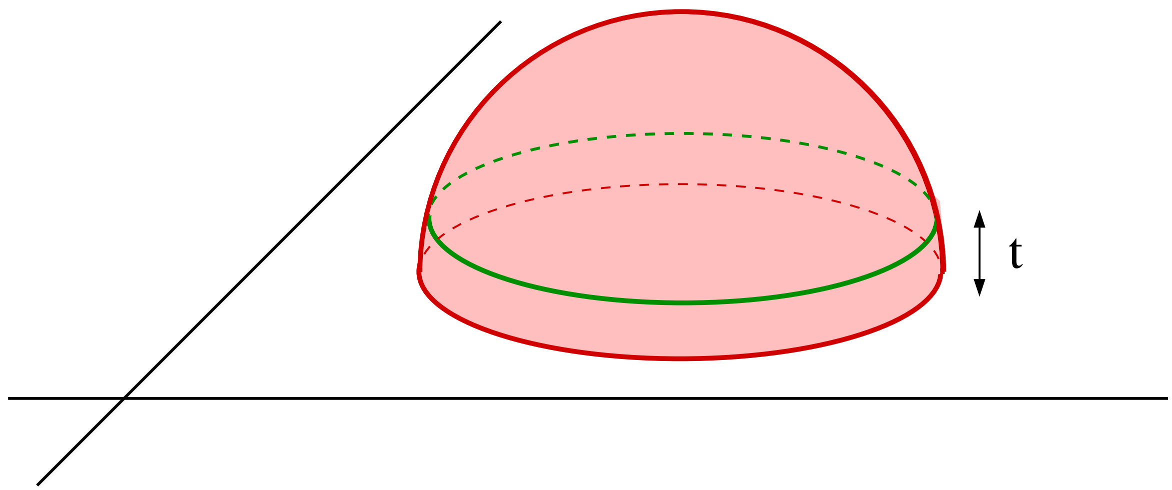


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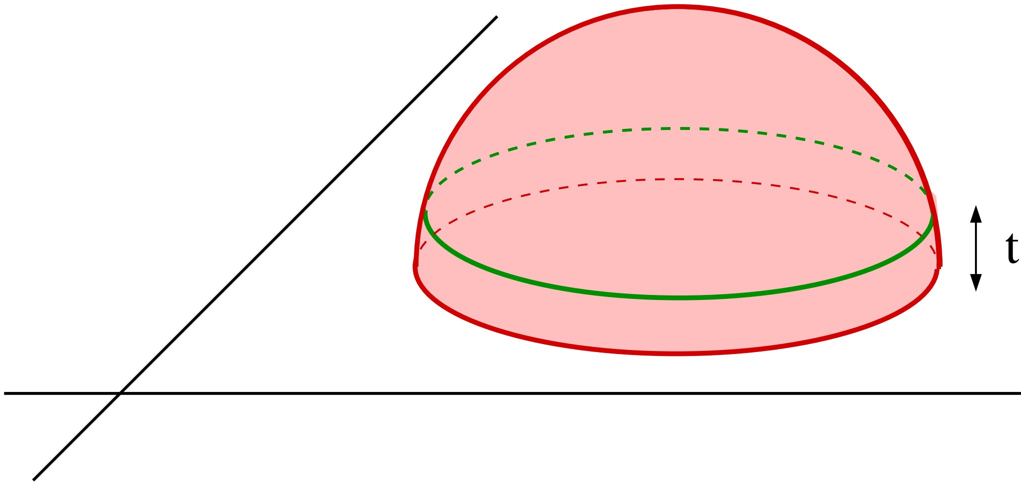
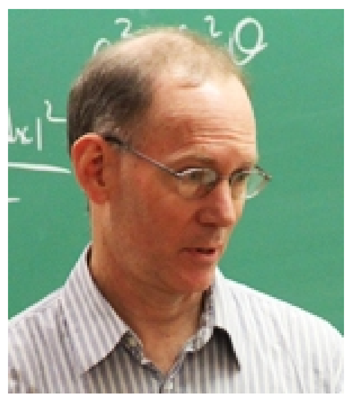
Additive upper bound? (Cheeger constant $= 1 + O(\frac{1}{A})$, asymptotically.)

For surface in upper half-space with boundary on $\mathbb{R}^2$, we can form sub-domains by cutting at a certain height.




**Renormalized area**: $\mathcal{A}_R(S) = \lim_{t\searrow 0}\left[\mathrm{A}_\rho(S_t) - \ell_\rho(\partial S_t)\right].$

Graham and Witten proved well defined.



Related to quantum entanglement, AdS/CFT correspondence.

**Theorem:** $S$ has finite renormalized area iff $\Gamma$ is Weil-Petersson.

Use isoperimetric inequalities for curved surfaces to show

$$\mathcal{A}_R(S) < \infty \quad \Rightarrow \quad \int_S \kappa^2 dA_\rho < \infty.$$

Two proofs of Weil-Petersson $\Rightarrow \mathcal{A}_R(S) < \infty$.

- Use Gauss-Bonnet, Seppi's estimate and $\int \delta^2 < \infty$.
- Use "dyadic cylinder", a discrete version of minimal surface $S$.

Using the Gauss-Bonnet theorem

$$\begin{aligned}
\mathrm{A}_\rho(S_t) - \ell_\rho(\partial S_t) &= \int_{S_t} 1 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= \int_{S_t} (1+\kappa^2) d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= -\int_{S_t} K d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= -2\pi\chi(S_t) + \int_{\partial S_t} \kappa_g d\ell_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= -2\pi\chi(S_t) - \int_{S_t} \kappa^2 d\mathrm{A}_\rho + \int_{\partial S_t} (\kappa_g - 1) d\ell_\rho
\end{aligned}$$

Can prove $\kappa_g(z) = 1 + O(\delta^2(z))$, so WP implies last term $\to 0$.

