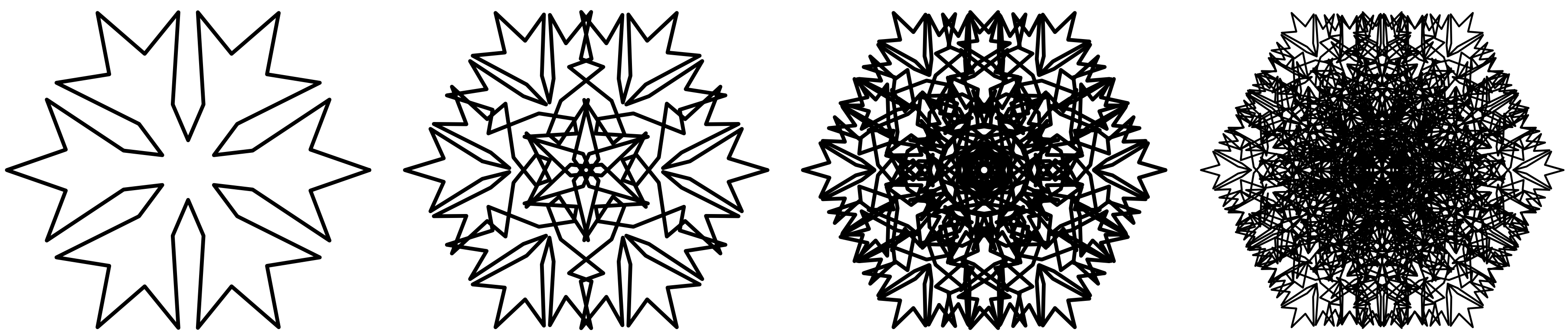
A RANDOM WALK RUNS THROUGH IT: A PORTFOLIO OF PROBABILISTIC PICTURES

Christopher Bishop, Stony Brook

Illustrating dynamics and probability
ICERM, Providence RI, Nov 11-15

`www.math.sunysb.edu/~bishop/lectures`

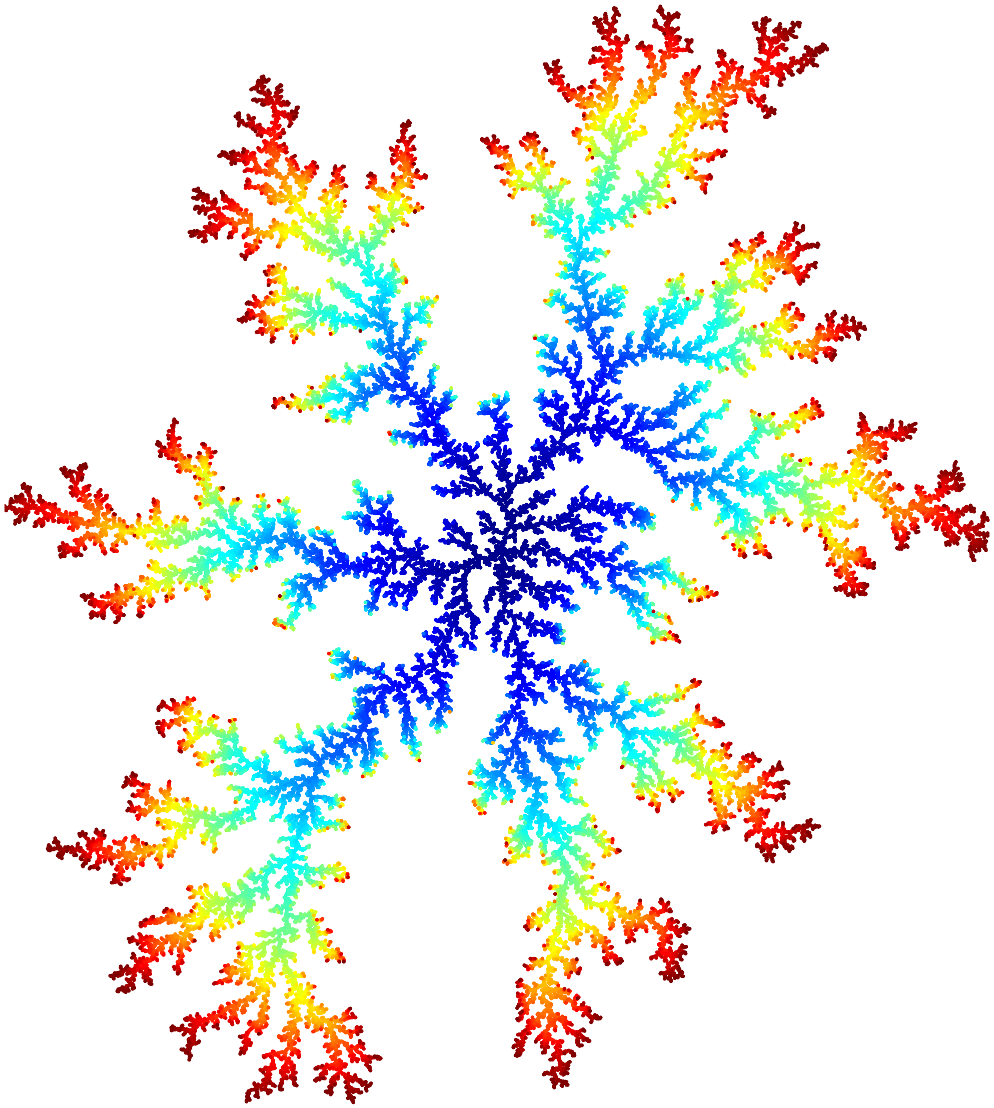


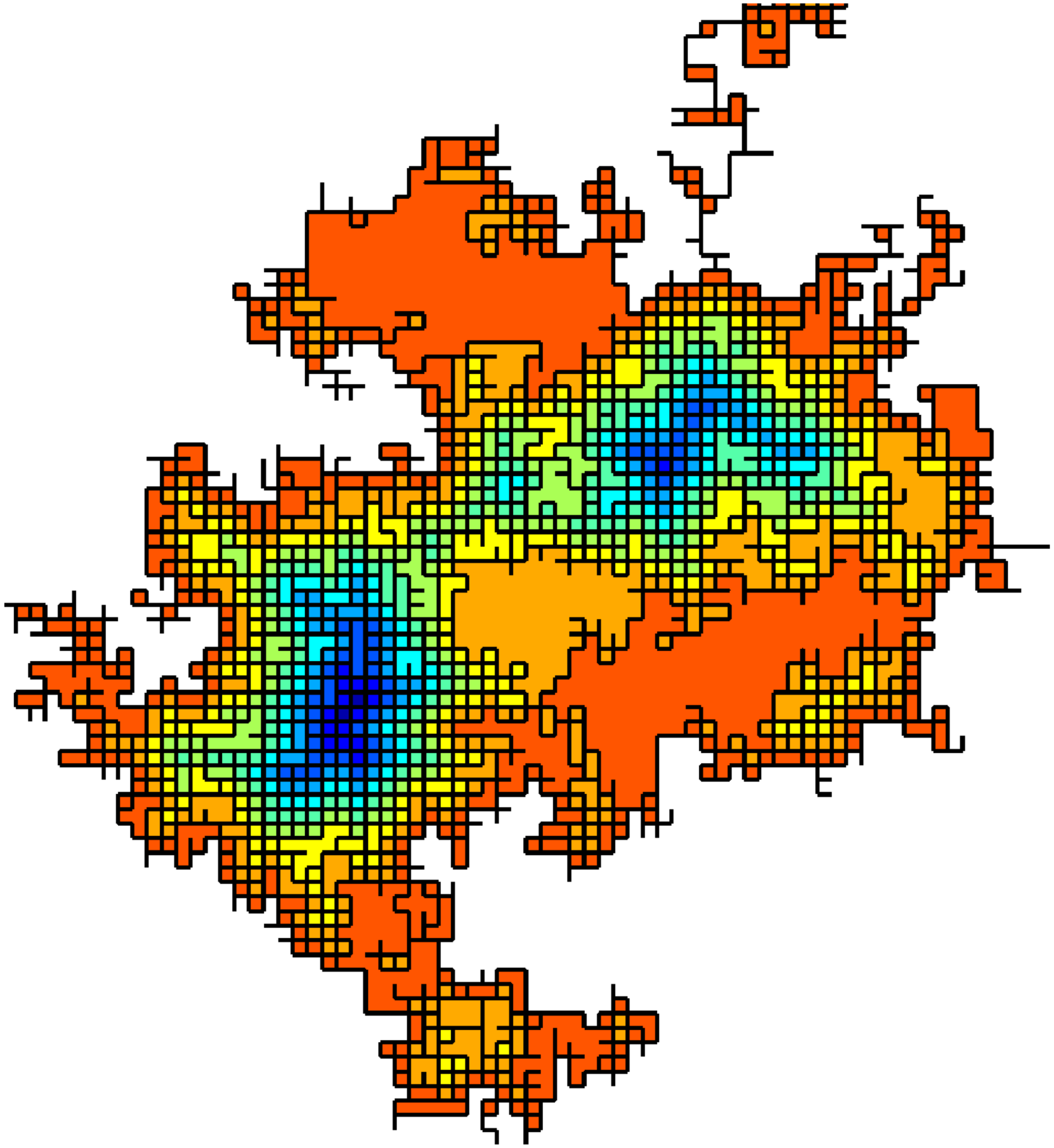
"As for my father, I never knew whether he believed God was a mathematician but he certainly believed God could count and that only by picking up God's rhythms were we able to regain power and beauty. Unlike many Presbyterians, he often used the word "beautiful"."

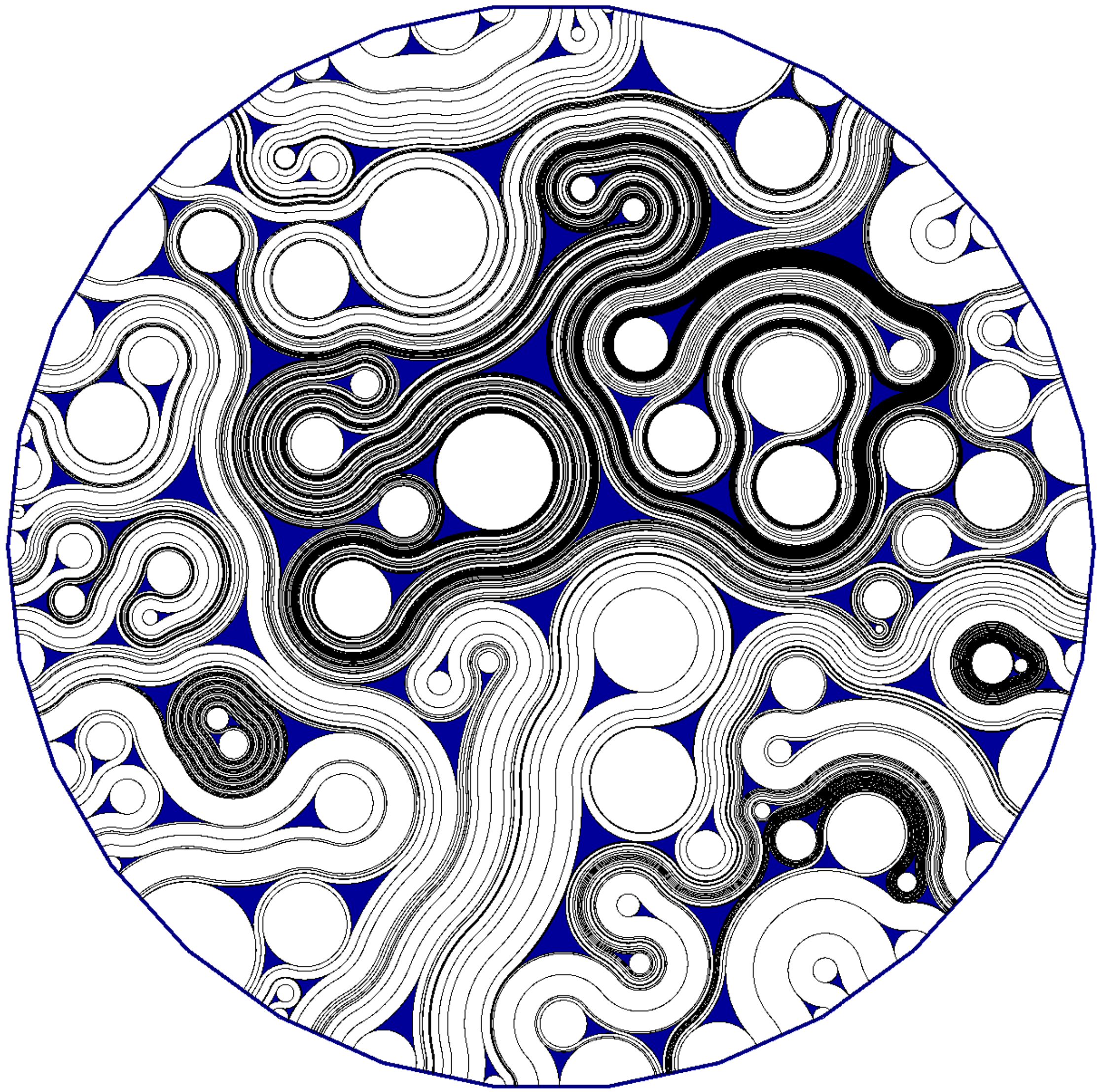
– Norman Maclean, *A River Runs Through It*

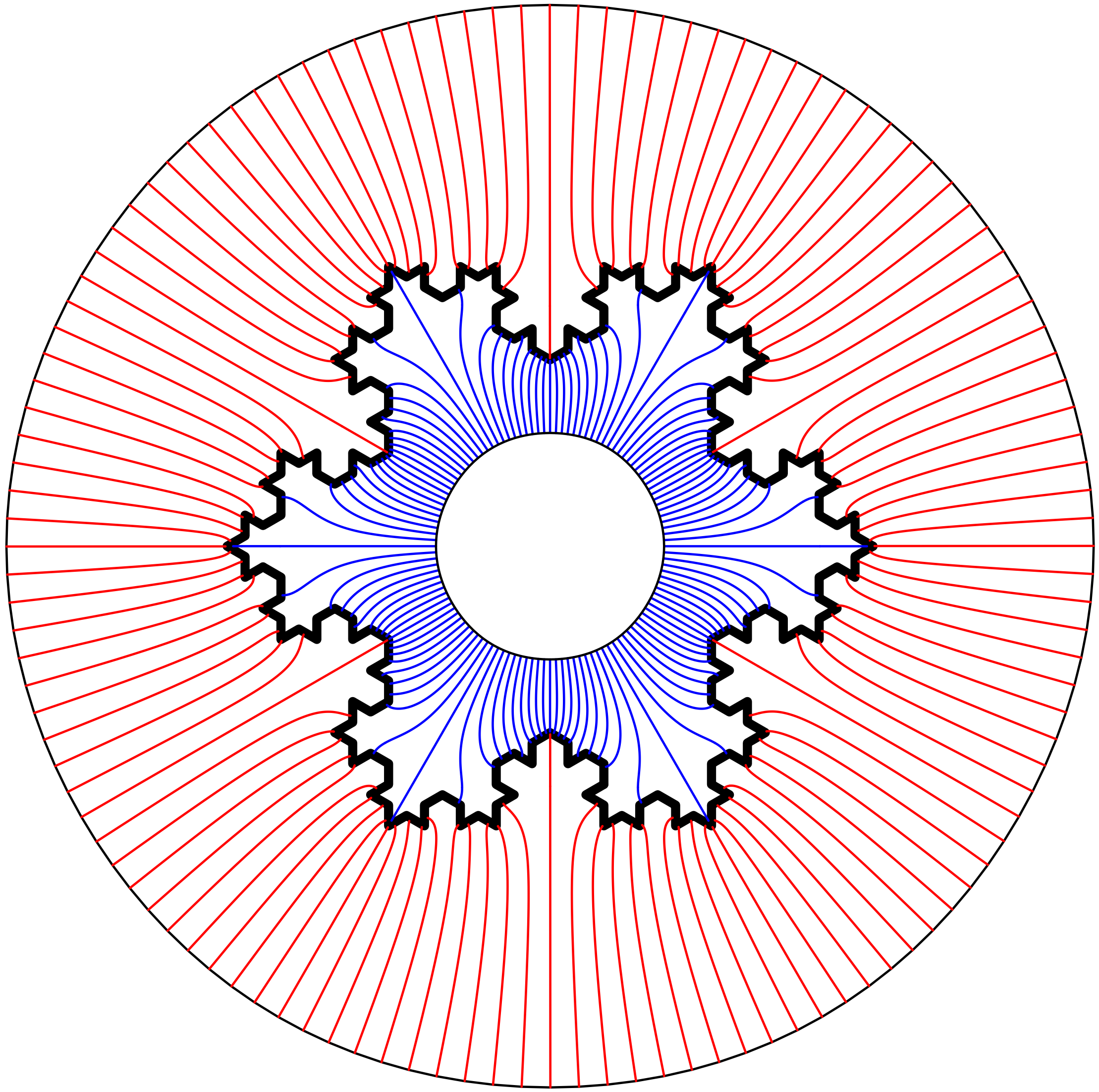
"A cinematographer … is the chief over the camera and light crews working on a film, … and is responsible for making artistic and technical decisions related to the image. … The cinematographer selects the camera, film stock, lenses, filters, etc., to realize the scene in accordance with the intentions of the director."

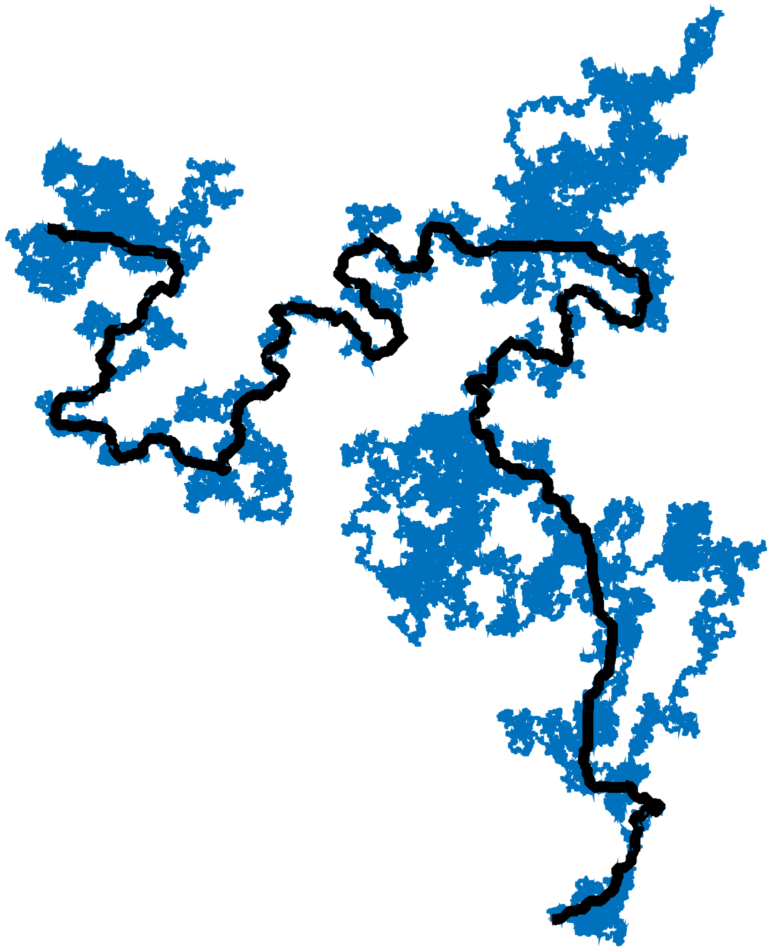
– Wikipedia

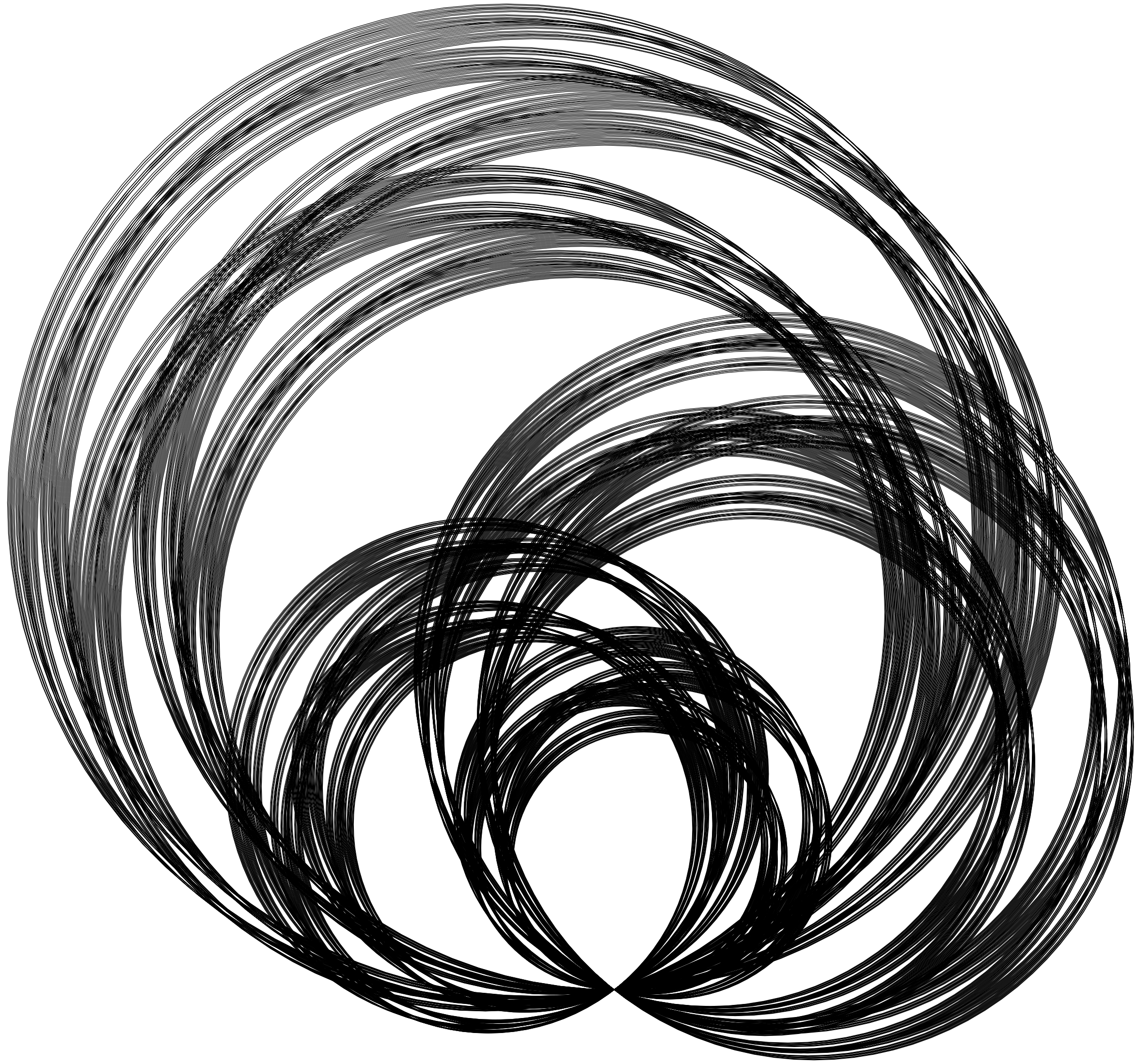


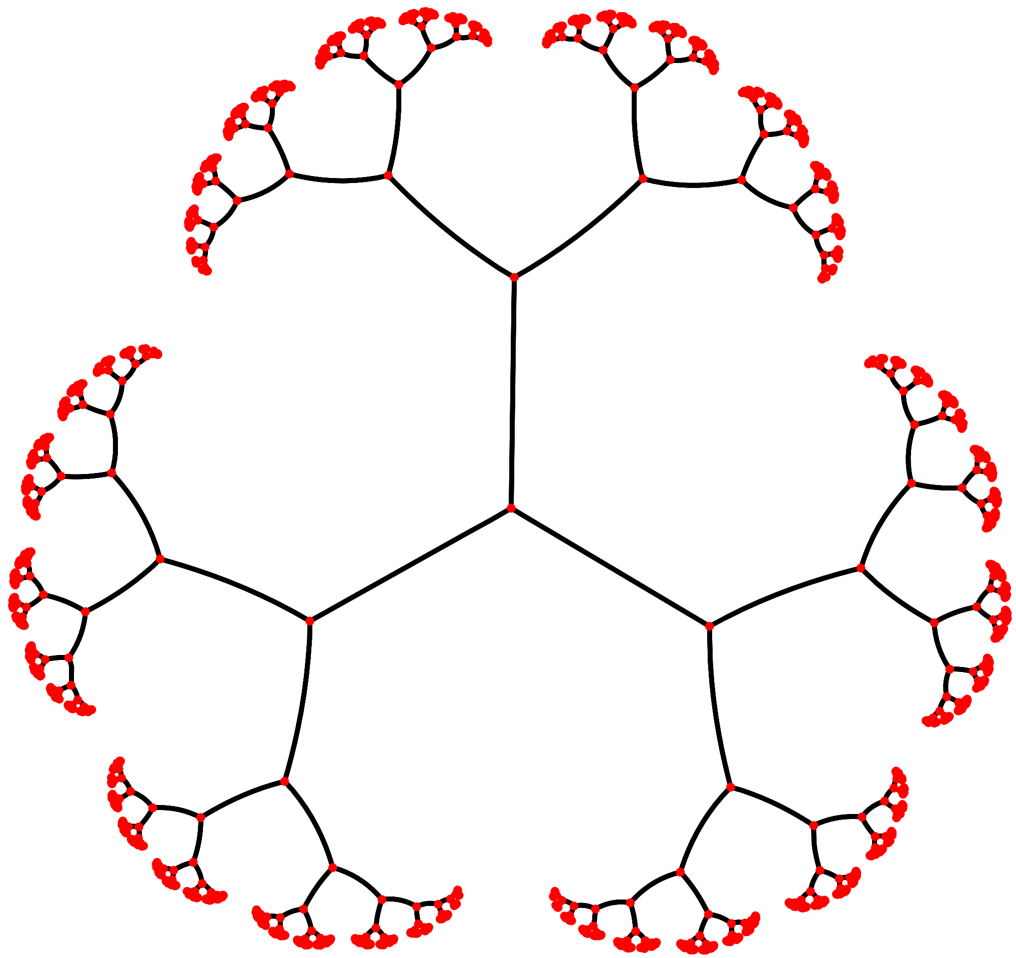


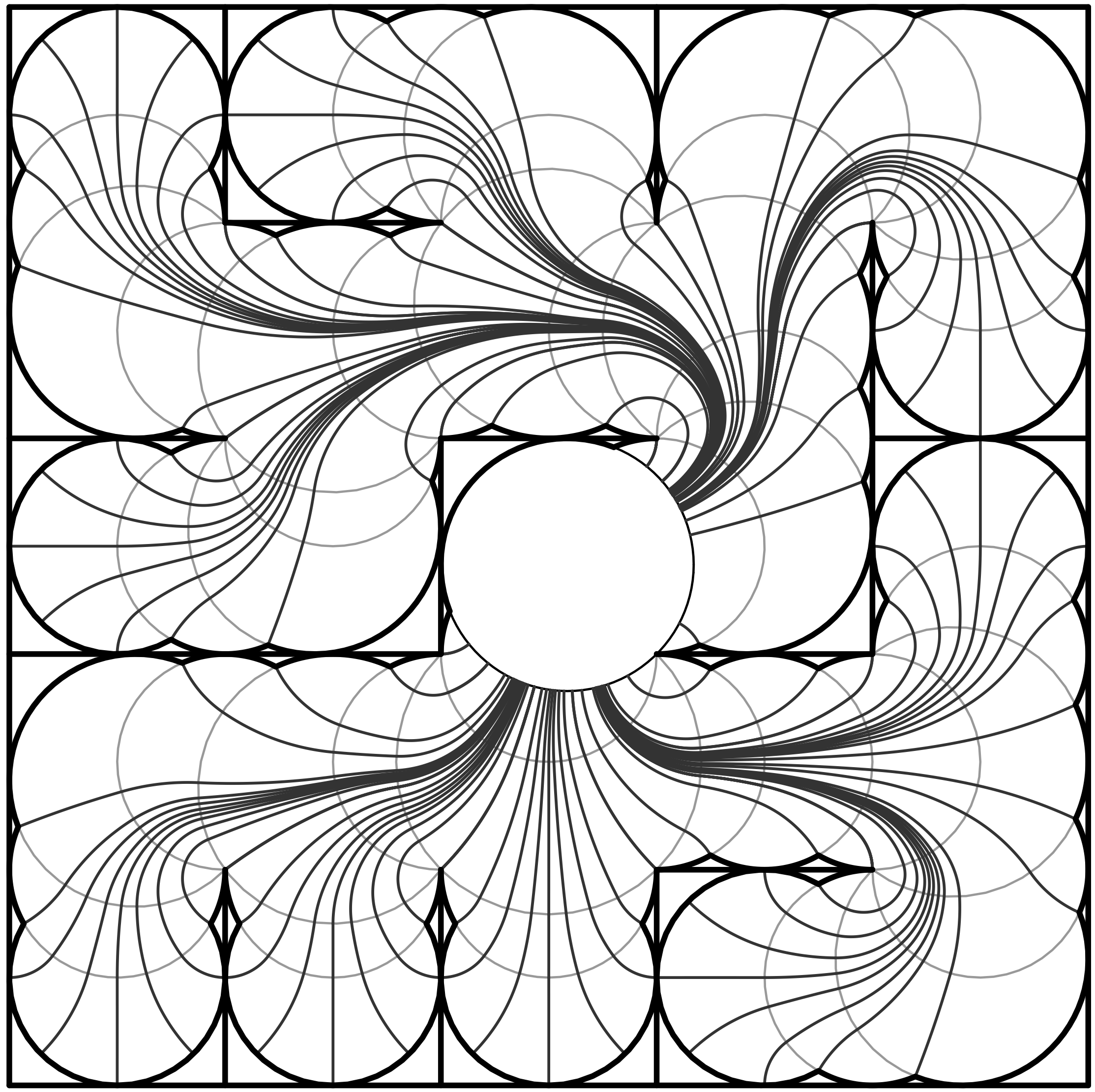


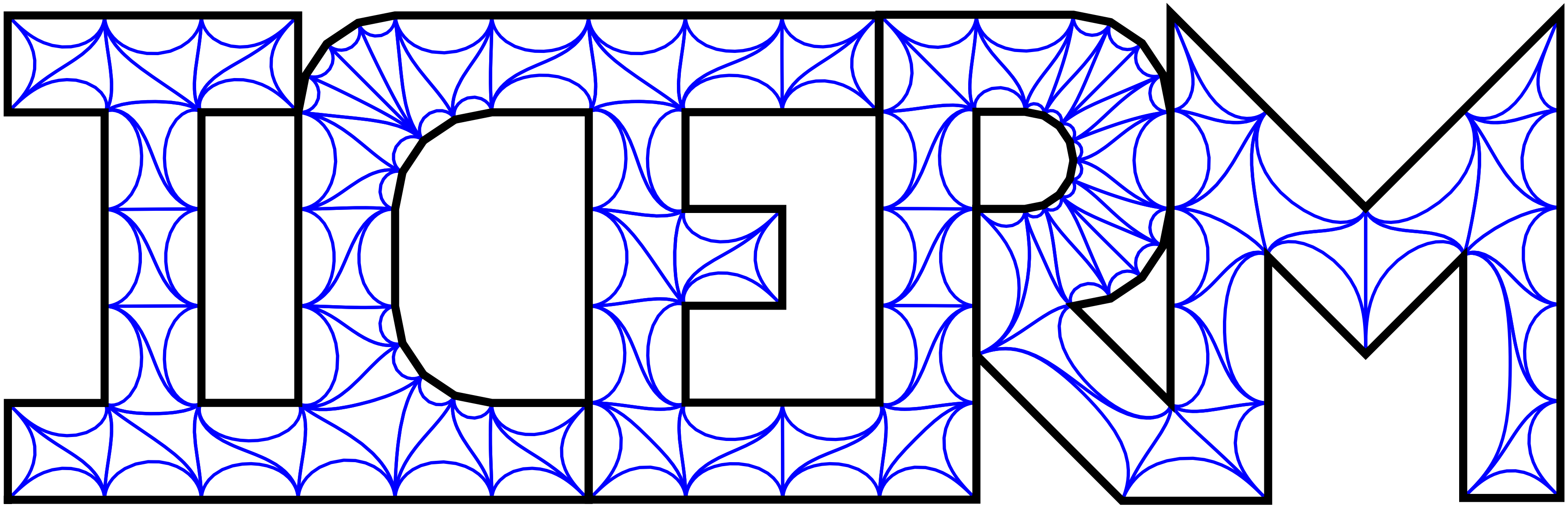








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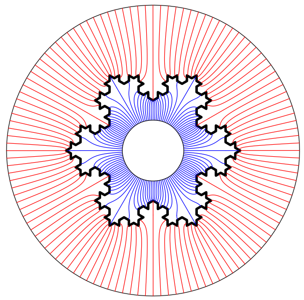

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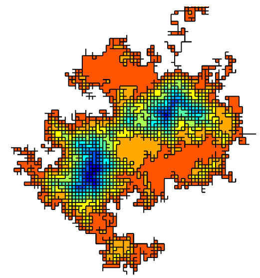
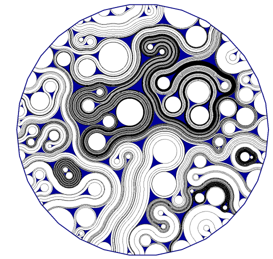


Deep Blue



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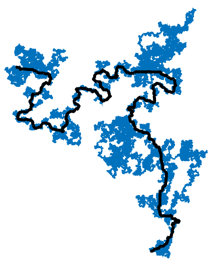
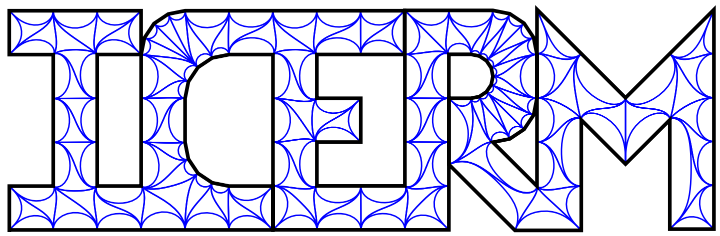


Shortcuts



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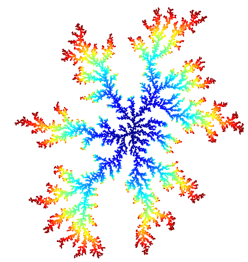
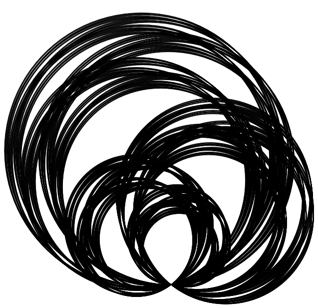
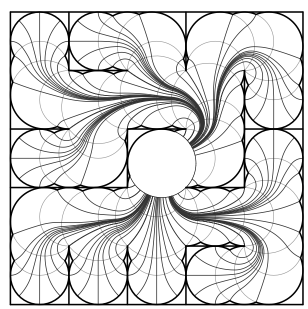