**Theorem:** For any closed curve $\Gamma \subset \mathbb{R}^2$ and for any minimal surface $S \subset \mathbb{R}^3_+$ with finite Euler characteristic and asymptotic boundary $\Gamma$,

$$\mathcal{A}_R(S) = -2\pi\chi(S) - \int_S \kappa^2(z) d\mathrm{A}_\rho,$$

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Due to Alexakis and Mazzeo assuming that $\Gamma$ is $C^{3,\alpha}$.

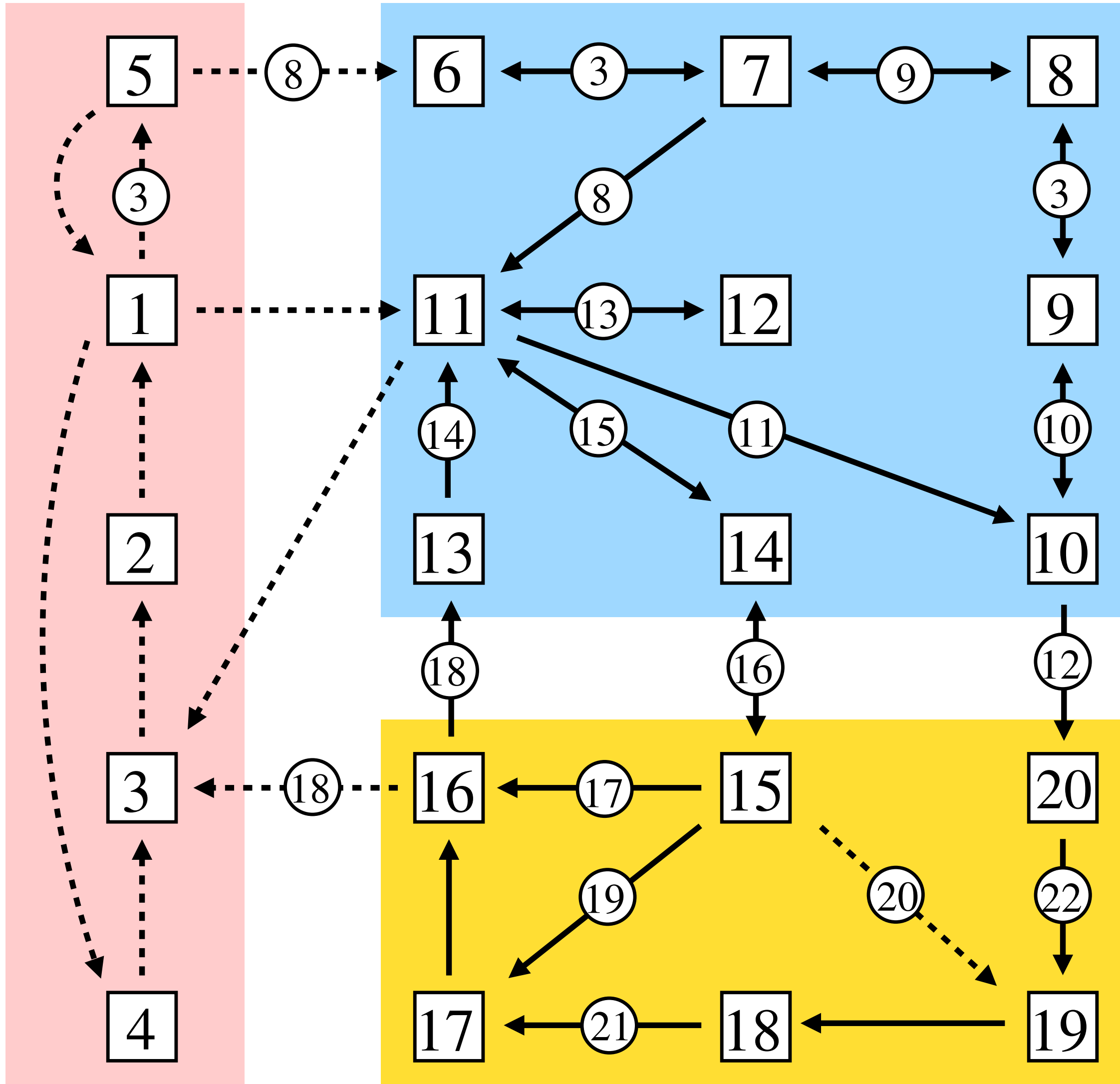
| Definition | Description |
| --- | --- |
| 1 | $\log f'$ in Dirichlet class |
| 2 | Schwarzian derivative |
| 3 | QC dilatation in $L^2$ |
| 4 | conformal welding midpoints |
| 5 | $\exp(i \log f')$ in $H^{1/2}$ |
| 6 | arclength parameterization in $H^{3/2}$ |
| 7 | tangents in $H^{1/2}$ |
| 8 | finite Möbius energy |
| 9 | Jones conjecture |
| 10 | good polygonal approximations |
| 11 | $\beta^2$-sum is finite |
| 12 | Menger curvature |
| 13 | biLipschitz involutions |
| 14 | between disjoint disks |
| 15 | thickness of convex hull |
| 16 | finite total curvature surface |
| 17 | minimal surface of finite curvature |
| 18 | additive isoperimetric bound |
| 19 | finite renormalized area |
| 20 | dyadic cylinder |

Weil-Petersson curves

André Weil

Hans Petersson

# THE THEOREM




# THANKS FOR LISTENING. QUESTIONS?

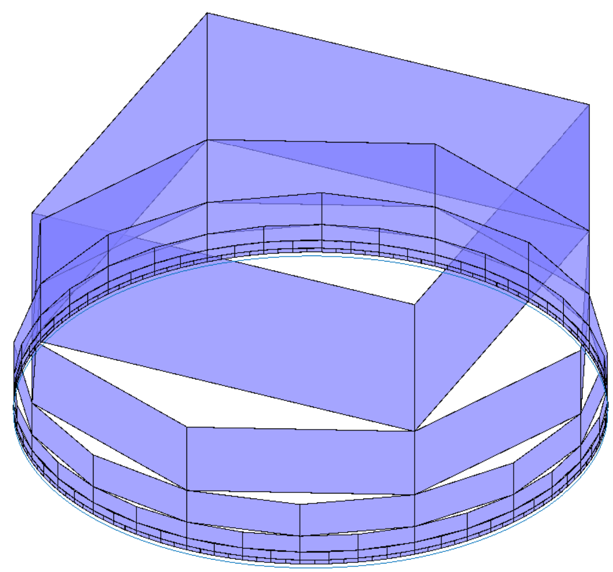
An idea connecting Euclidean and hyperbolic results.