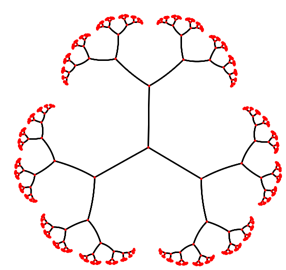
Diabo**l**ic**al**



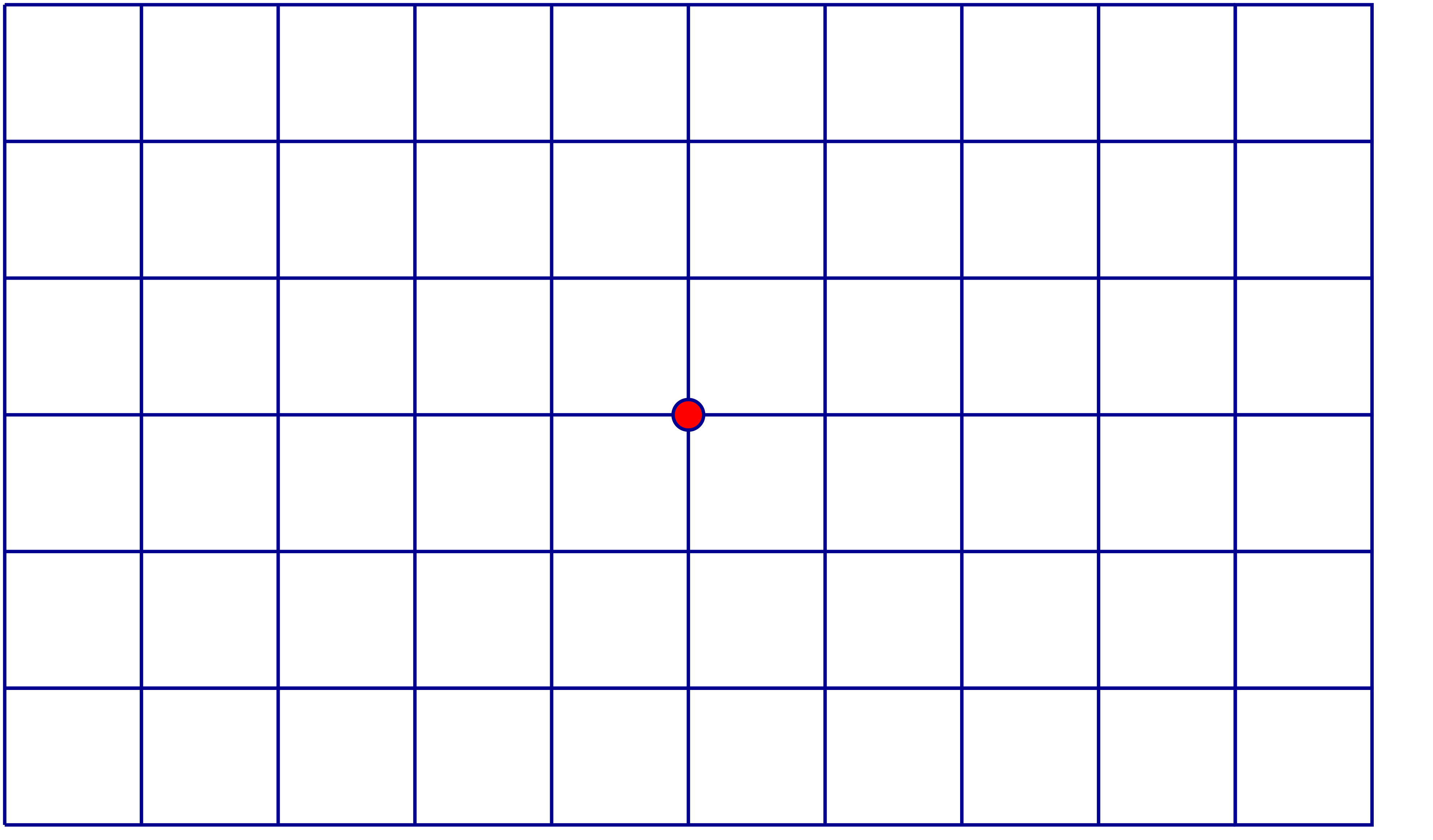
Circles

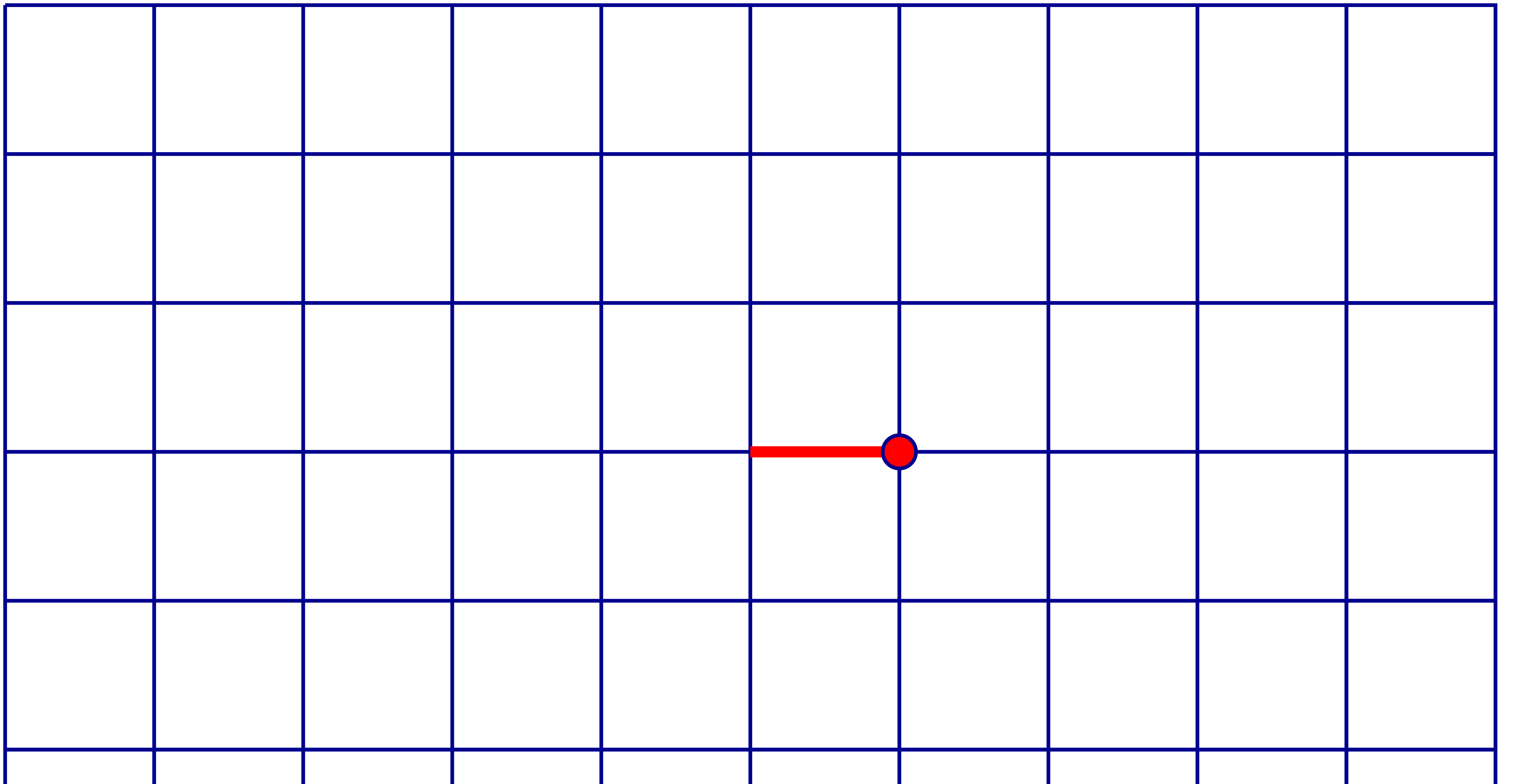


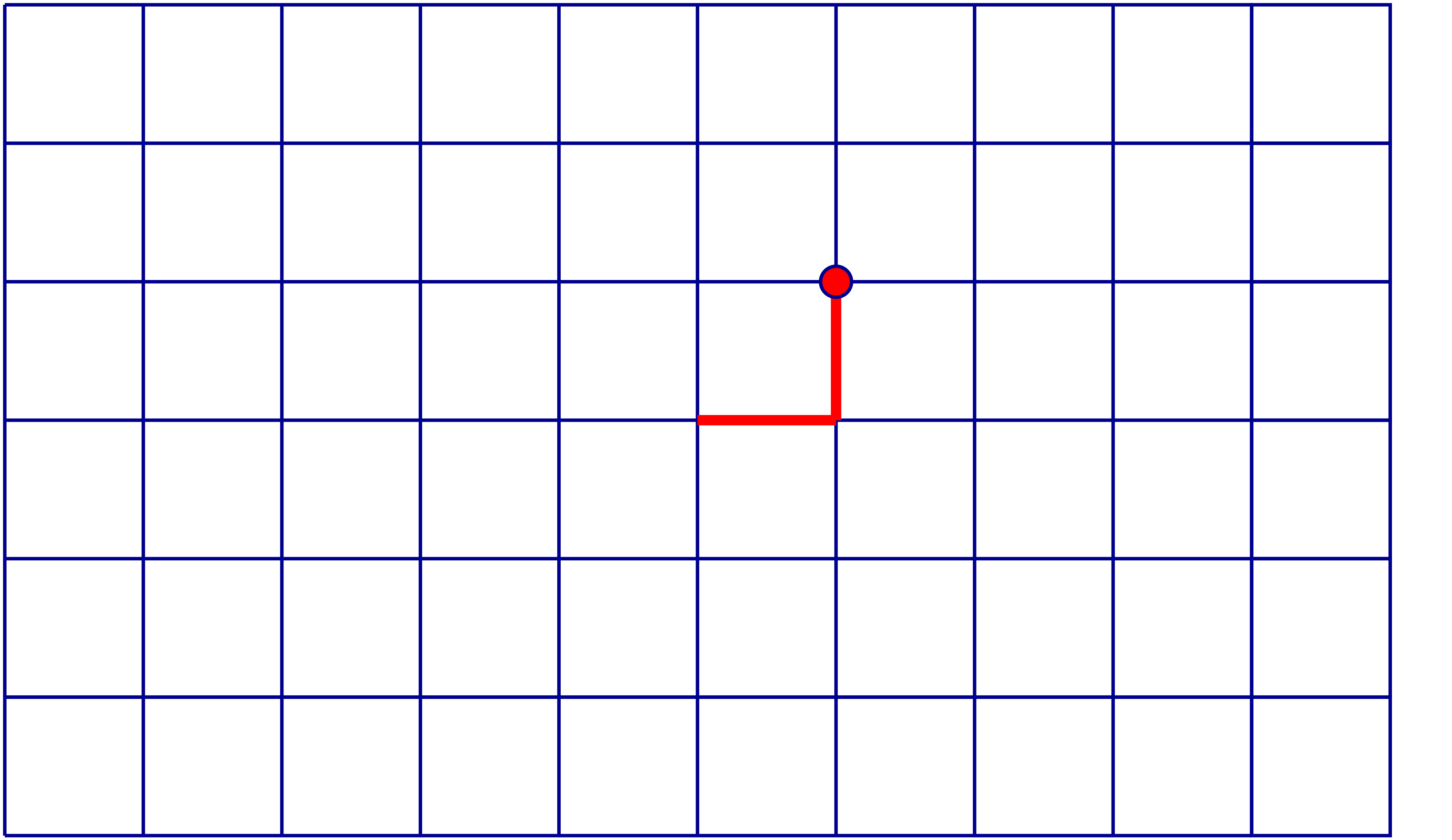
Fast Map

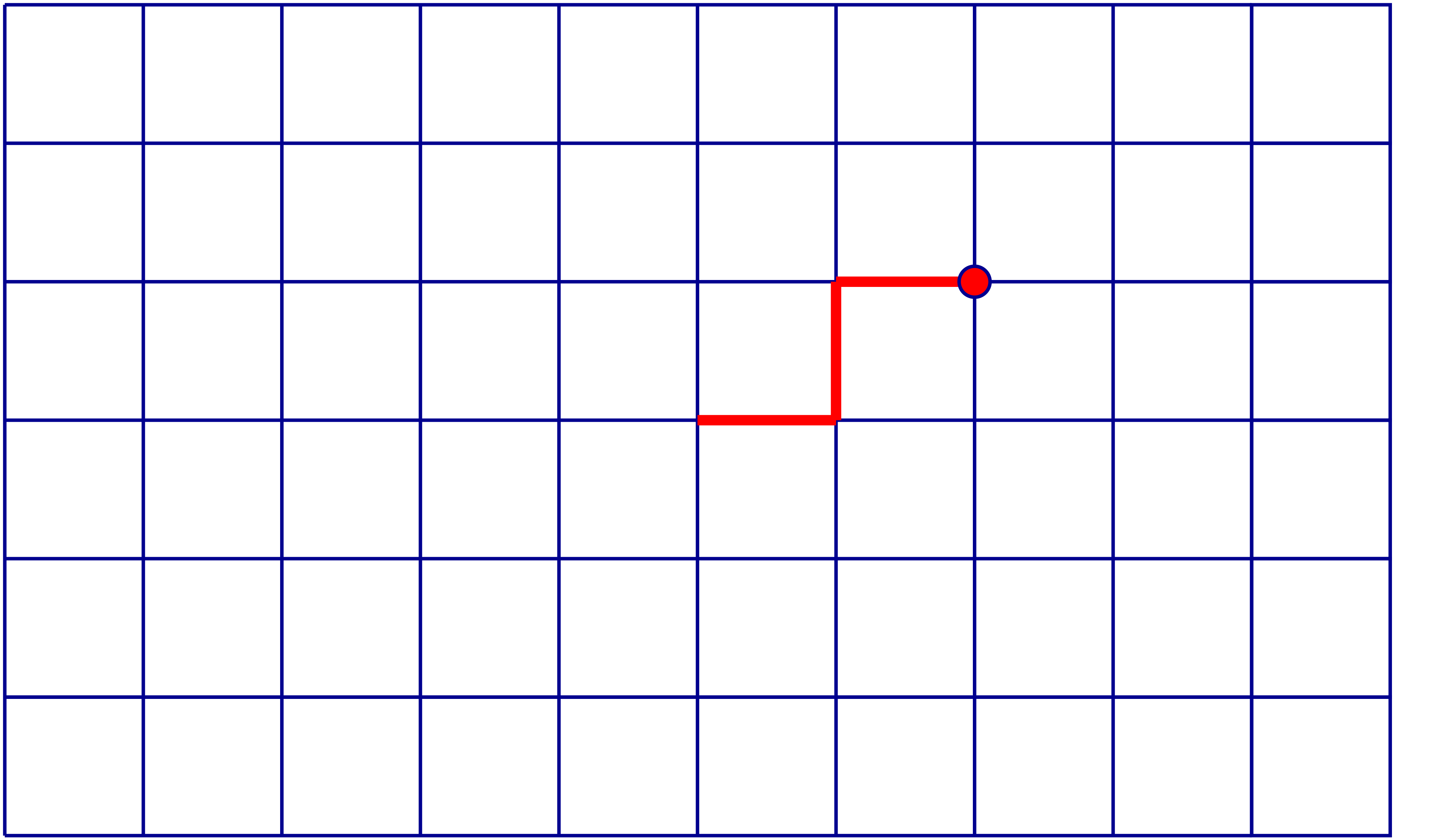


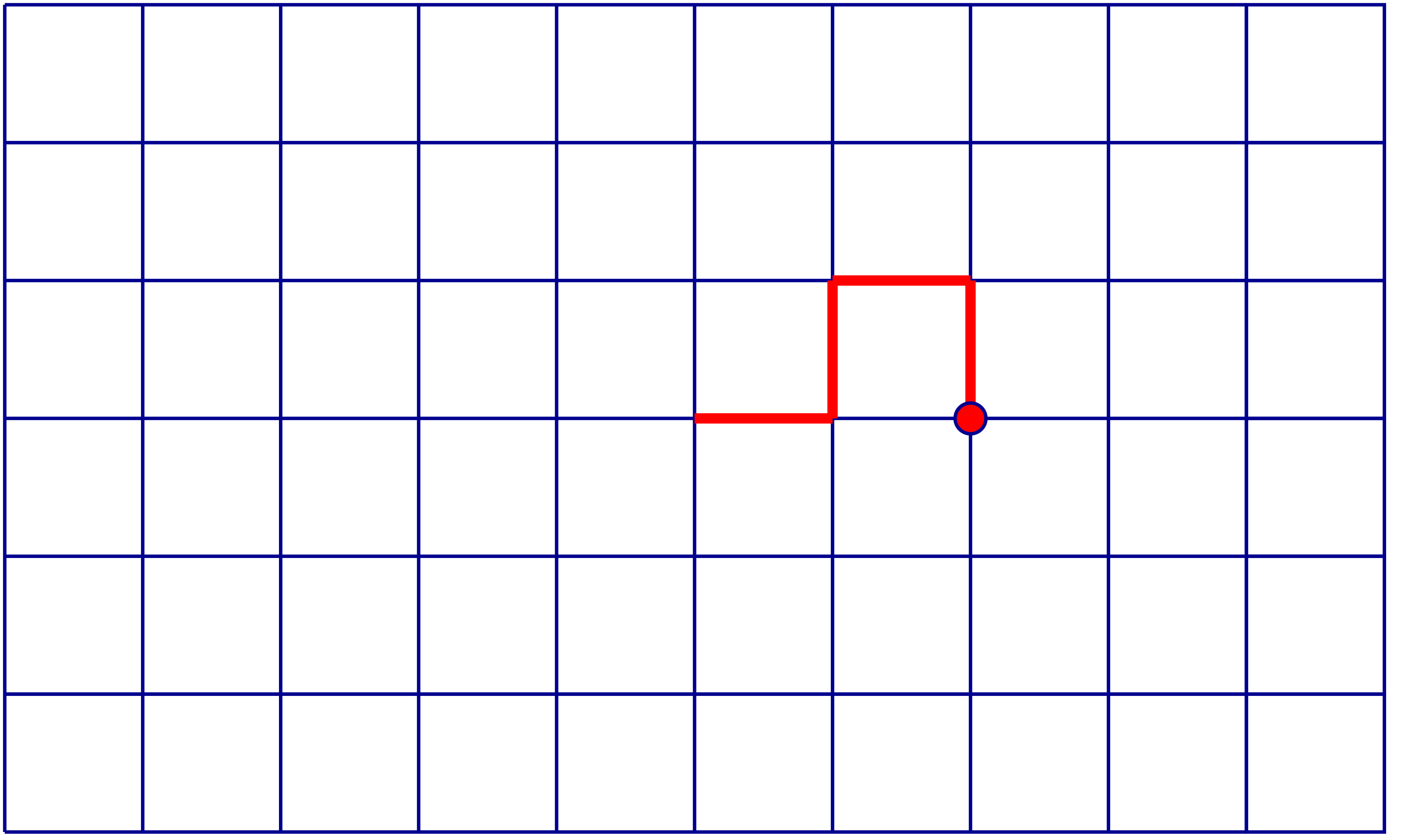
Regular Truth

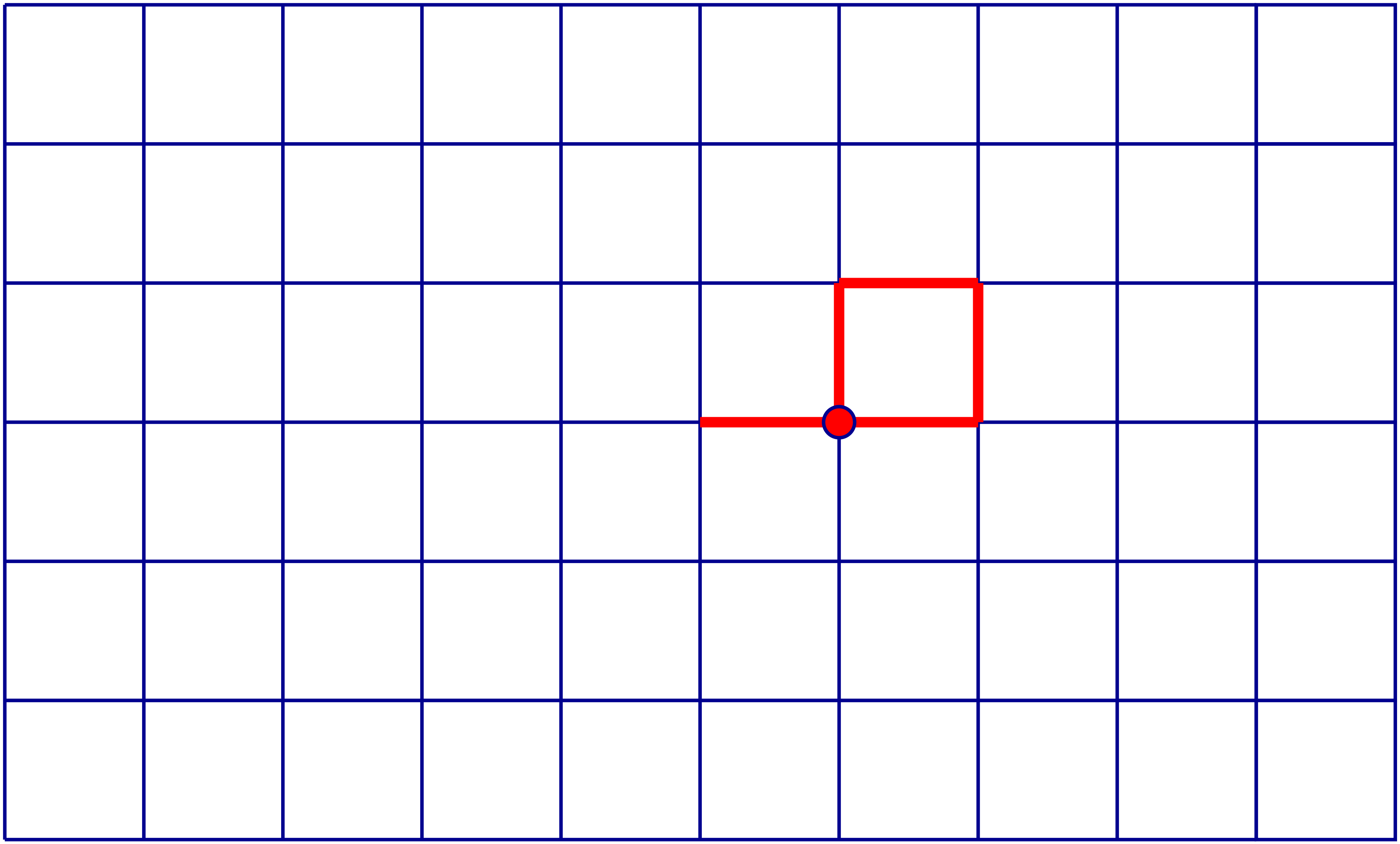


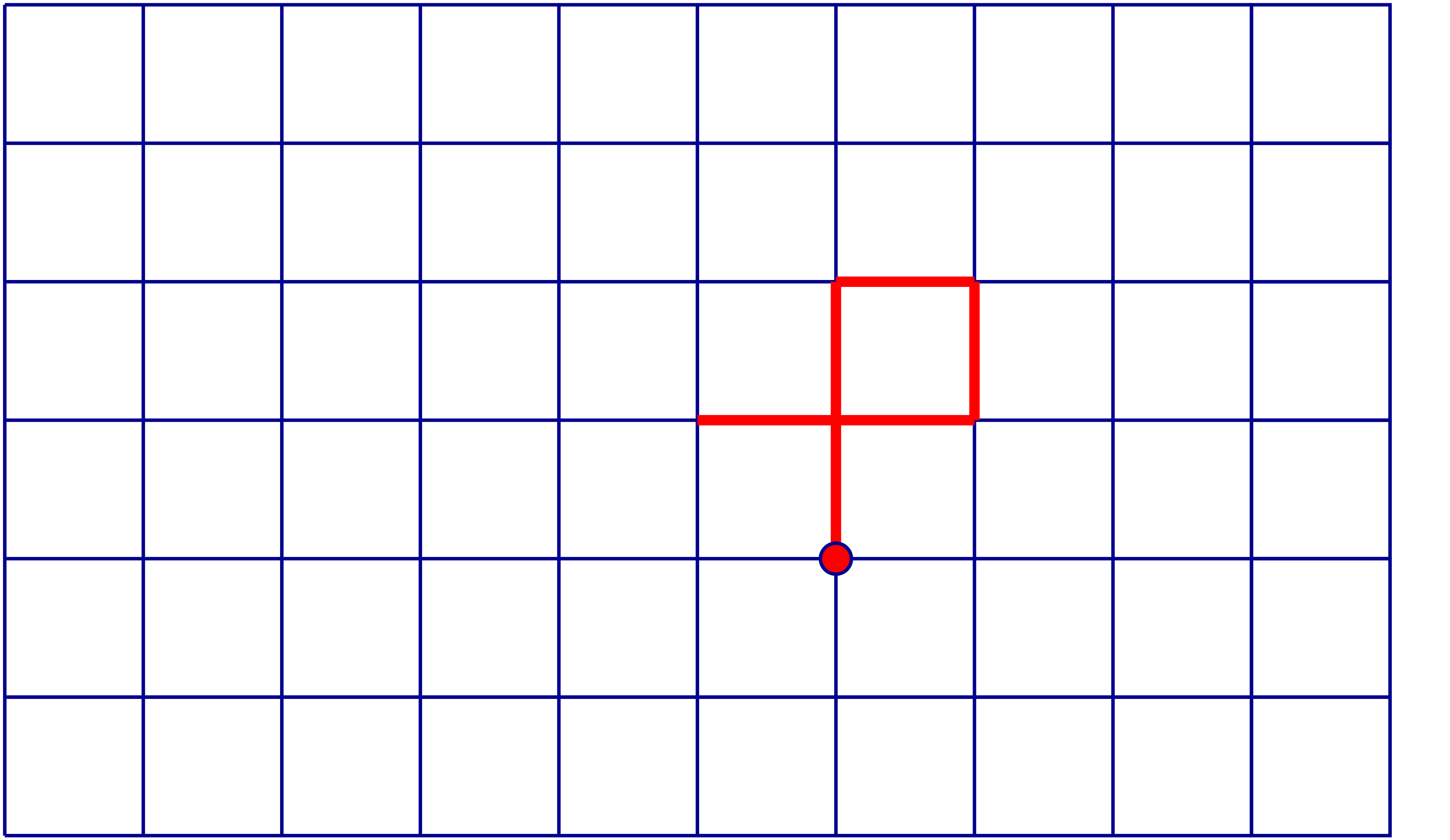


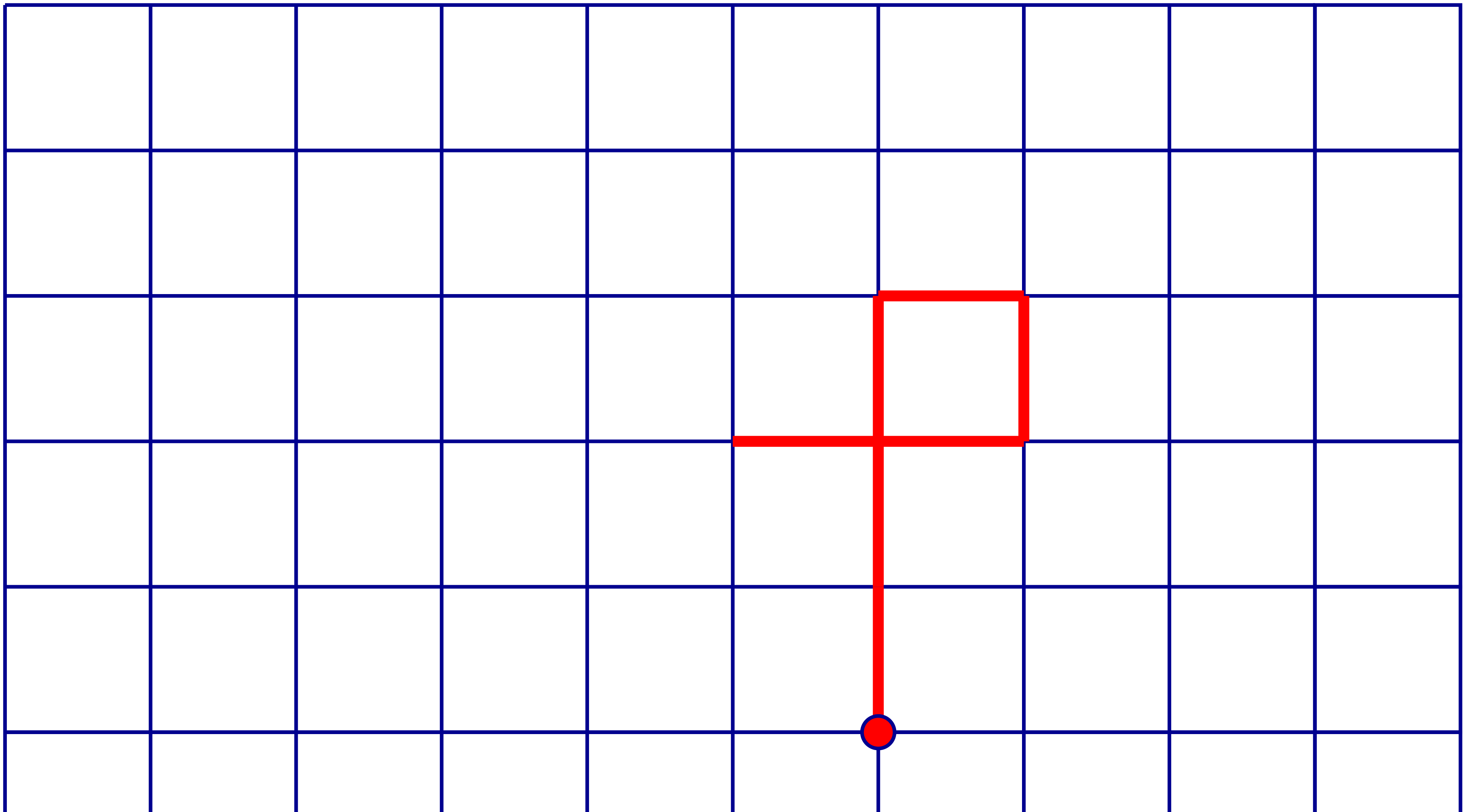


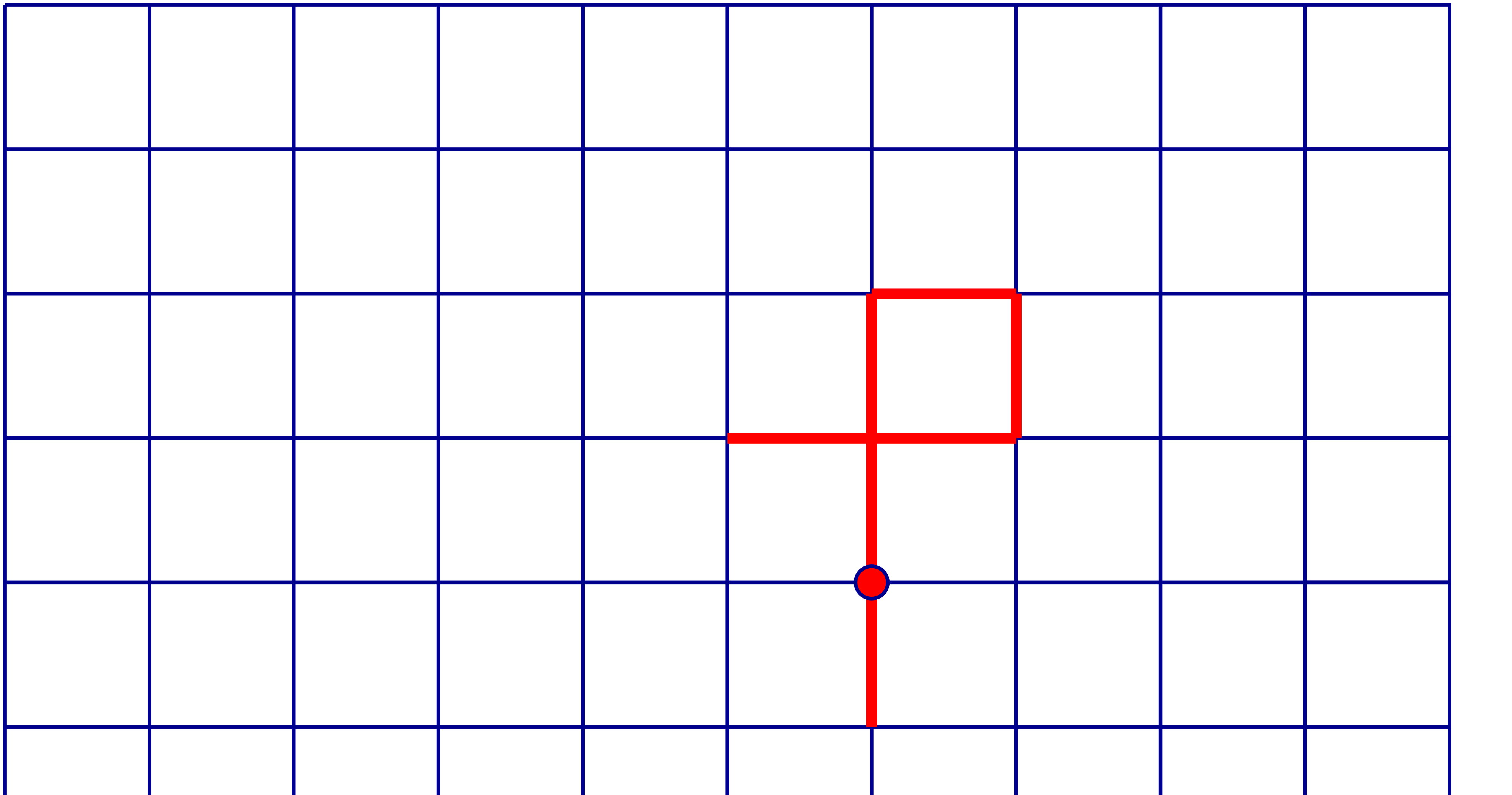


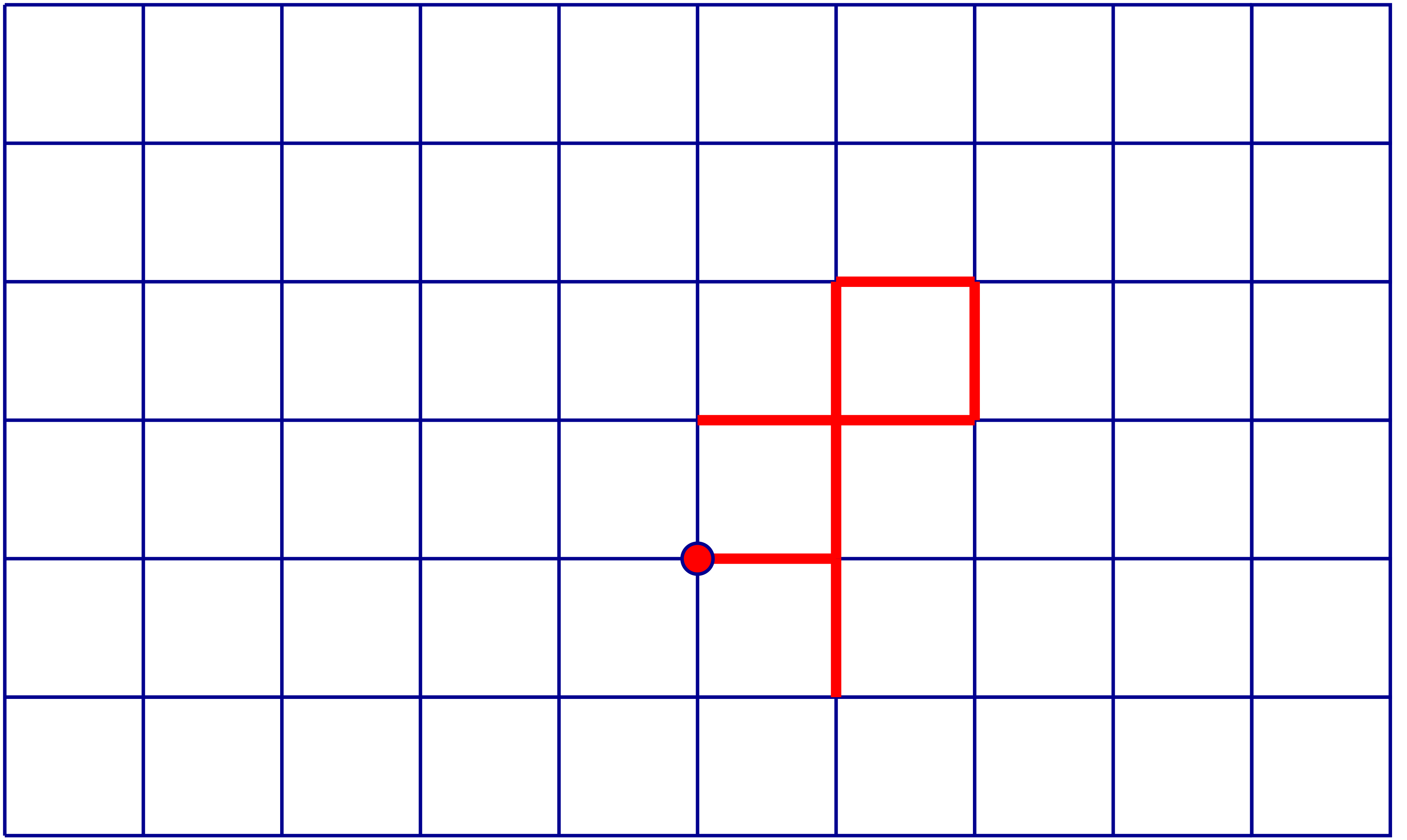


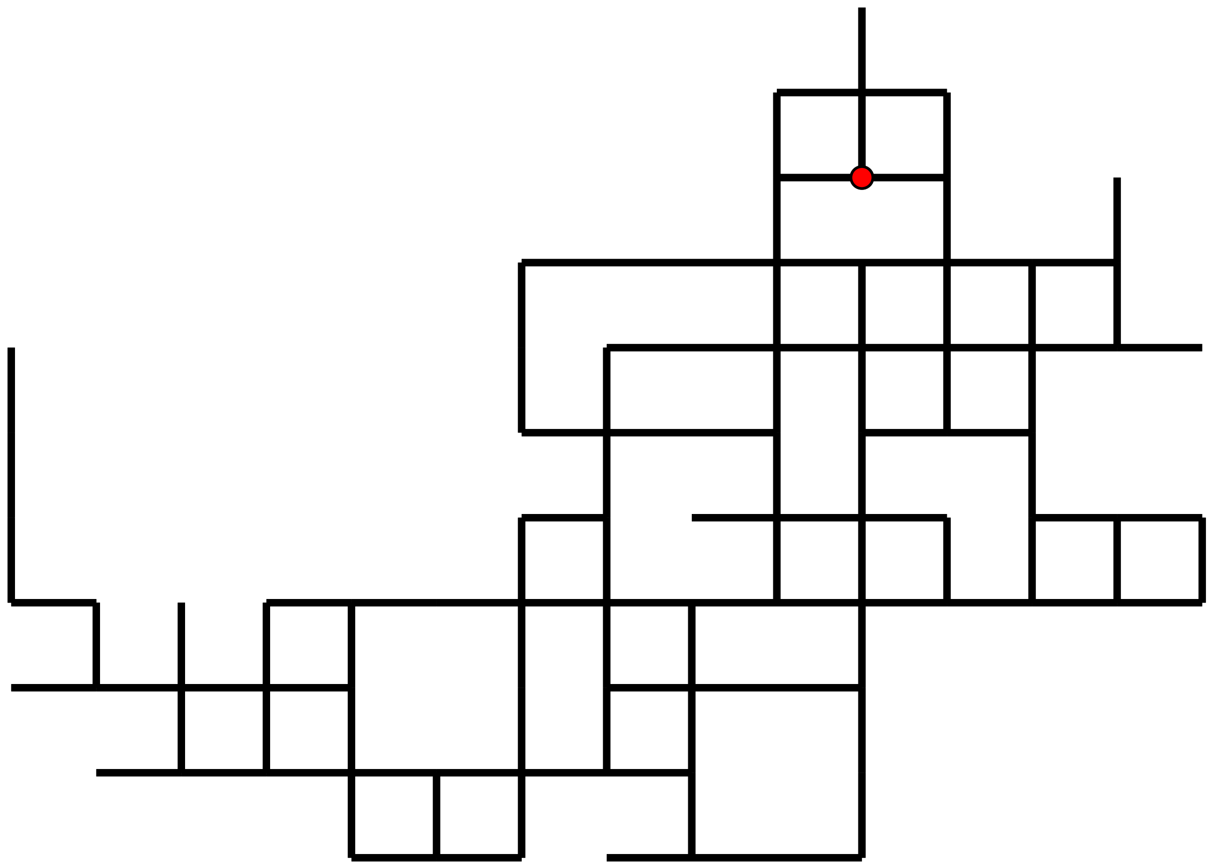




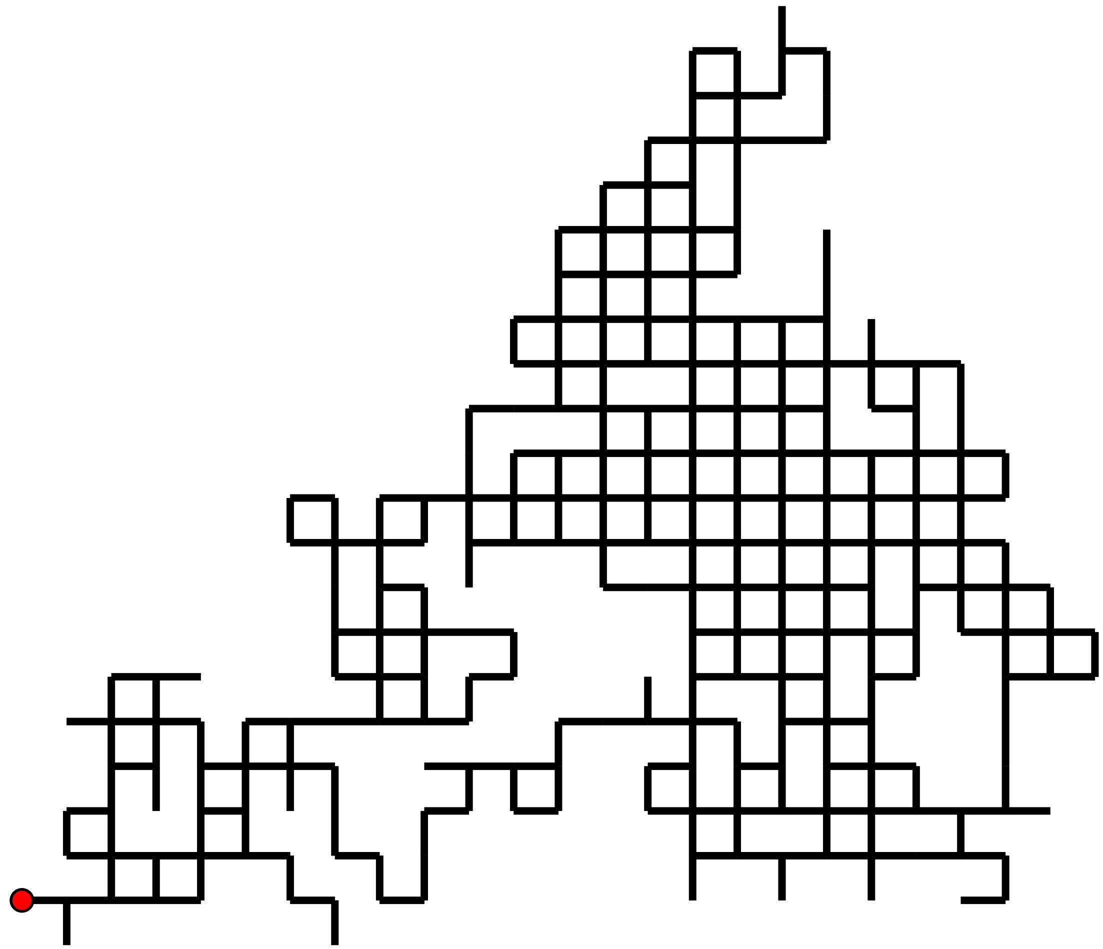








200 step random walk.



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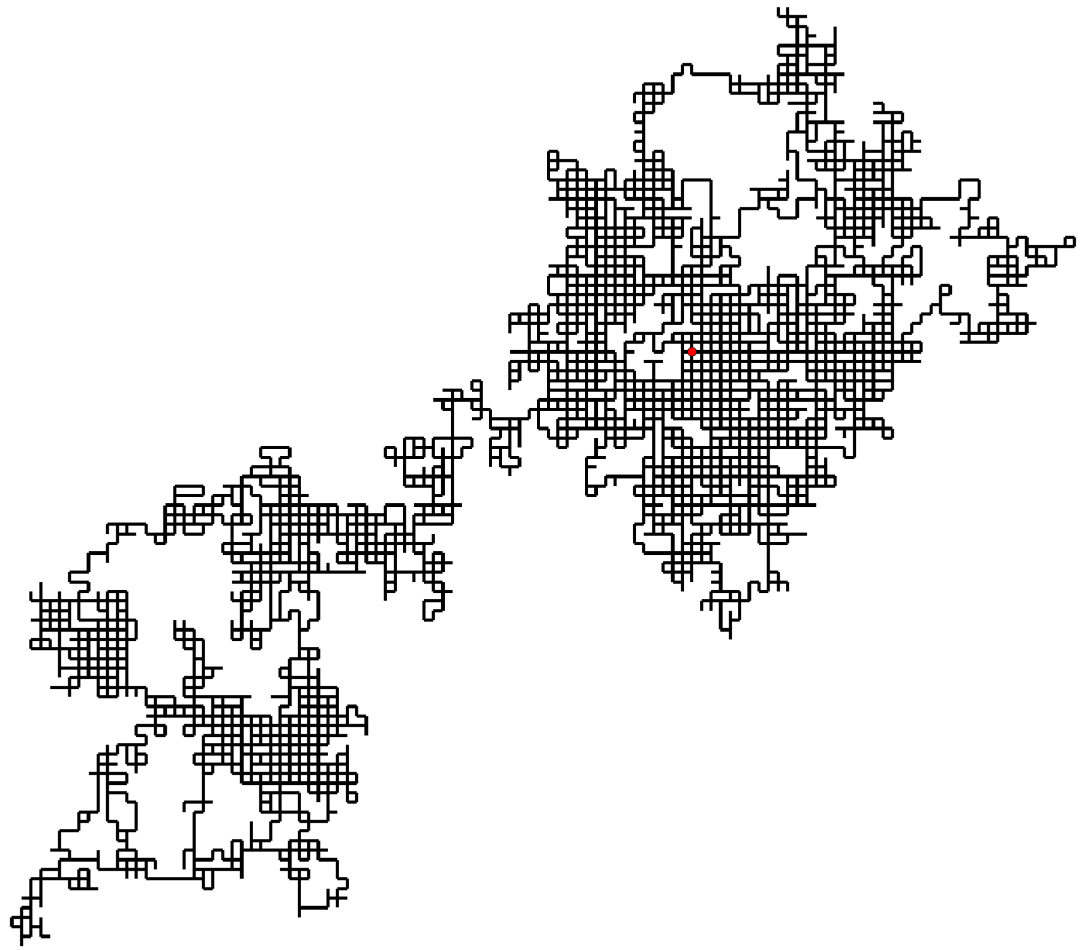
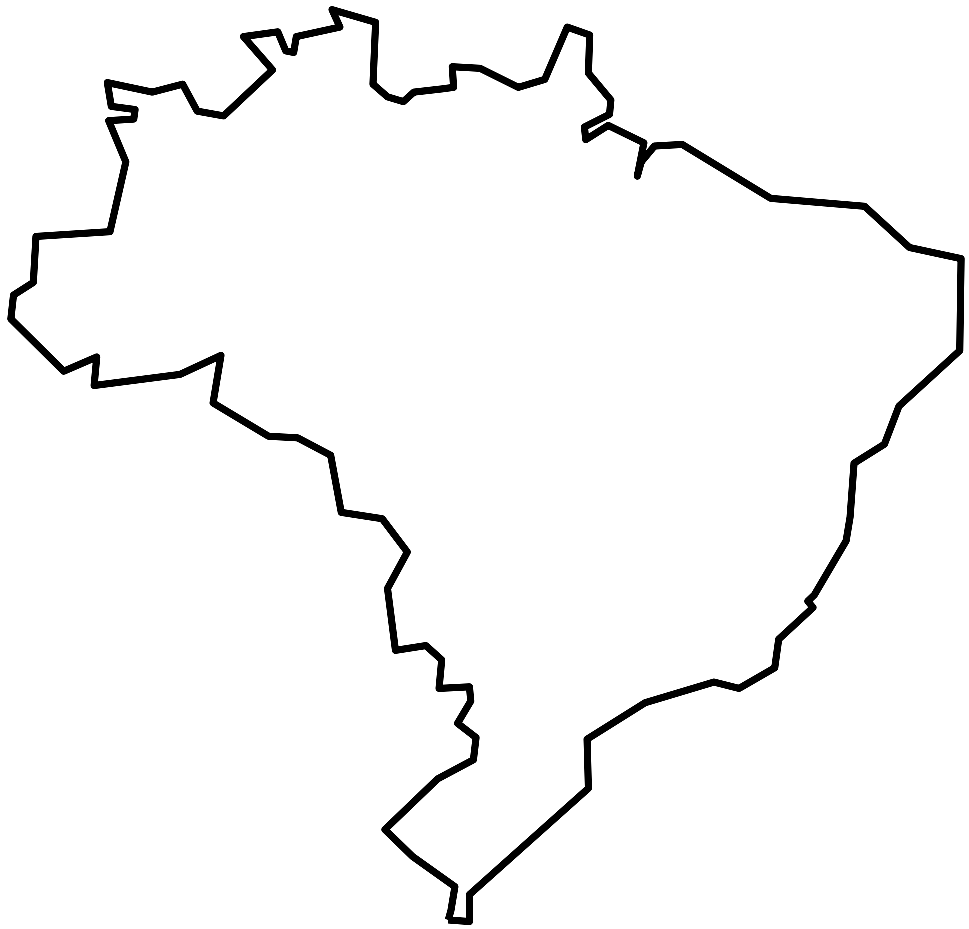


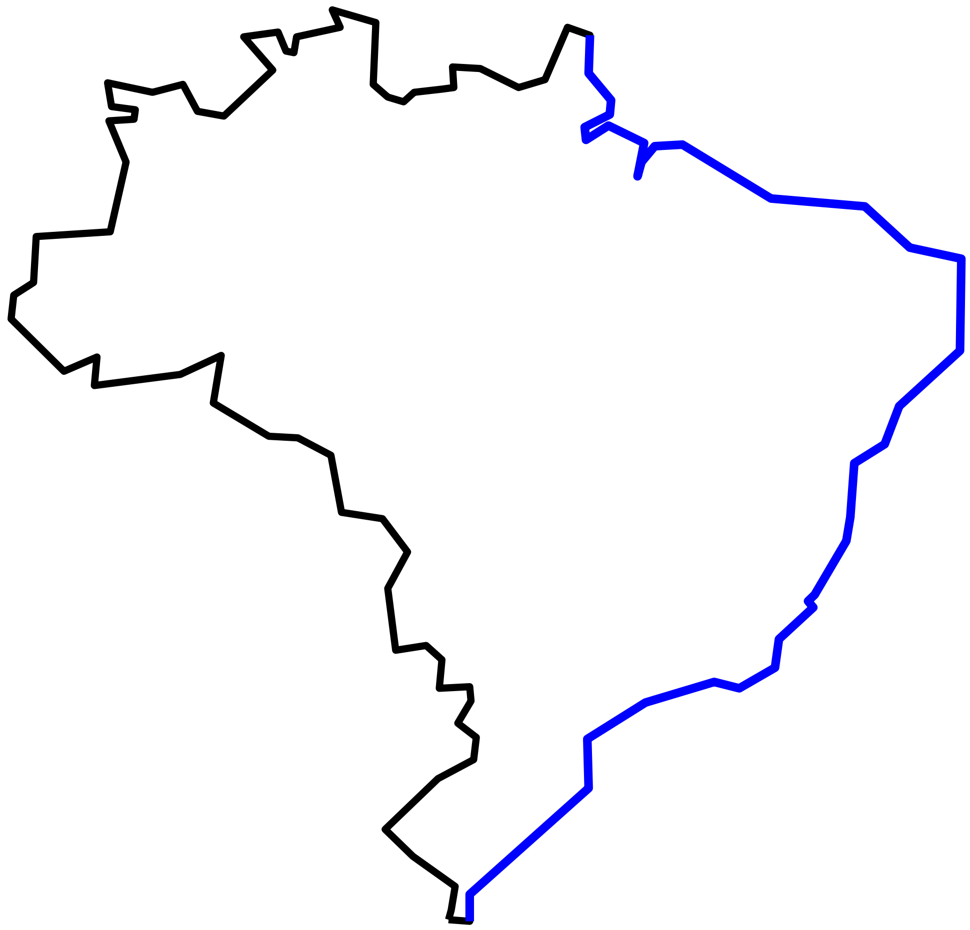
10000 step random walk.

Harmonic measure = hitting distribution of Brownian motion



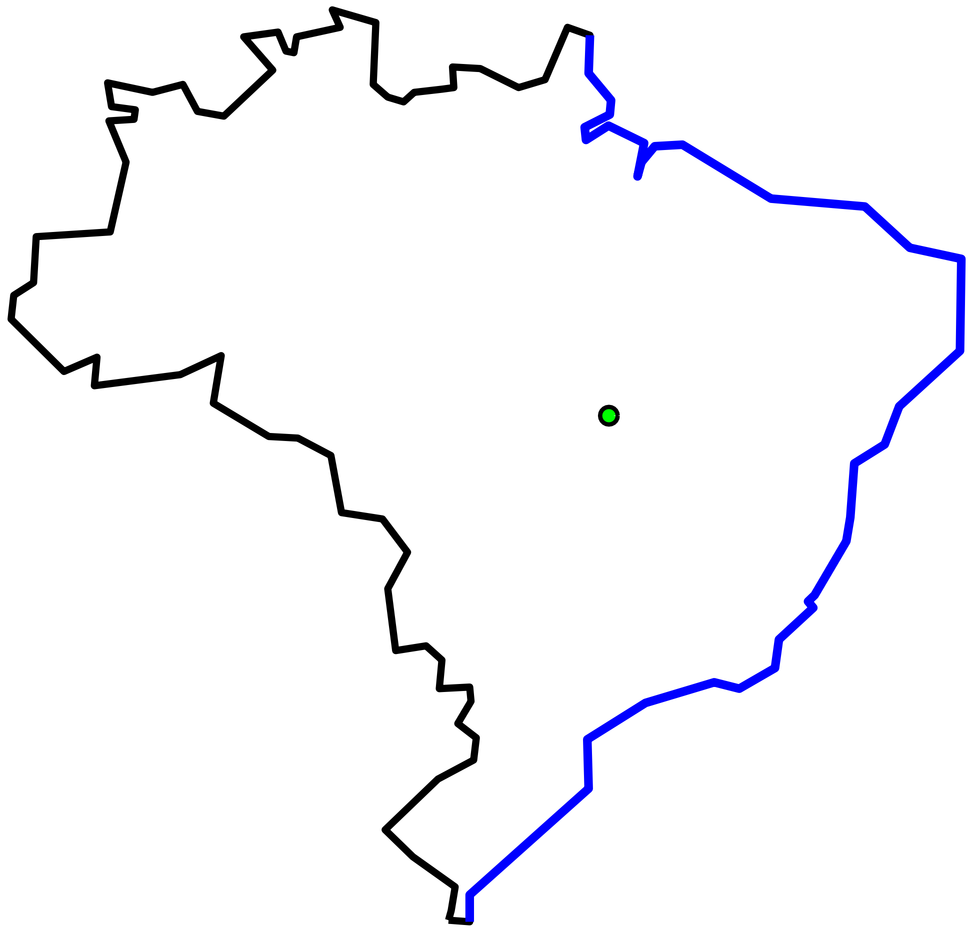
Suppose Ω is a planar Jordan domain.

Harmonic measure = hitting distribution of Brownian motion



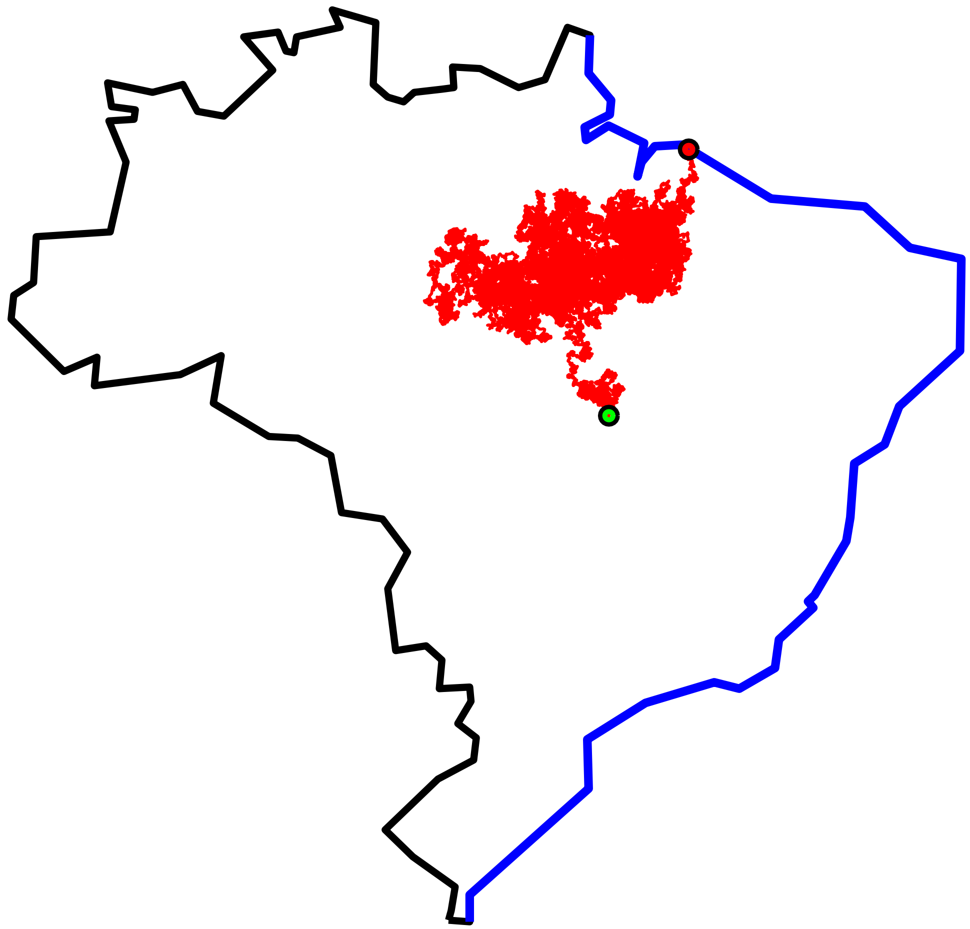
Let E be a subset of the boundary, $\partial\Omega$.

Harmonic measure = hitting distribution of Brownian motion



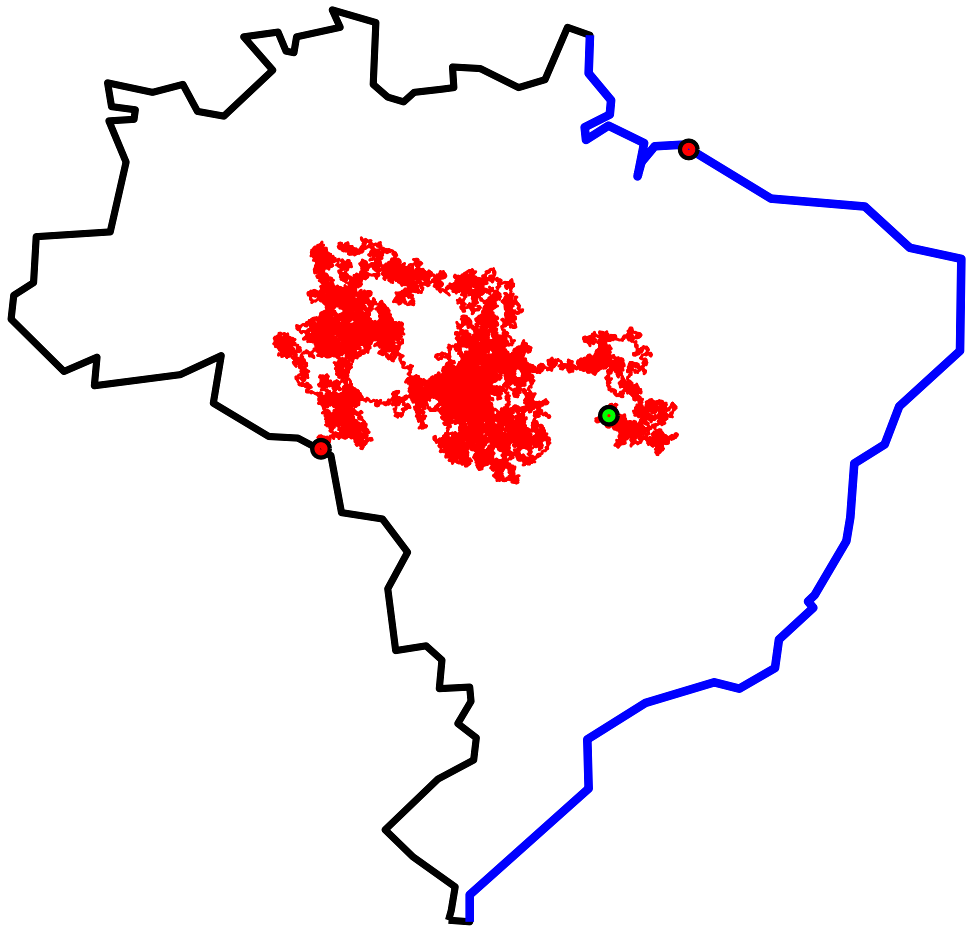
Choose an interior point $z \in \Omega$.

Harmonic measure = hitting distribution of Brownian motion



$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



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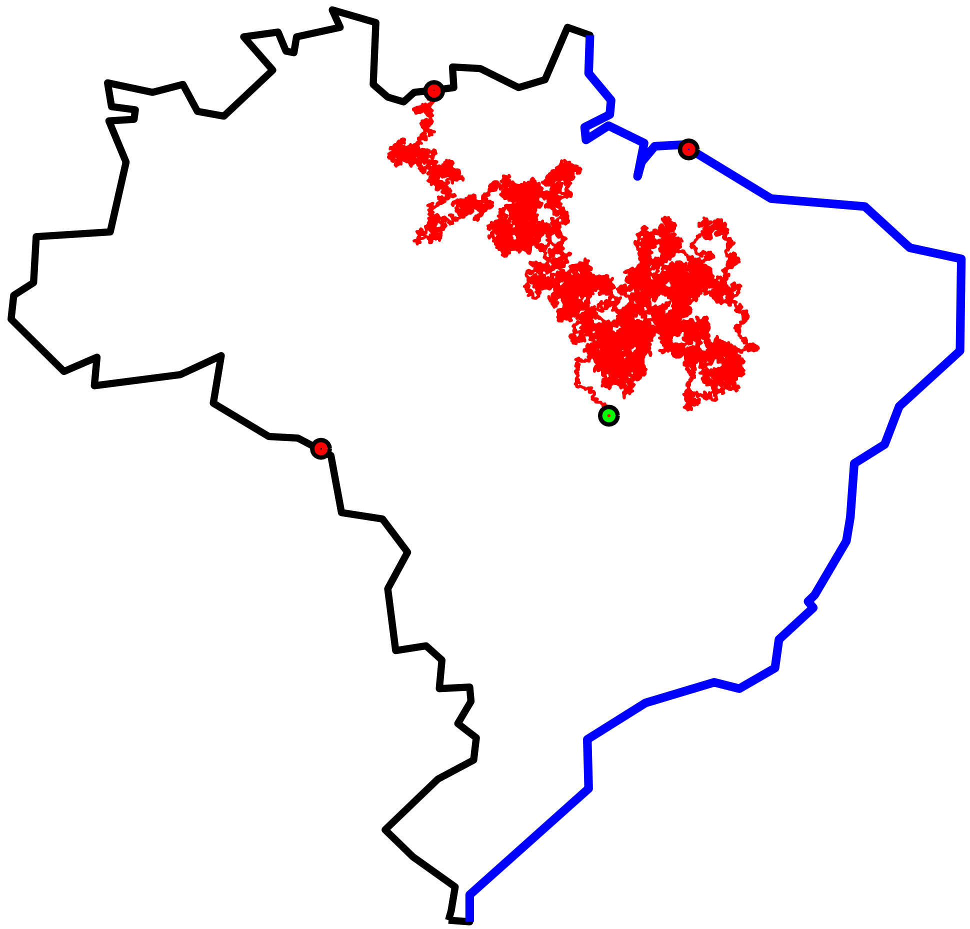

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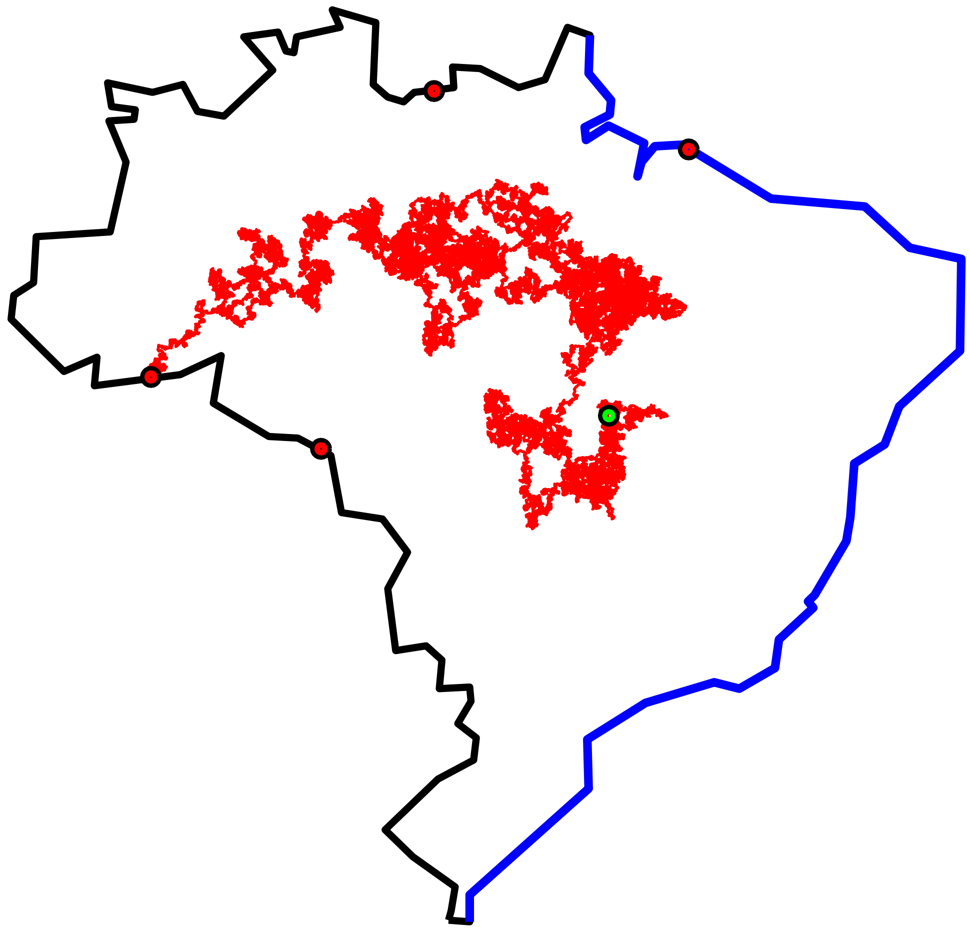

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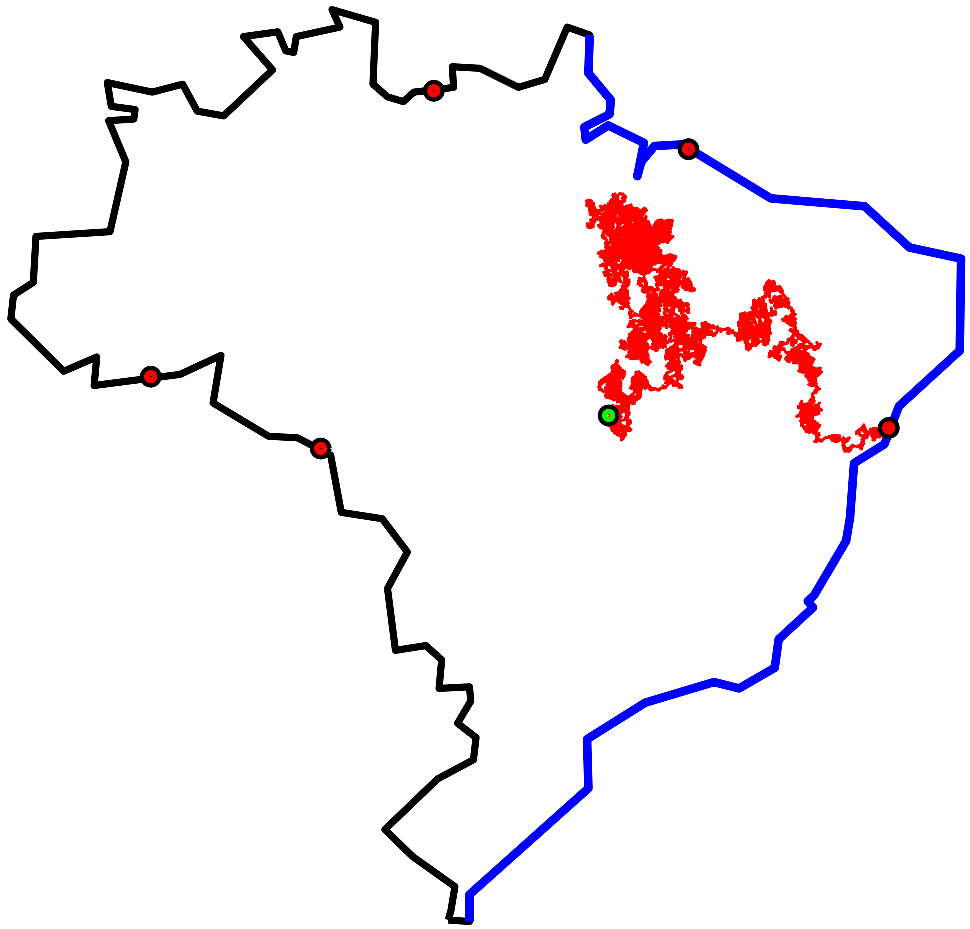

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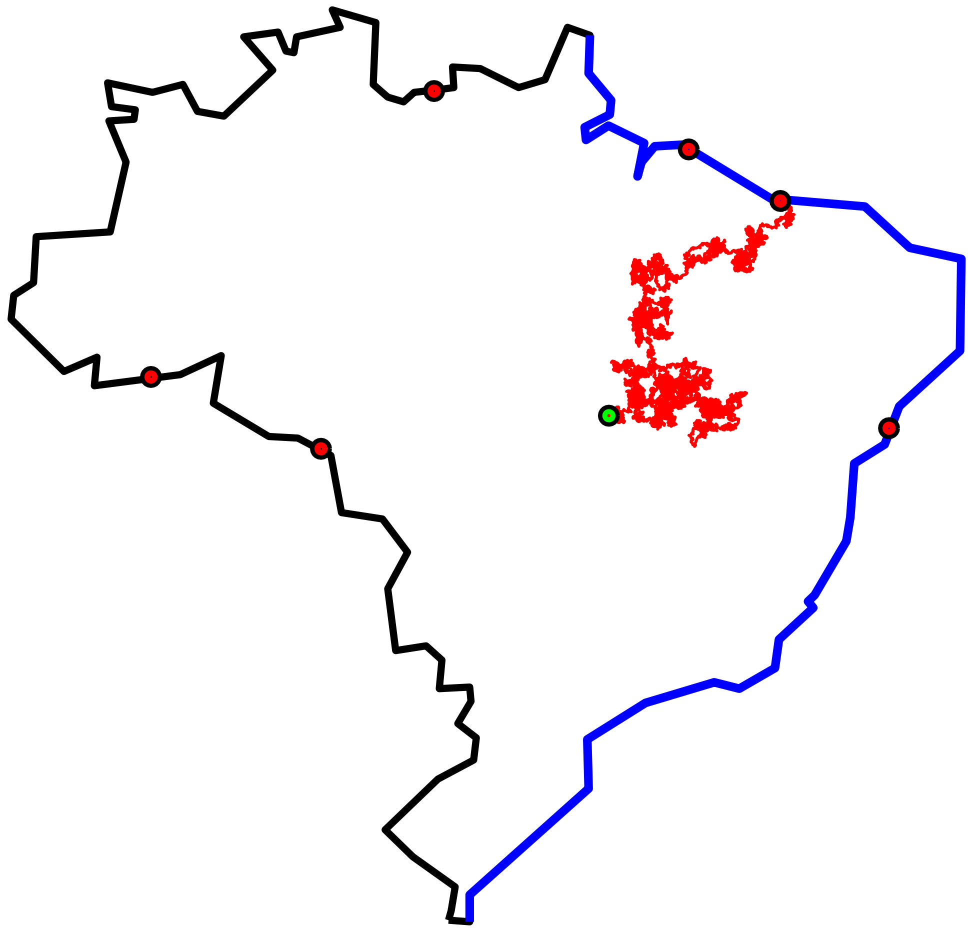

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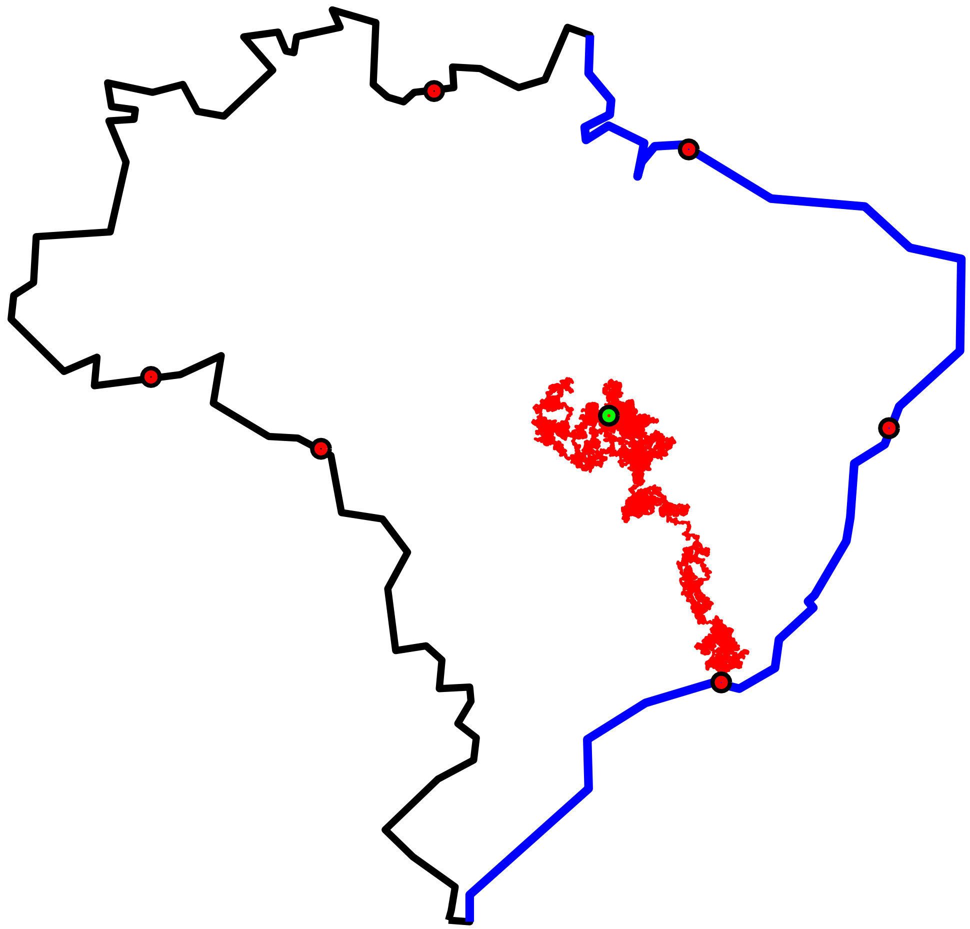

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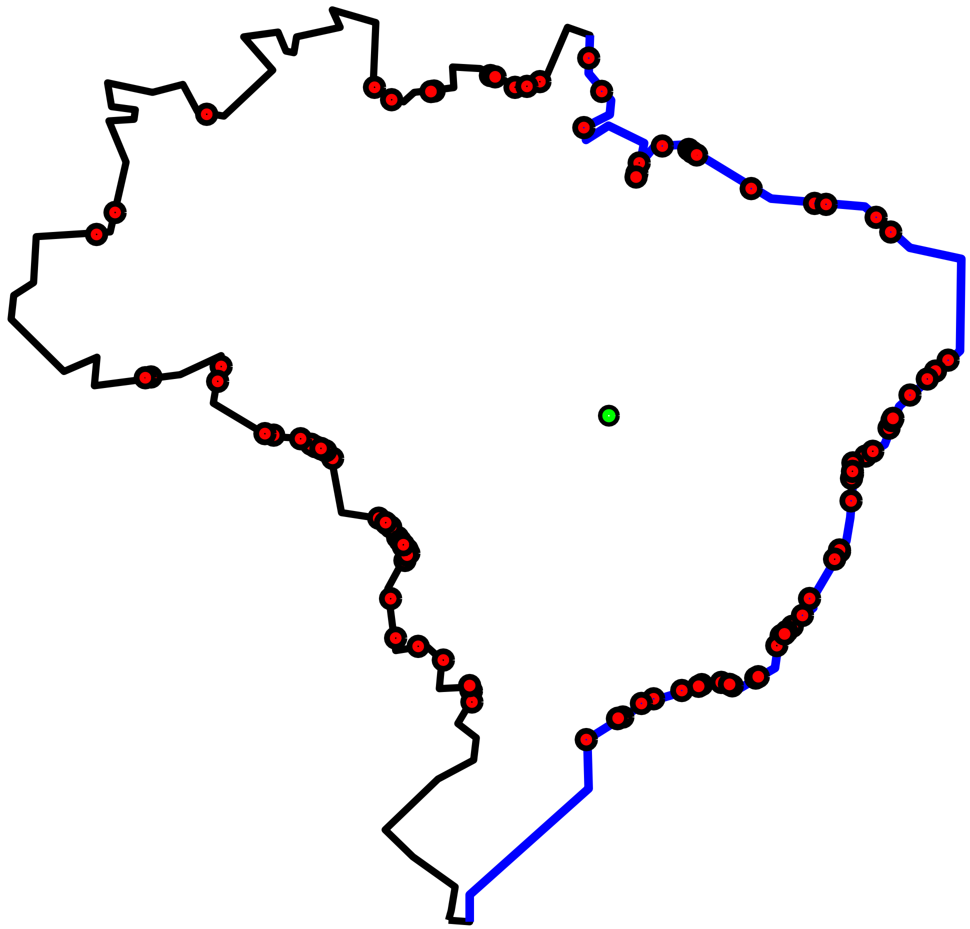
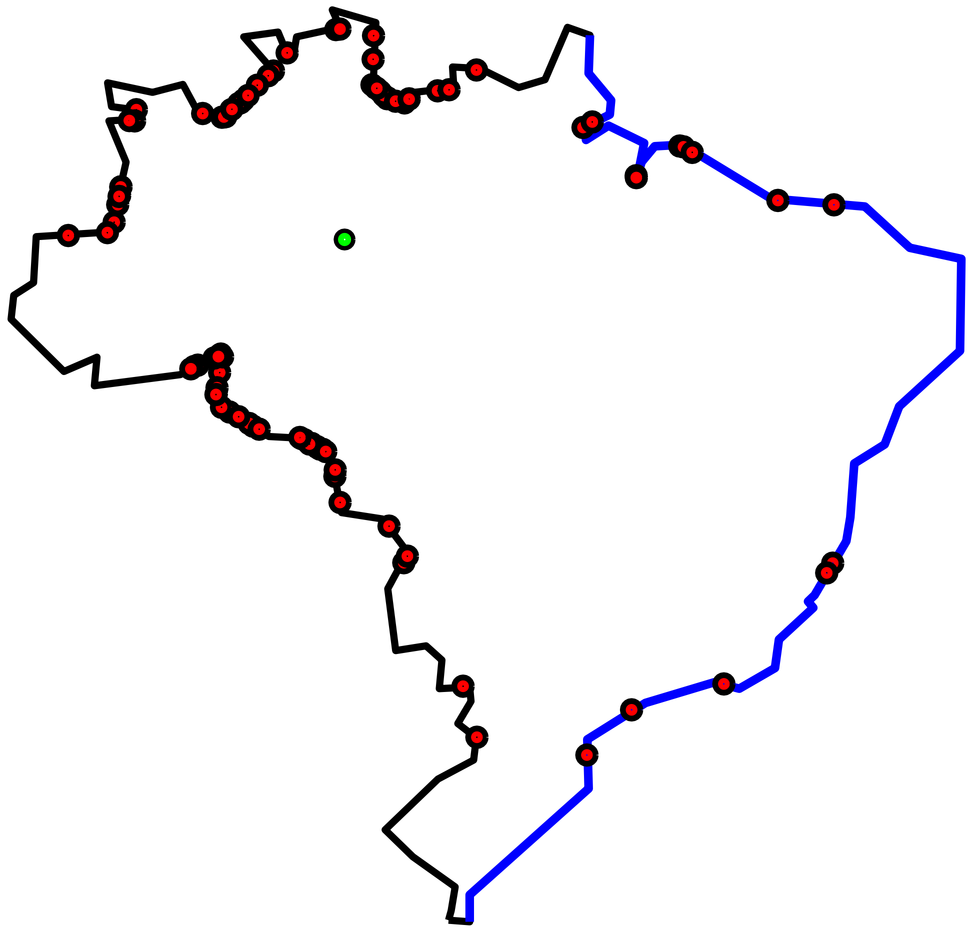


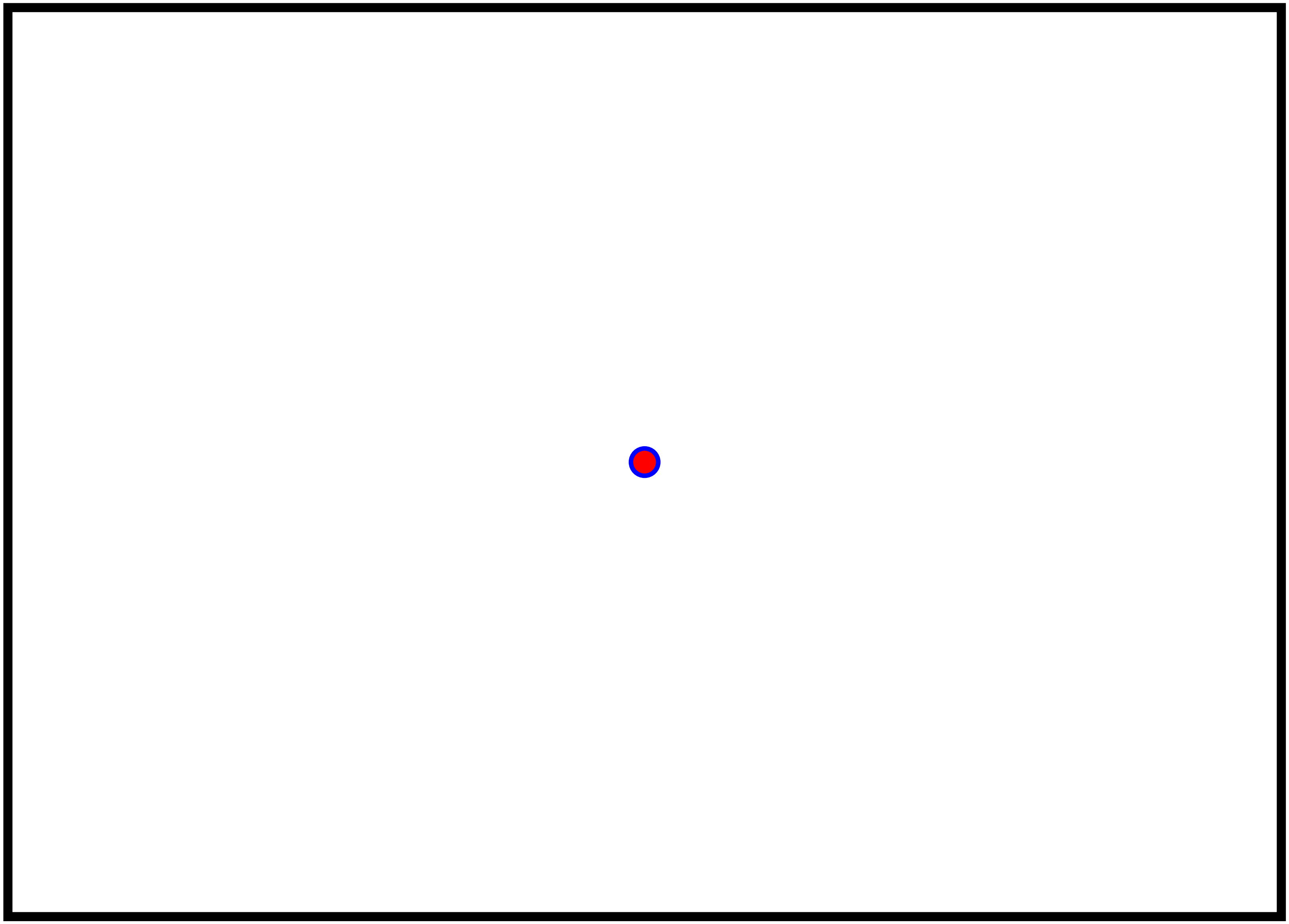
$\omega(z, E, \Omega) \approx 64/100$. What if we move starting point z?

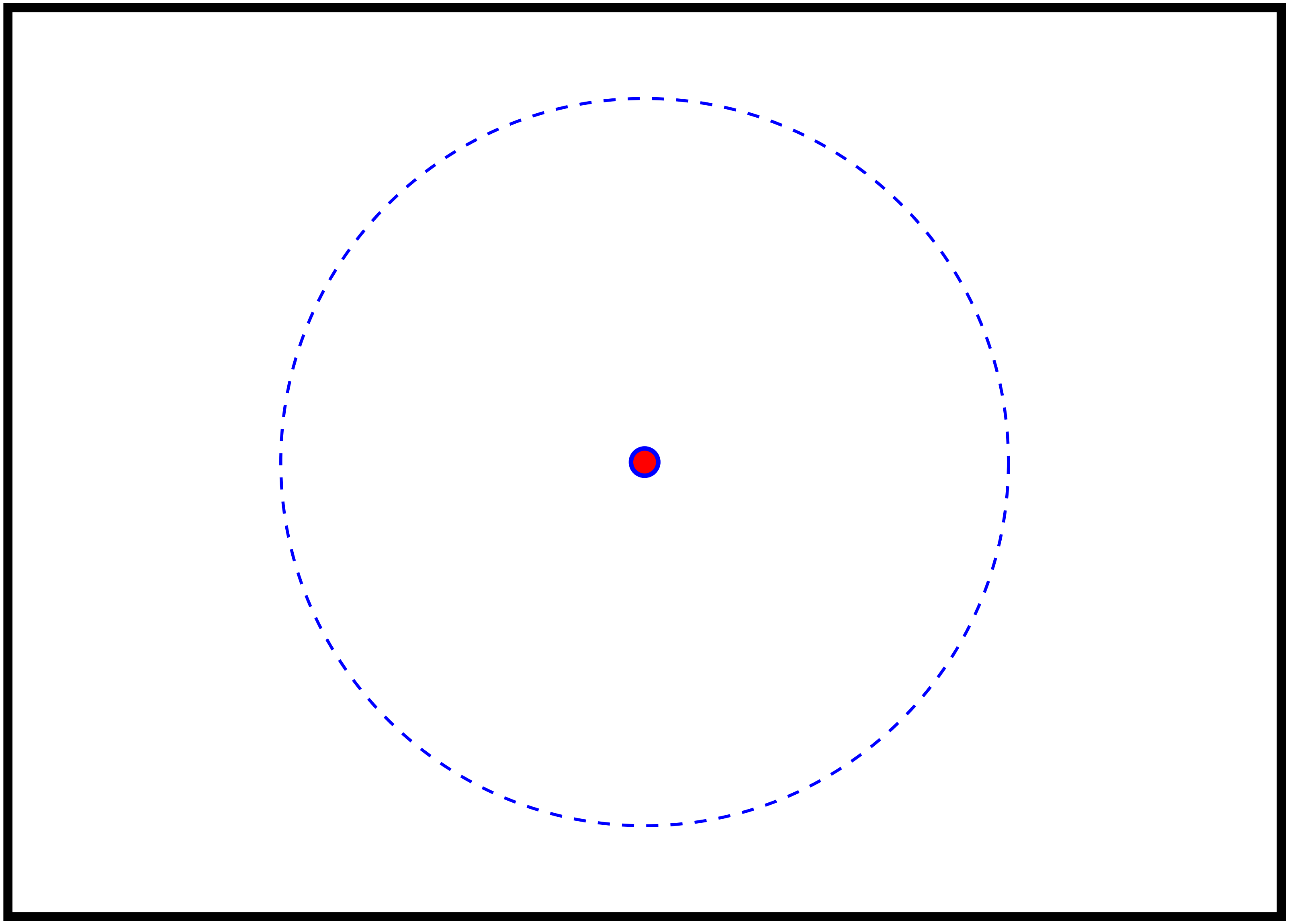
Harmonic measure = hitting distribution of Brownian motion

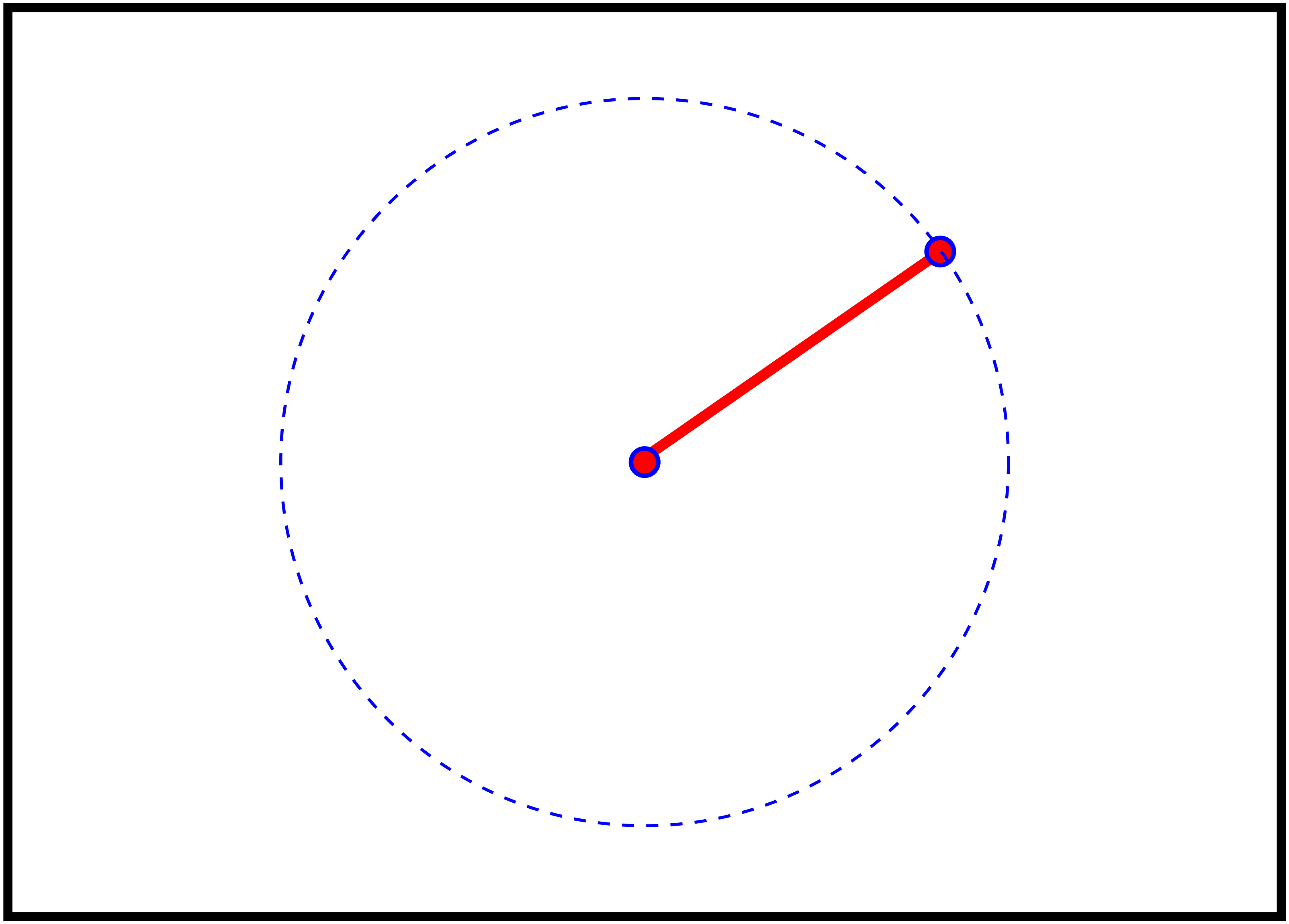


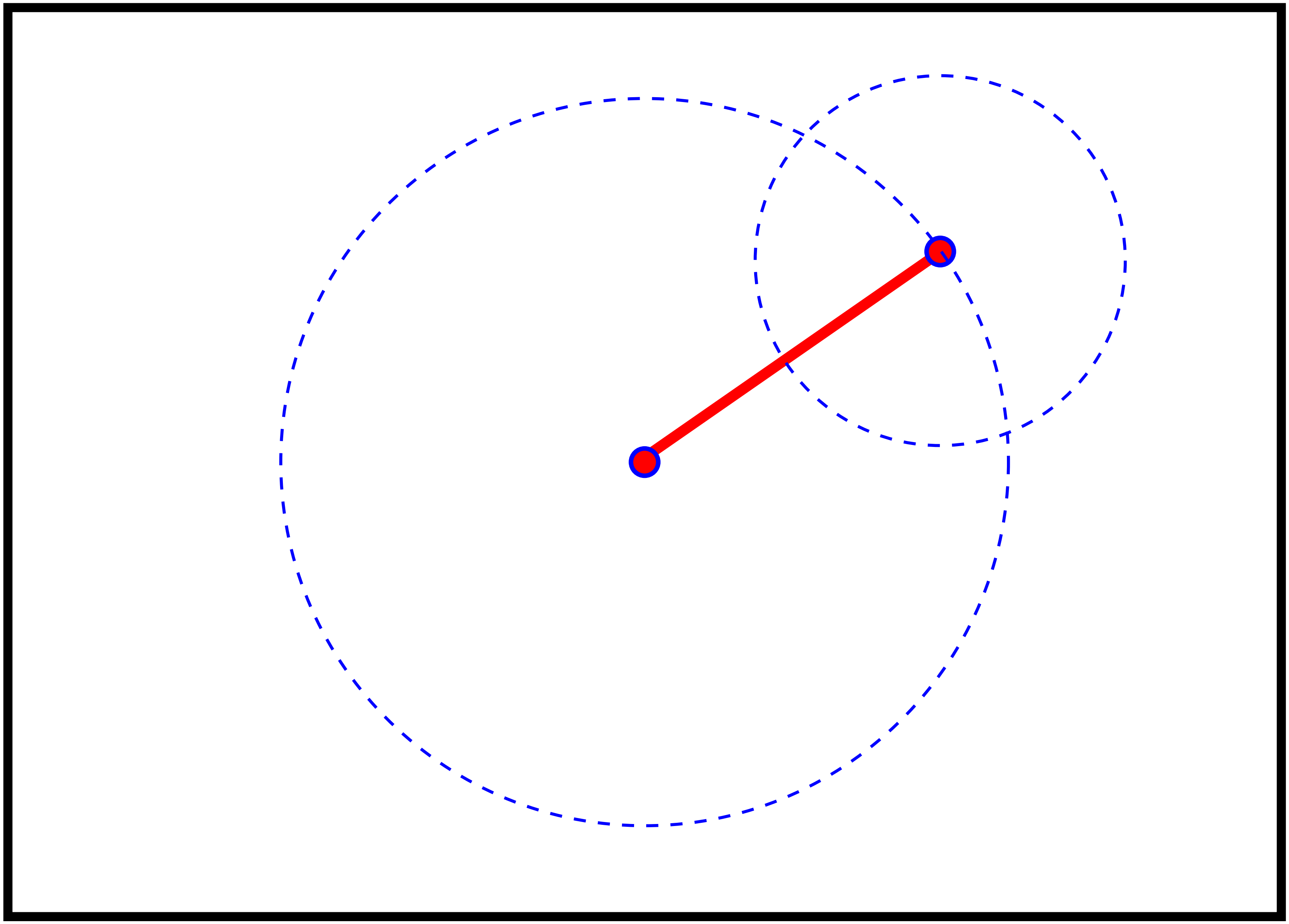
Different z gives $\omega(z, E, \Omega) \approx 12/100$.

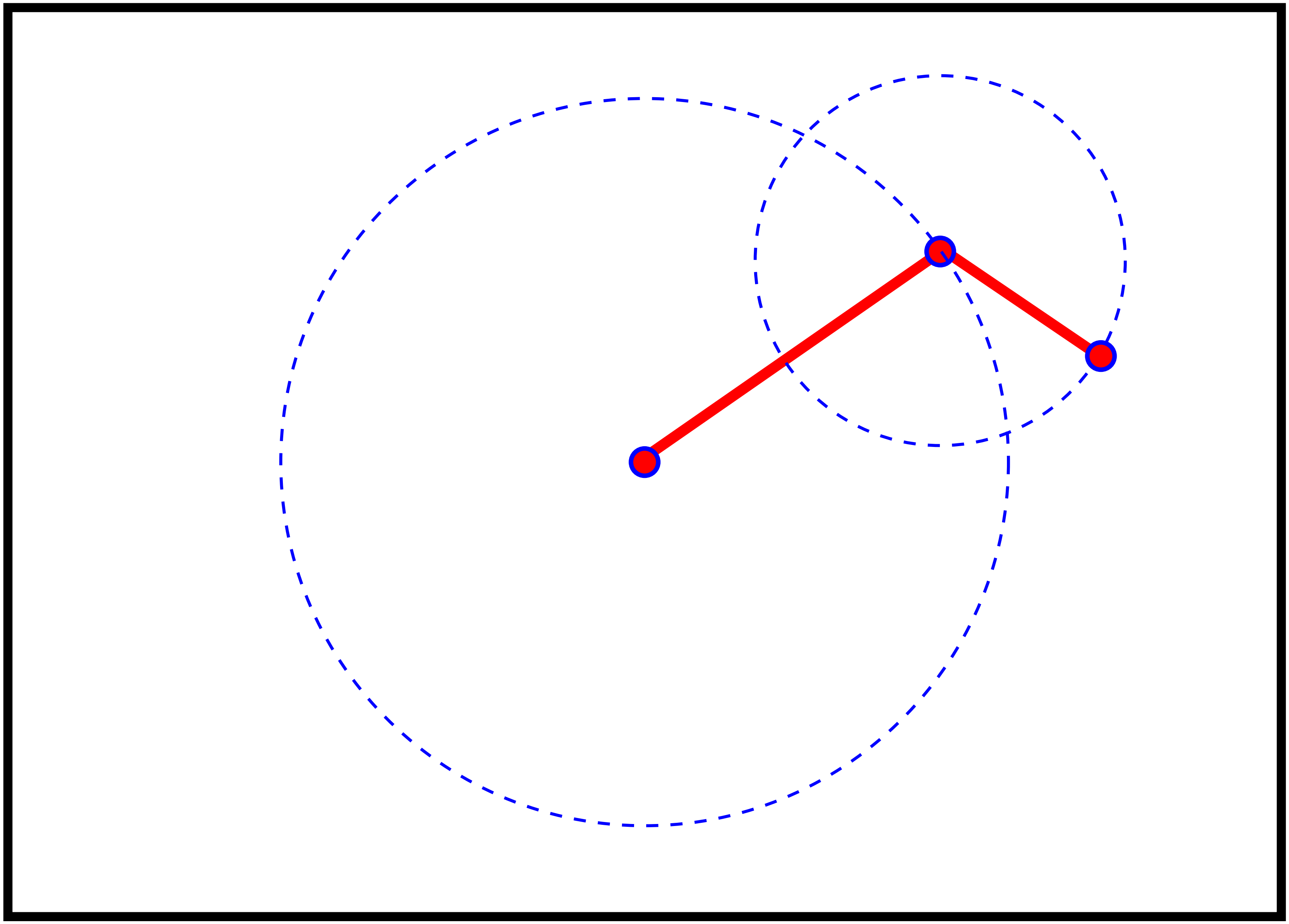
ω is harmonic in z with boundary values $\omega = 1$ on E, $\omega = 0$ off E.