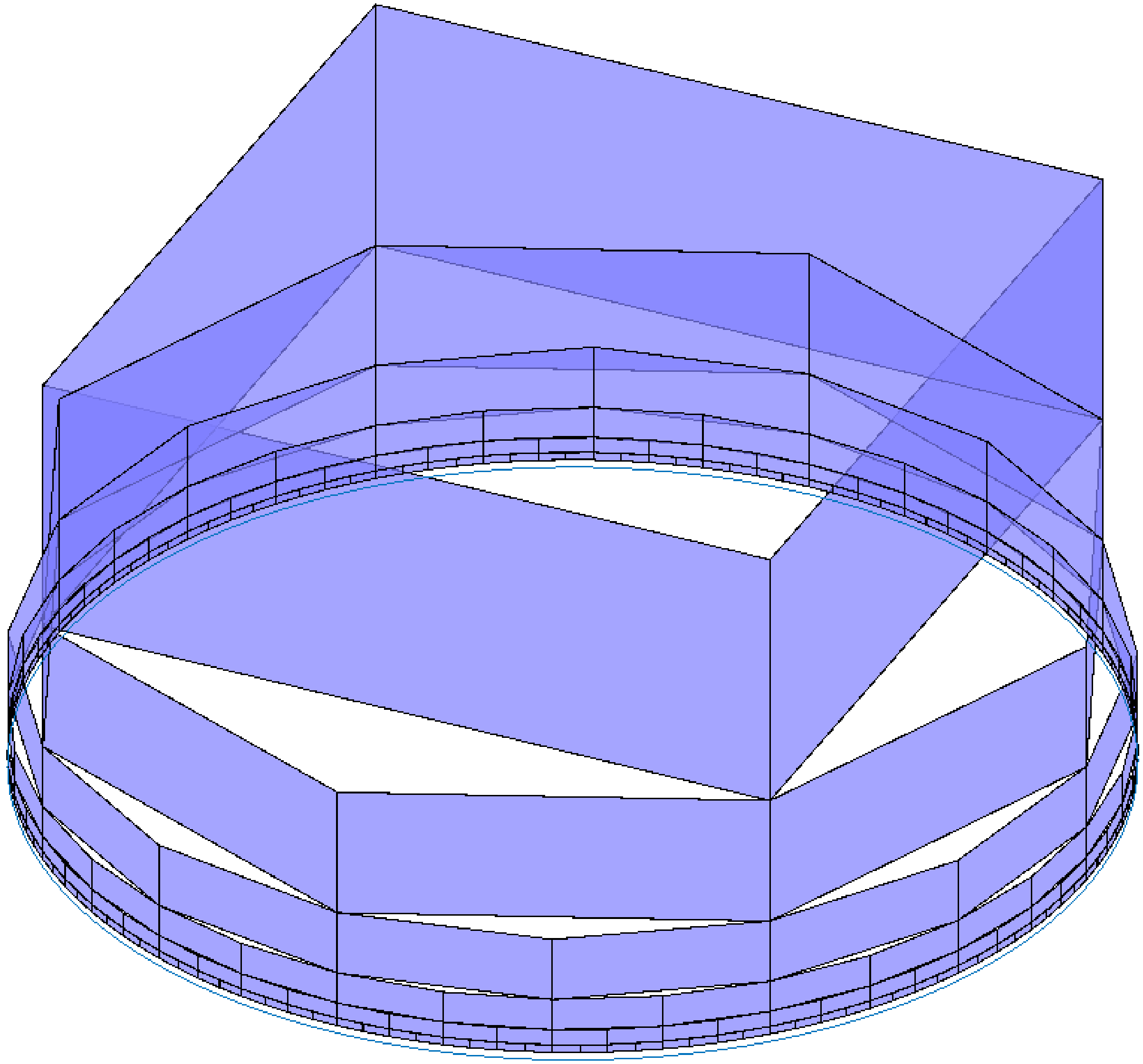
Define a dyadic cylinder in the upper half-space:

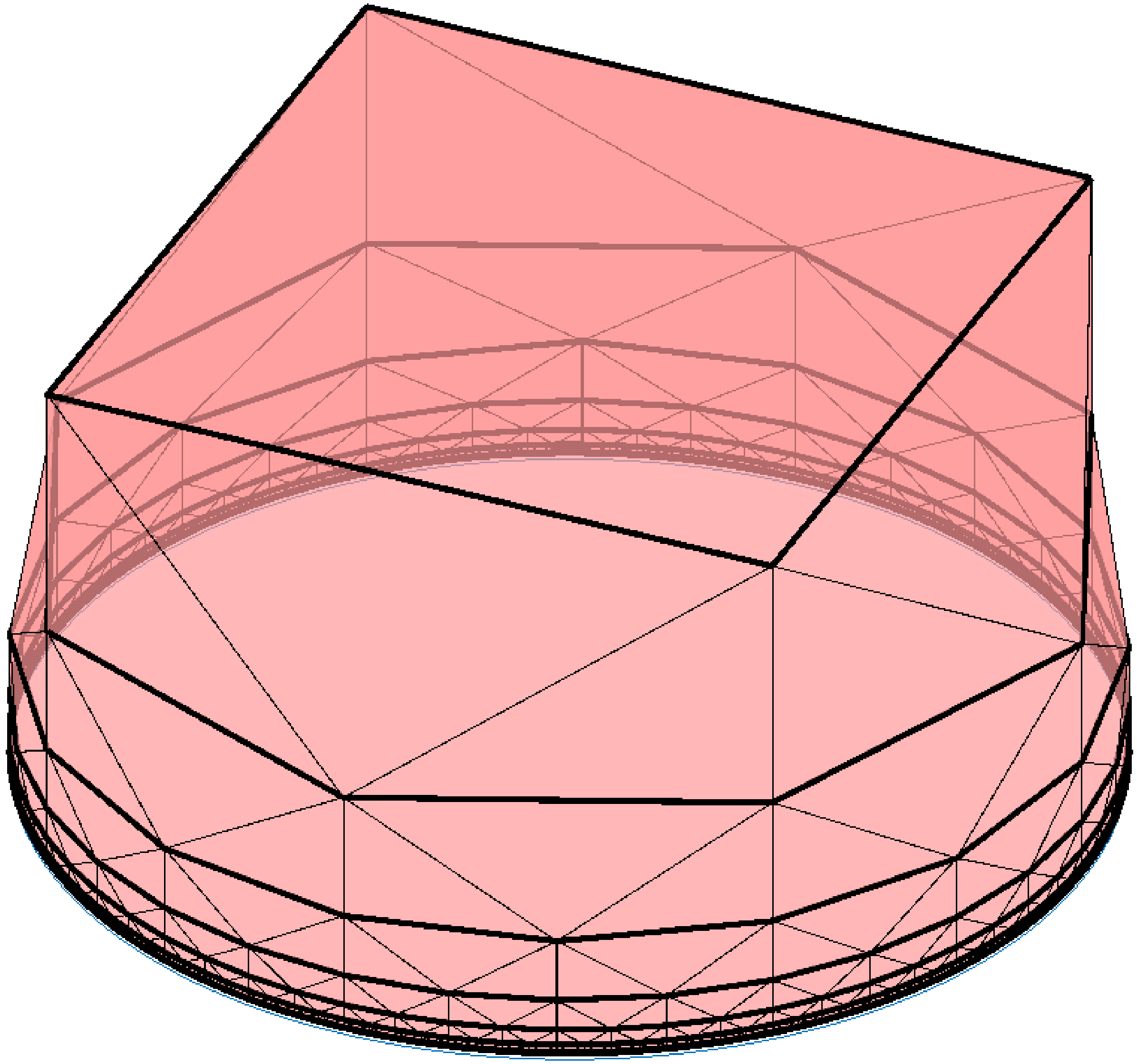
$$X = \bigcup_{n=1}^{\infty} \Gamma_n \times [2^{-n}, 2^{-n+1}),$$

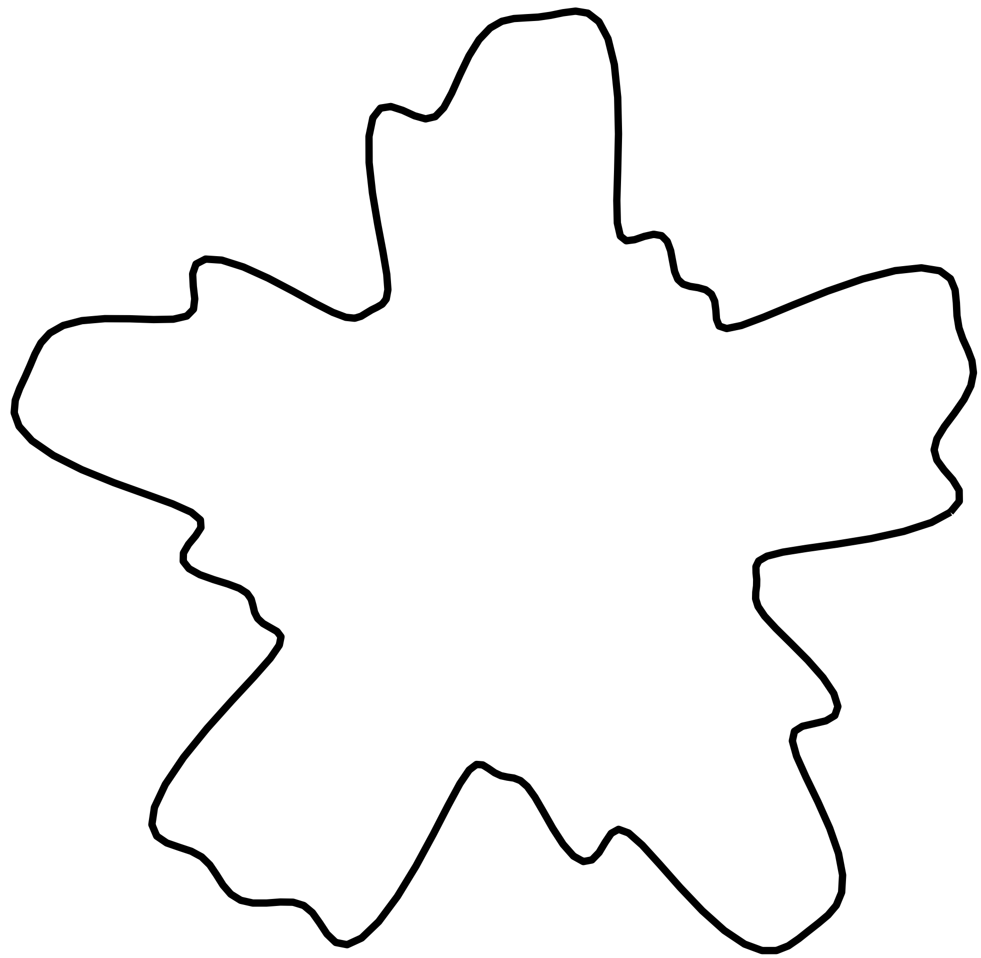
where $\{\Gamma_n\}$ are inscribed dyadic polygons in $\Gamma$.