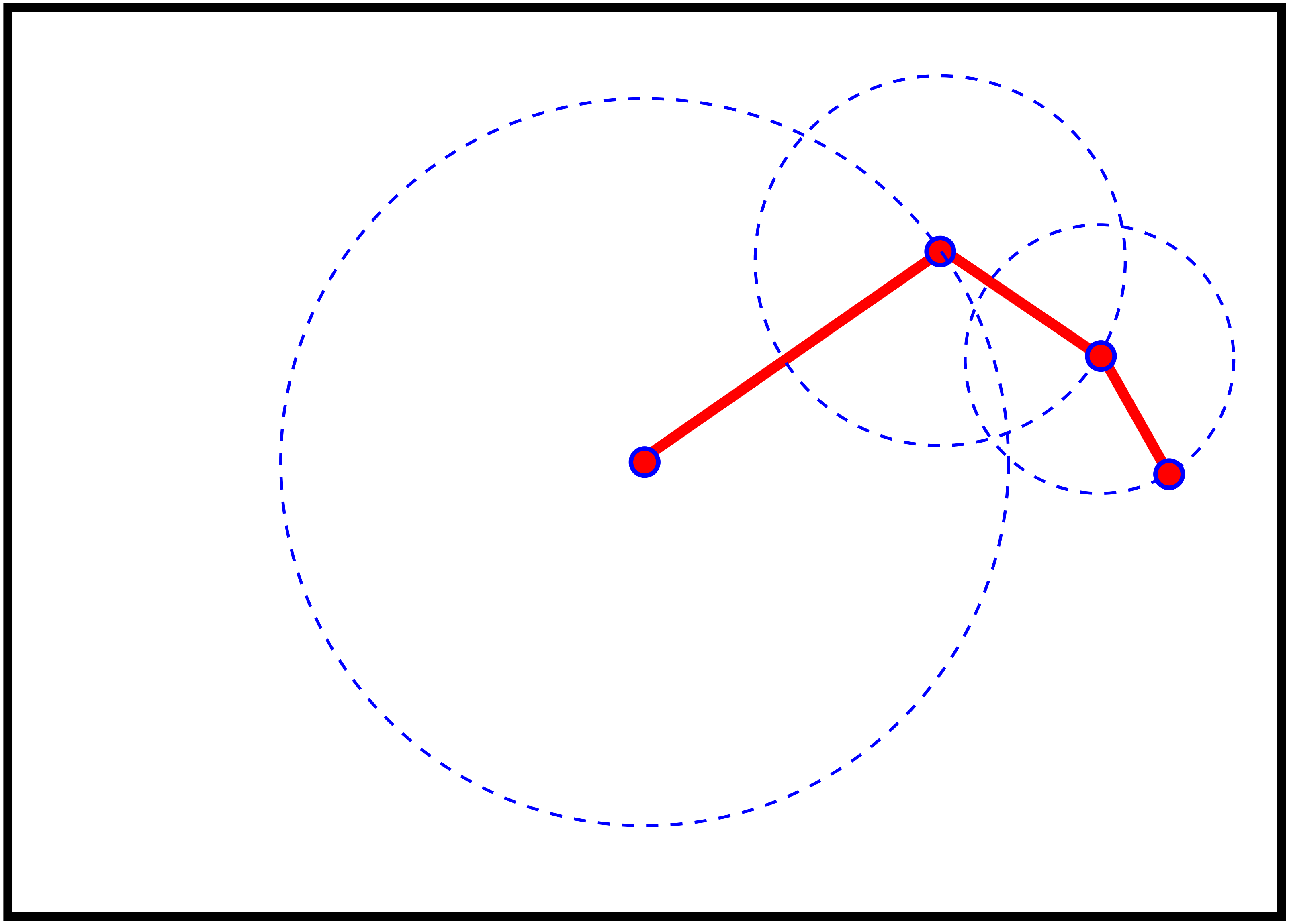


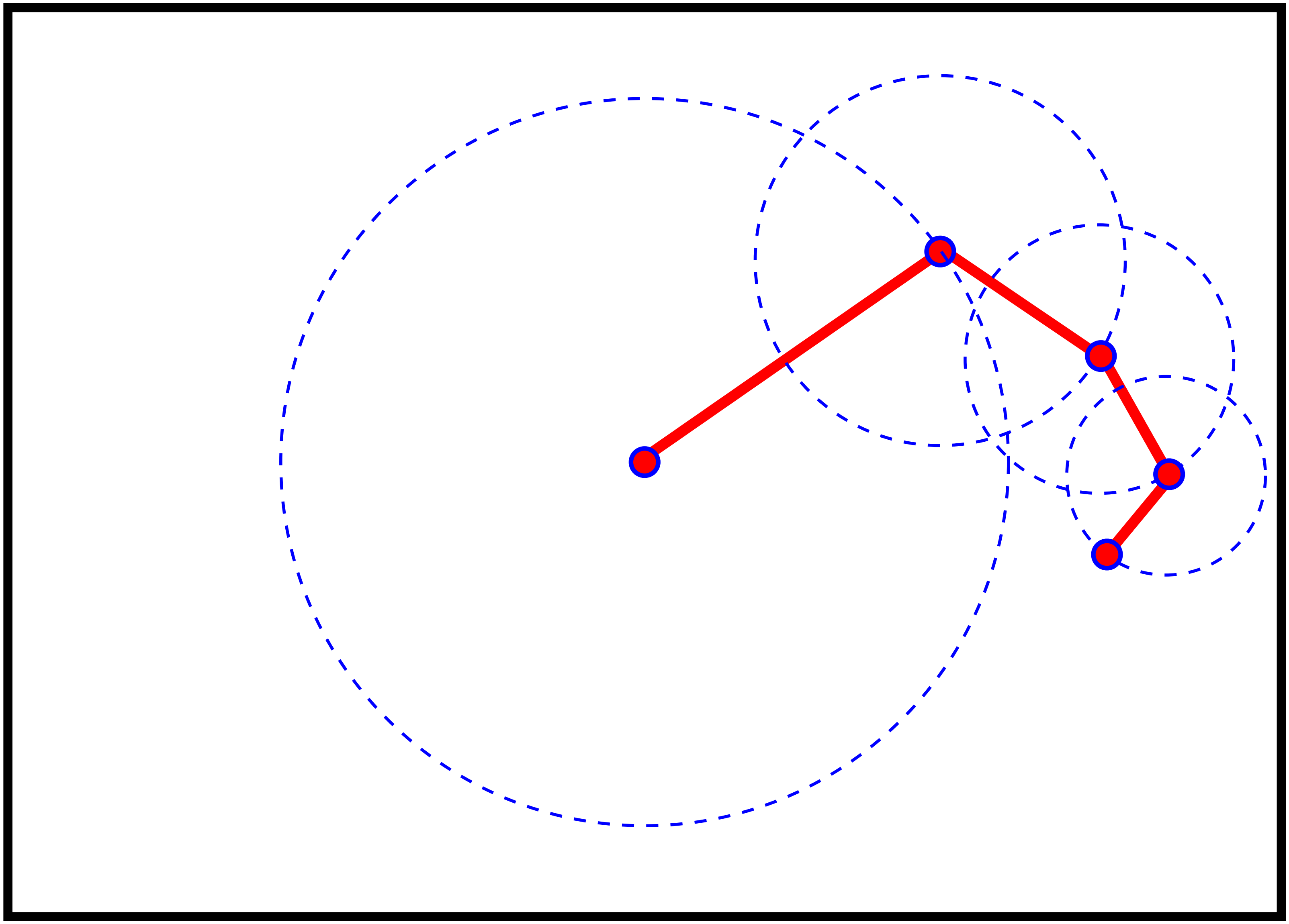


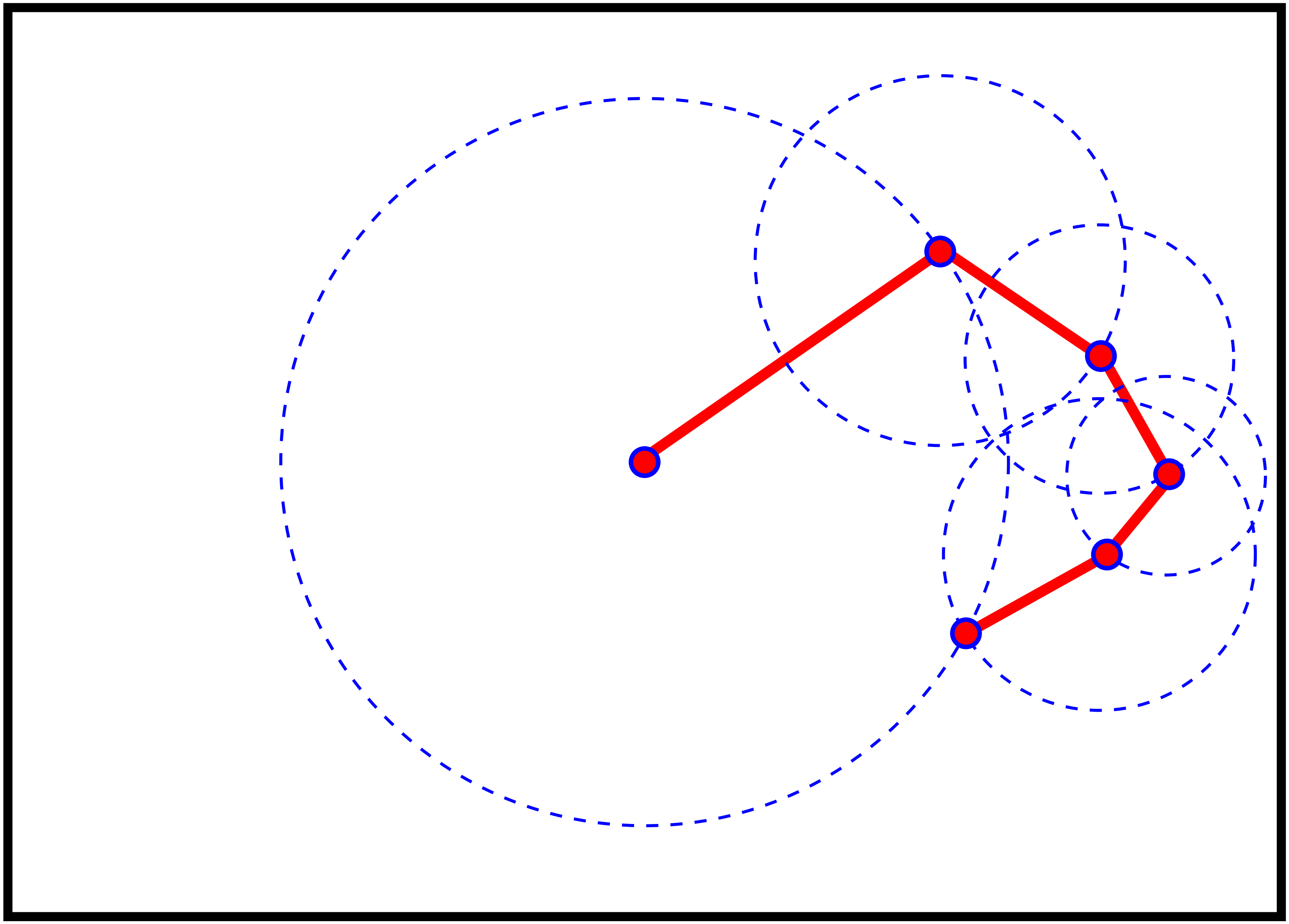


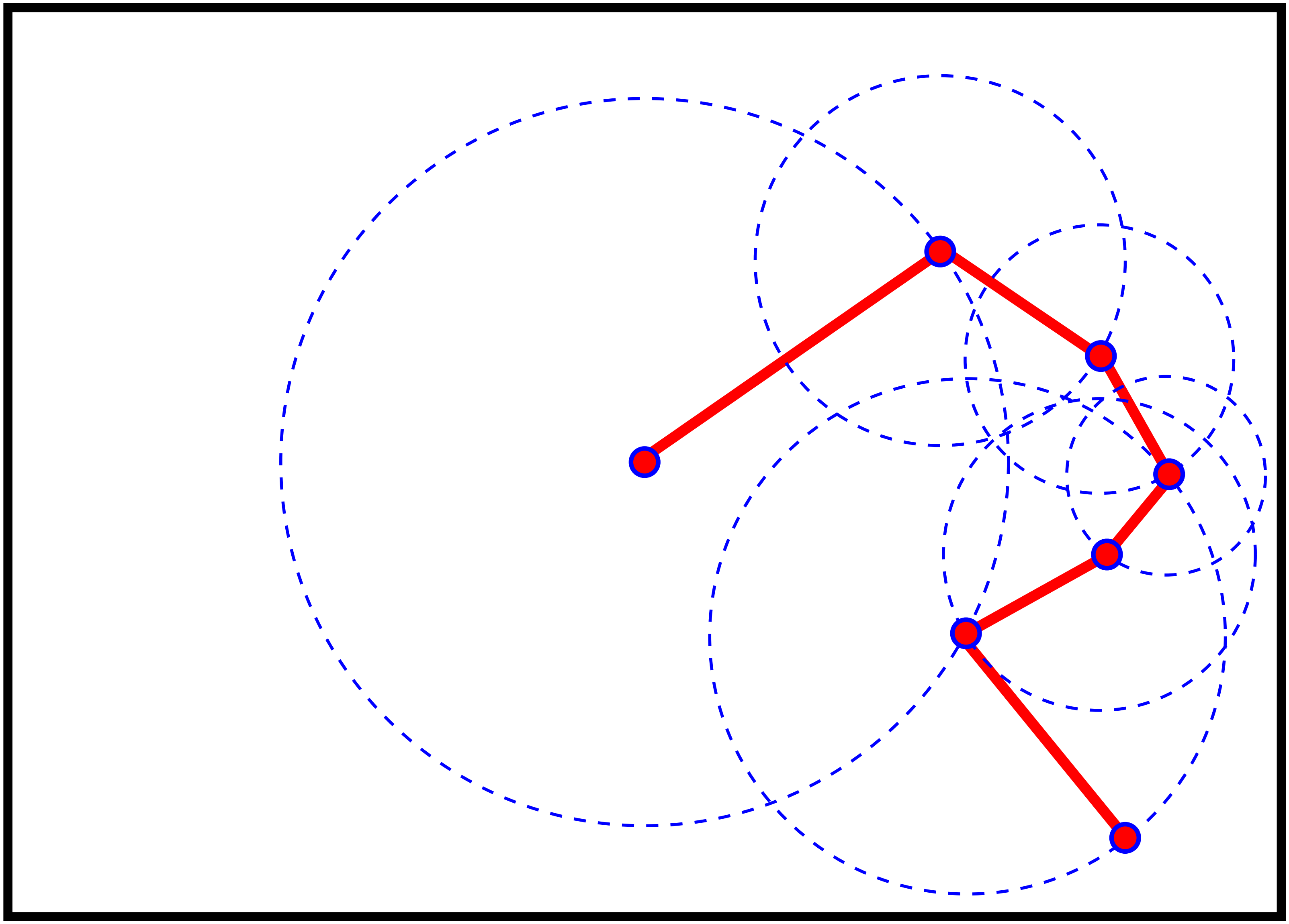


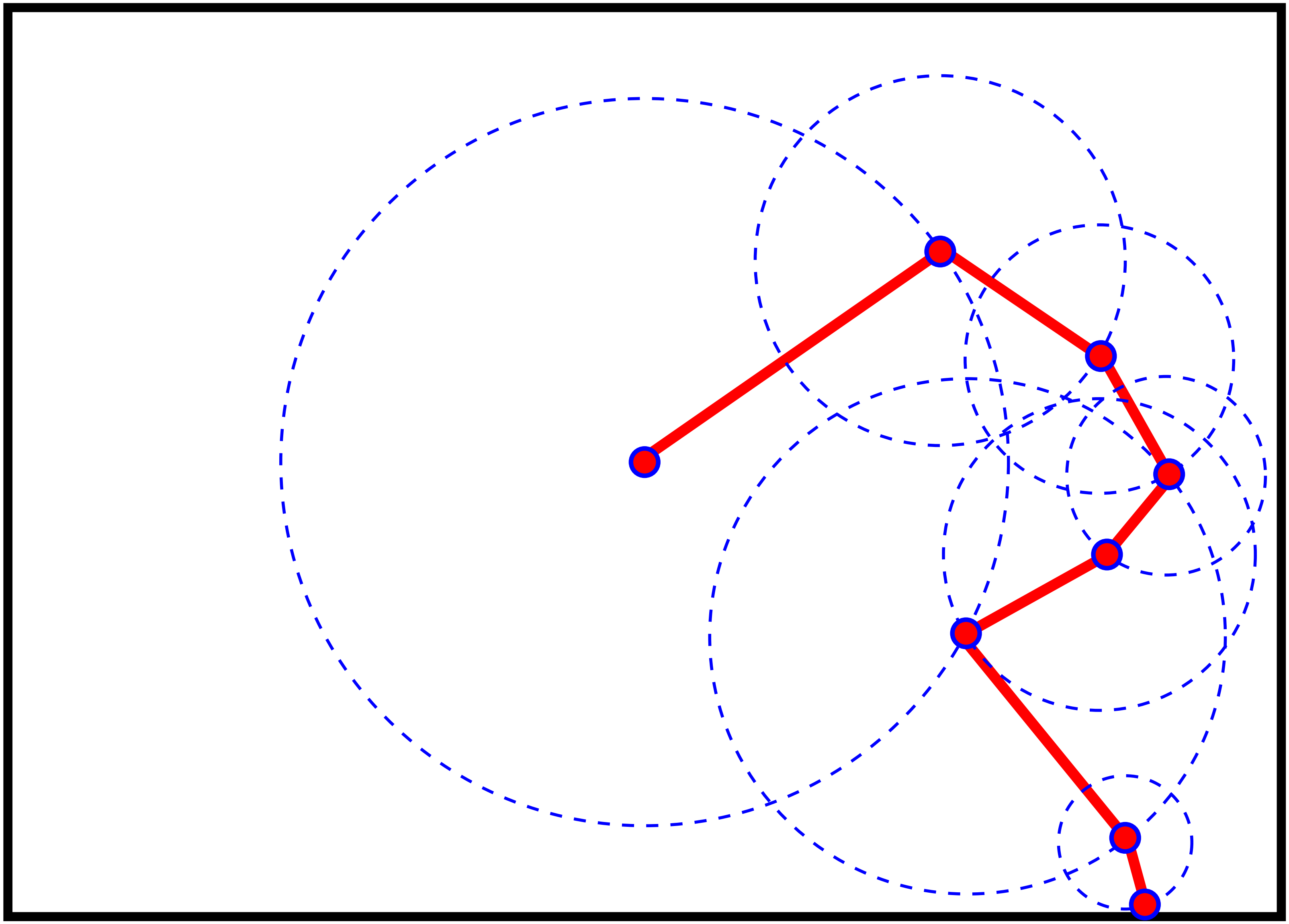




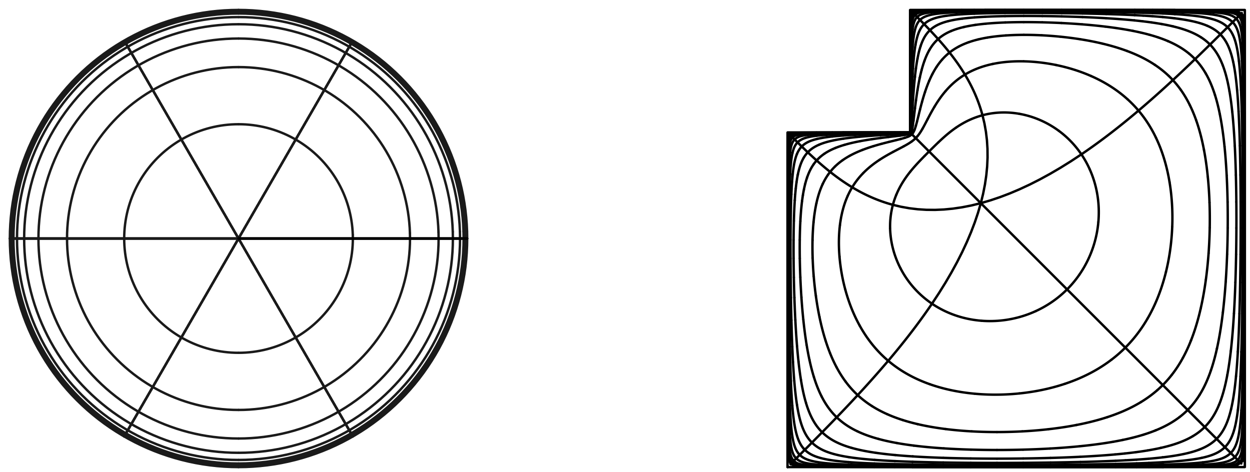




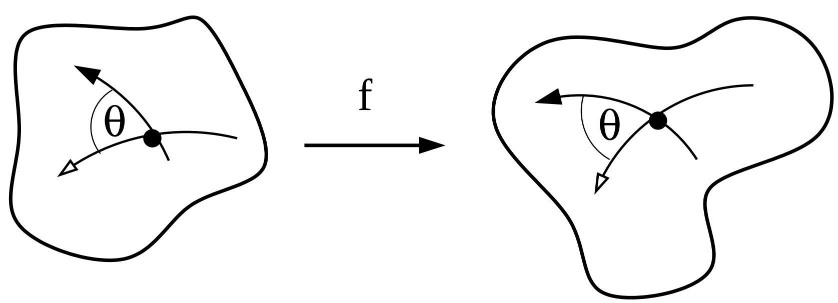


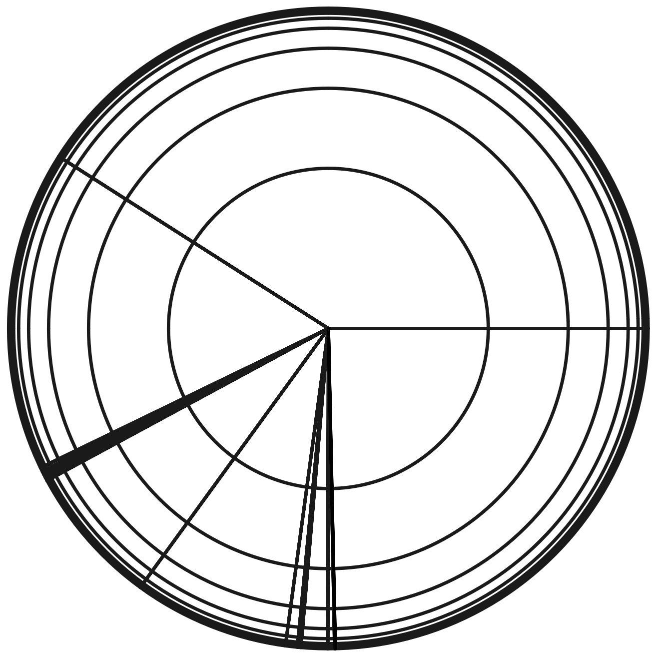


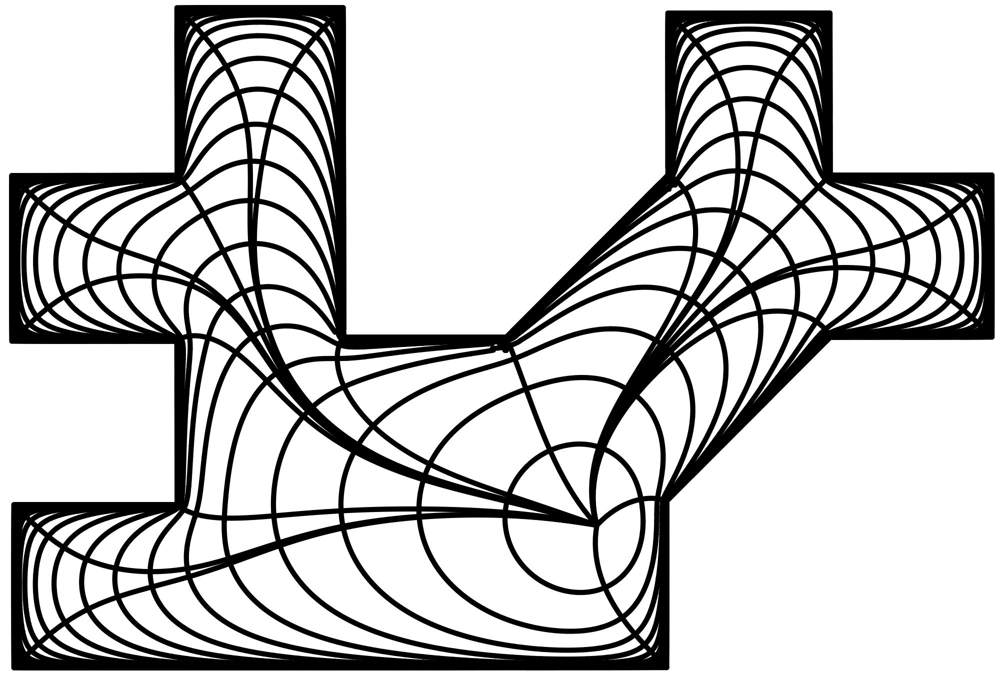
Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected, then there is a conformal map $f : \mathbb{D} \to \Omega$.

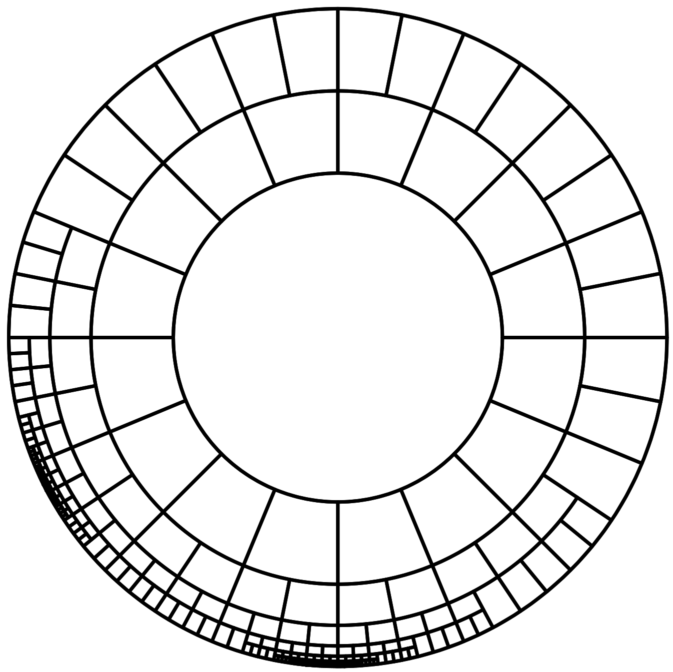
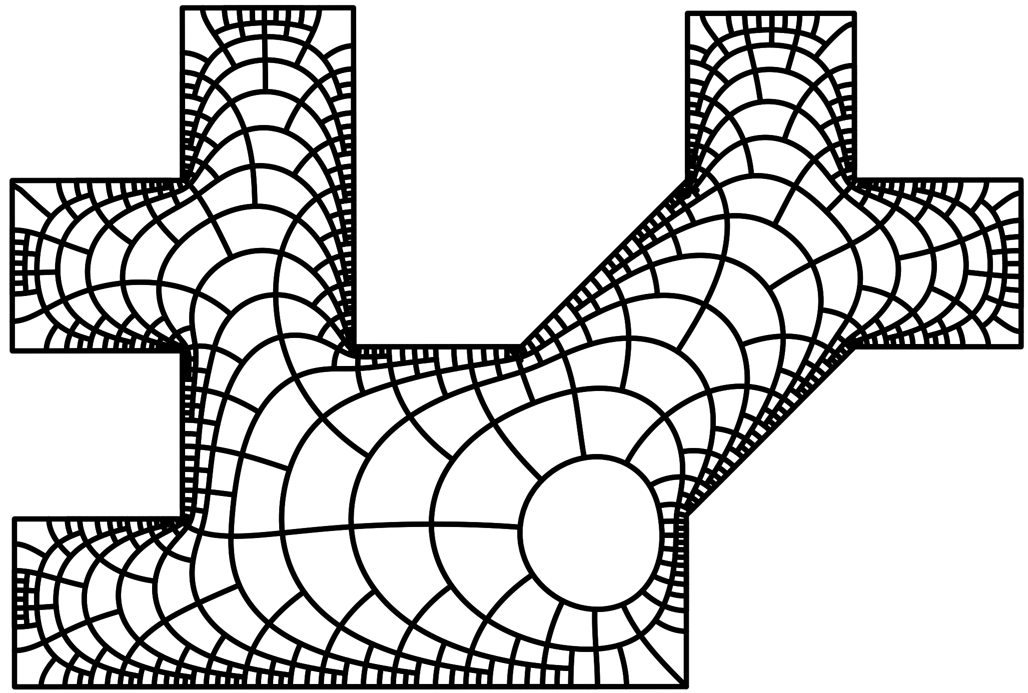


Conformal = angle preserving

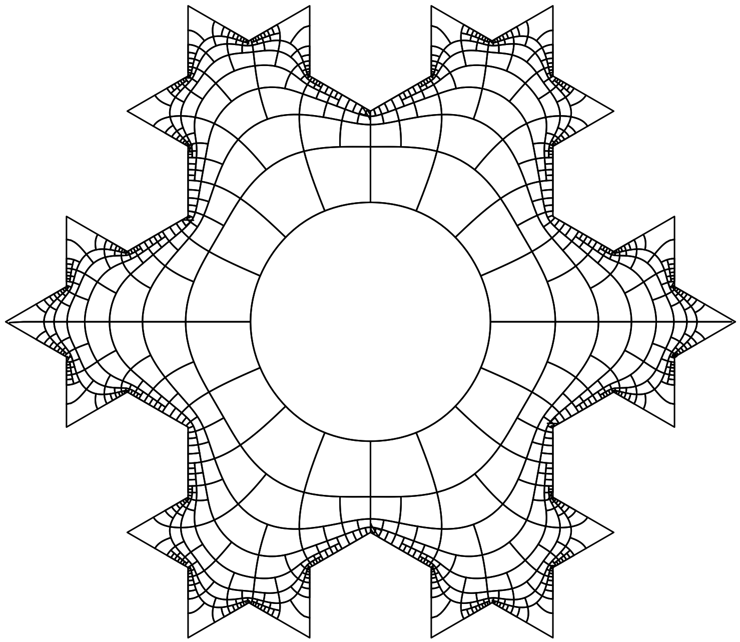
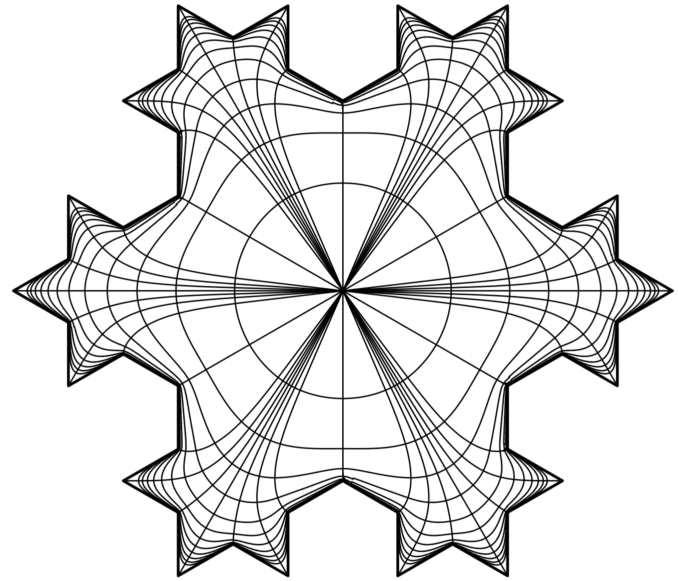






Brownian motion is conformally invariant, so normalized length measure maps to harmonic measure. Fastest way to compute harmonic measure.

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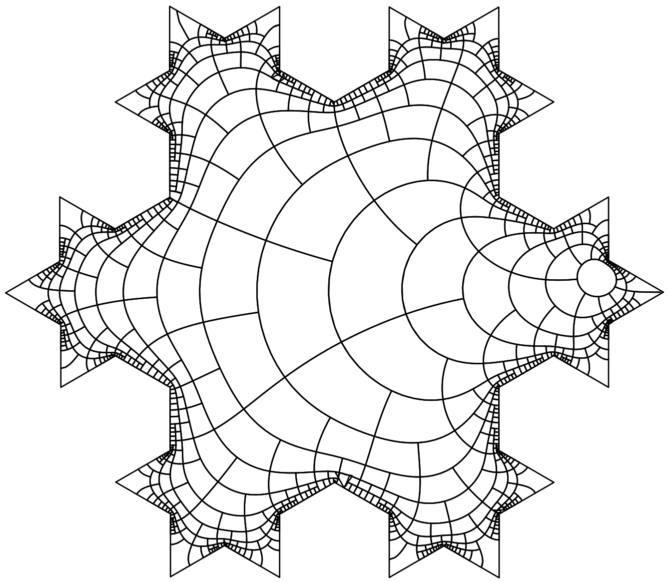
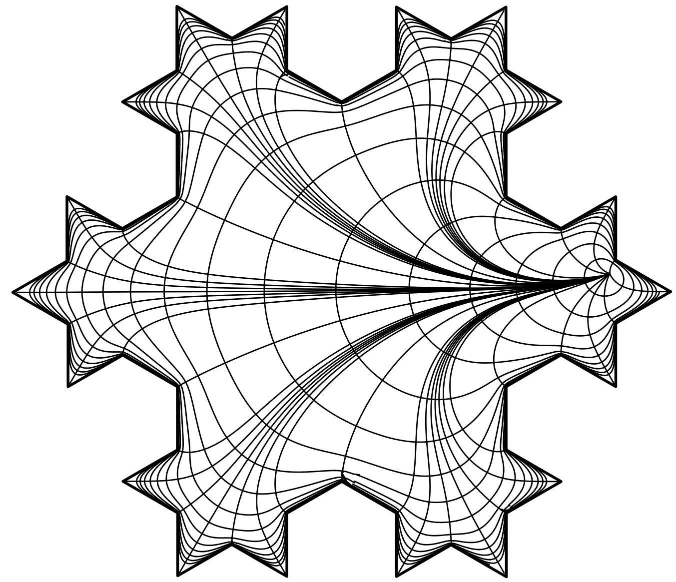

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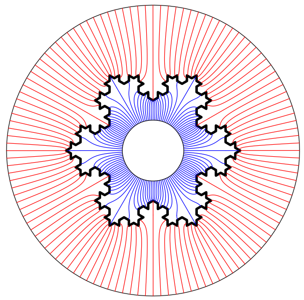

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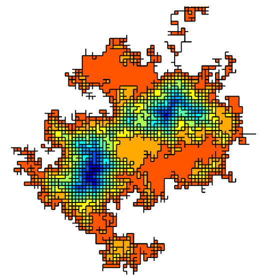
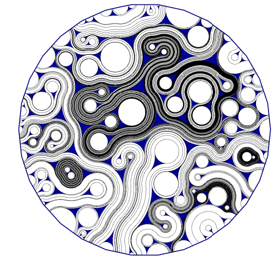


Deep Blue



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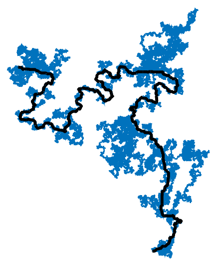
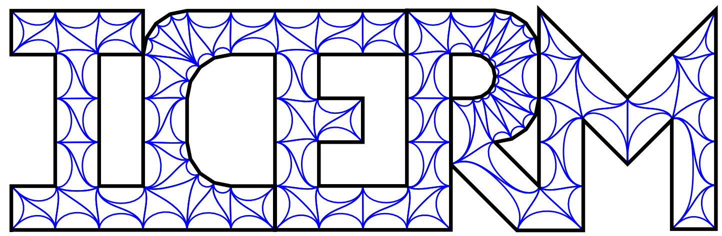


Shortcuts



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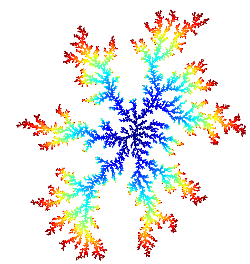
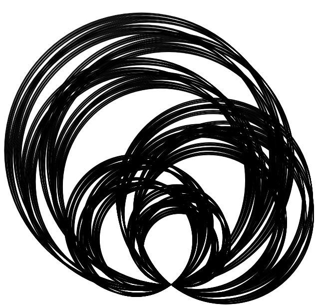
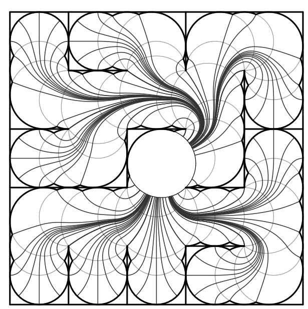


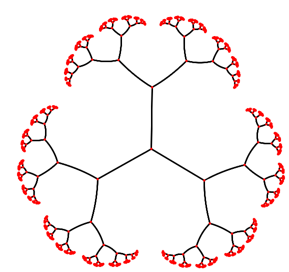
Diabo**l**ic**al**



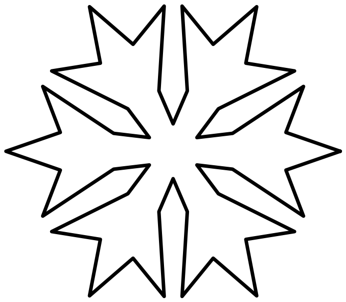
Circles



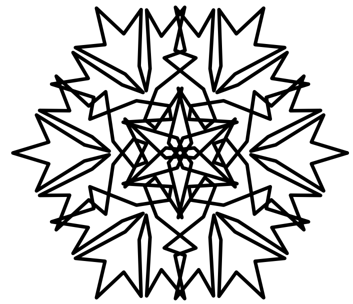
Fast Map



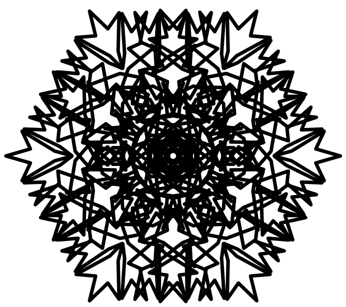
Regular Truth



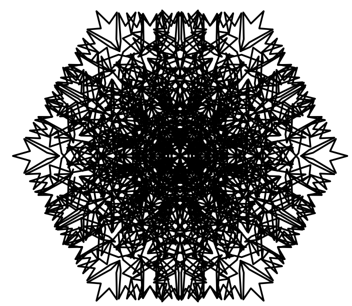
The portfolio



Wermer curves

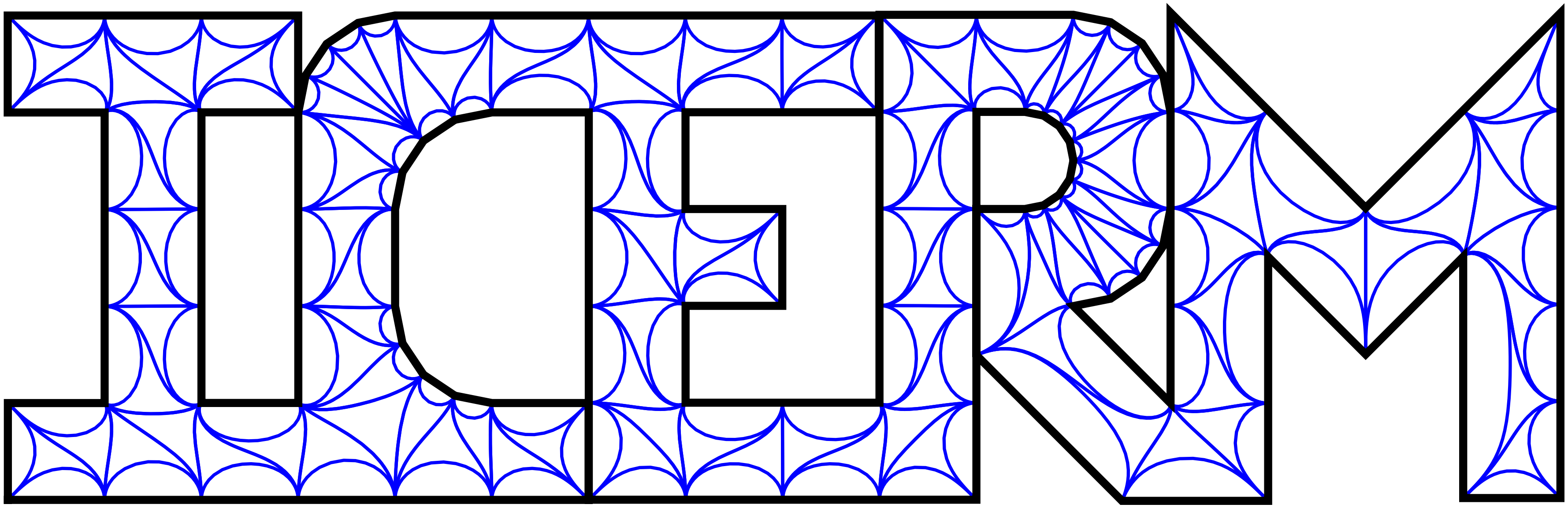


Devilish Quote 1



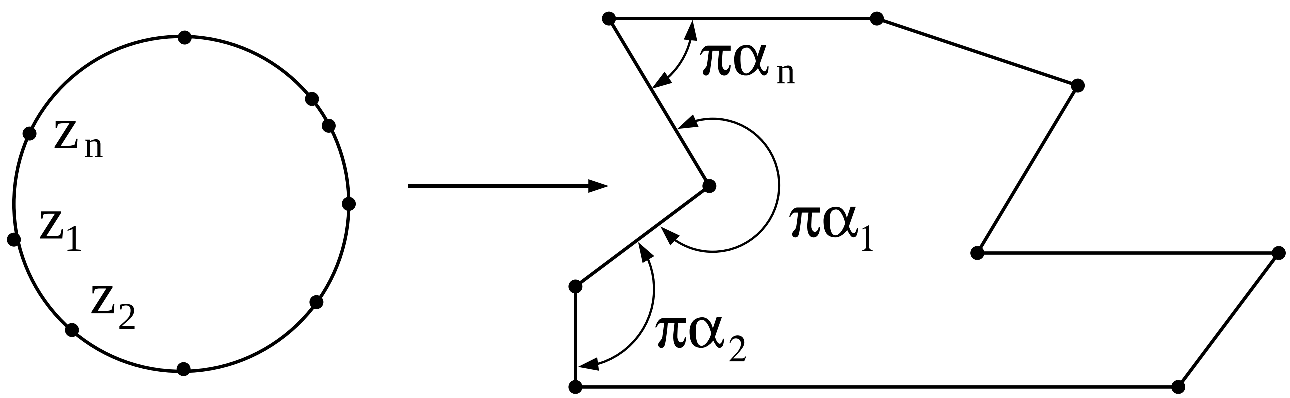
Devilish Quote 2

CONFORMAL MAPPING



Schwarz-Christoffel formula (1867):

$$f(z) = A + C \int^z \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw,$$

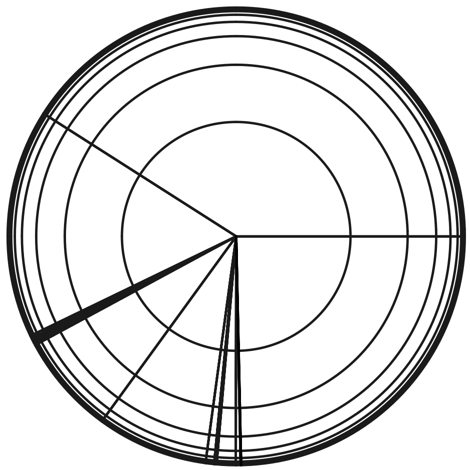


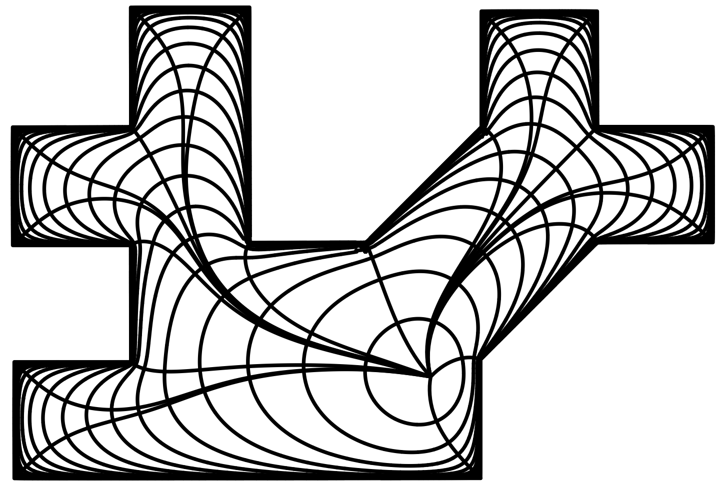
α's are known.

z's unknown (= **SC-parameters** = **pre-vertices**)

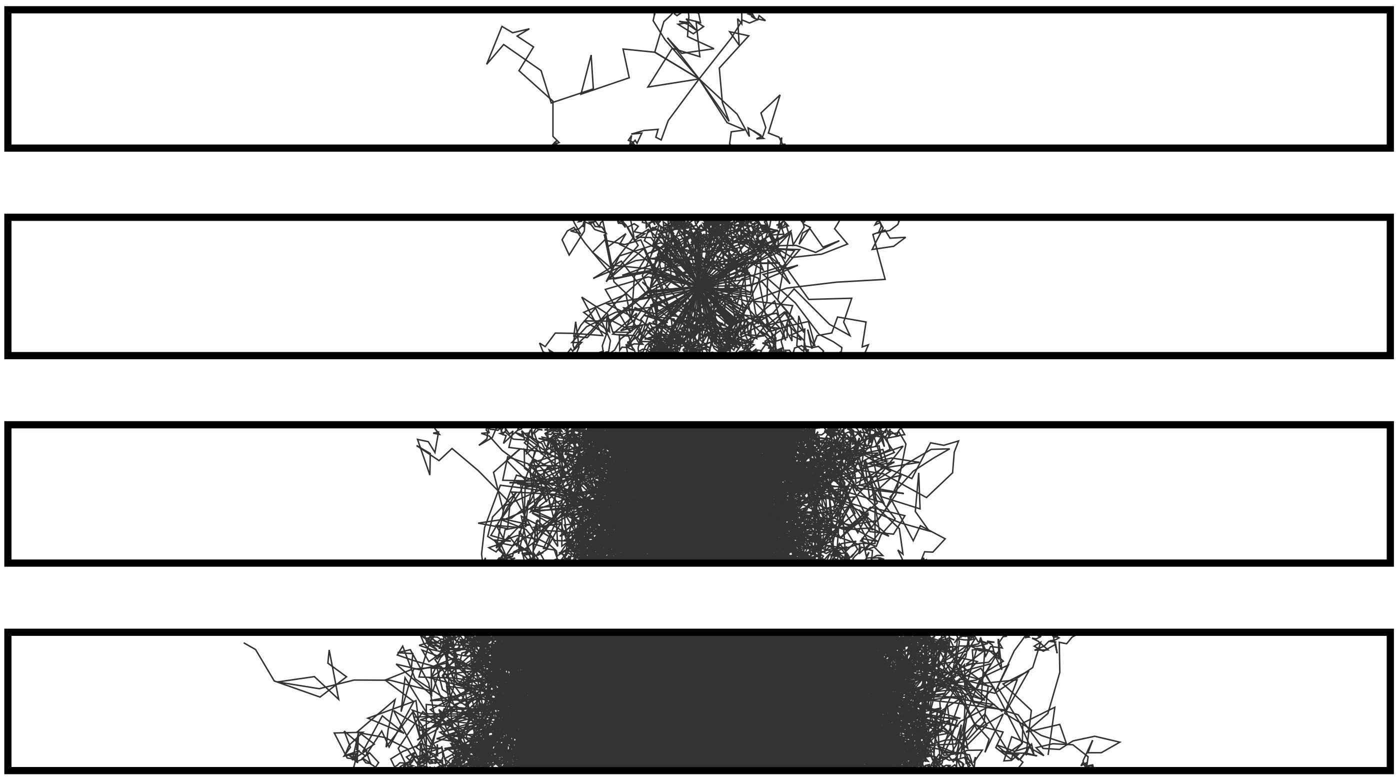
Knowing z's is same as knowing harmonic measure of edges.

How fast can we compute the harmonic measures for an n-gon?





Simulating random walks to estimate harmonic measure is terrible.



10, 100, 1000 and 10,000 paths in a 1×10 rectangle.

Probability of traversing $1 \times r$ corridor is $\simeq \exp(-\pi r)$.

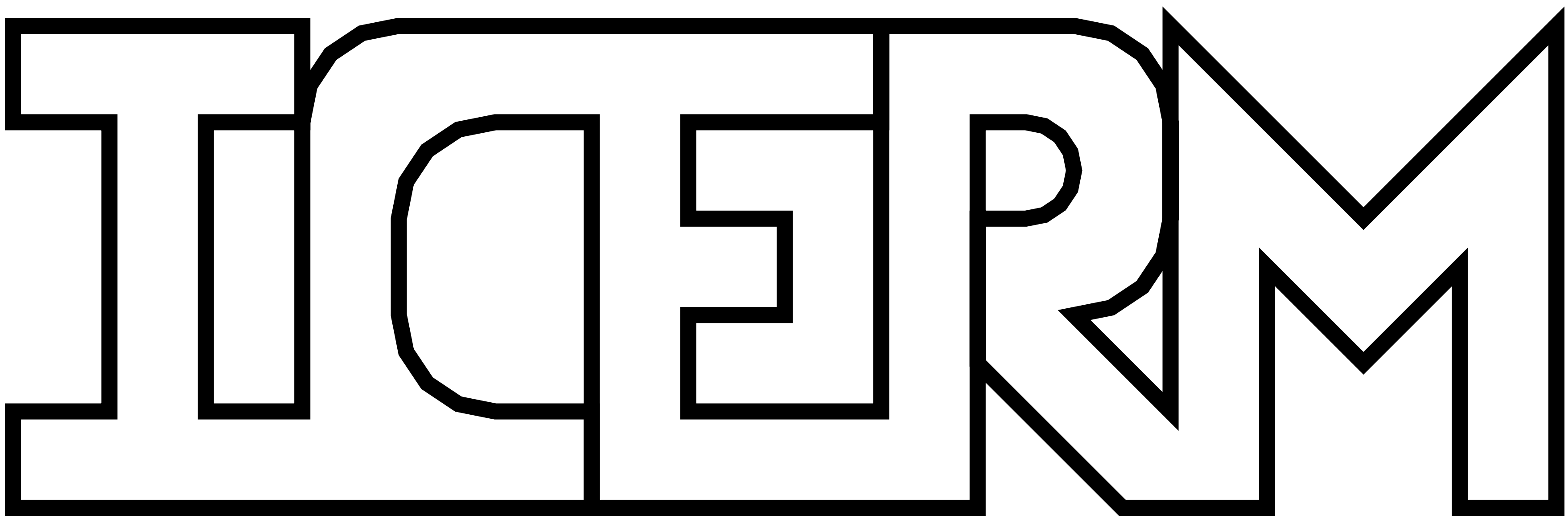
Theorem: Can compute harmonic measures of n-gon in time $C_\epsilon \cdot n$.

ϵ = error in quasiconformal sense.

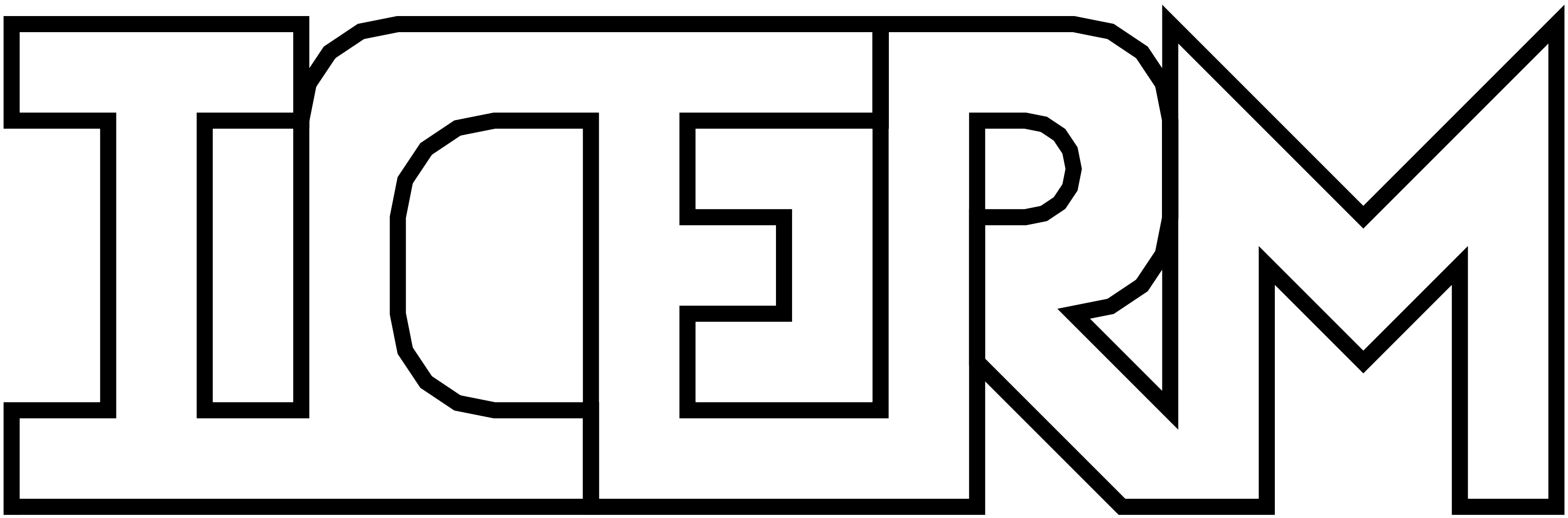
Implies uniform approximation.

$C_\epsilon = O(\log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$.

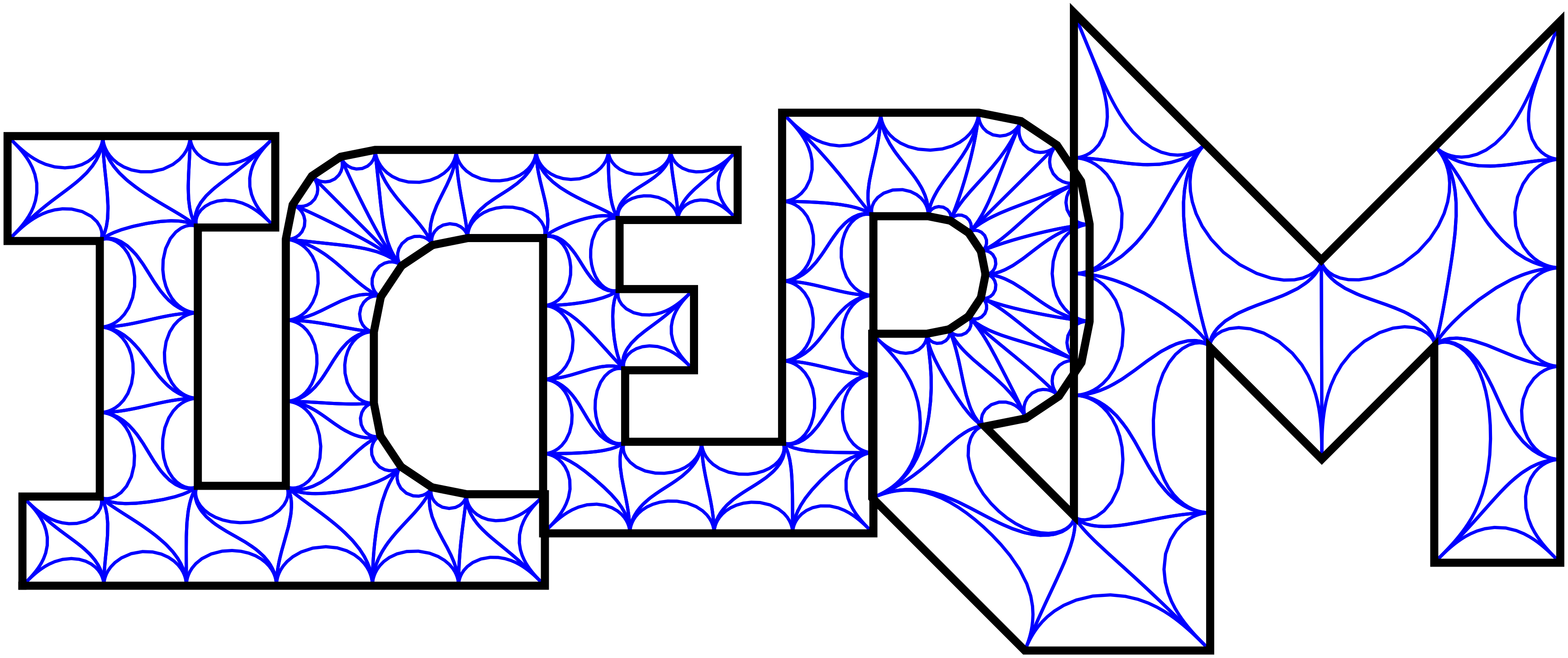
Other methods used in practice, but not proven to converge.



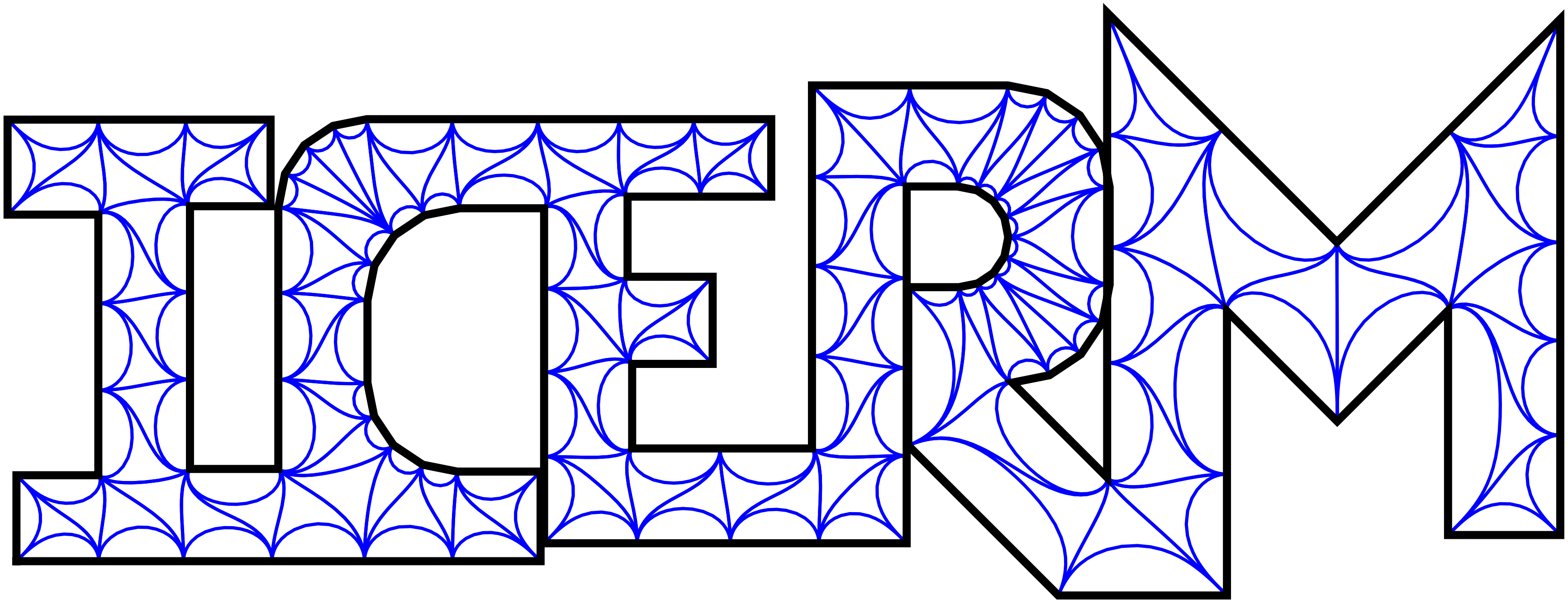
A randomly selected 105-gon.



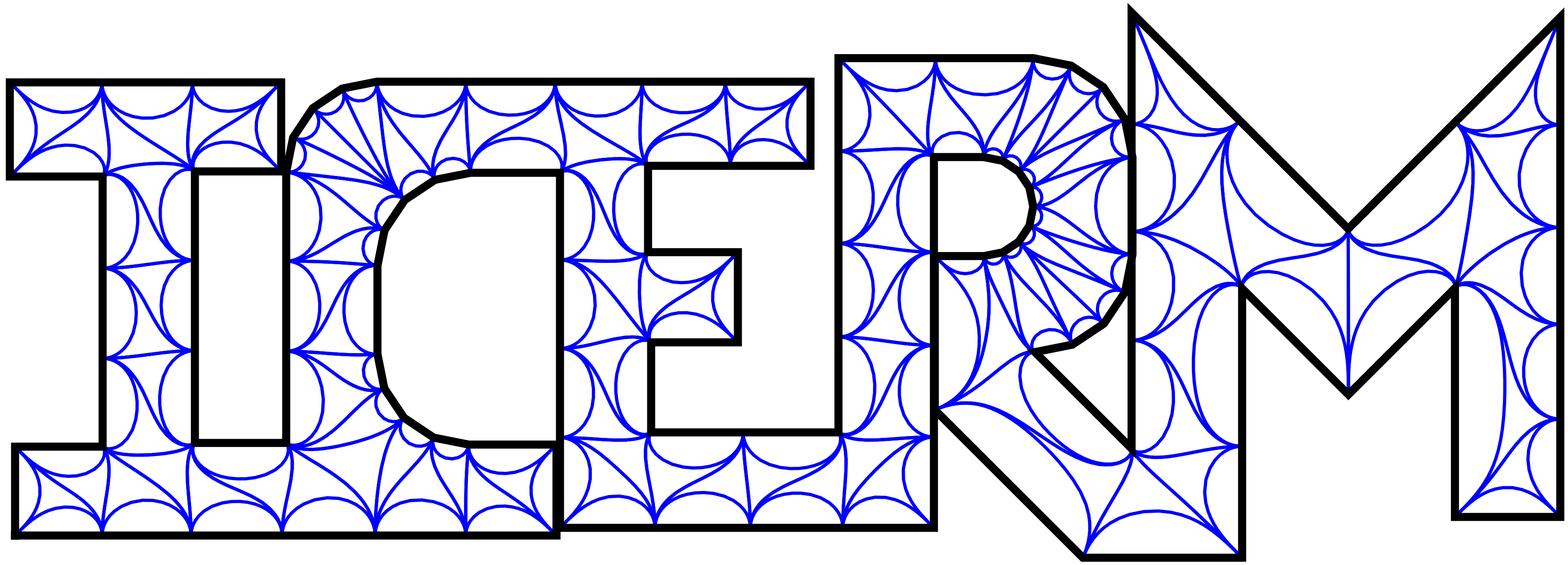
A randomly selected 105-gon.
Roughly looks like 1×50 tube
$\exp(50\pi) \approx 1.6 \times 10^{68}$



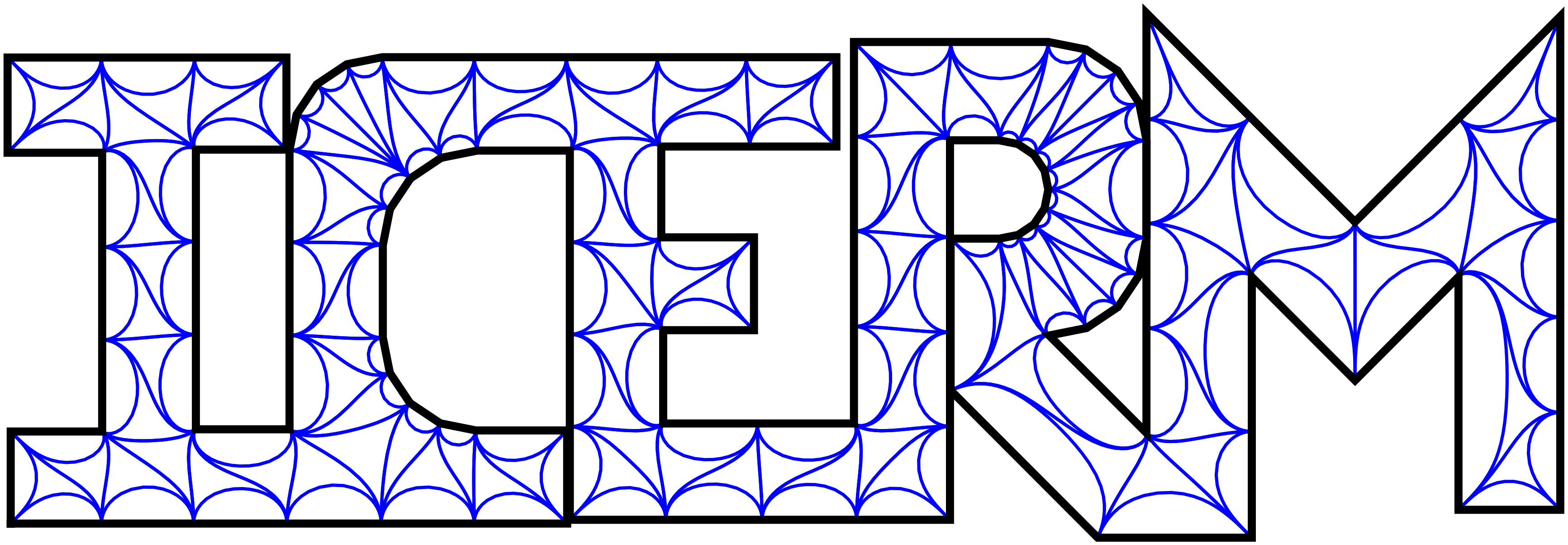
Iteration 1



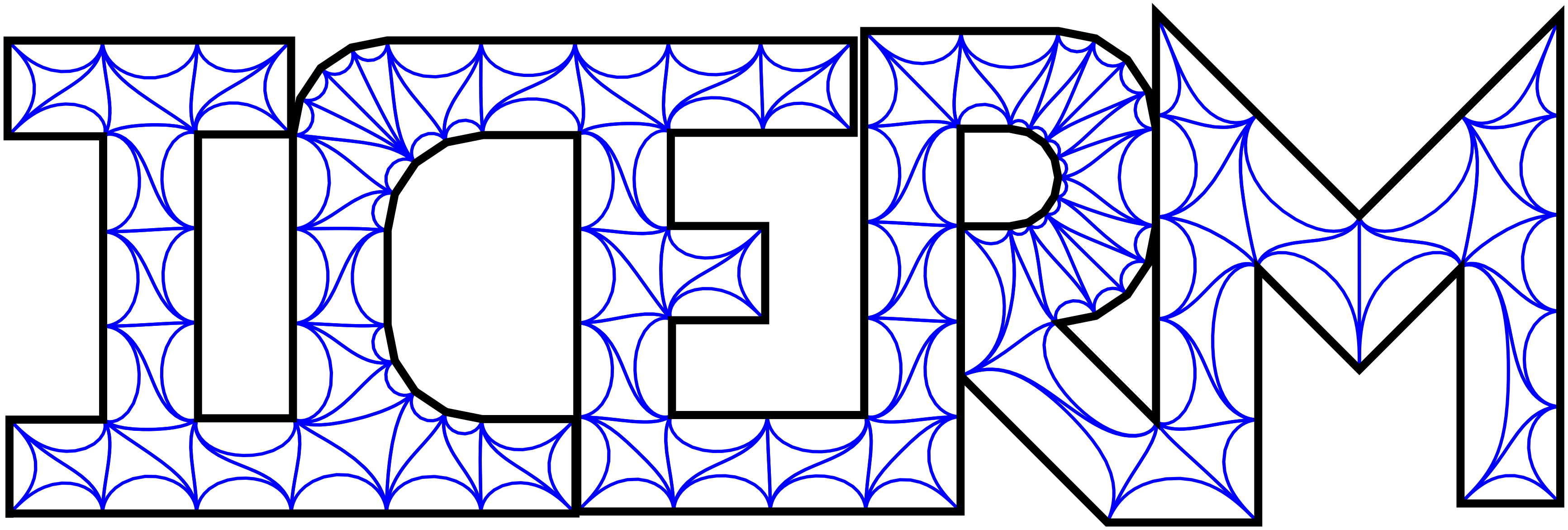
Iteration 2



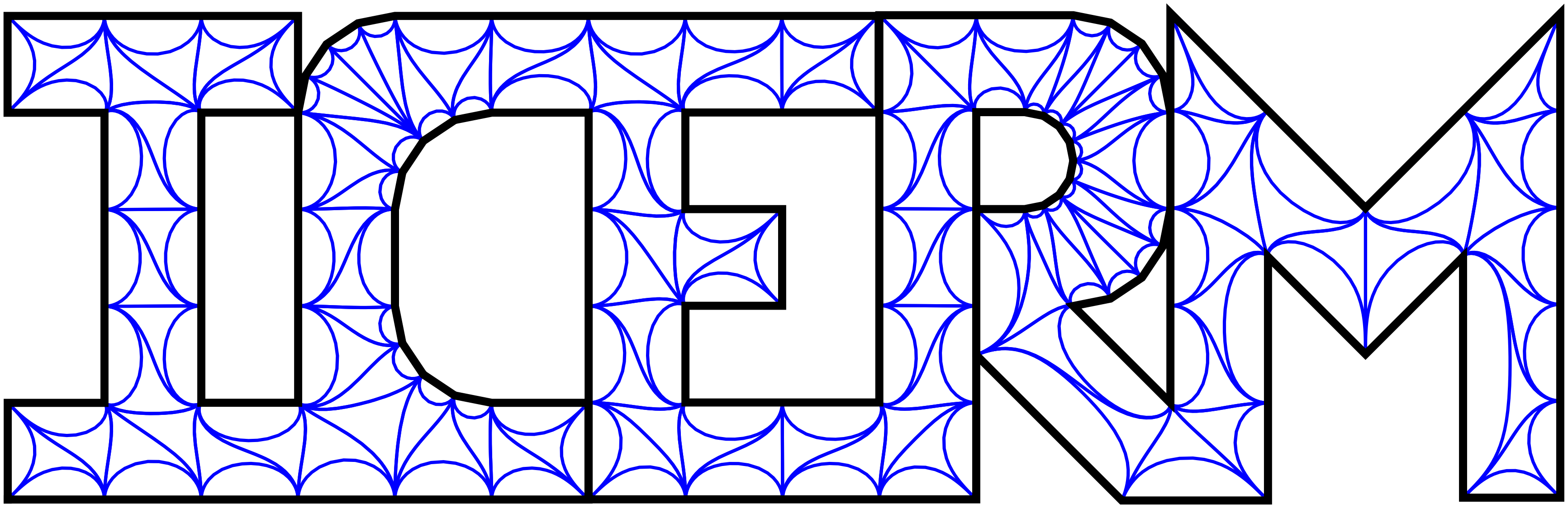
Iteration 3



Iteration 4

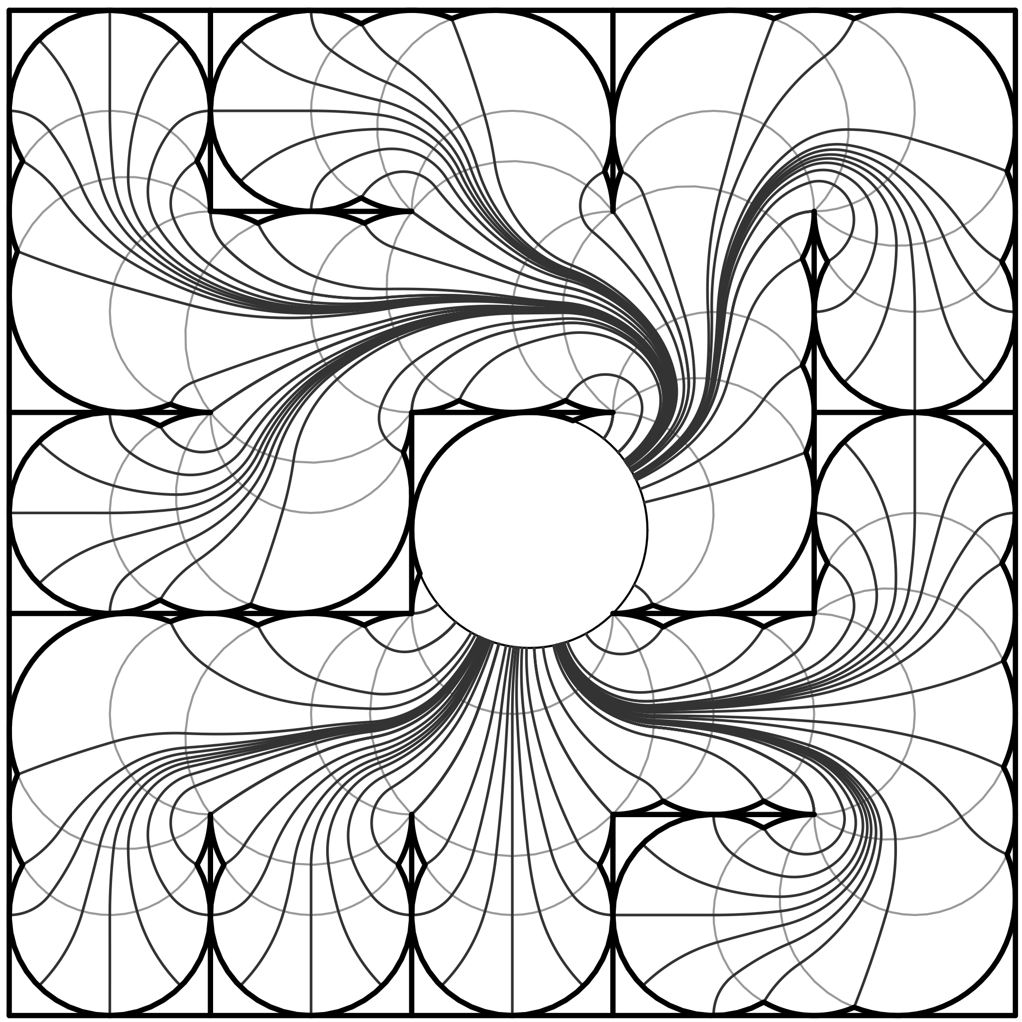


Iteration 5



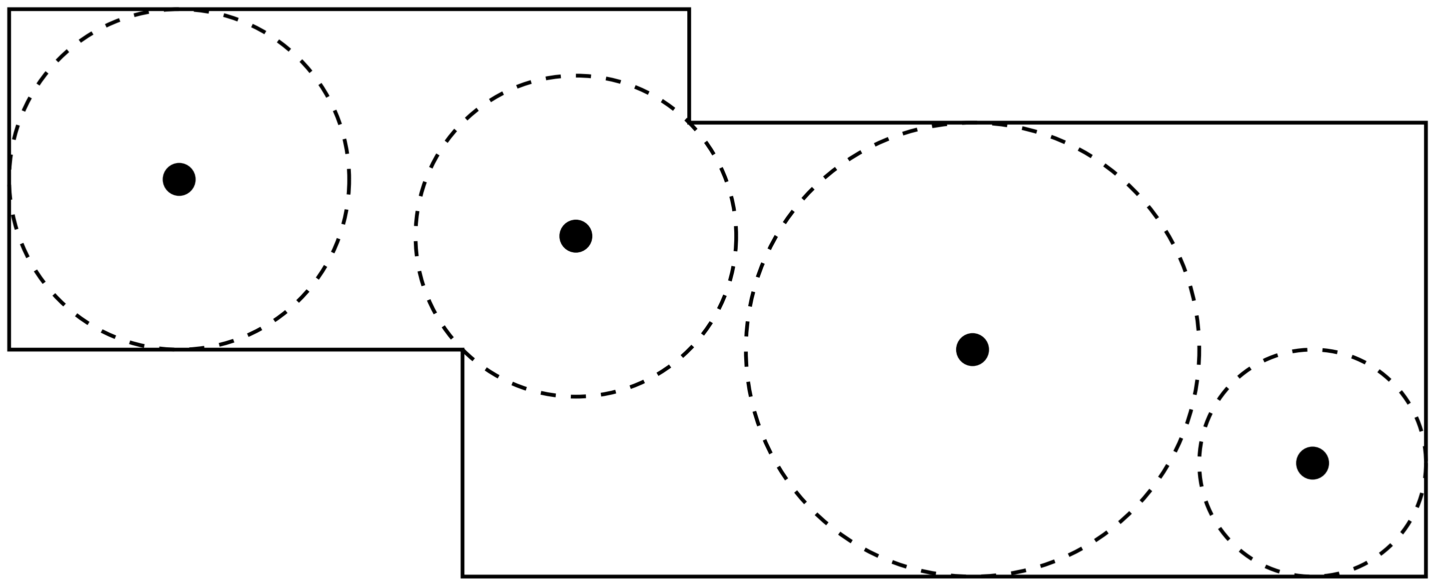
Iteration 10

THE MEDIAL AXIS FLOW
(ESTIMATING HARMONIC MEASURE QUICKLY)

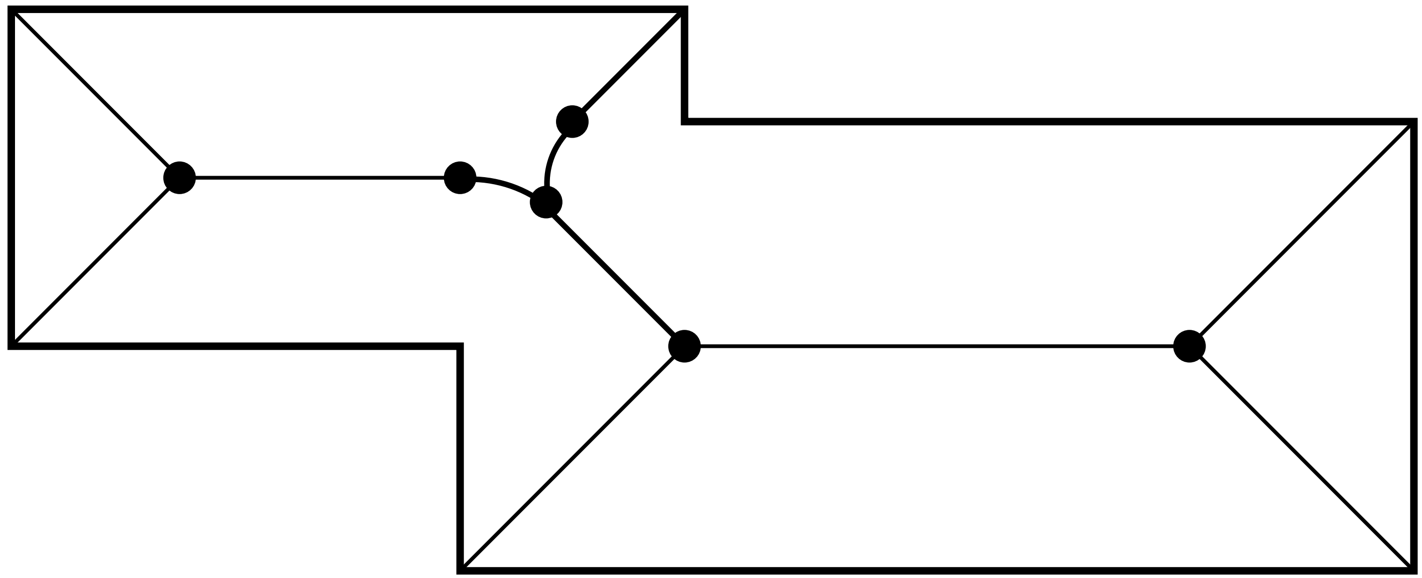


Medial axis:

centers of disks that hit boundary in at least two points.



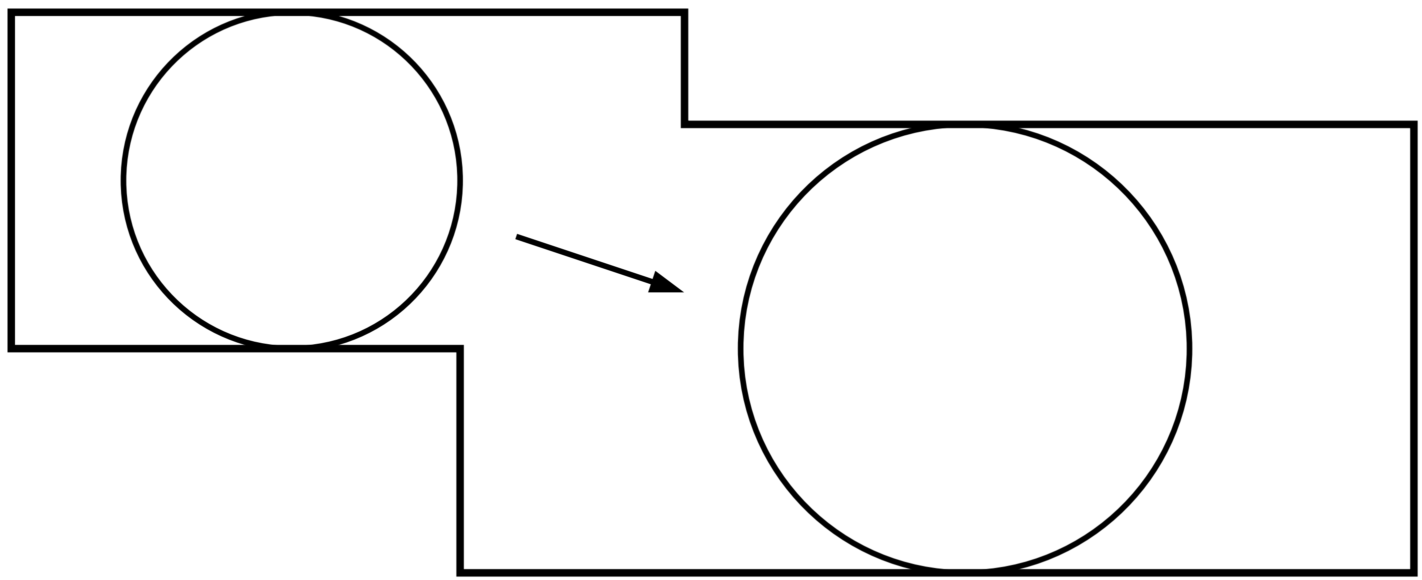
Medial axis:
centers of disks that hit boundary in at least two points.



Medial axis of a polygon is a finite tree.

Related to Voronoi diagrams: medial axis divides polygon interior according to nearest edge.

Medial axis:
centers of disks that hit boundary in at least two points.



Claim: there is a "natural" map between any two medial axis disks.

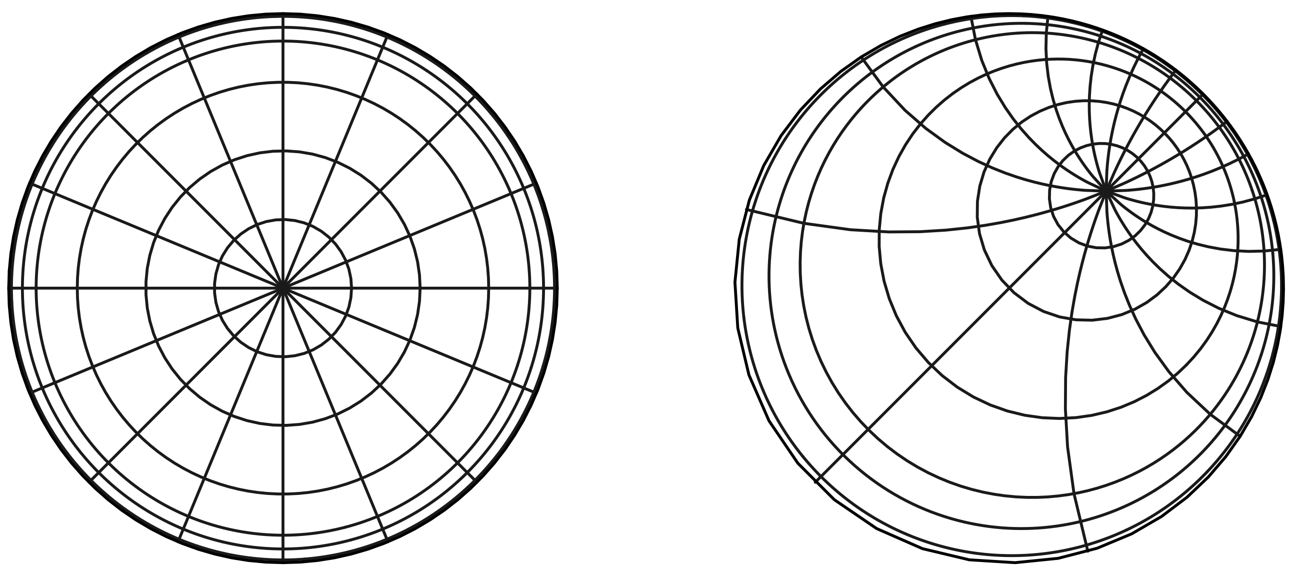
A **Möbius transformation** is a map of the form

$$z \to \frac{az+b}{cz+d}.$$

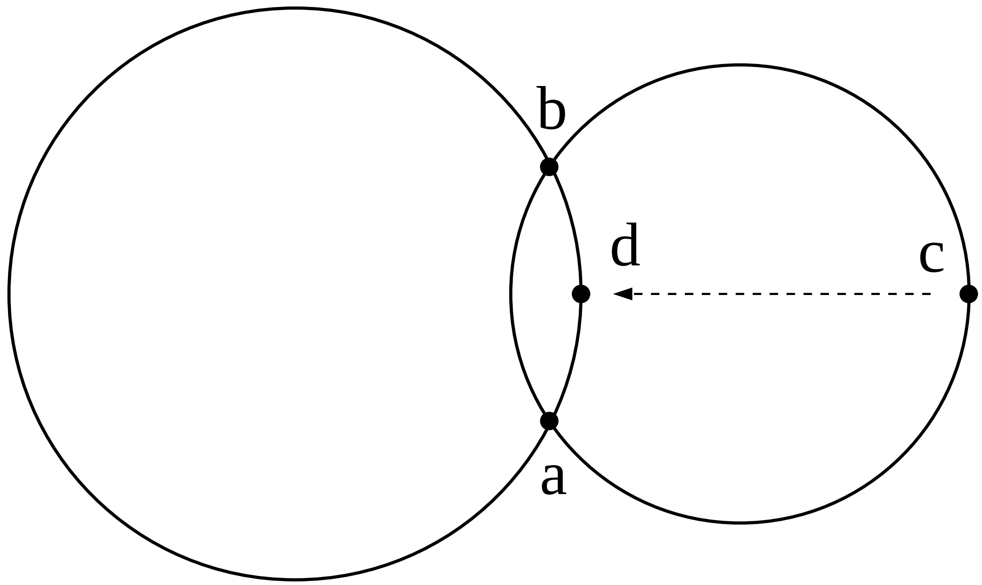
Conformally maps disks to disks (or half-planes).

Form a group under composition.

Uniquely determined by images of 3 distinct points.



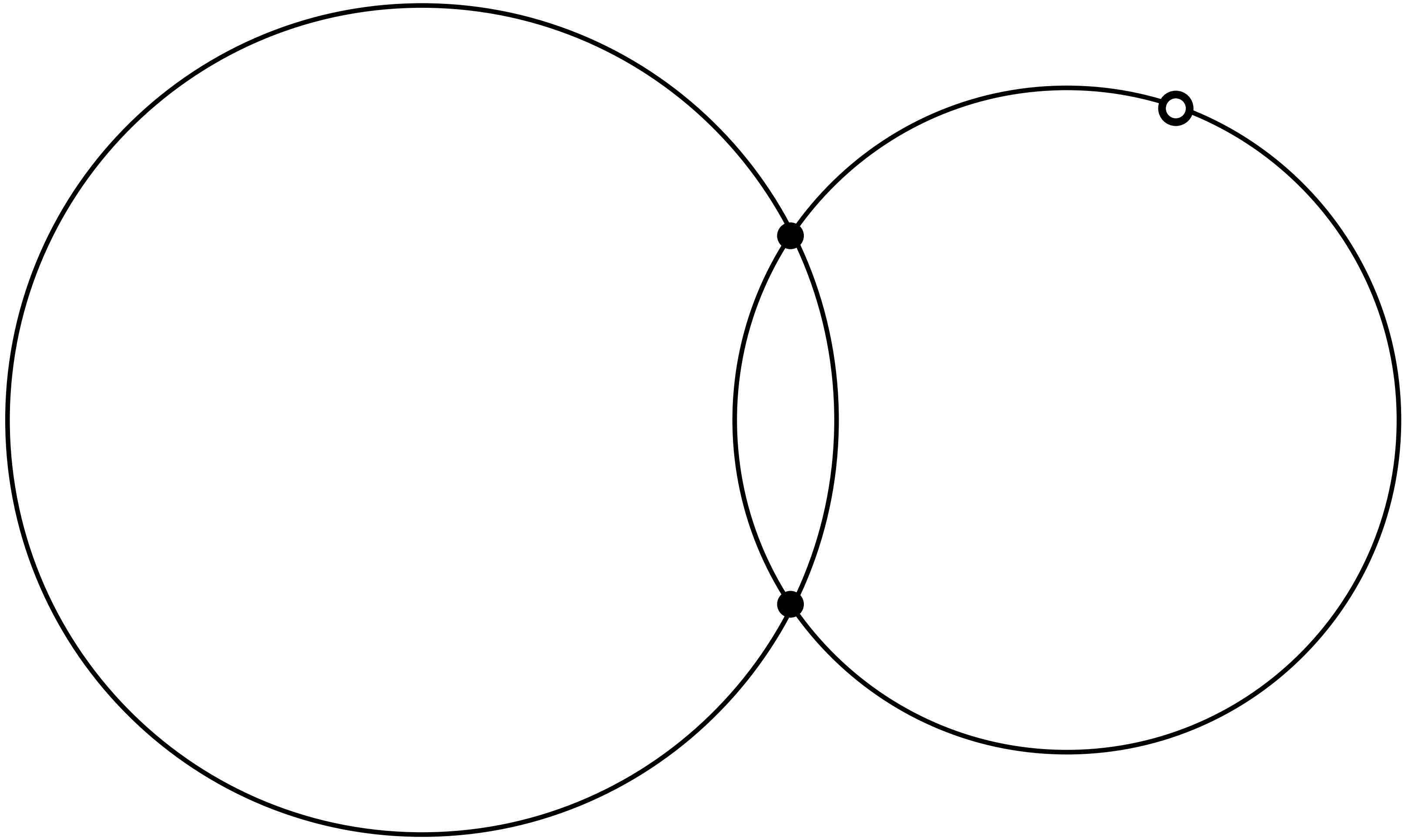
Intersecting circles:

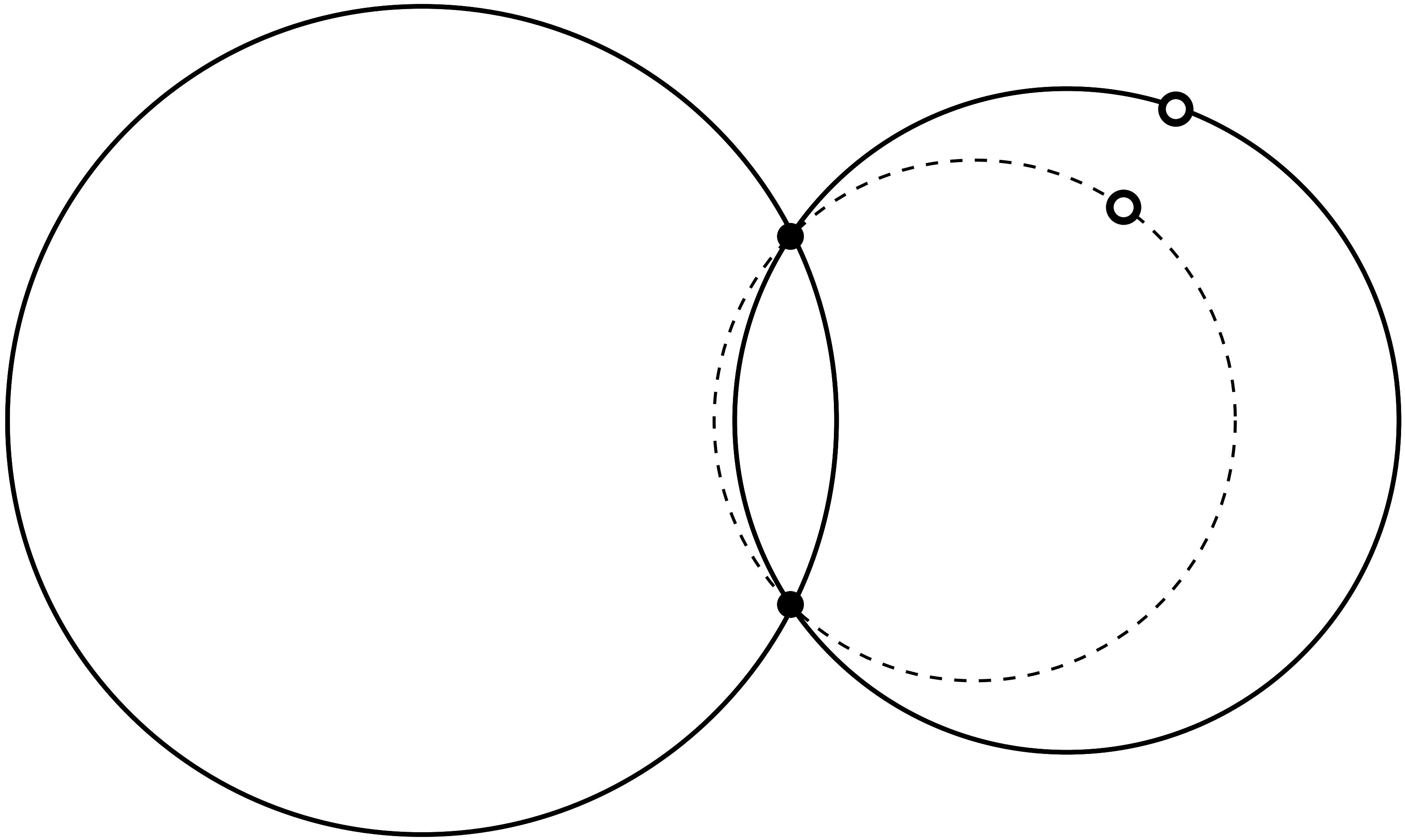


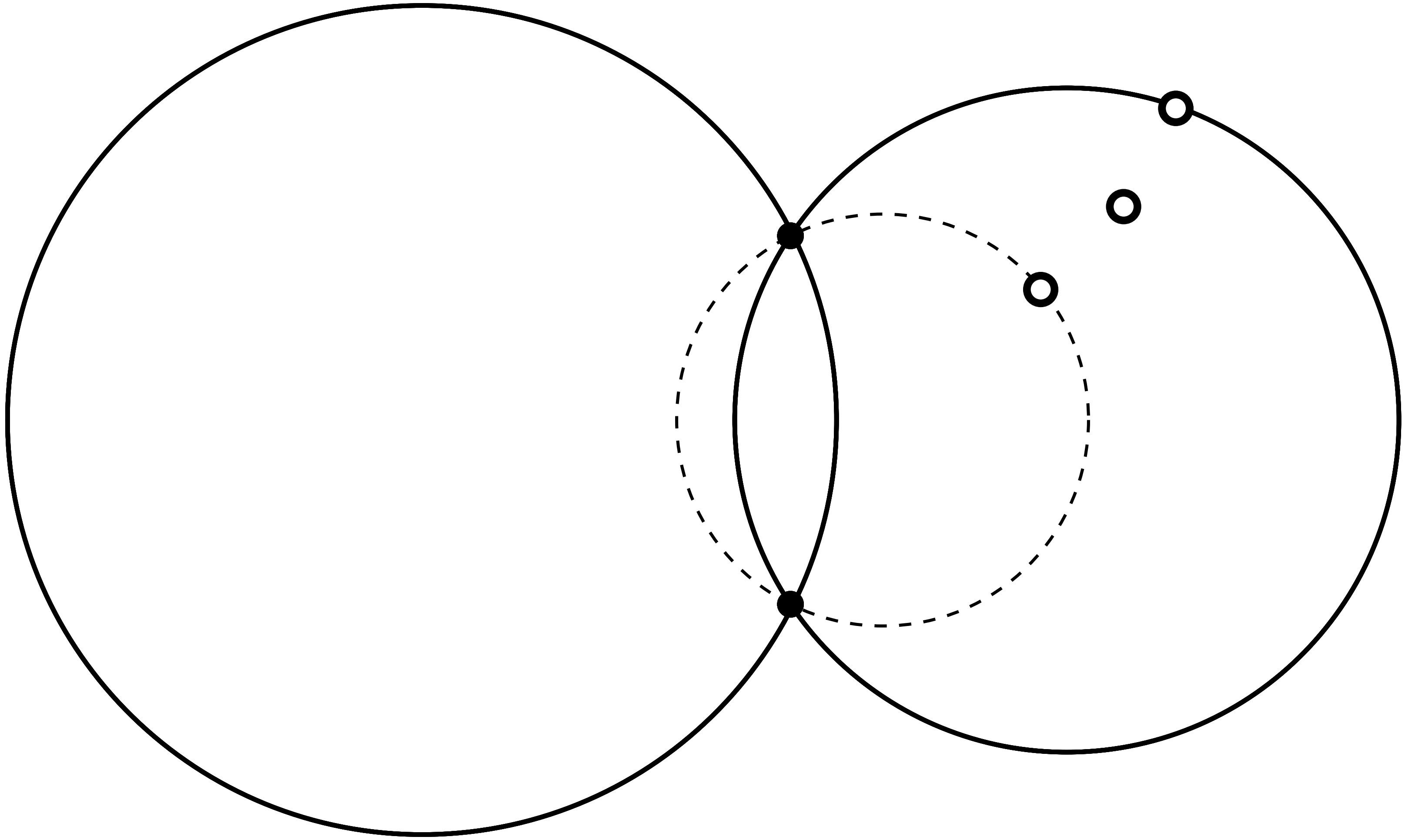
Fix intersection points a, b and map $c \to d$ as shown.

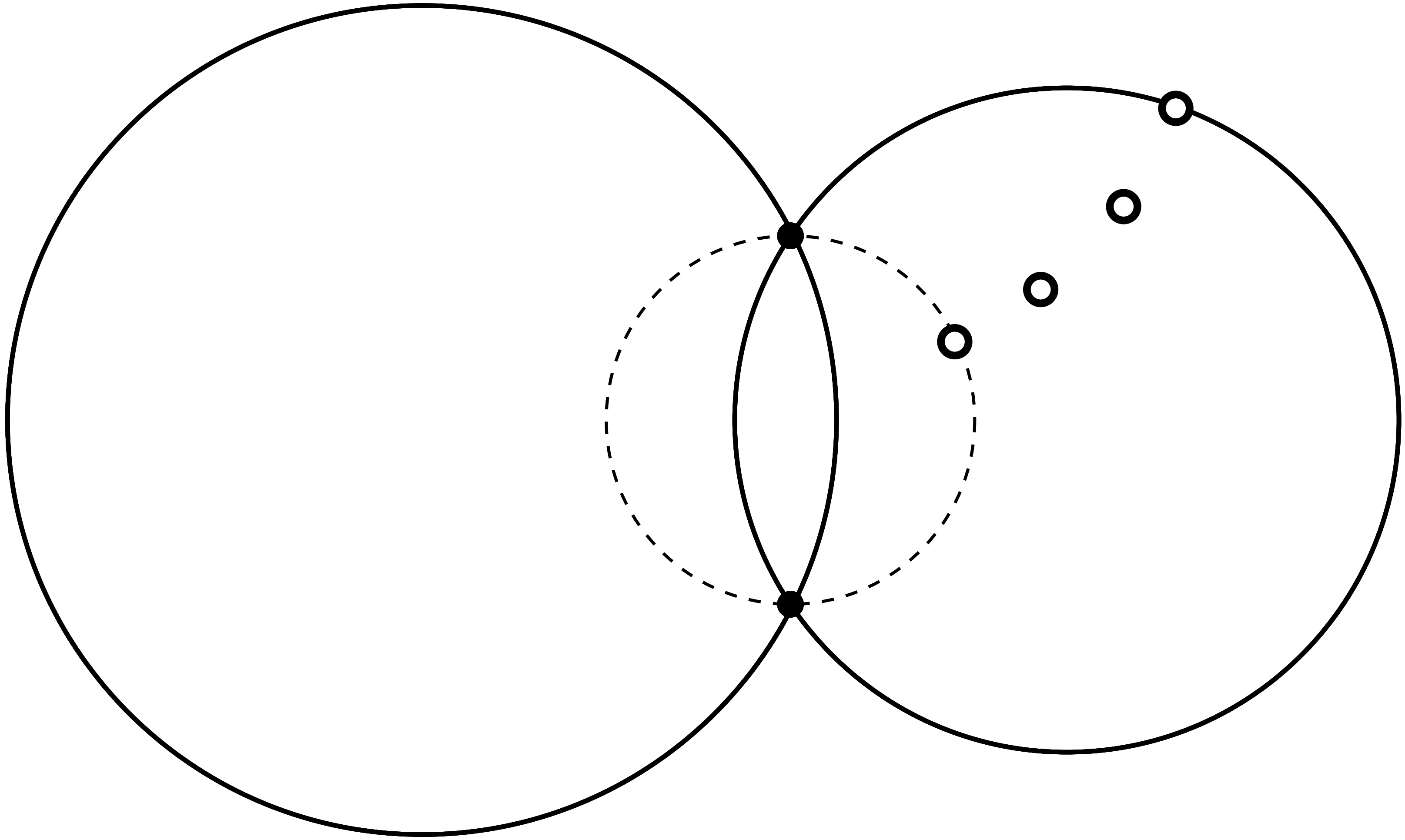
Determines unique Möbius map between disks.