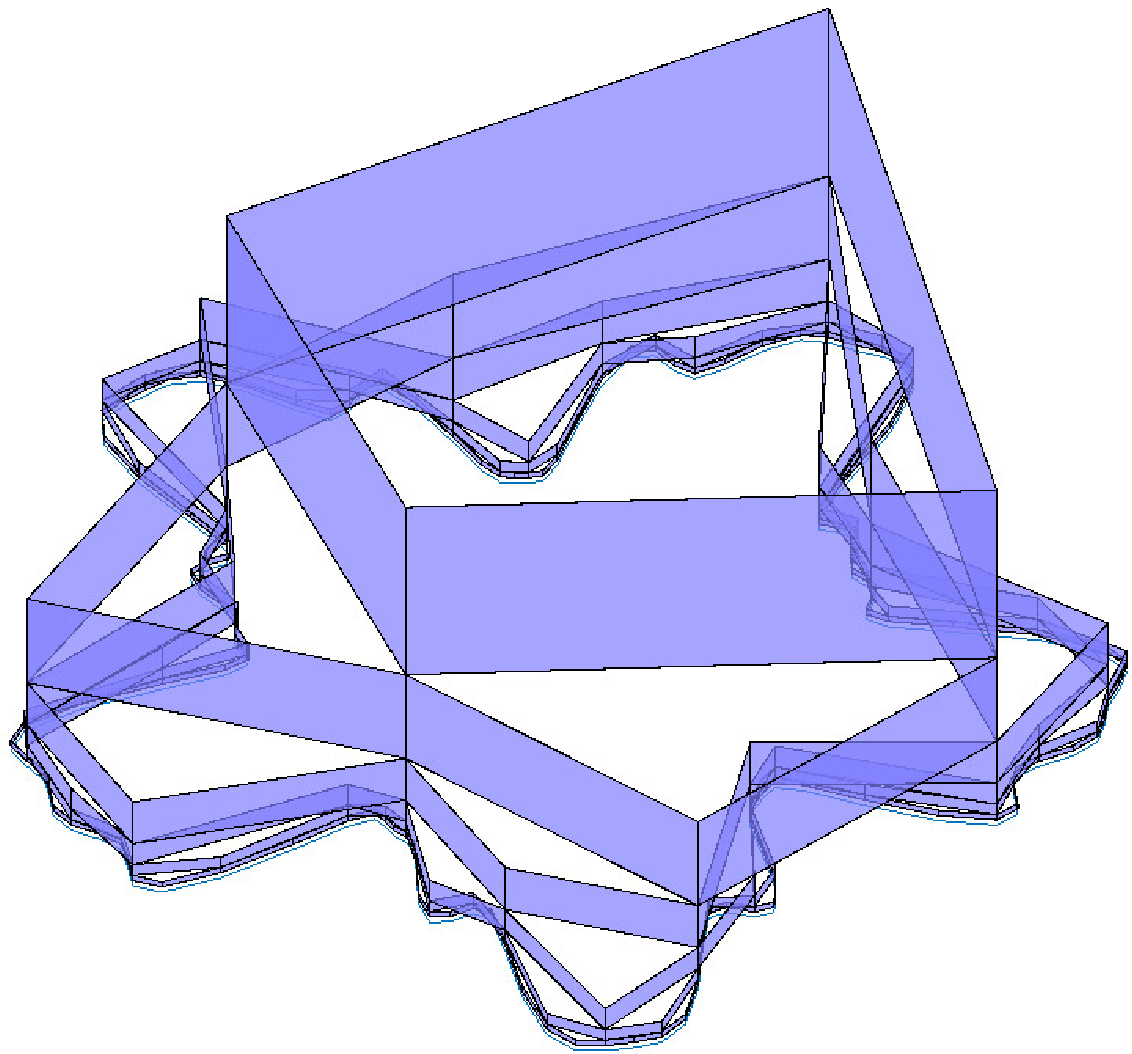
Discrete analog of minimal surface with boundary $\Gamma$.