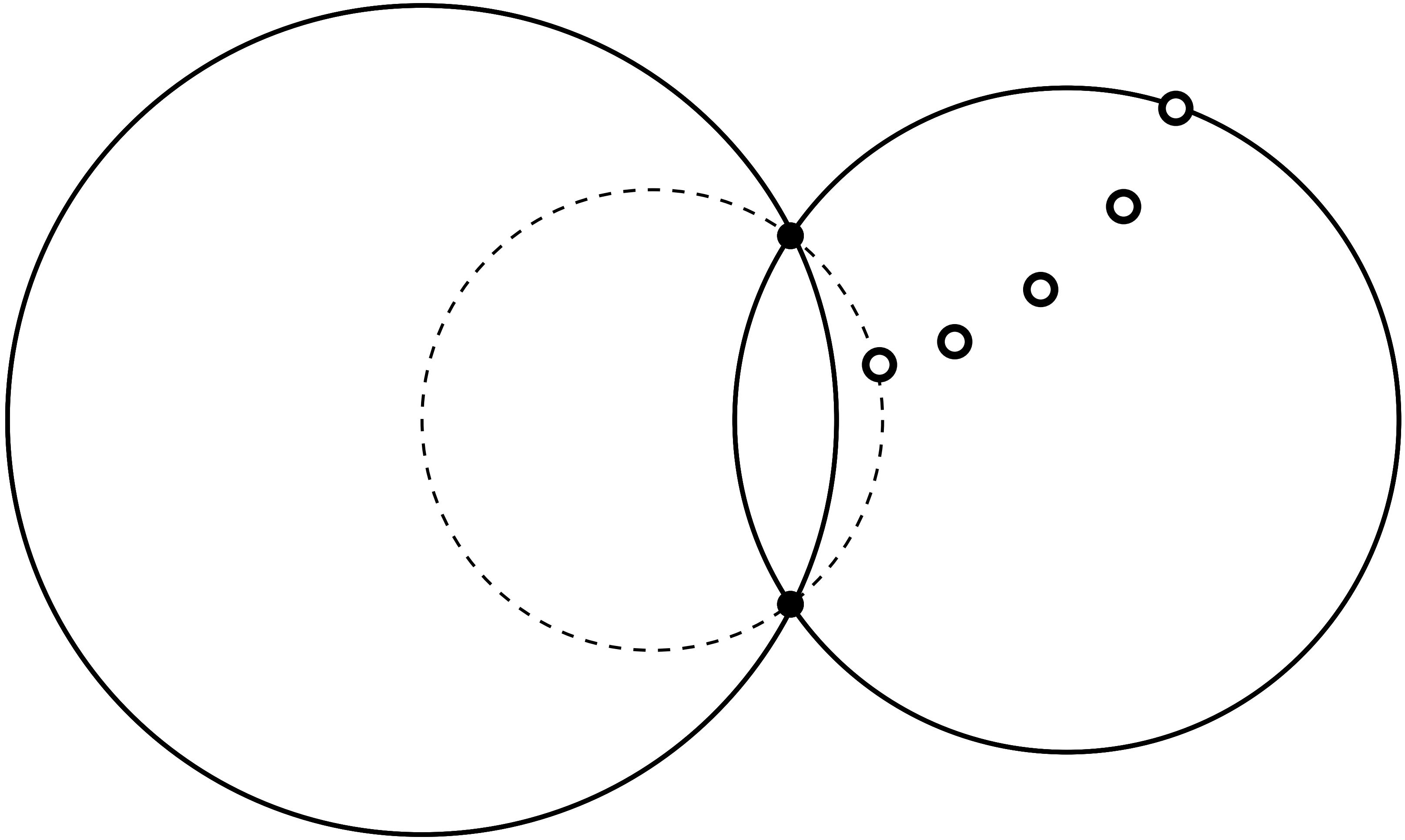
Part of 1-parameter symmetric family fixing a, b.

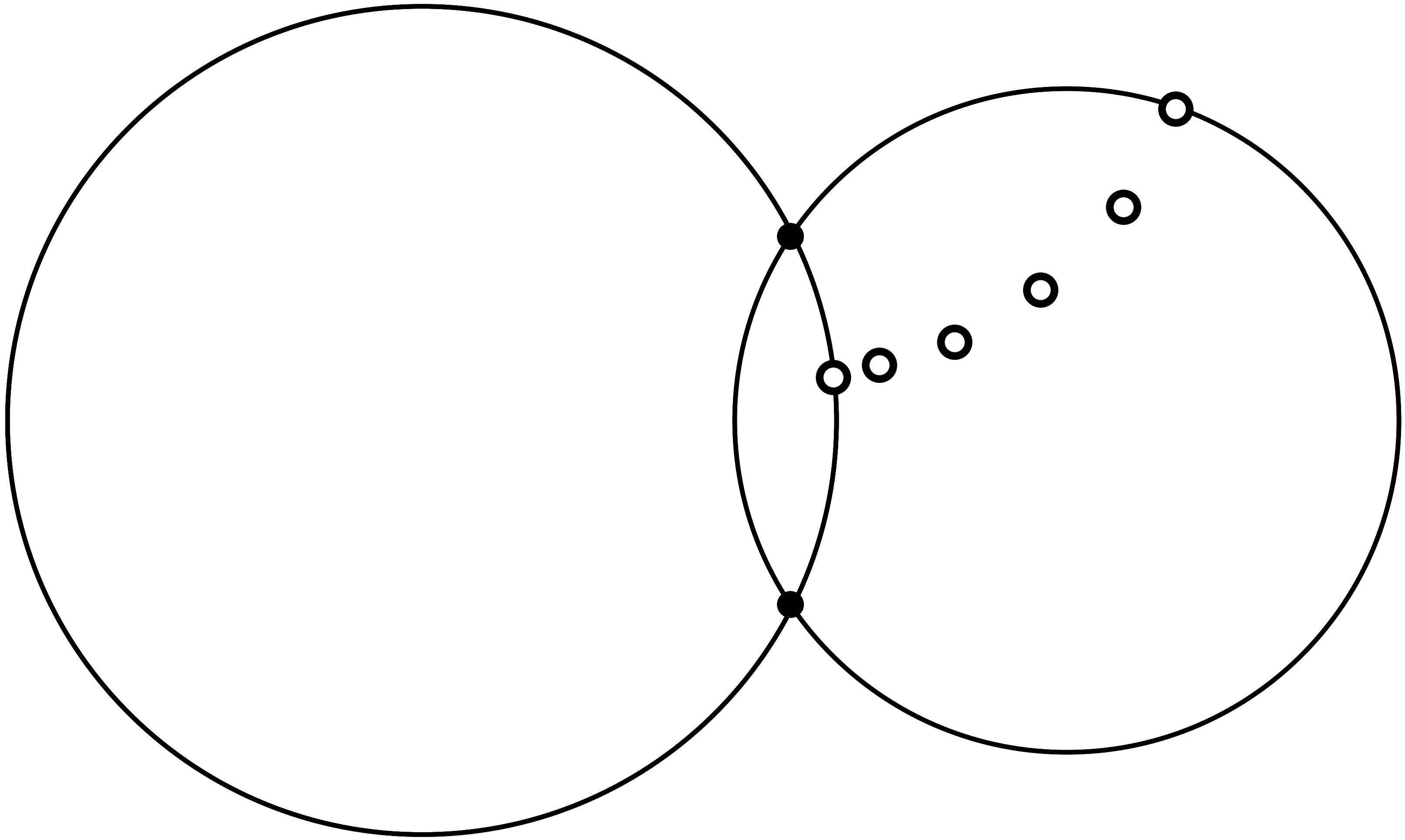


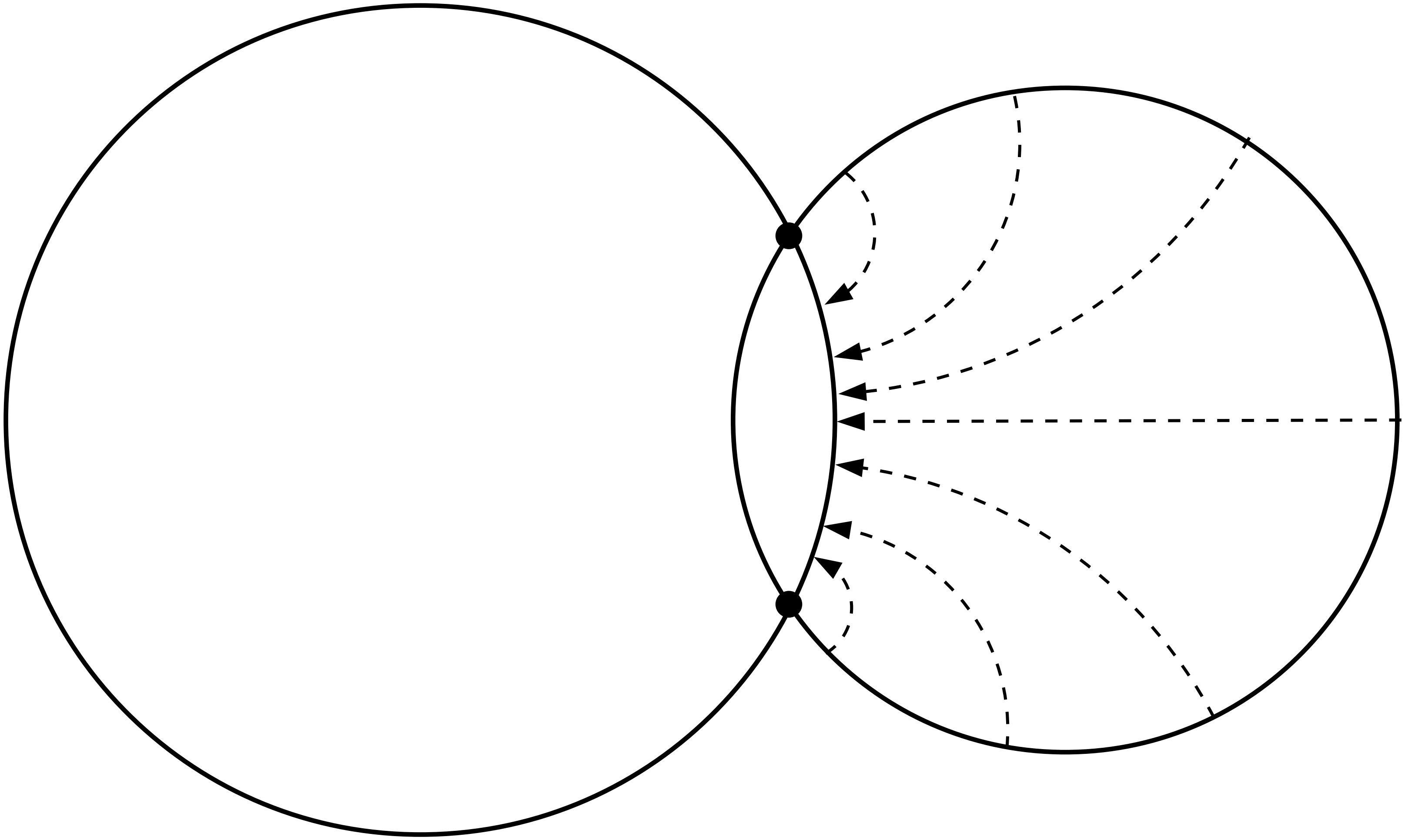






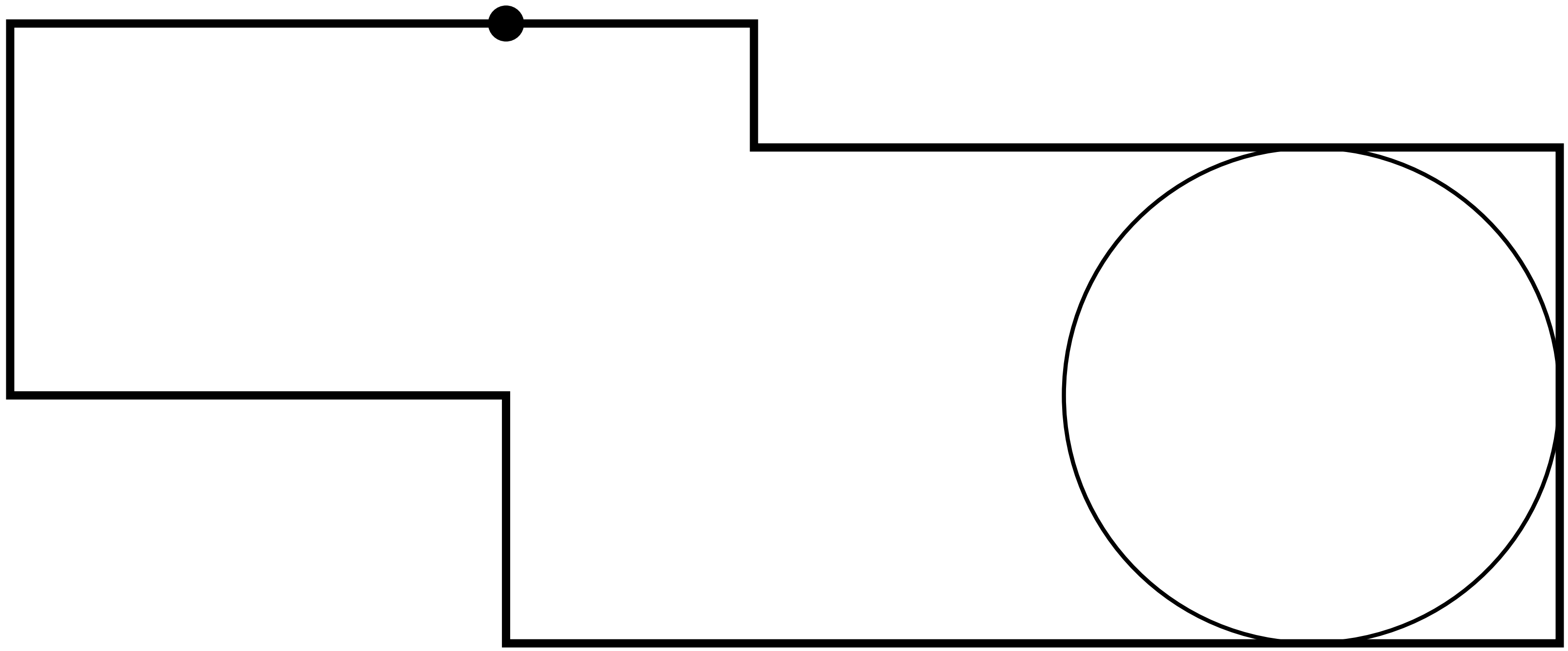




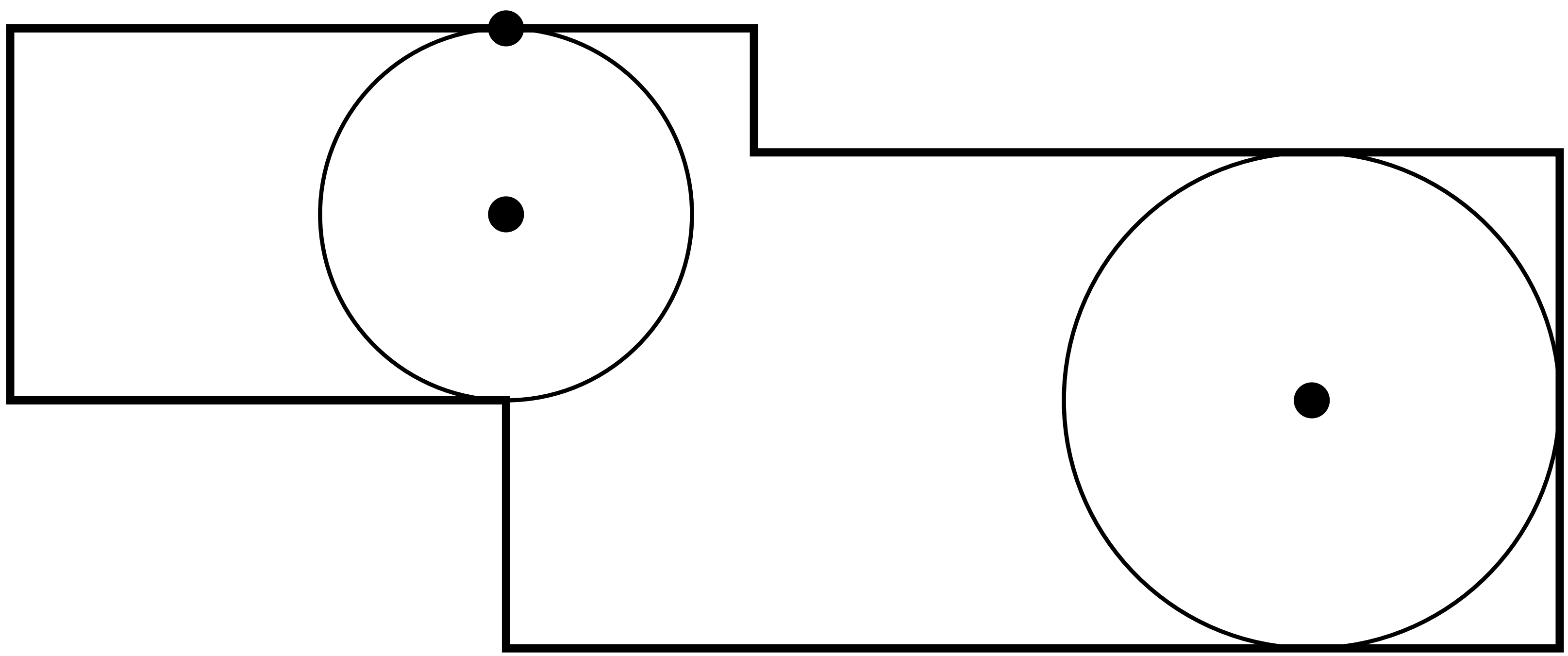


Points follow circular paths, perpendicular to boundary.

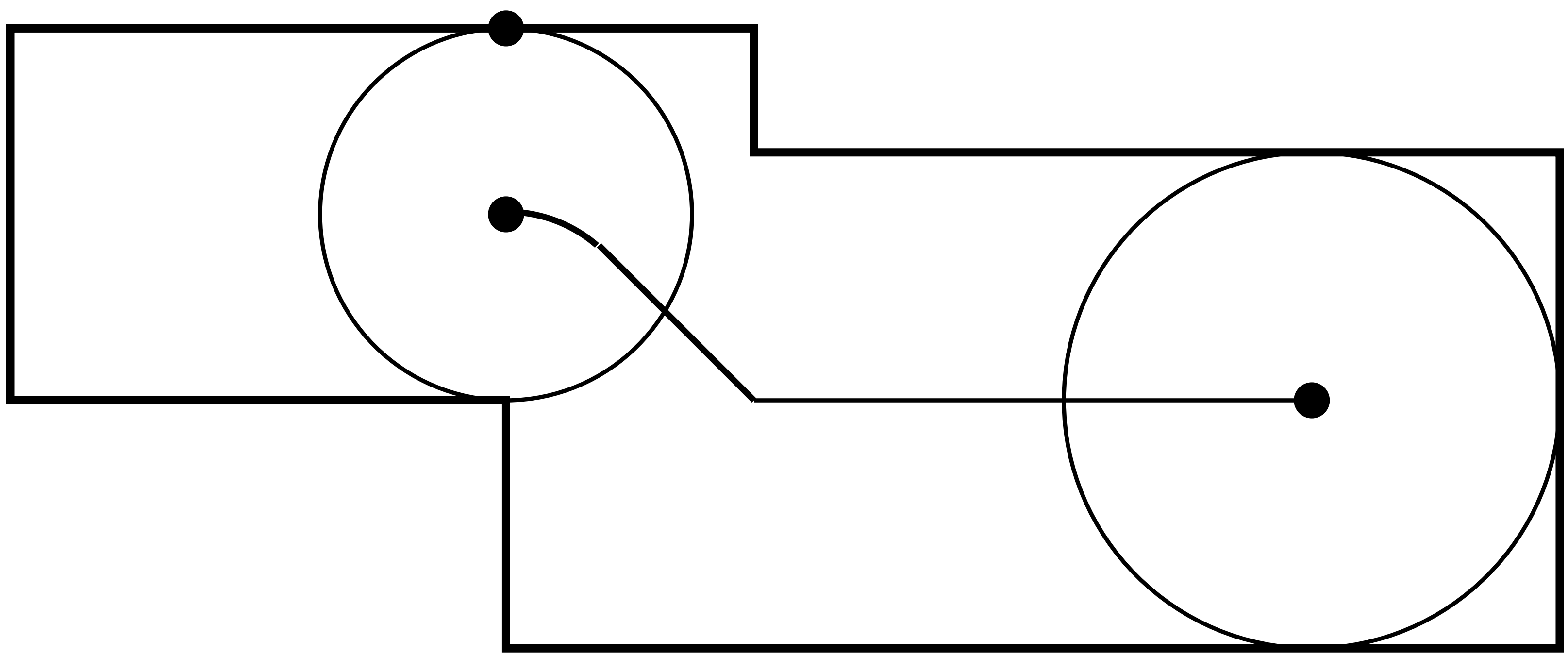
How does this give a map from polygon P to a circle?



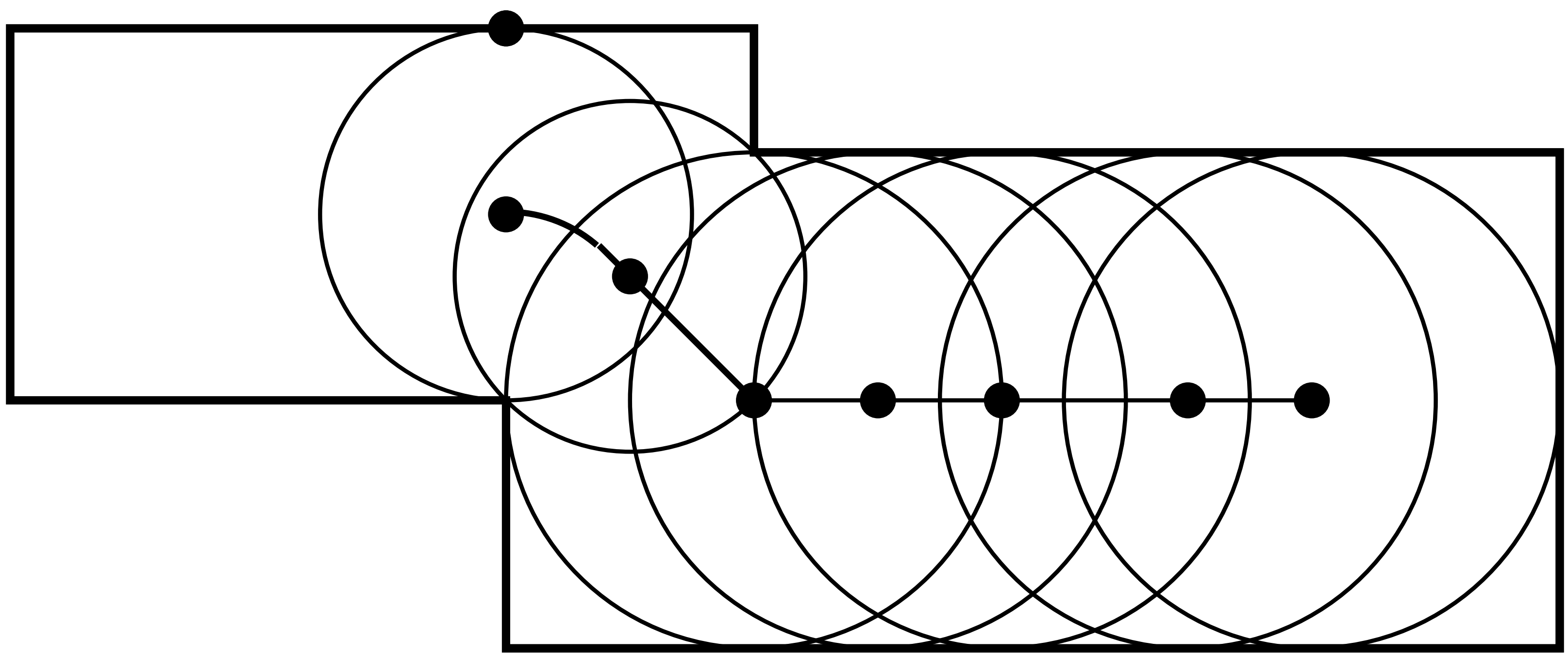
- Fix a “root” MA disk D.

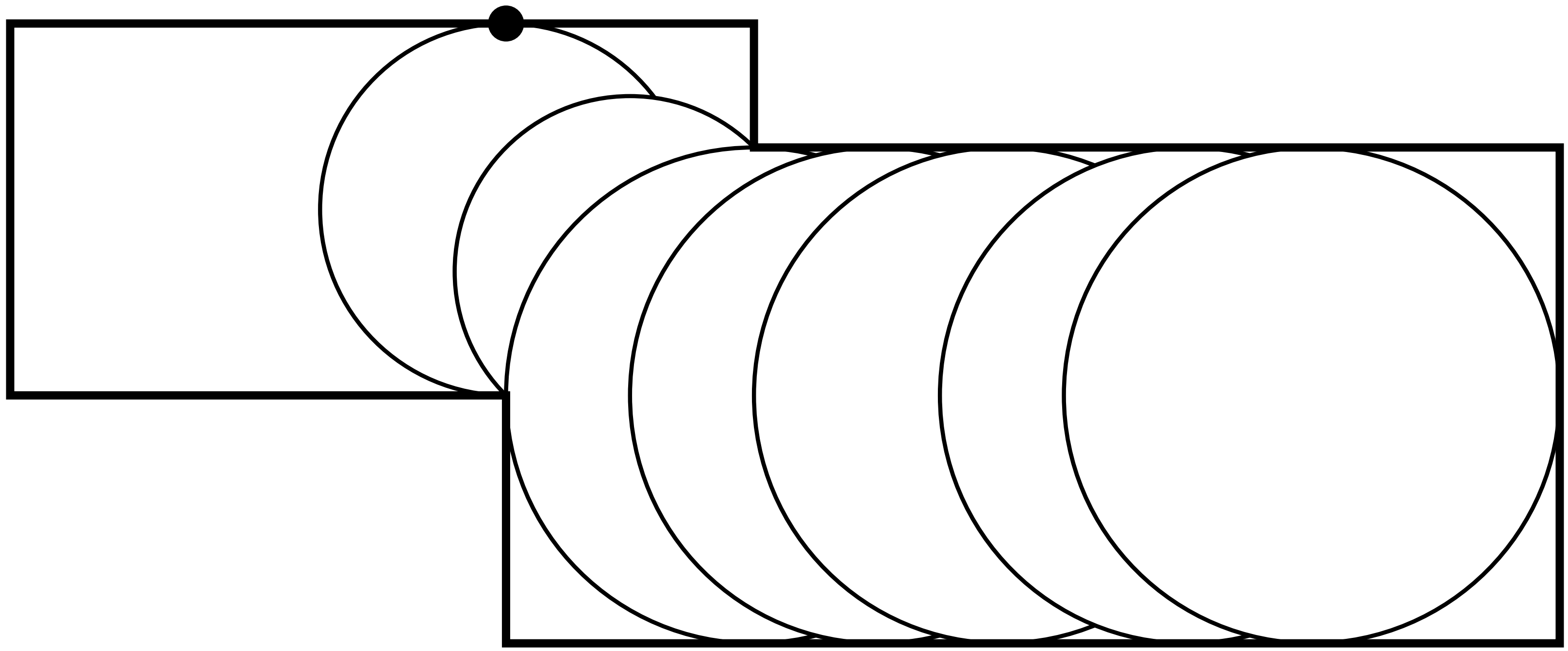


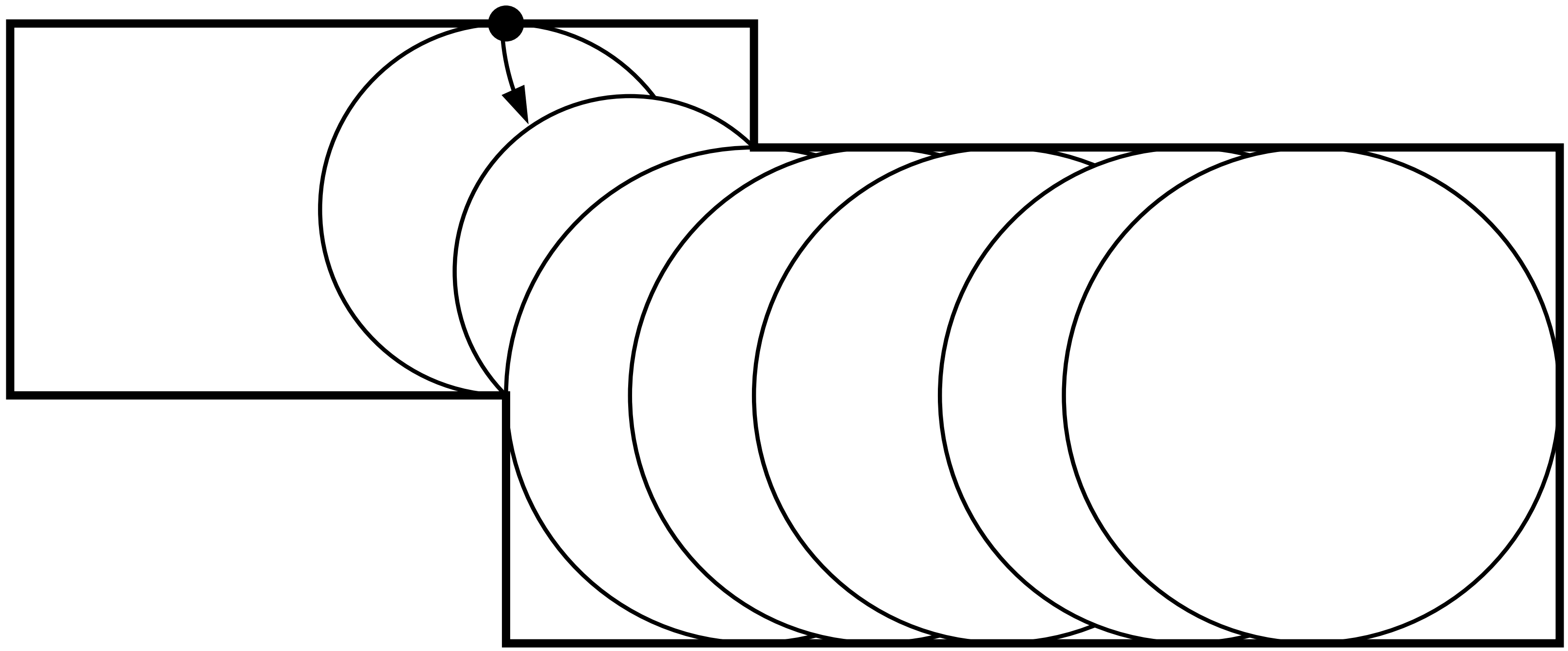
- For any $z \in P$, take MA disk D_z touching z.

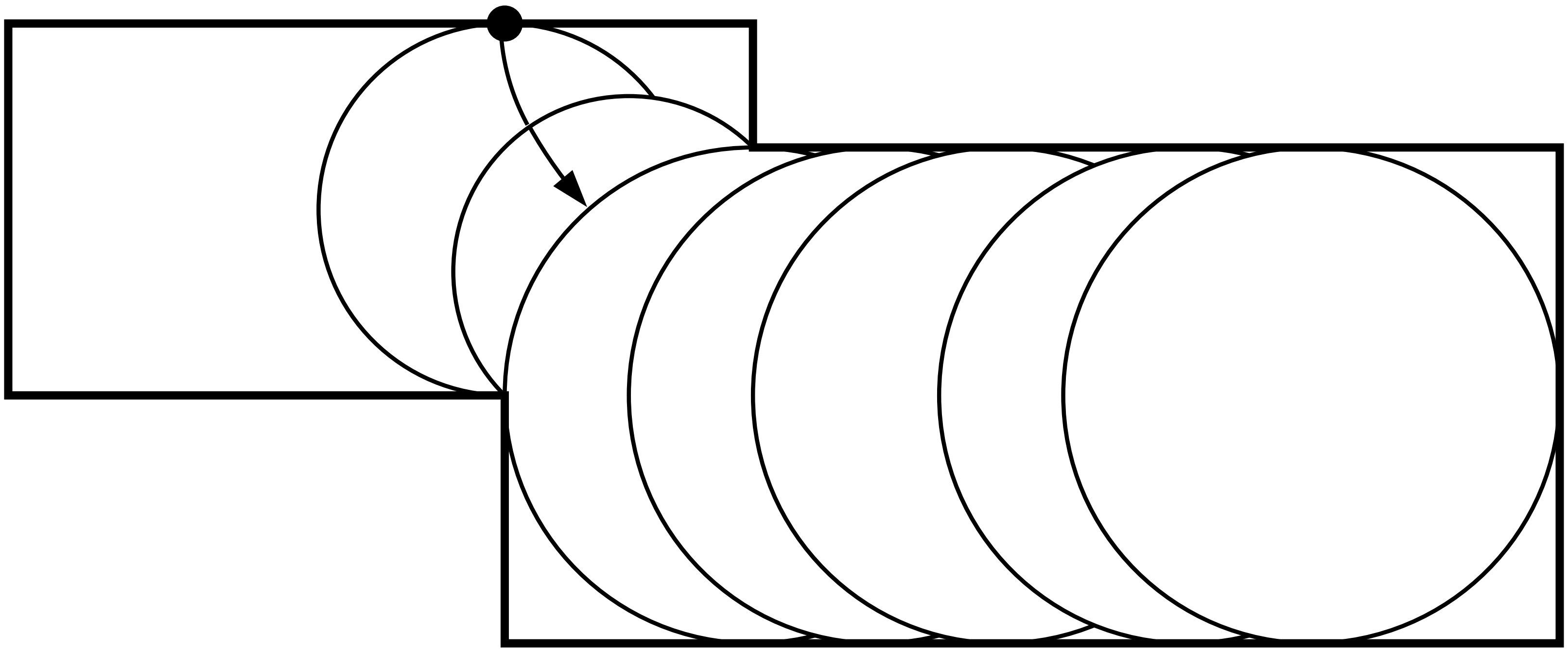


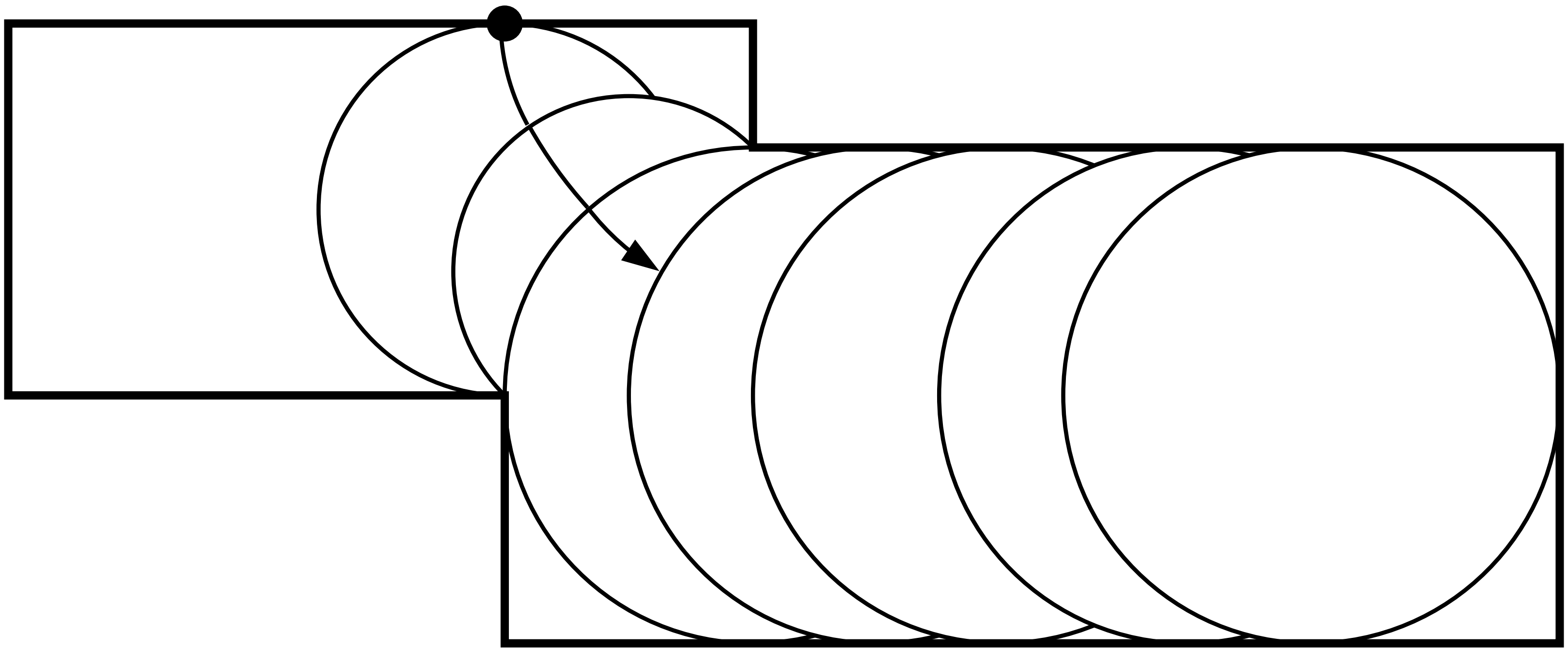
- Connect D_z to D on MA.

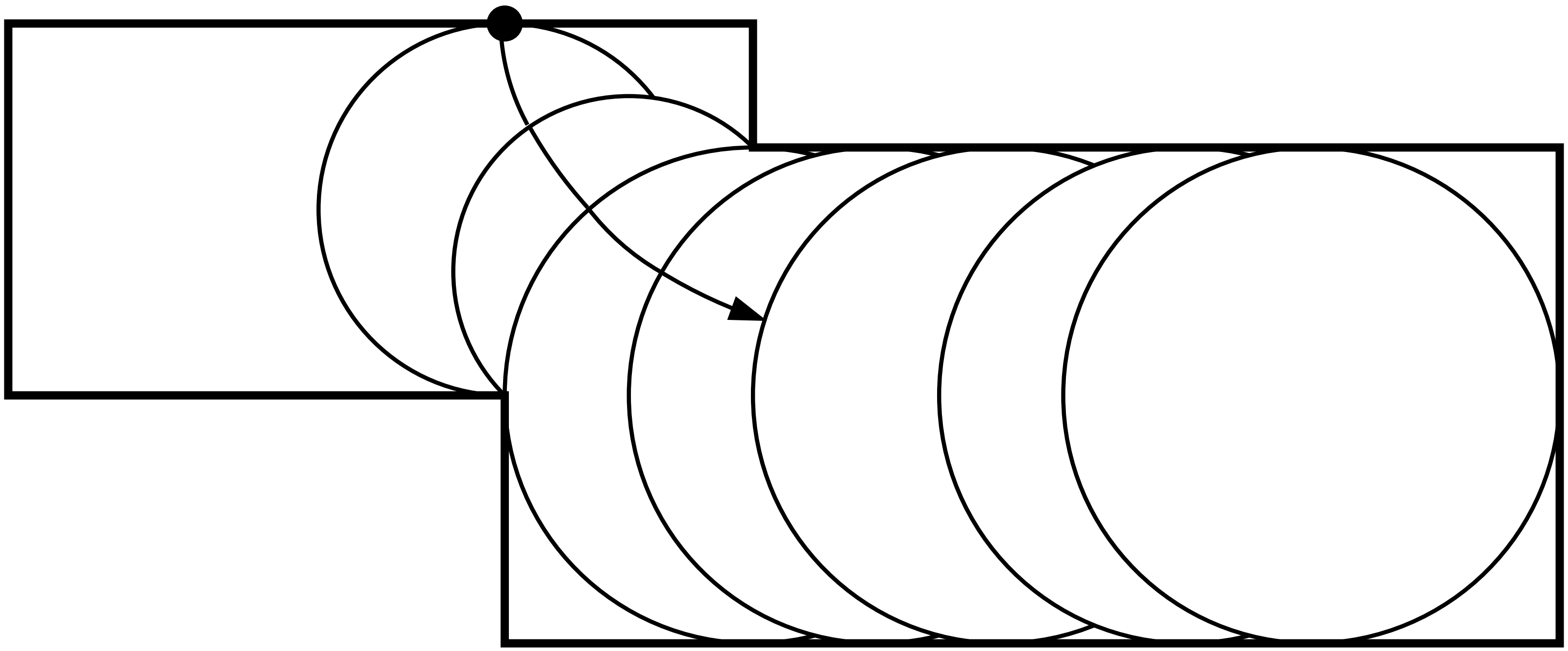


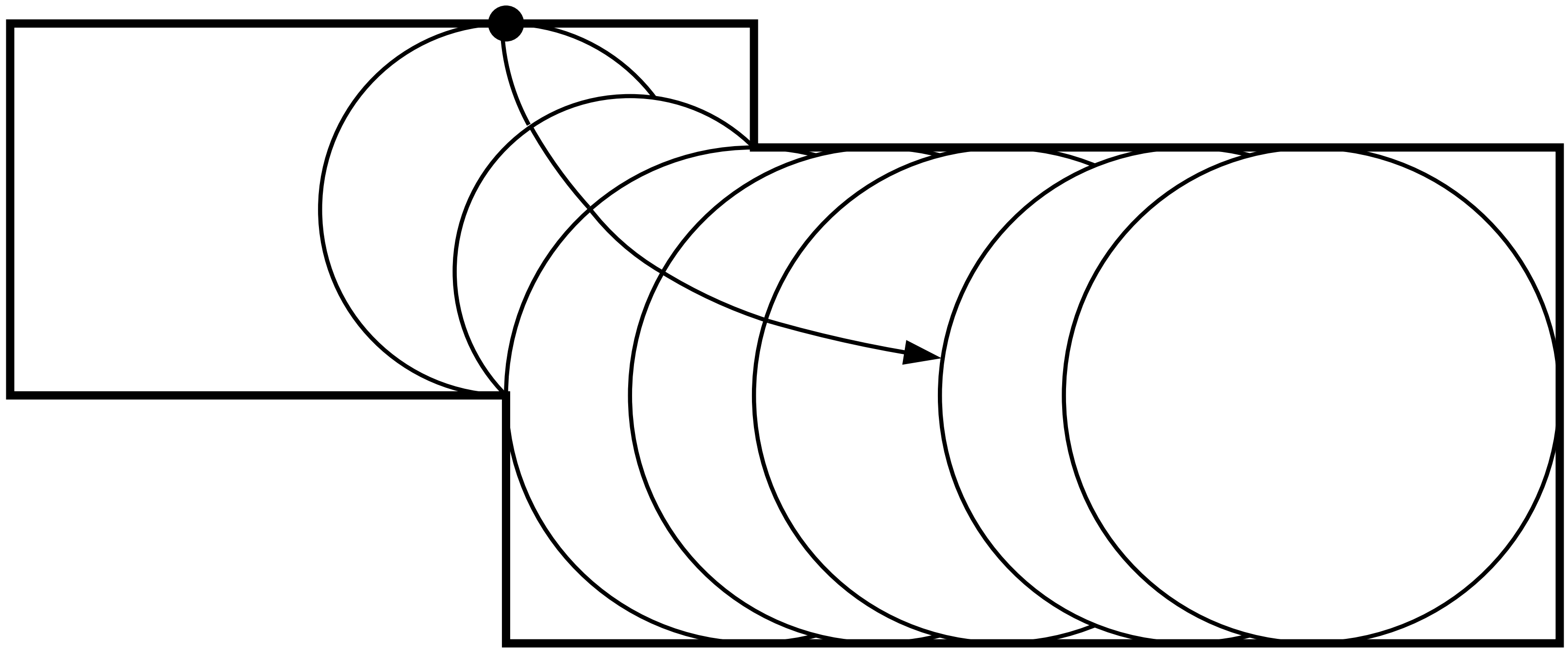


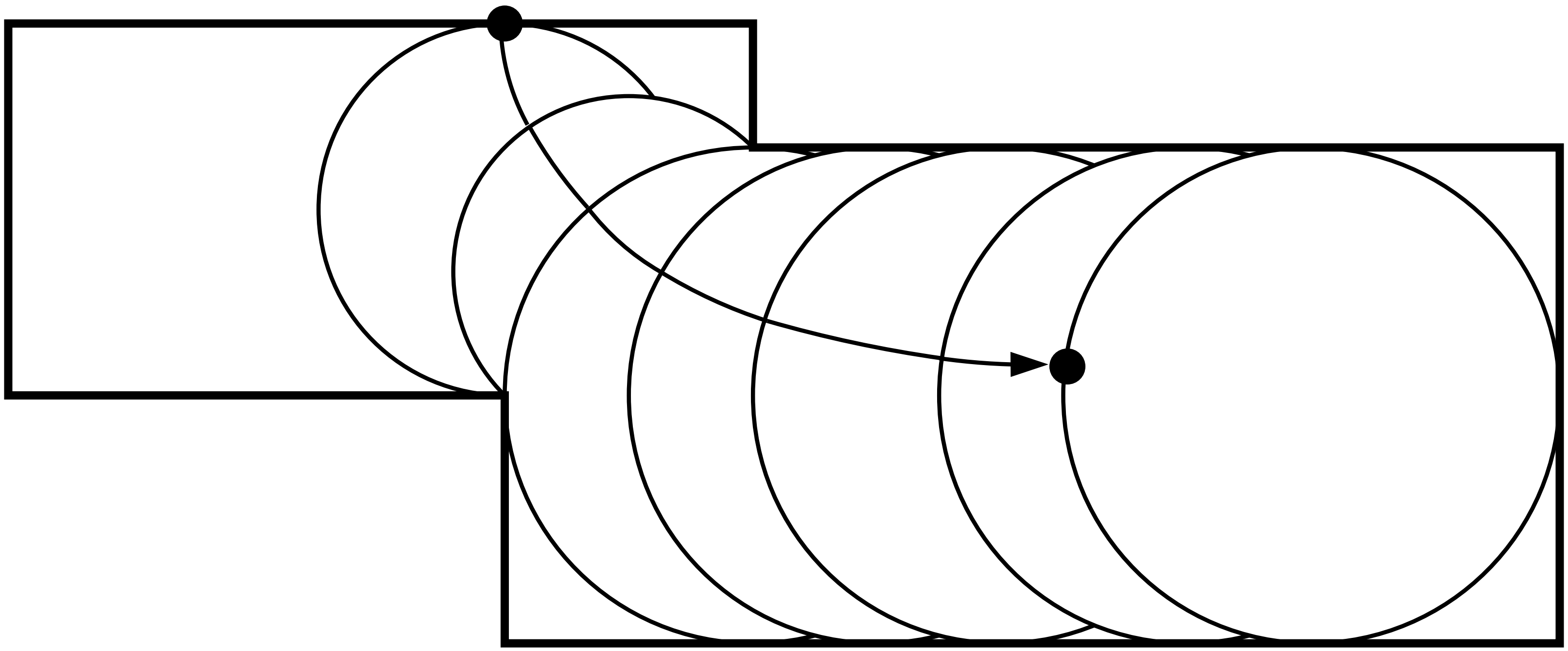


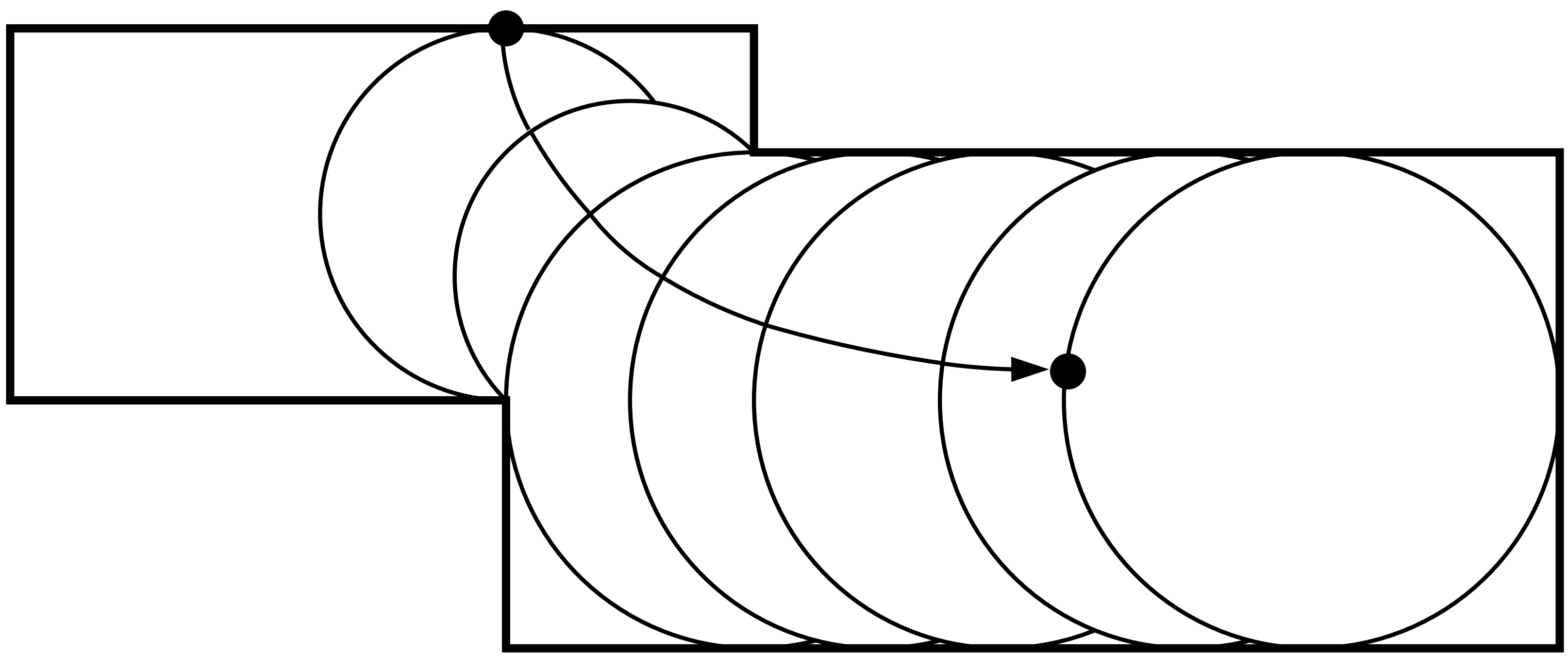






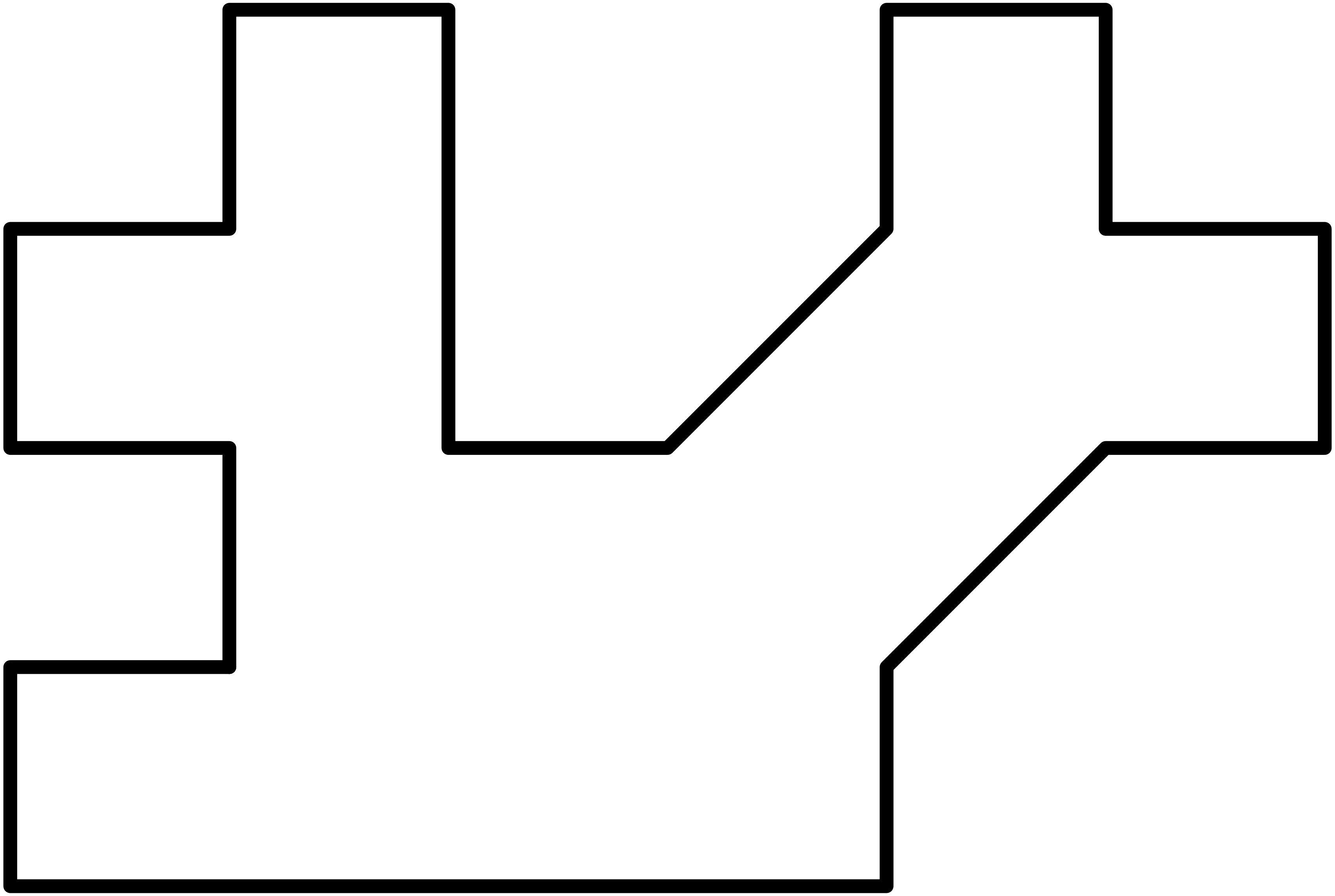


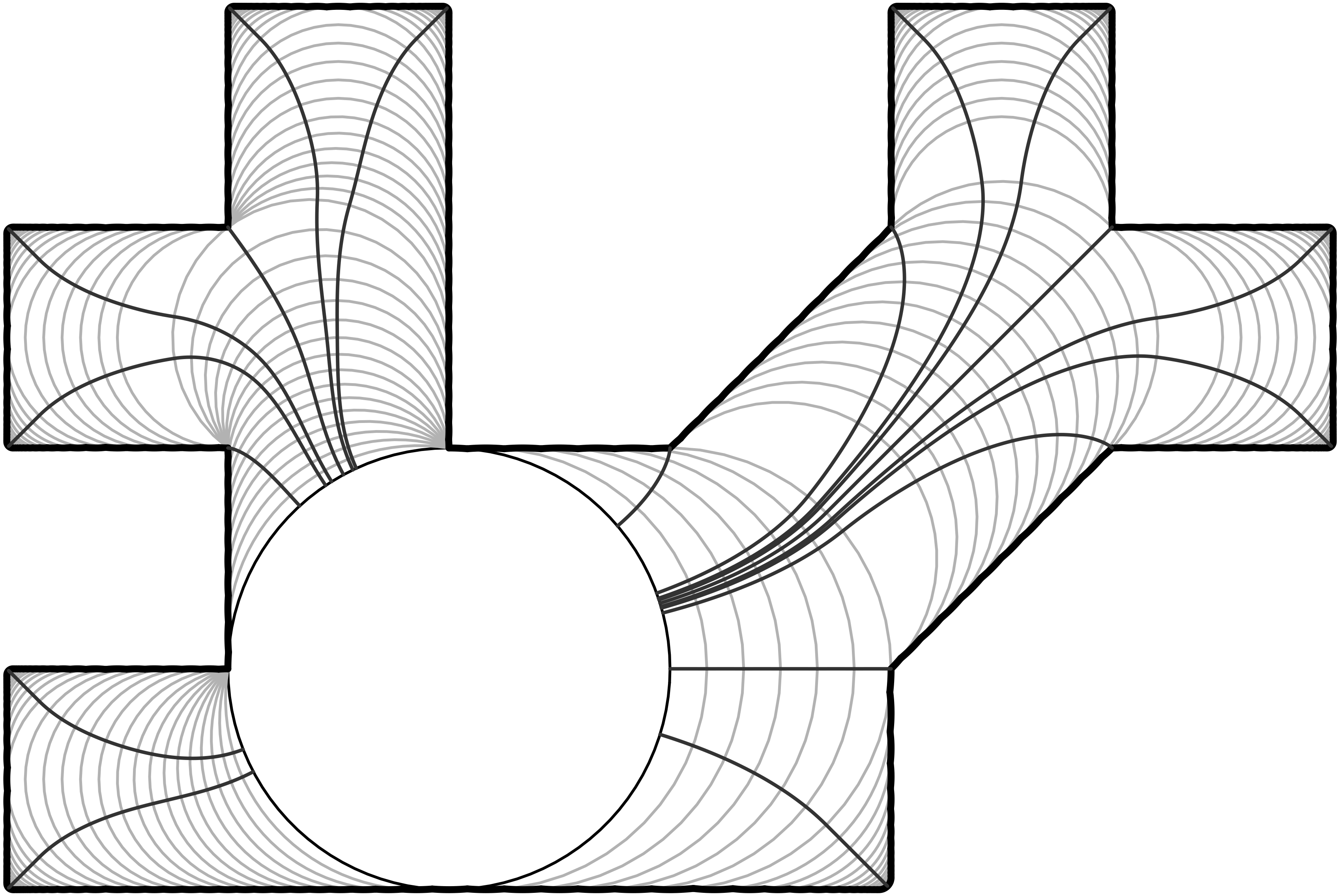


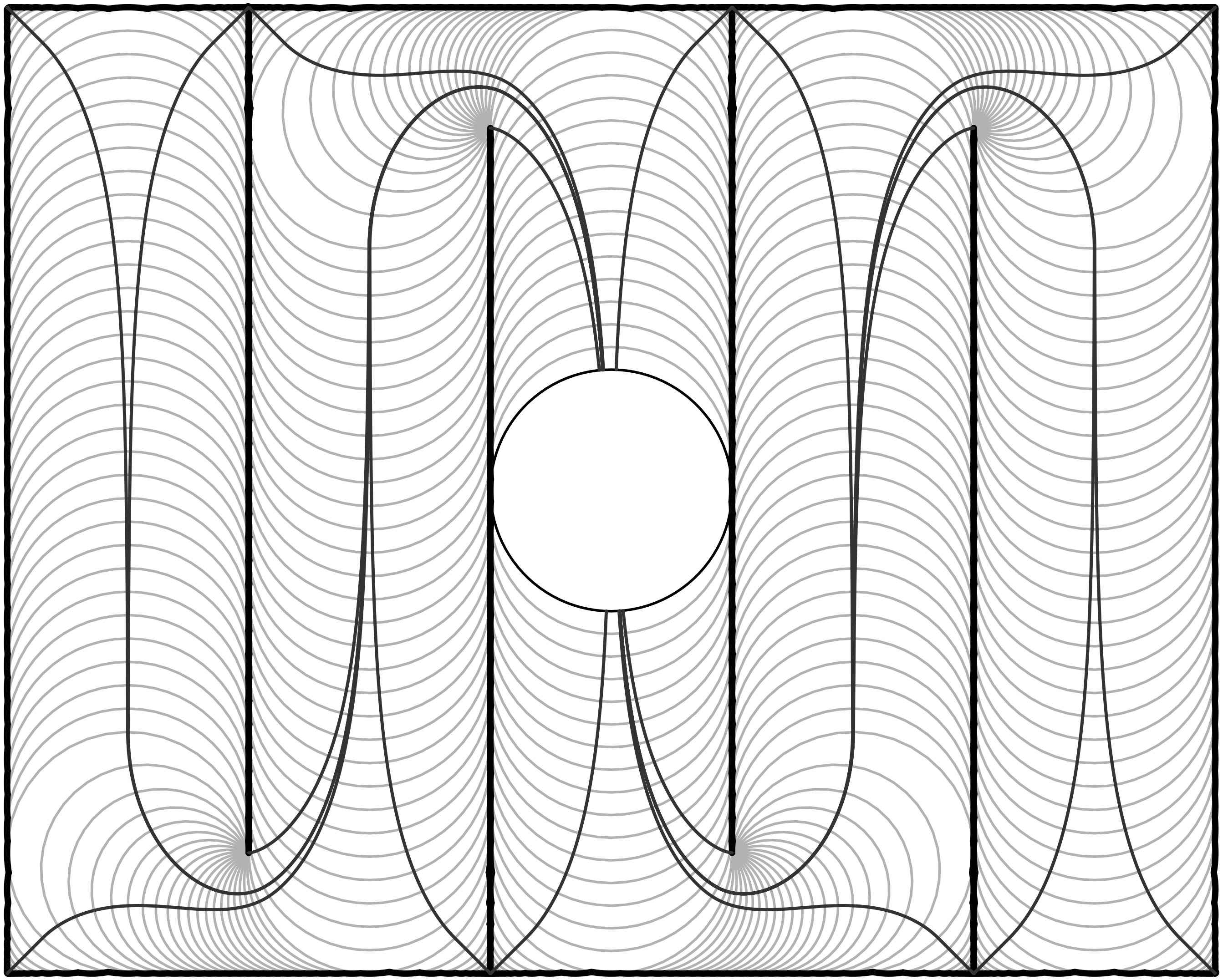


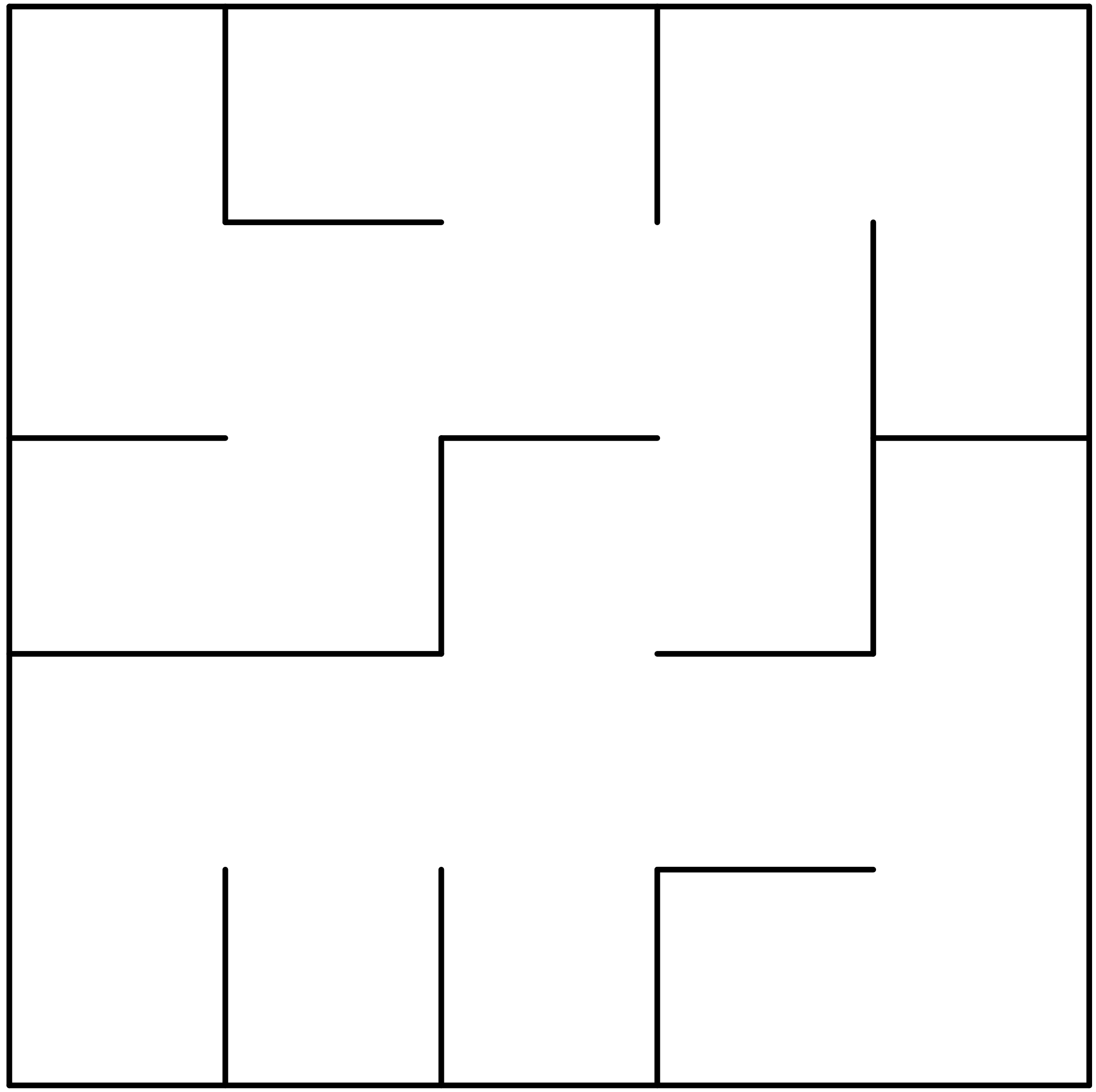
We discretize only to draw picture.

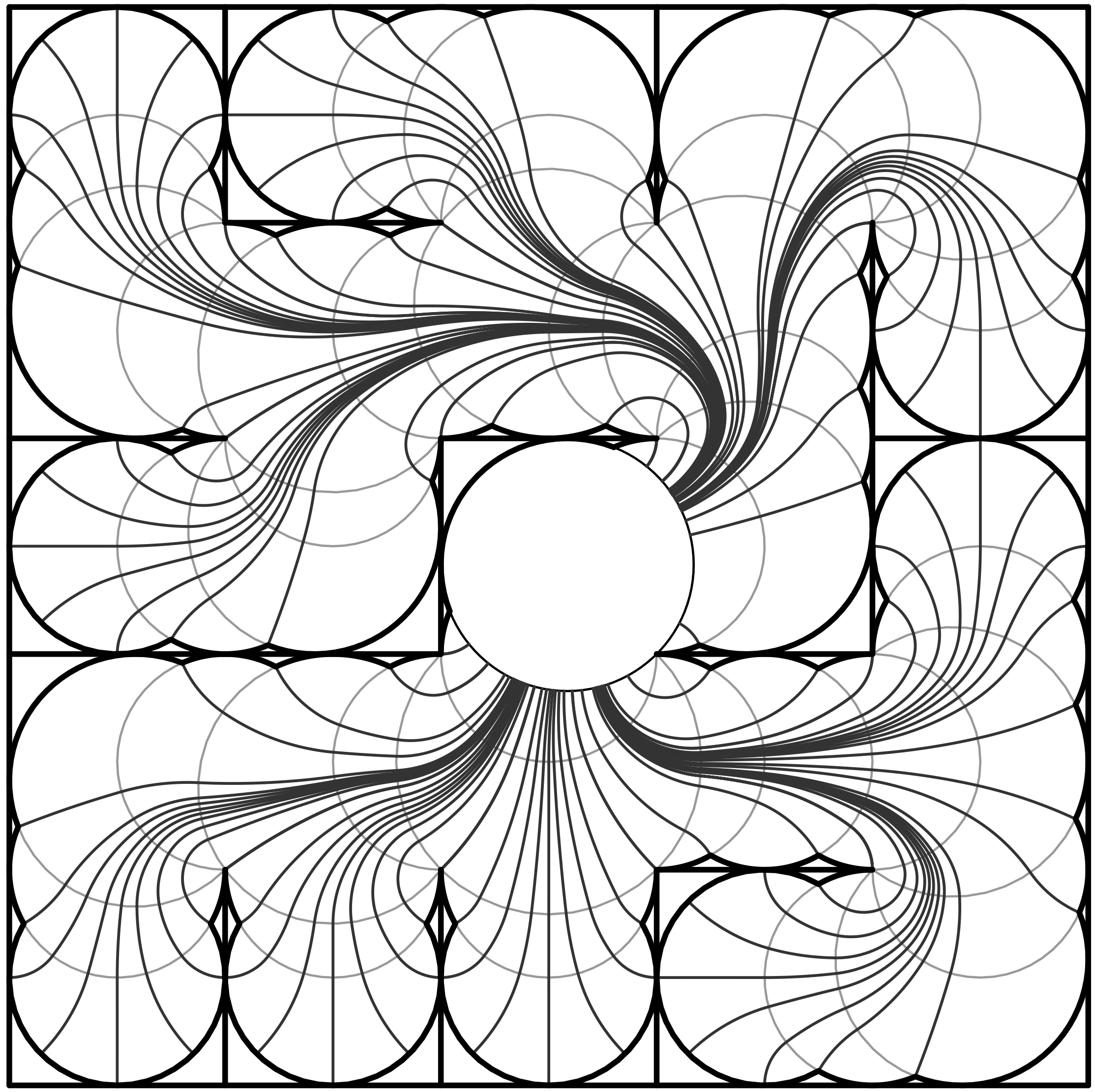
Limiting map has **formula** in terms of medial axis.





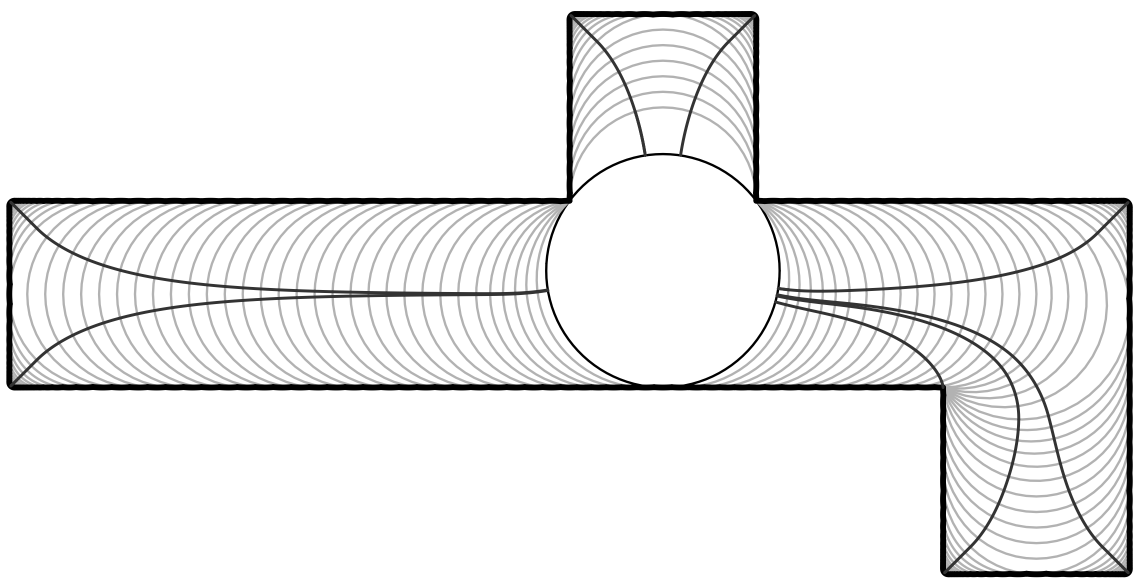






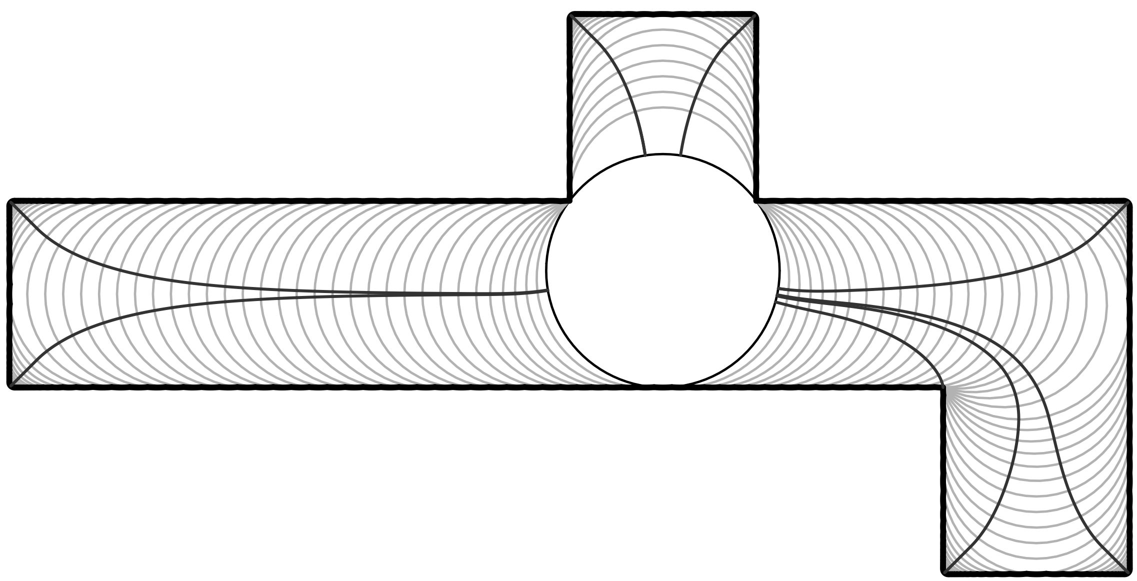
Theorem: Mapping all n vertices takes $O(n)$ time.

Uses linear time computation of MA (Chin-Snoeyink-Wang) and book-keeping with cross ratios.



Theorem: Medial axis flow is good approximation of conformal map.

Boundary map is quasiconformal with uniform bound. Proof motivated by hyperbolic 3-manifolds (convex cores, pleated surfaces).

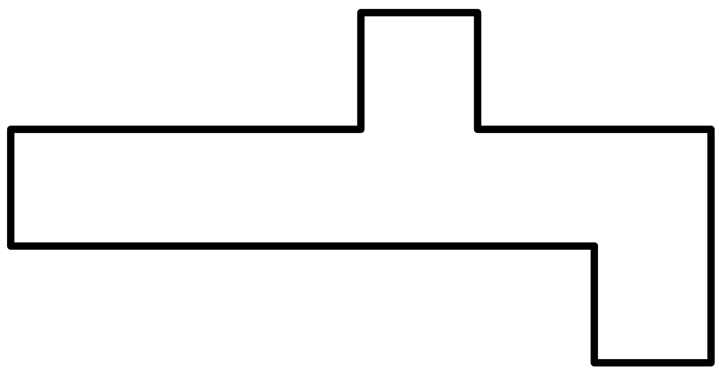


Some Euclidean problems are best approached via hyperbolic geometry.

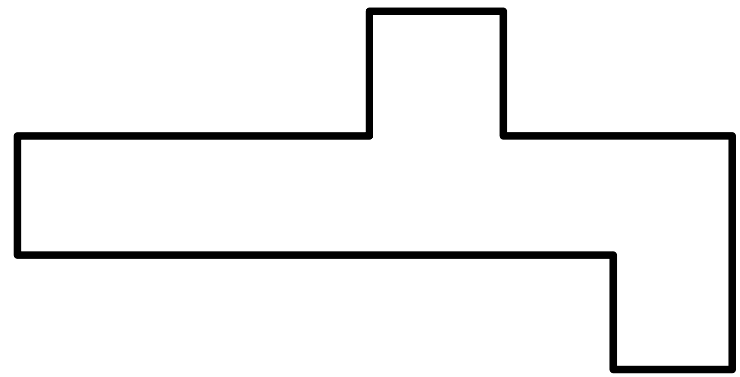
How close is medial axis map to conformal map?

How close is medial axis map to conformal map?

Use “MA-parameters” in Schwarz-Christoffel formula.

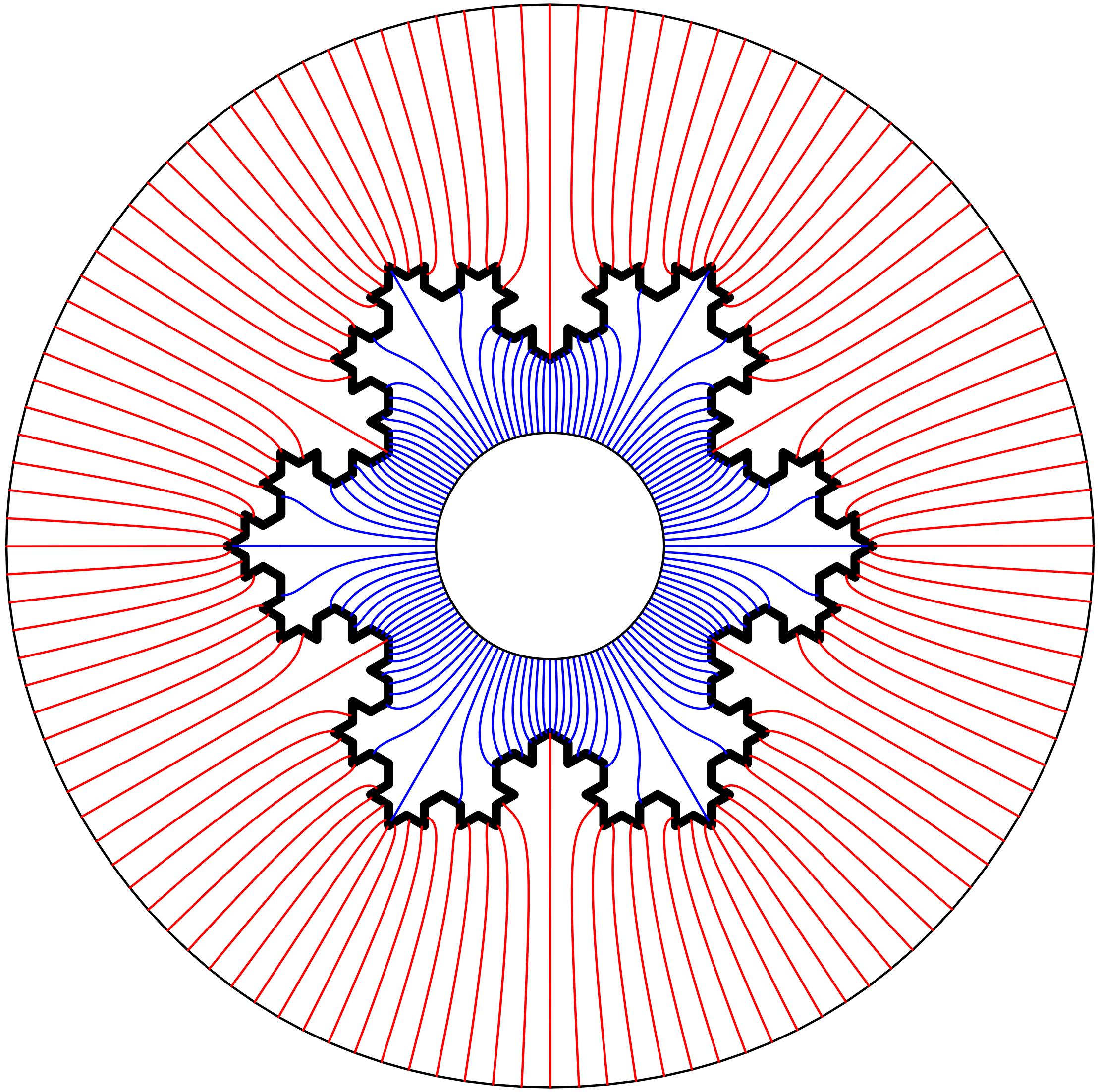


Target Polygon



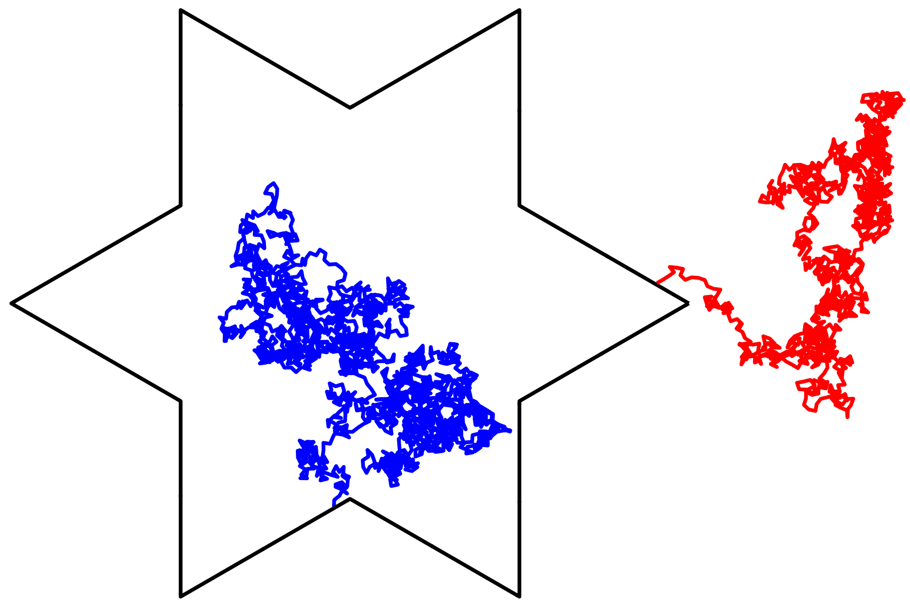
MA Parameters

SINGULAR MEASURES



Thm (F & M Riesz 1916):

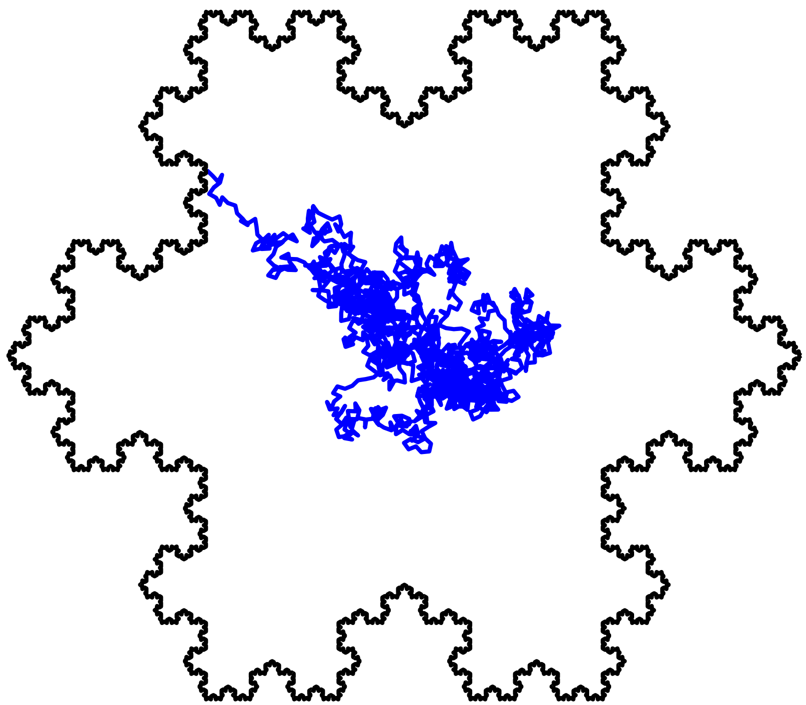
For rectifiable boundaries, $\omega(E) = 0$ iff E has zero length.



"Inside" and "outside" harmonic measures have same null sets.

Measures are mutually absolutely continuous. Same measure class.

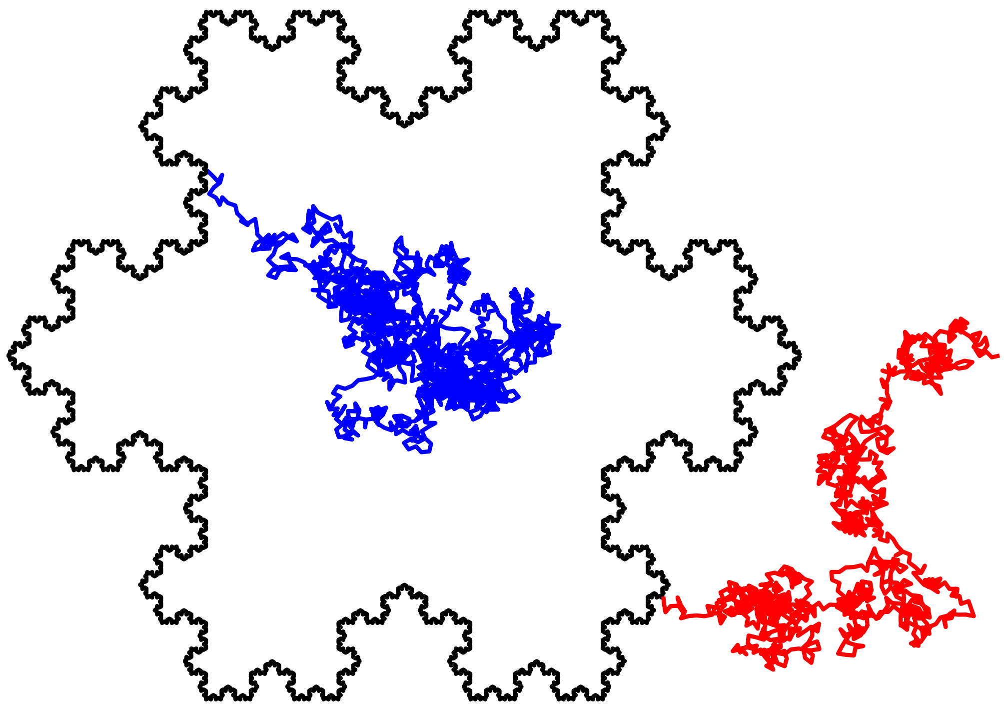
Thm (Makarov 1985):
For fractal domains, ω gives full measure to a set of zero length.



First such examples due to Lavrentiev (1936).

Thm (Makarov 1985):

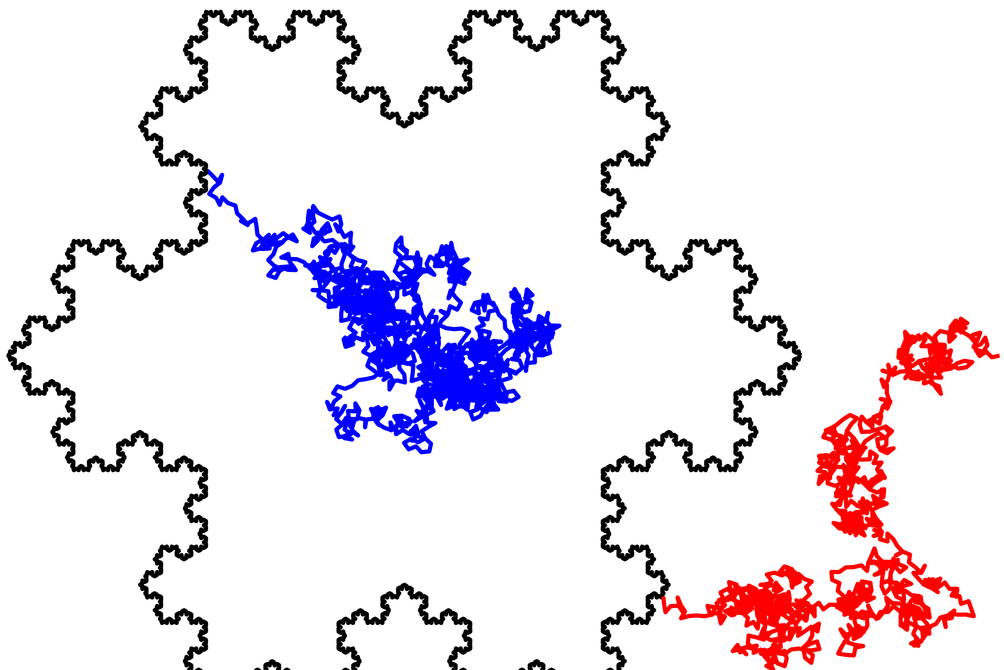
For fractal domains, ω gives full measure to a set of zero length.



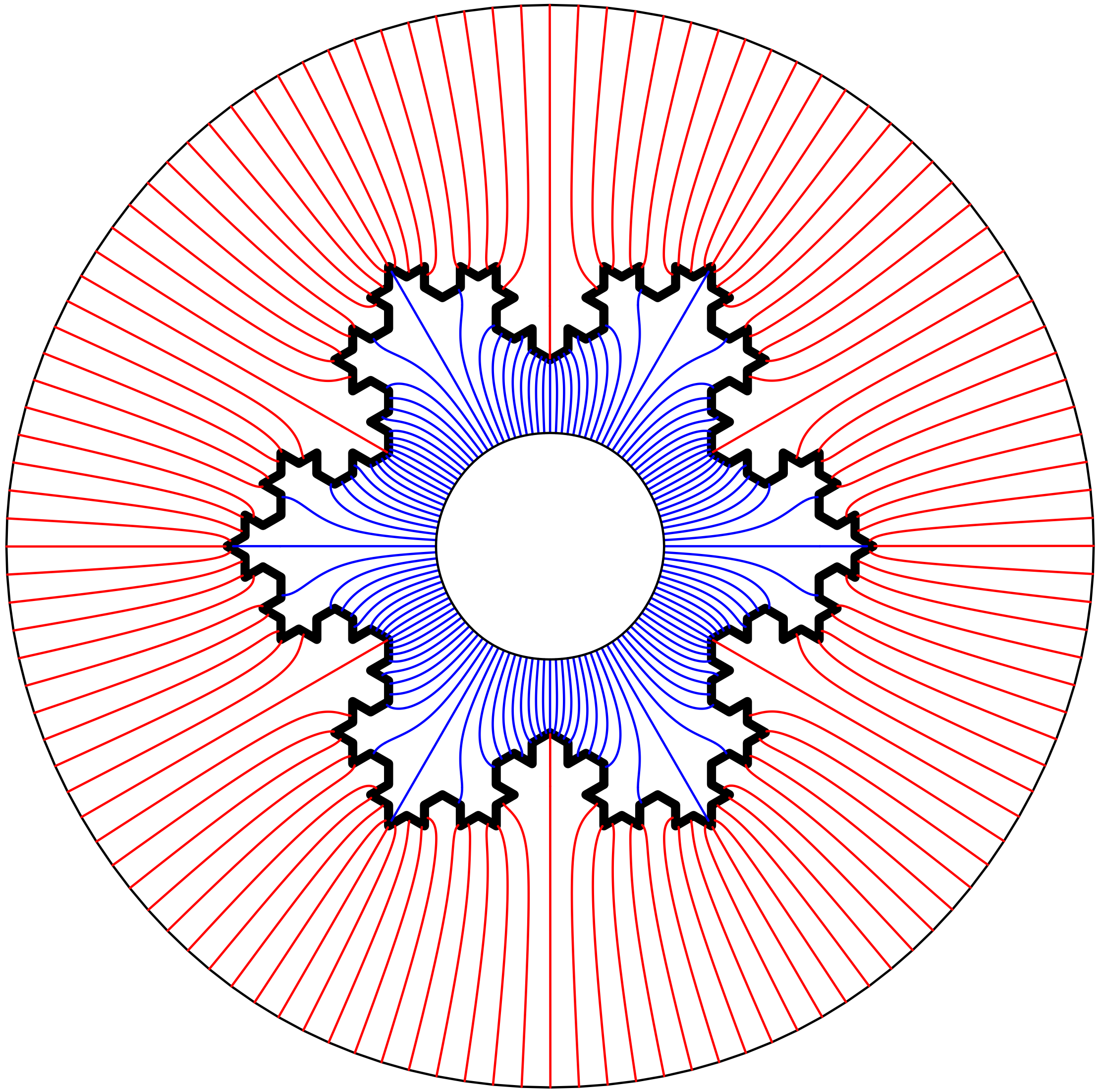
Outside is also fractal. Same set of length zero?

Theorem (B. 1987):

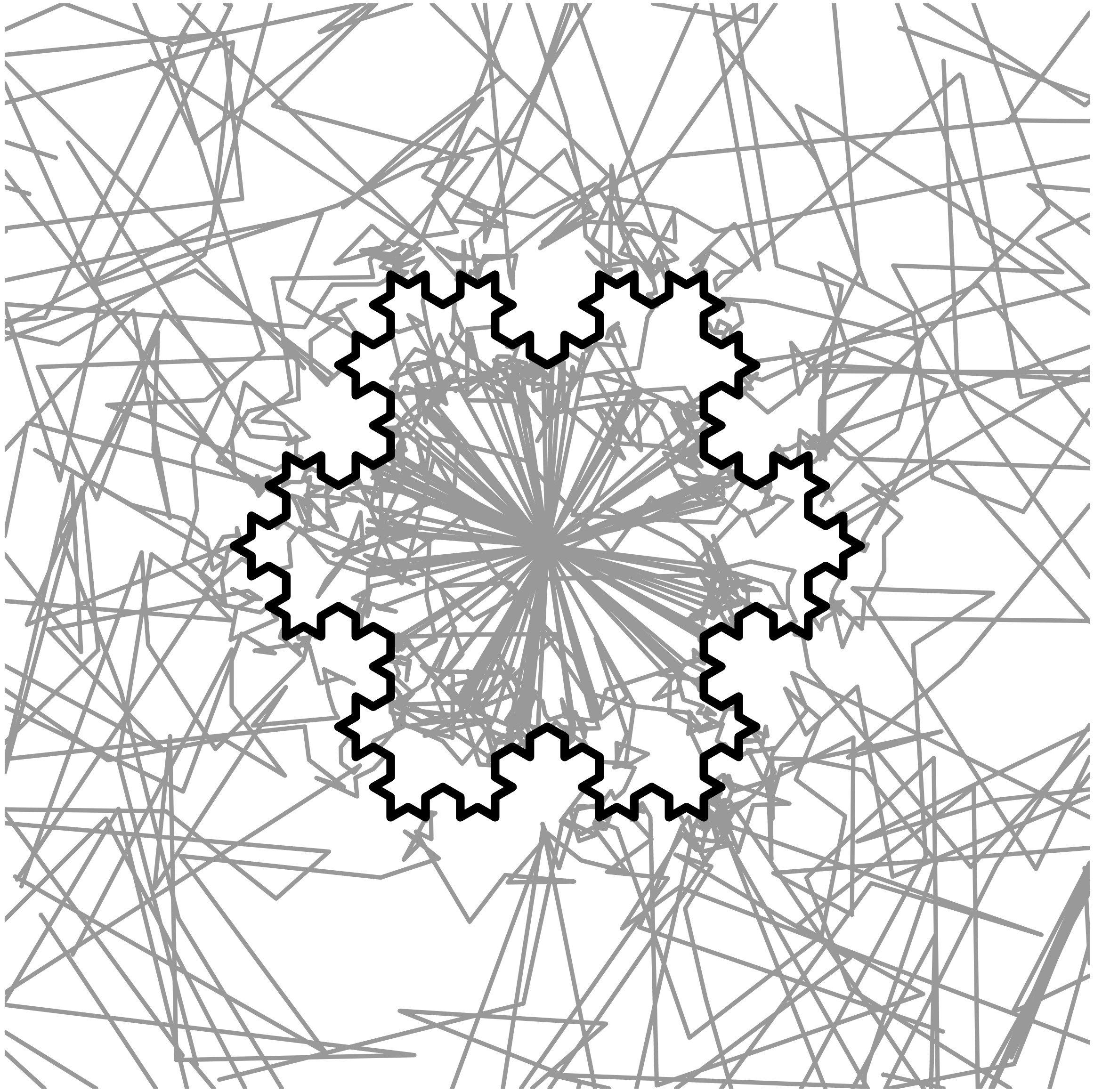
$\omega_1 \perp \omega_2$ iff tangents points have zero length.