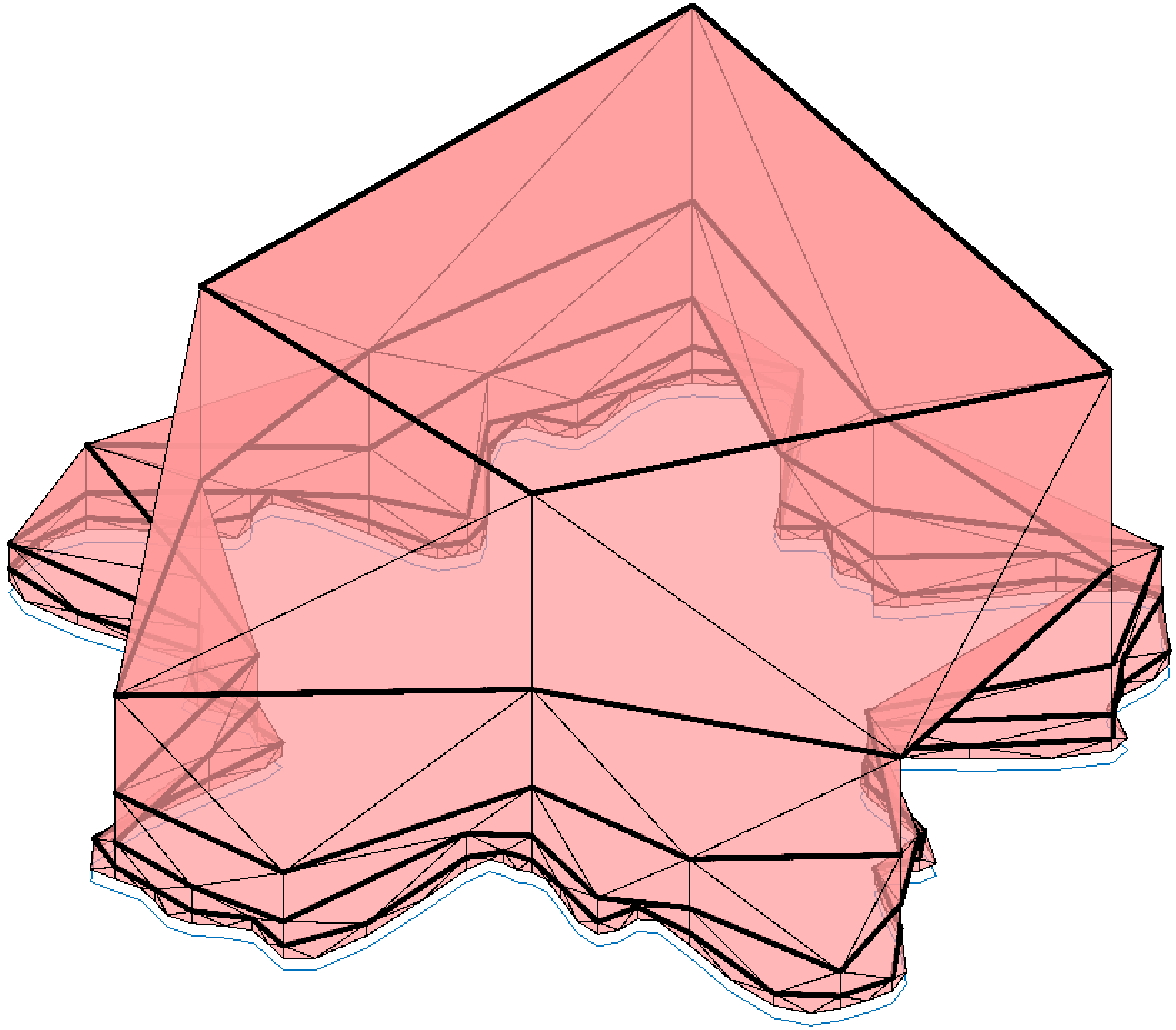












Our earlier estimate

$$\sum_n 2^n(\ell(\Gamma) - \ell(\Gamma_n)) < \infty$$

is equivalent to the dyadic cylinder having finite renormalized area.

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$$\sum_n 2^n(\ell(\Gamma) - \ell(\Gamma_n)) < \infty$$

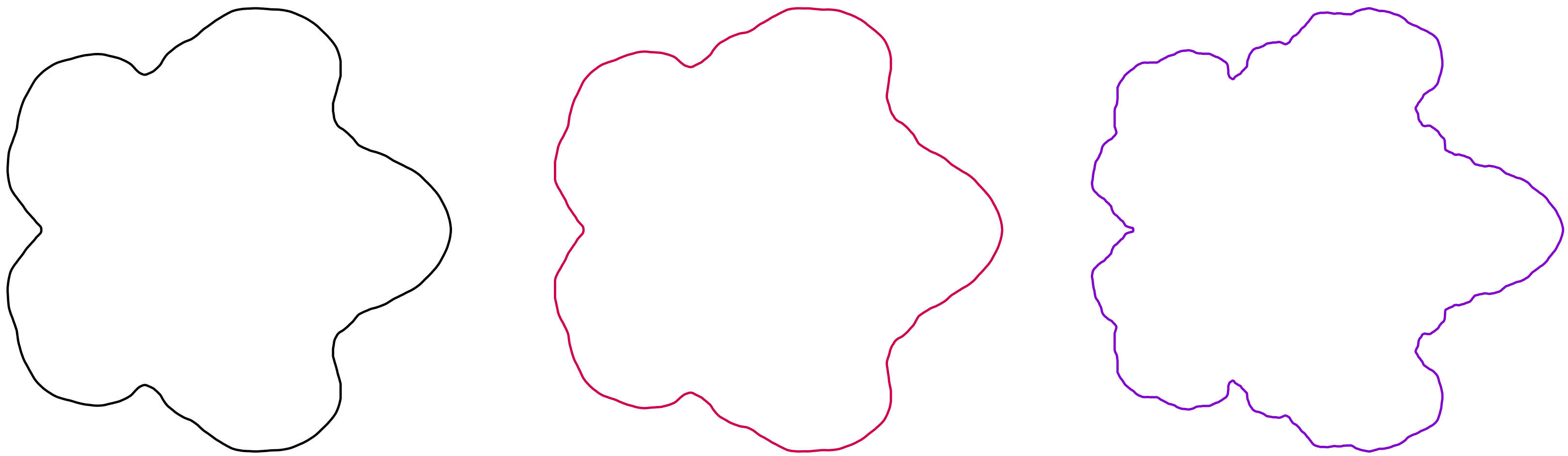
is equivalent to the dyadic cylinder having finite renormalized area.

Obvious "normal projection" from the dyadic cylinder to minimal surface, distorts length and area each by a bounded additive error.

We can deduce finite renormalized area for the minimal surface from the same result for the dyadic cylinder.

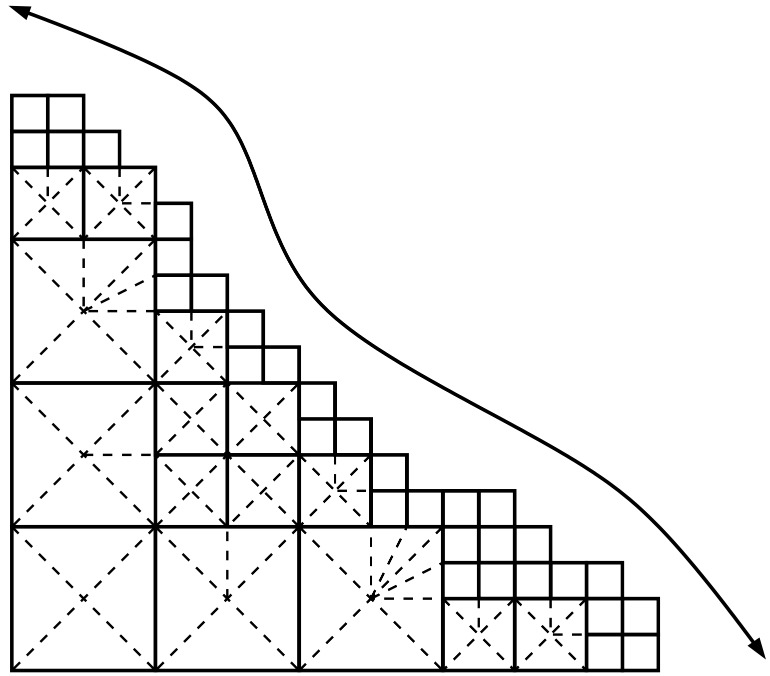
$F(z) = \sum_1^\infty a_n z^n$ is Dirichlet class iff $\sum n|a_n|^2 < \infty$.

If $\log f' = \sum \sqrt{\frac{b_k}{\lambda^k}}\, z^{\lambda^k}$ then $\Gamma = f(\mathbb{T})$ is WP iff $\sum b_k < \infty$.



$\sum \frac{1}{k \log^2 k} < \infty$ $\qquad$ $\sum \frac{1}{k \log k} = \infty$ $\qquad$ $\sum 1 = \infty$

Easy to see $\sum \beta^2 < \infty$ implies Weil-Petersson.



- Triangulate one side of $\Gamma$ (e.g., triangulate Whitney squares).

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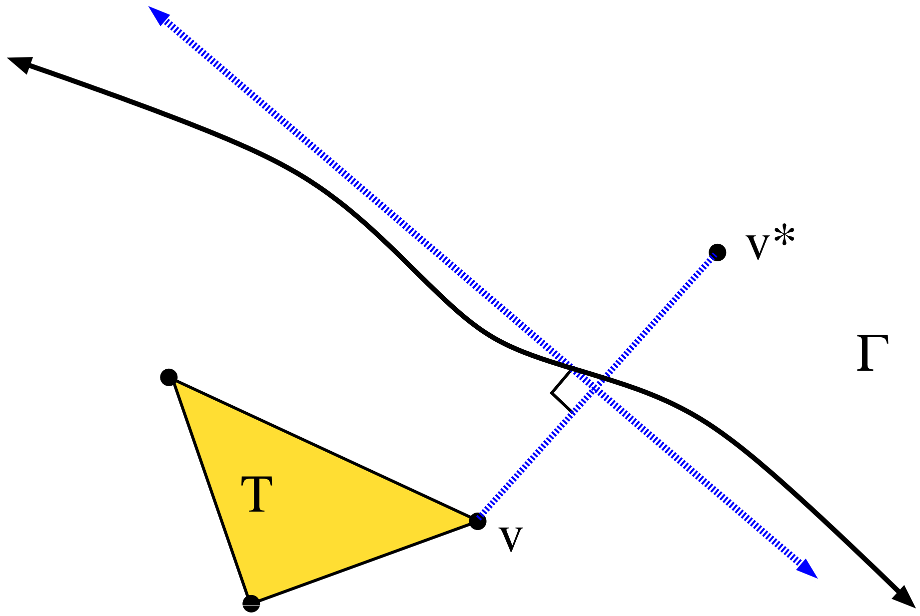


- Triangulate one side of $\Gamma$ (e.g., triangulate Whitney squares).
- Use approximating lines to reflect vertices.

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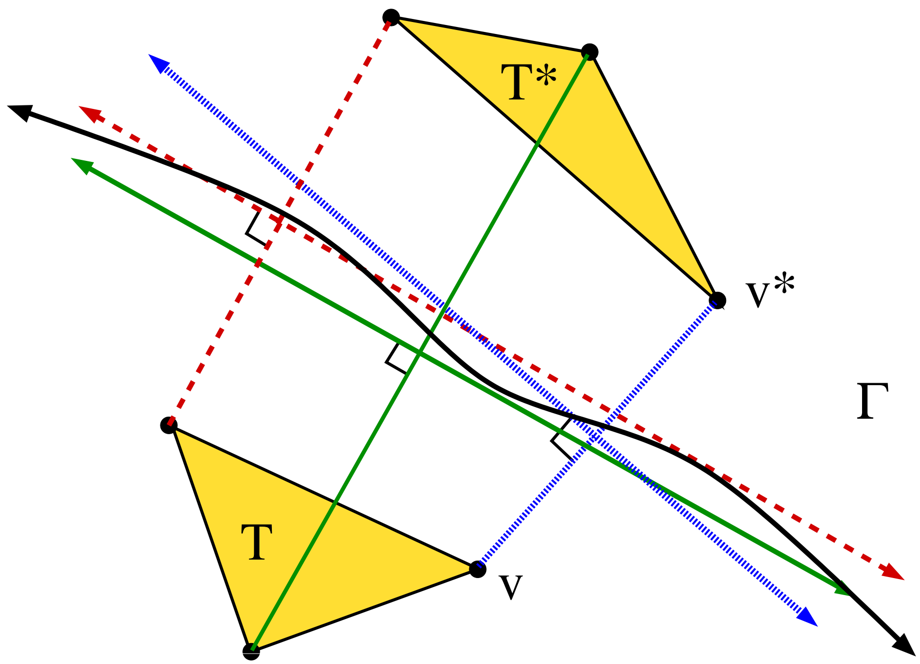
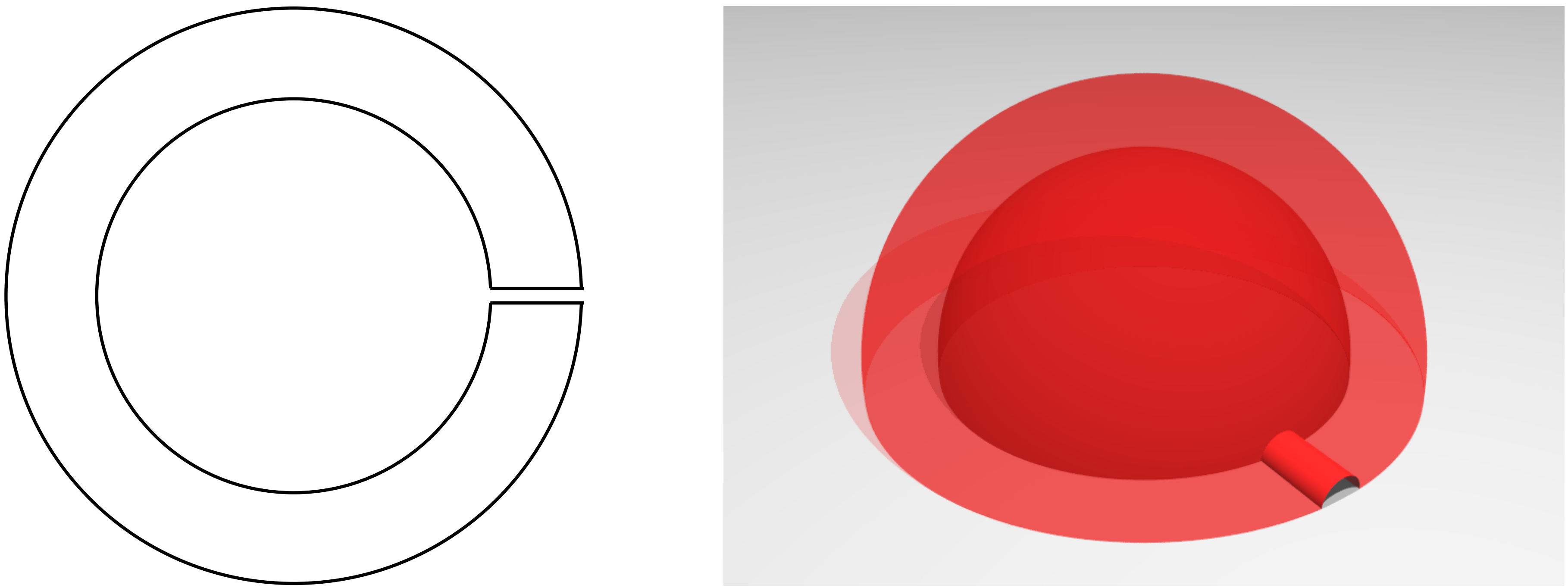



- Triangulate one side of $\Gamma$ (e.g., triangulate Whitney squares).
- Use approximating lines to reflect vertices.
- Define piecewise linear map.
- $|\mu| = O(\beta)$.
- Get involution fixing $\Gamma$ with $|\mu| \in L^2(dA_\rho) \Rightarrow$ Weil-Petersson.

**Theorem (Anderson, 1983):** Every closed Jordan curve $\Gamma \subset \mathbb{R}^2$ bounds a minimal disk $S \subset \mathrm{CH}(\Gamma) \subset \mathbb{H}^3$.



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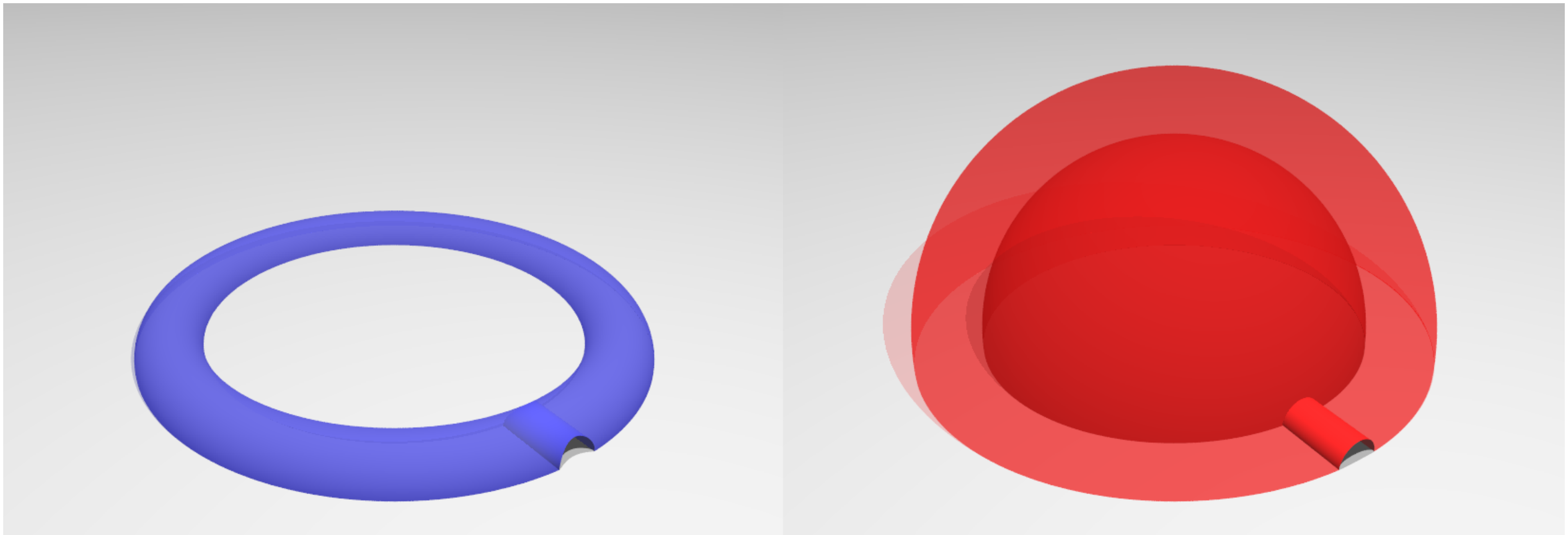

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