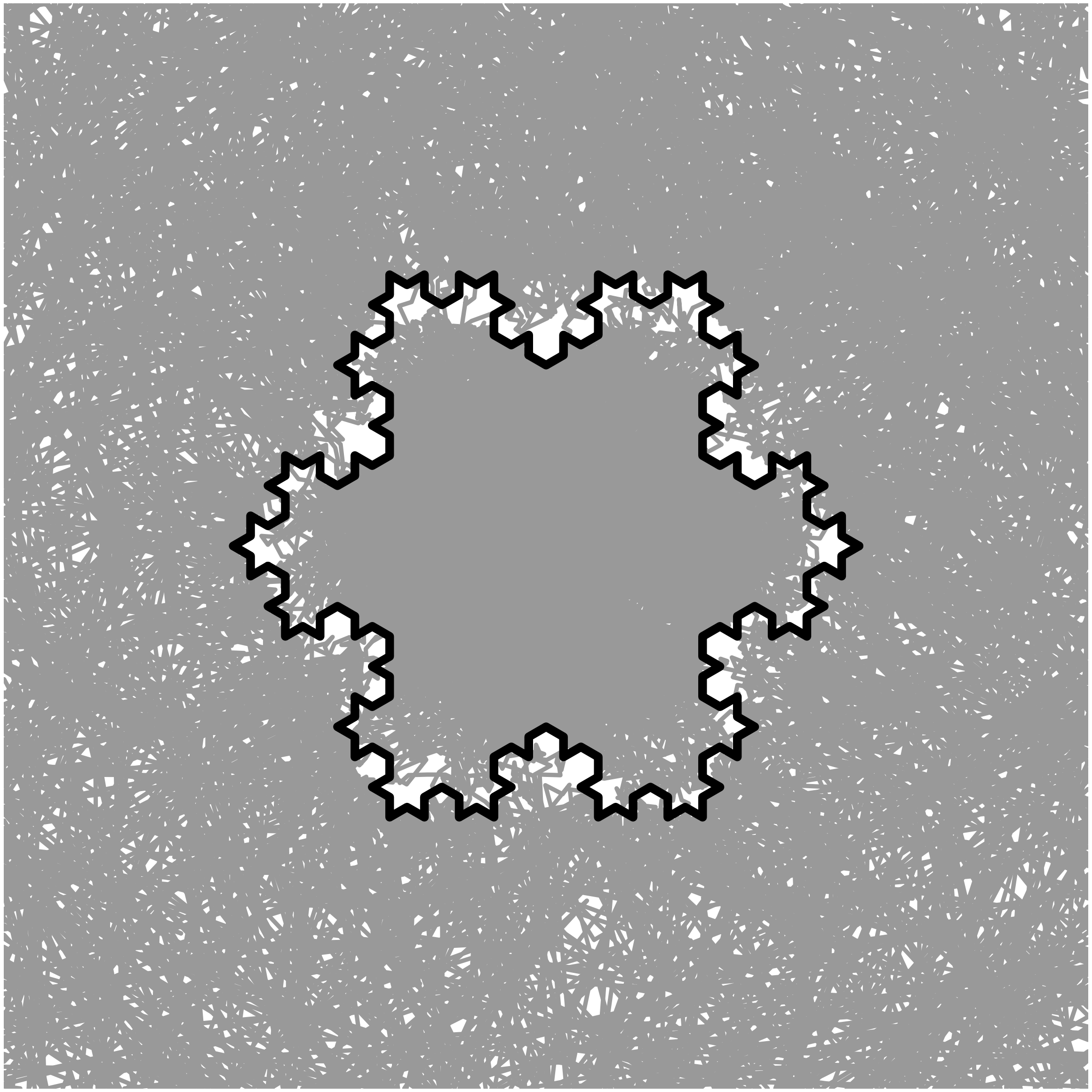


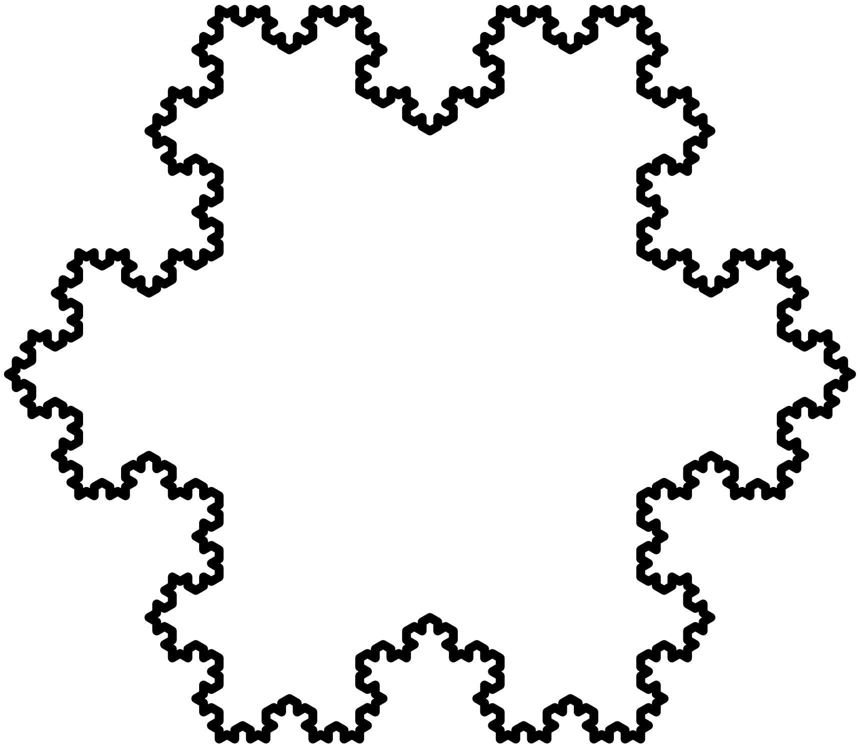
Inside and outside harmonic measures are singular

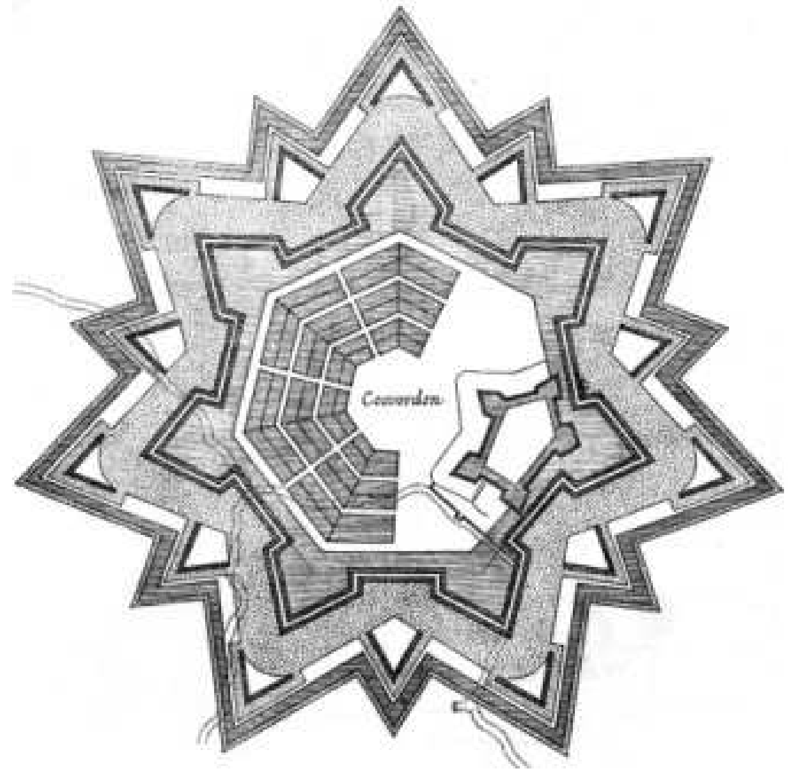


Images of radial lines for conformal maps to inside and outside. ⠿



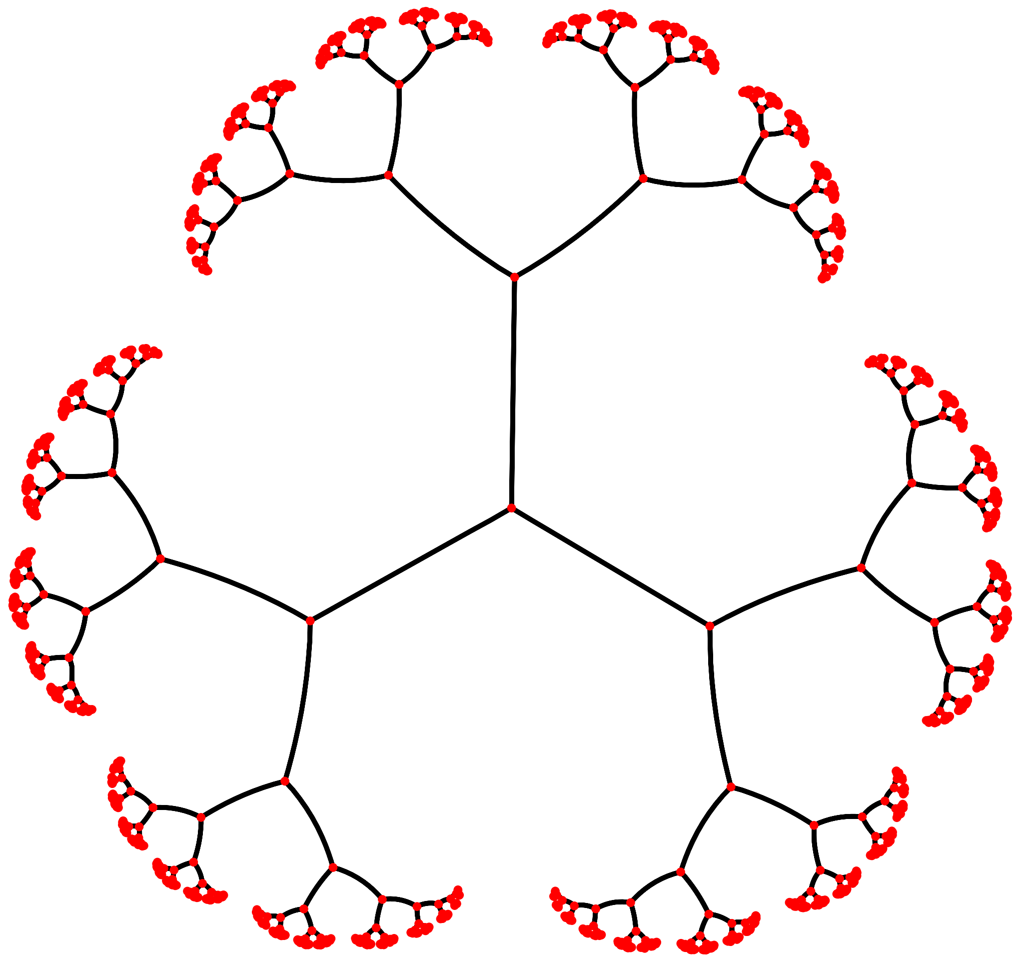


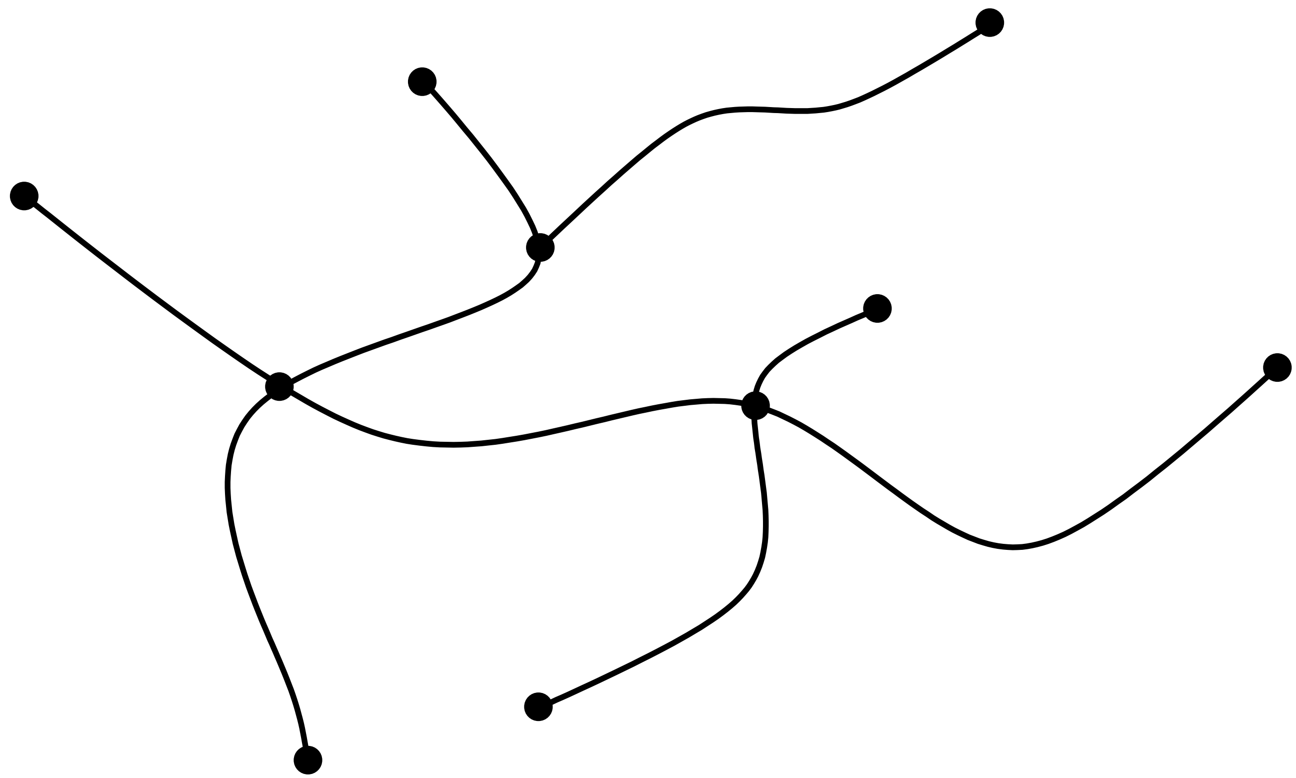




Snowflake and star fort ("trace italienne"). Coincidence?

TRUE TREES





A planar graph is a finite set of points connected by non-crossing edges.

It is a tree if there are no closed loops.

A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from ∞
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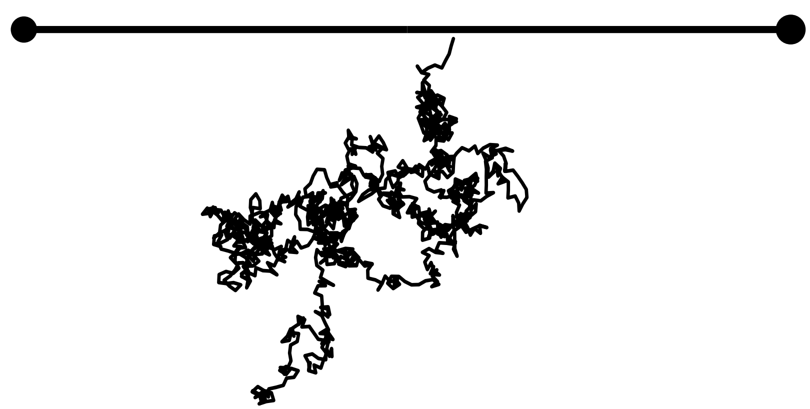
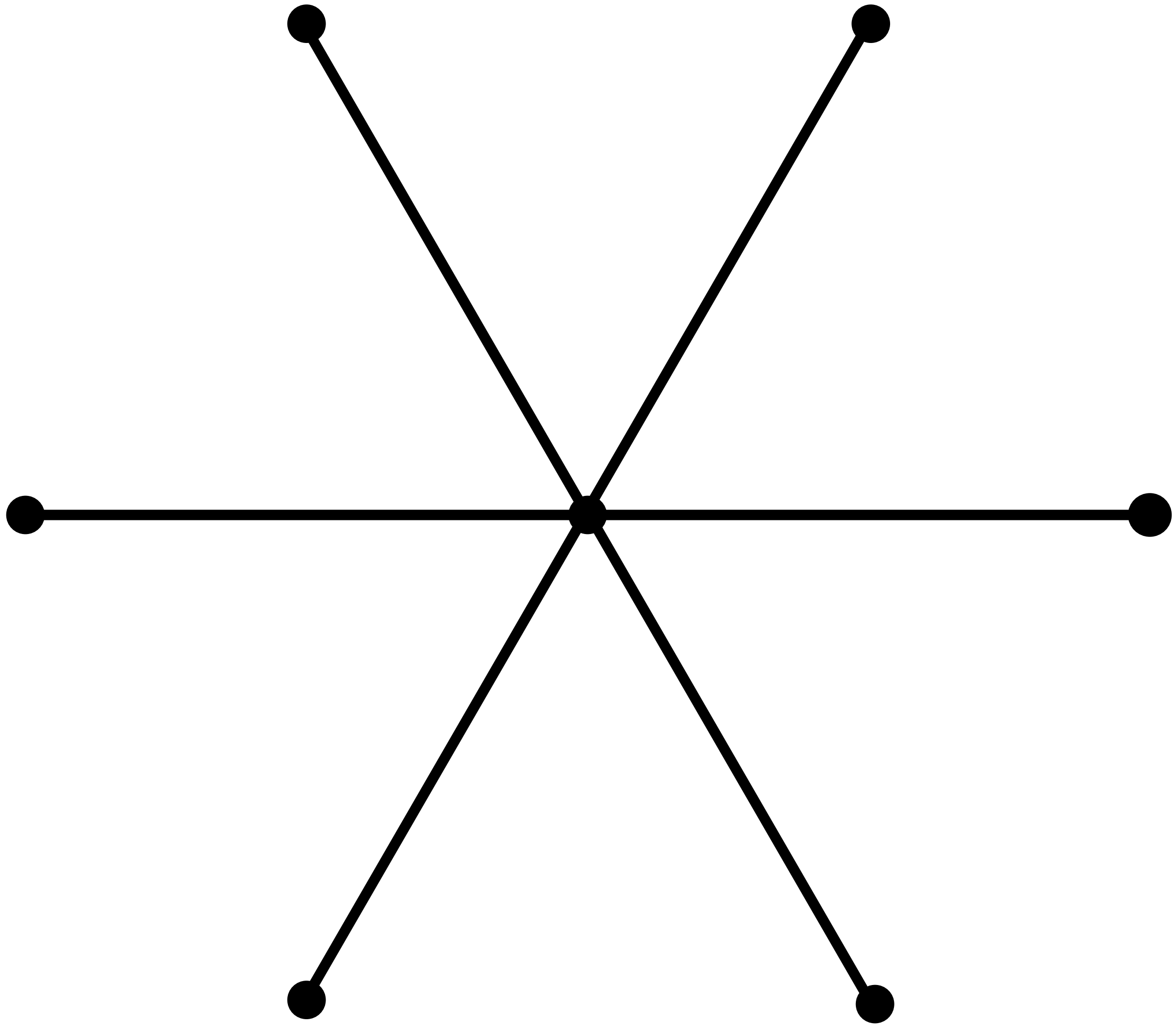
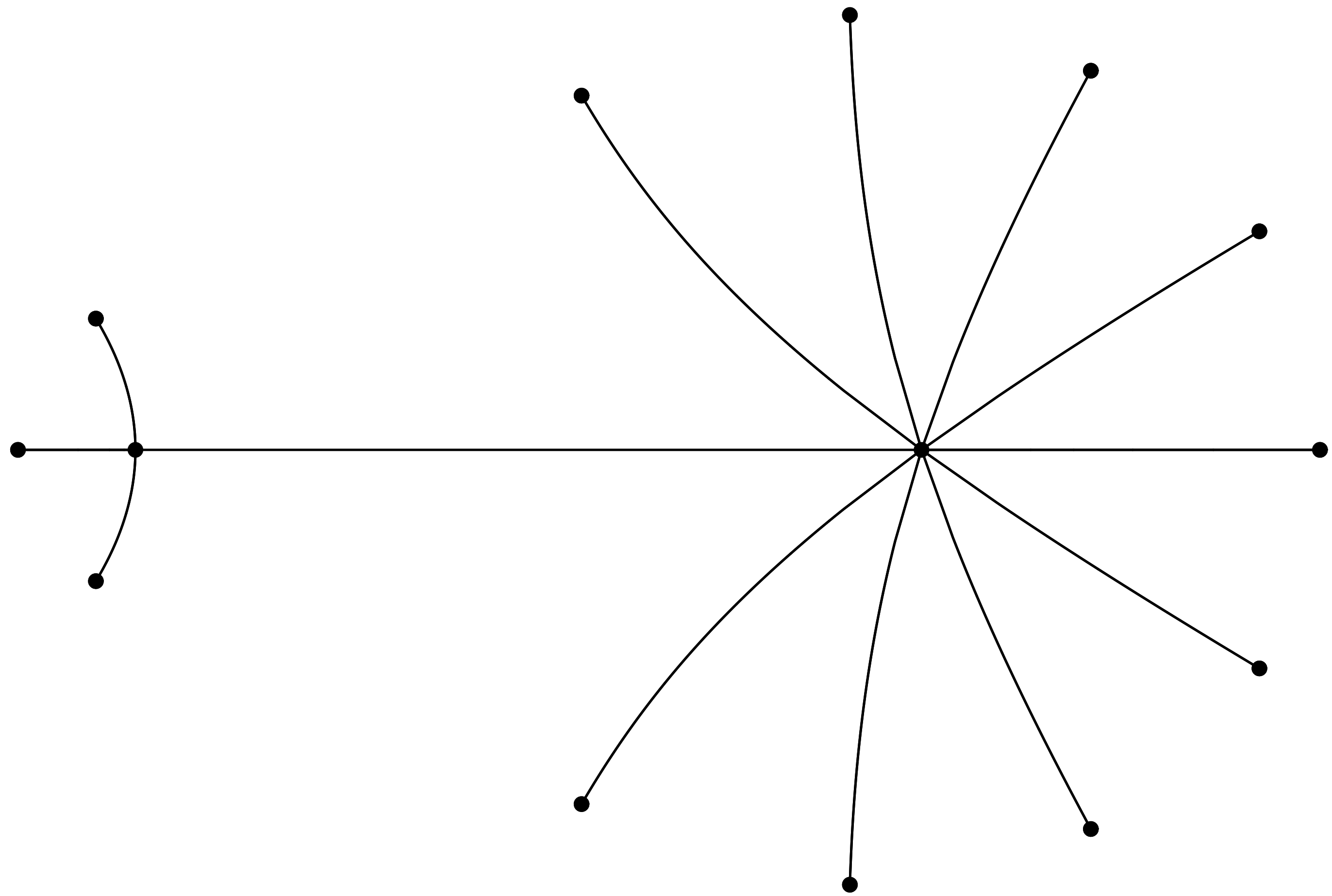




Trivially true by symmetry

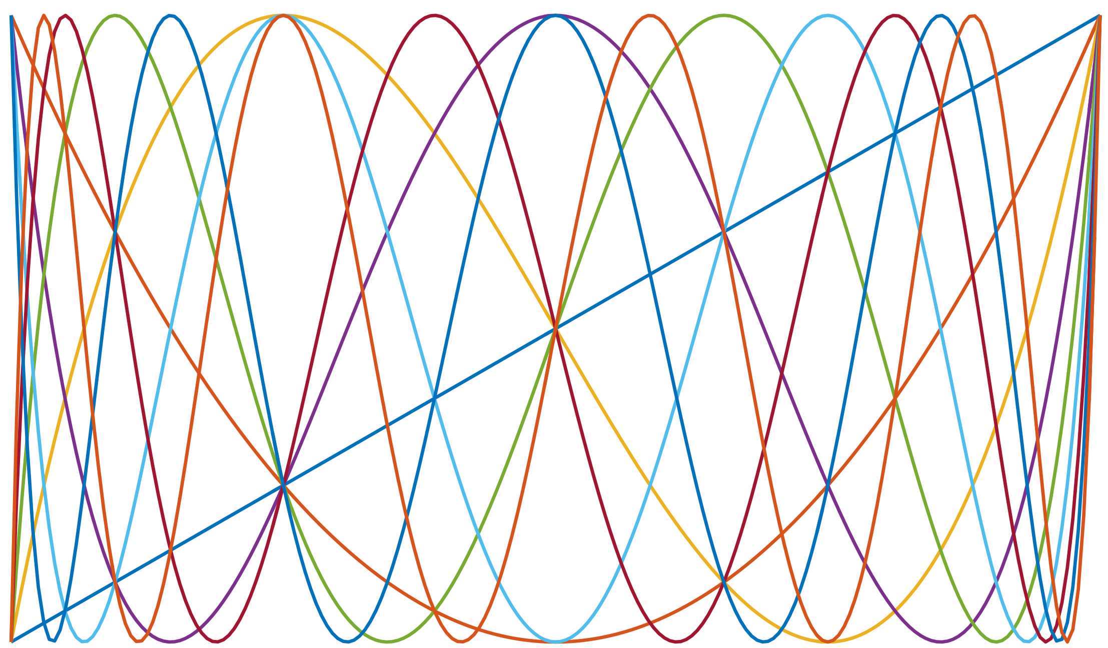


Non-obvious true tree

Definition of critical value: if p = polynomial, then

$$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

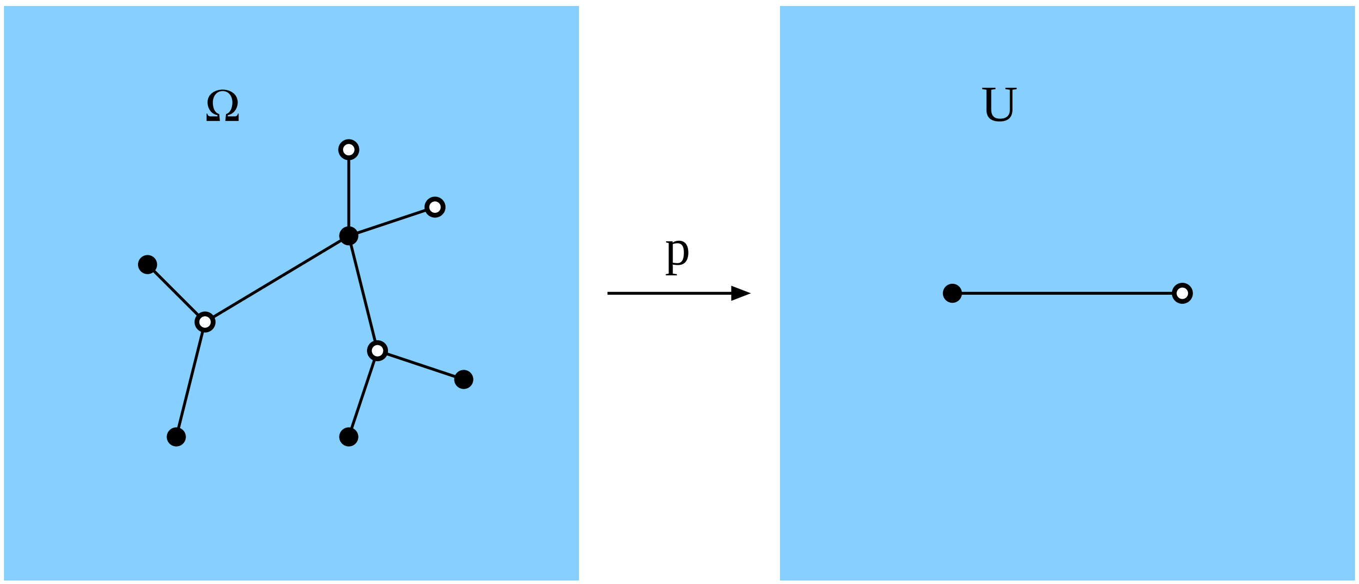
If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.



10 classical Chebyshev polynomials

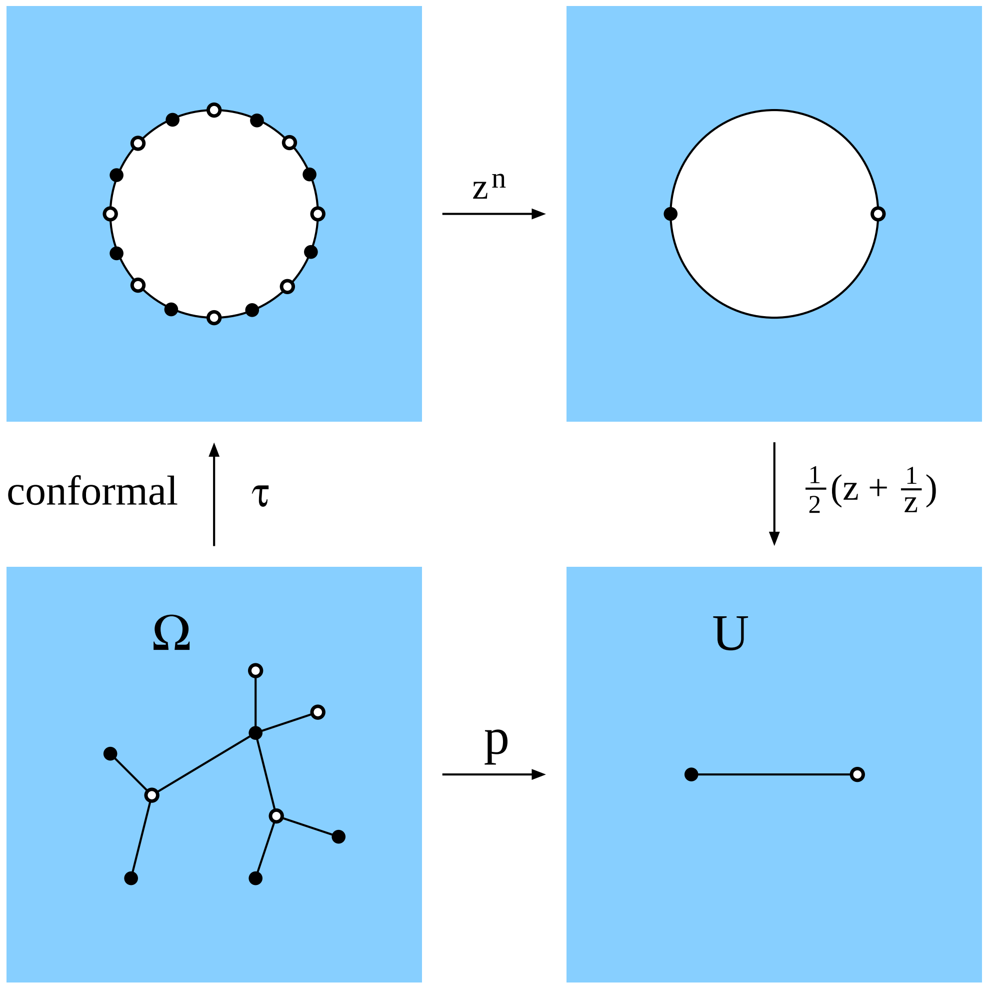
Balanced trees $\leftrightarrow$ Shabat polynomials

Fact: T is balanced iff $T = p^{-1}([-1, 1])$, $p =$ Shabat.



$\Omega = \mathbb{C} \setminus T$ $\qquad$ $U = \mathbb{C} \setminus [-1, 1]$

T balanced $\Leftrightarrow$ p Shabat.



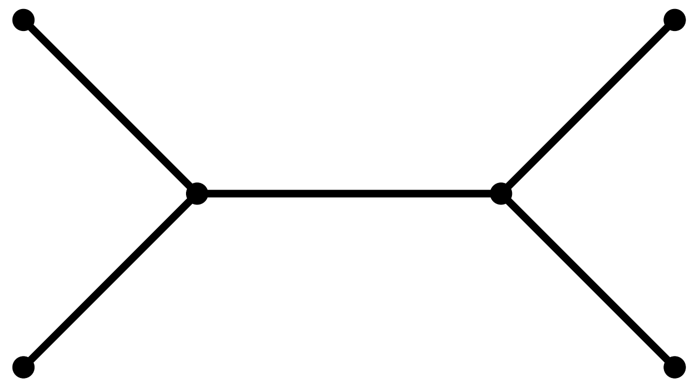
p is entire and n-to-1 $\Leftrightarrow$ p = polynomial.

$\mathrm{CV}(p) \notin U \leftrightarrow p : \Omega \to U$ is covering map.

Theorem: Every finite tree has a true form.

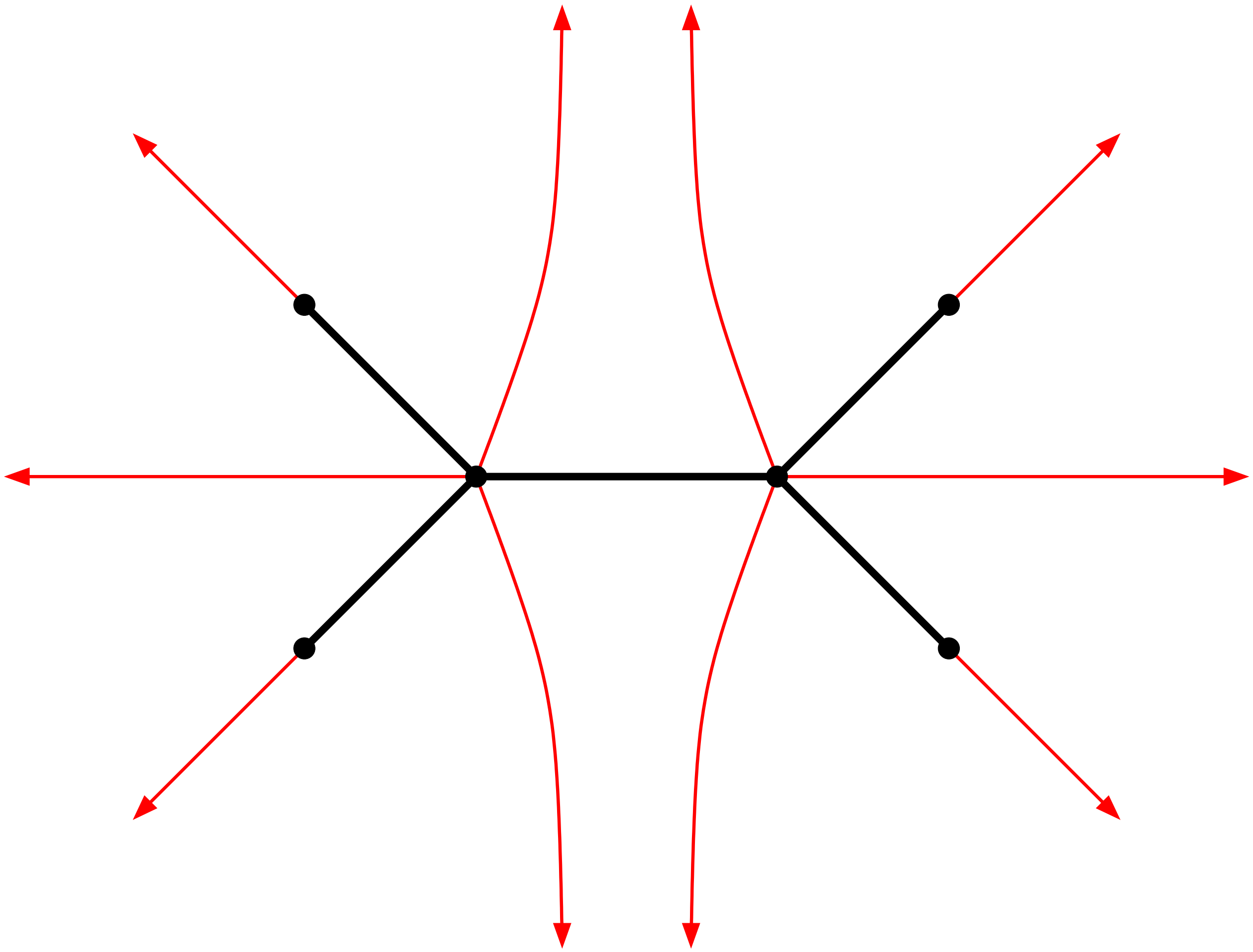
Standard proof uses the uniformization theorem.

Standard proof:



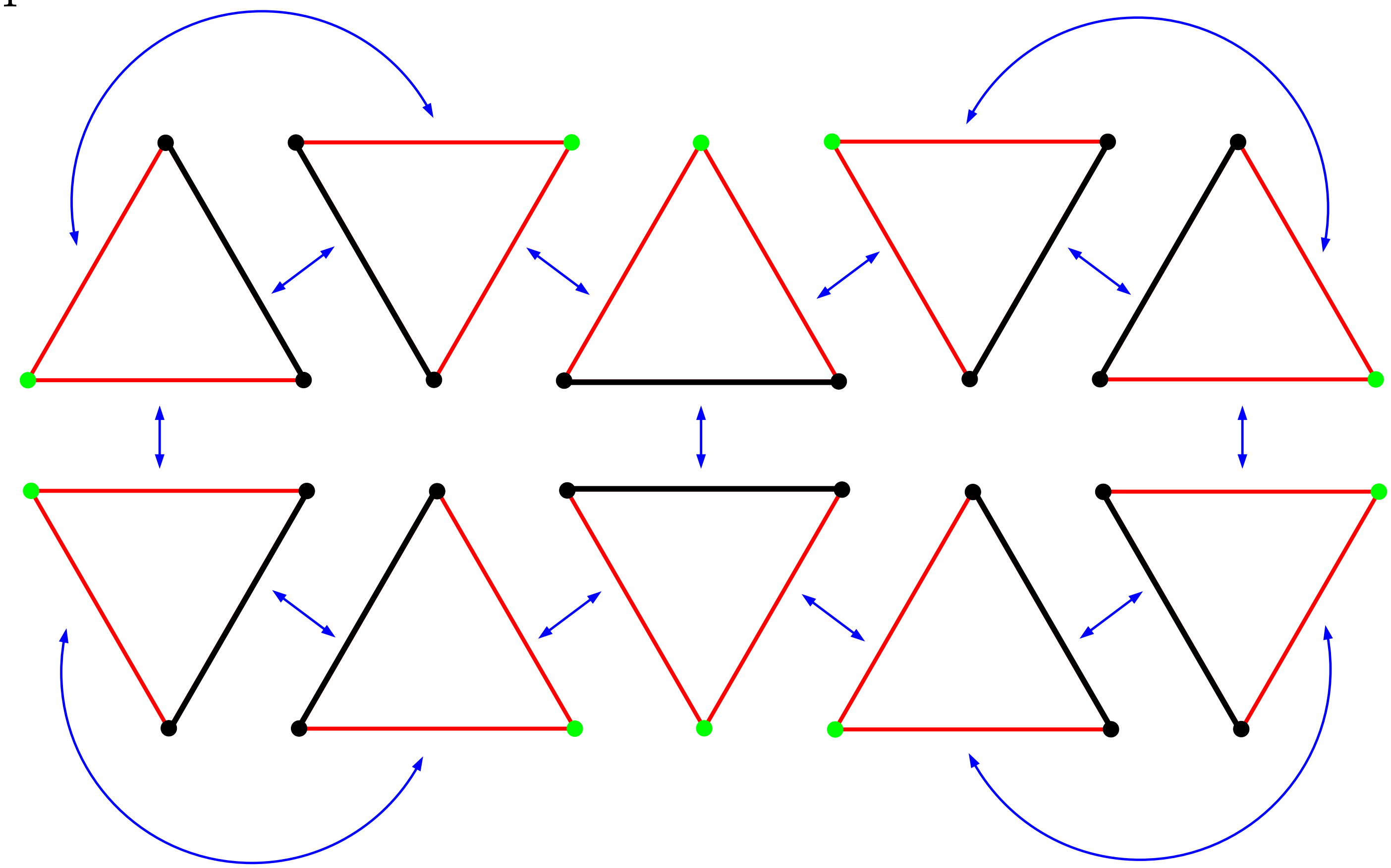
- start with a finite tree.

Standard proof:



- connect vertices of T to infinity; gives finite triangulation of sphere.
- Defines adjacencies between triangles.

Standard proof:

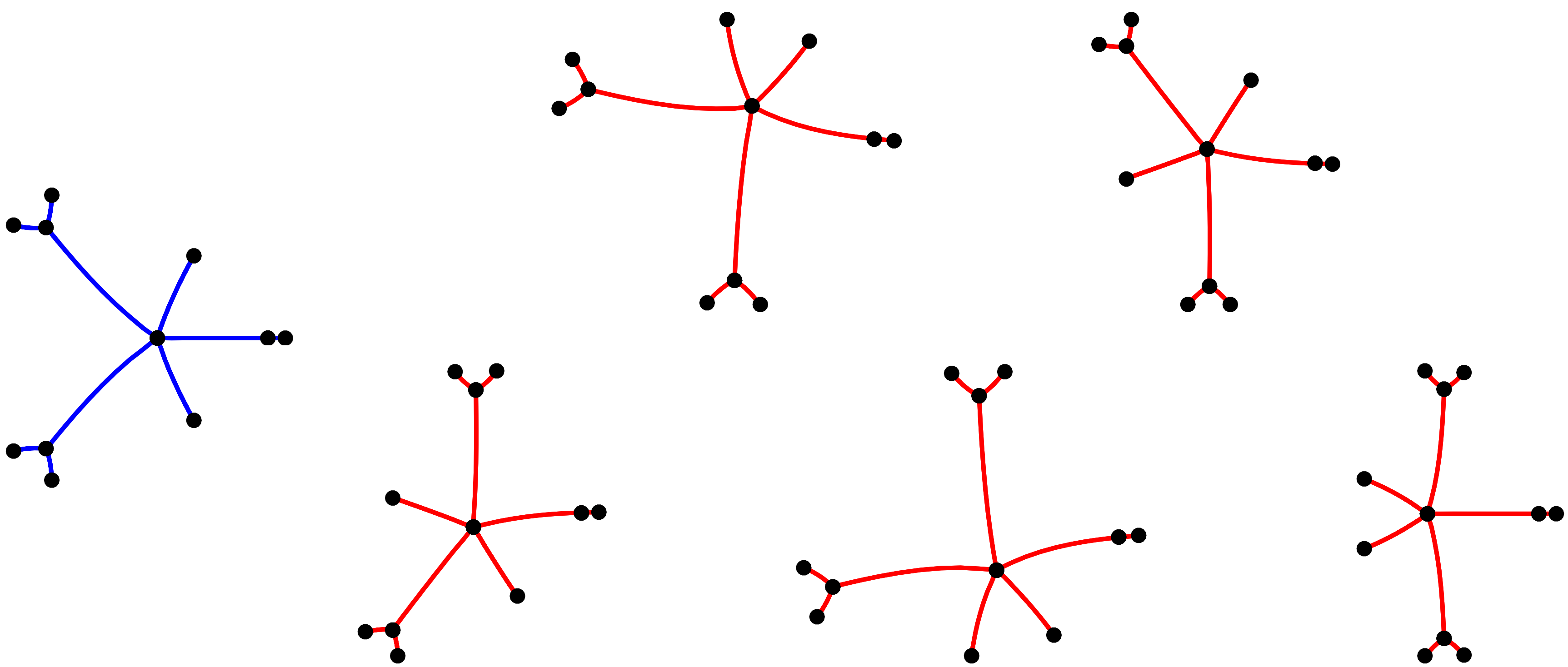


- Glue equilateral triangles using adjacencies: get a conformal 2-sphere.
- By uniformization theorem, conformal maps to Riemann sphere.
- Can check that tree maps to balanced tree.

Algebraic aside:

True trees are examples of Grothendieck's *dessins d'enfants* on sphere.

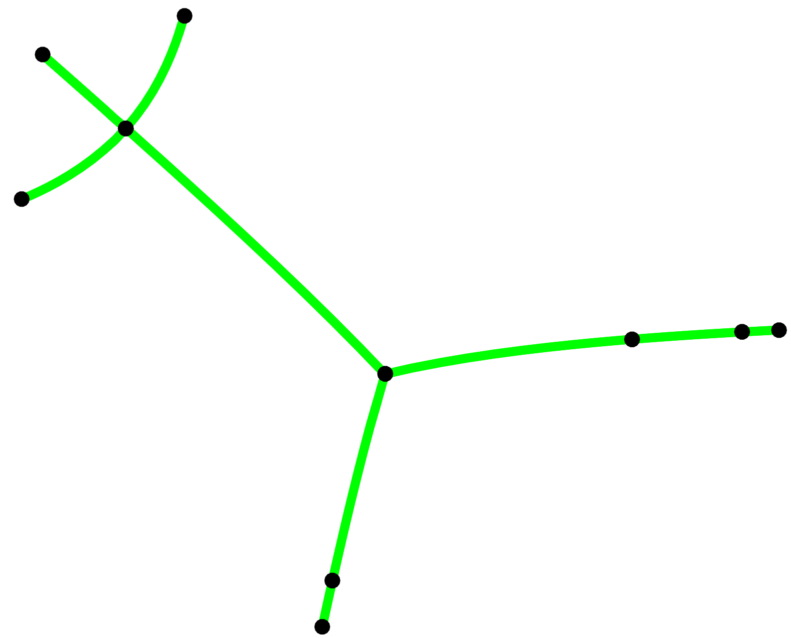
Normalized polynomials are algebraic, so planar trees correspond to number fields. Absolute Galois group acts on trees, but orbits unknown.



Six graphs of type 5 1 1 1 1 1 - 3 3 2 1 1, two orbits.

Even computing number field from tree is difficult.

Kochetkov (2009, 2014): did all trees with 9 and 10 edges.



For example, the polynomial for this 9-edge tree is

$$p(z) = z^4(z^2 + az + b)^2(z - 1),$$

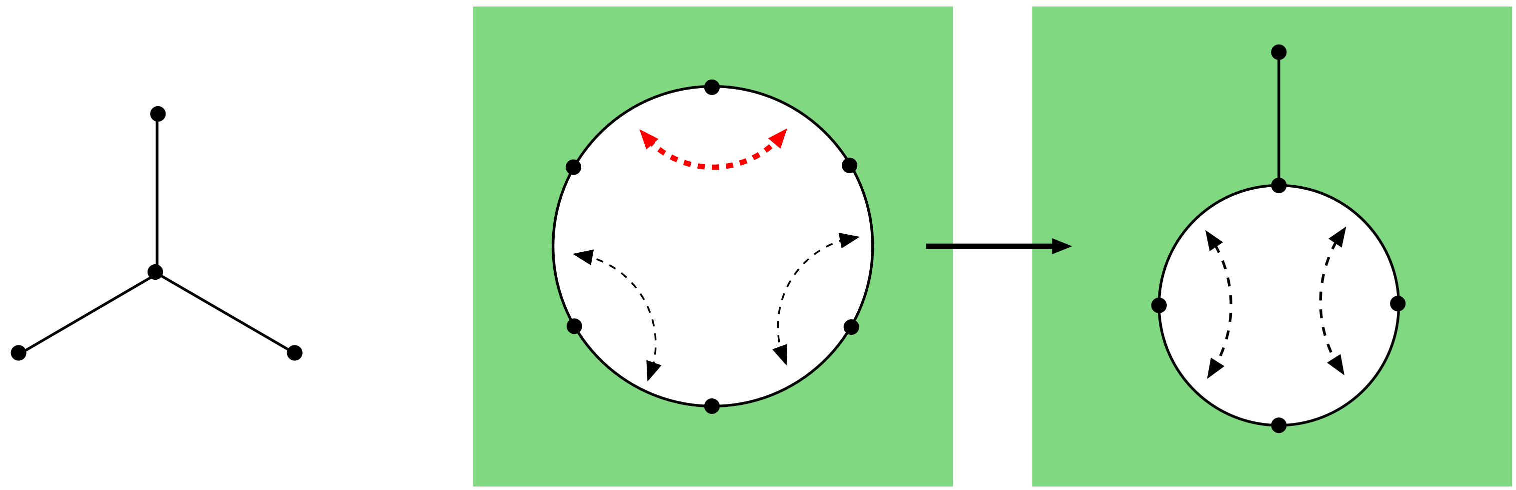
where a is a root of ...

$$
\begin{aligned}
0 = \quad & 126105021875\,a^{15} + 873367351500\,a^{14} \\
& +2340460381665\,a^{13} + 2877817869766\,a^{12} \\
& +3181427453757\,a^{11} - 68622755391456\,a^{10} \\
& -680918281137097\,a^{9} - 2851406436711330\,a^{8} \\
& -7139130404618520\,a^{7} - 12051656256571792\,a^{6} \\
& -14350515598839120\,a^{5} - 12058311779508768\,a^{4} \\
& -6916678783373312\,a^{3} - 2556853615656960\,a^{2} \\
& -561846360735744\,a - 65703906377728
\end{aligned}
$$

This is **not** the most complicated formula in Kochetkov's paper.

However, true form can be drawn without knowing the polynomial.

Don Marshall's `ZIPPER` uses conformal mapping to draw true trees.



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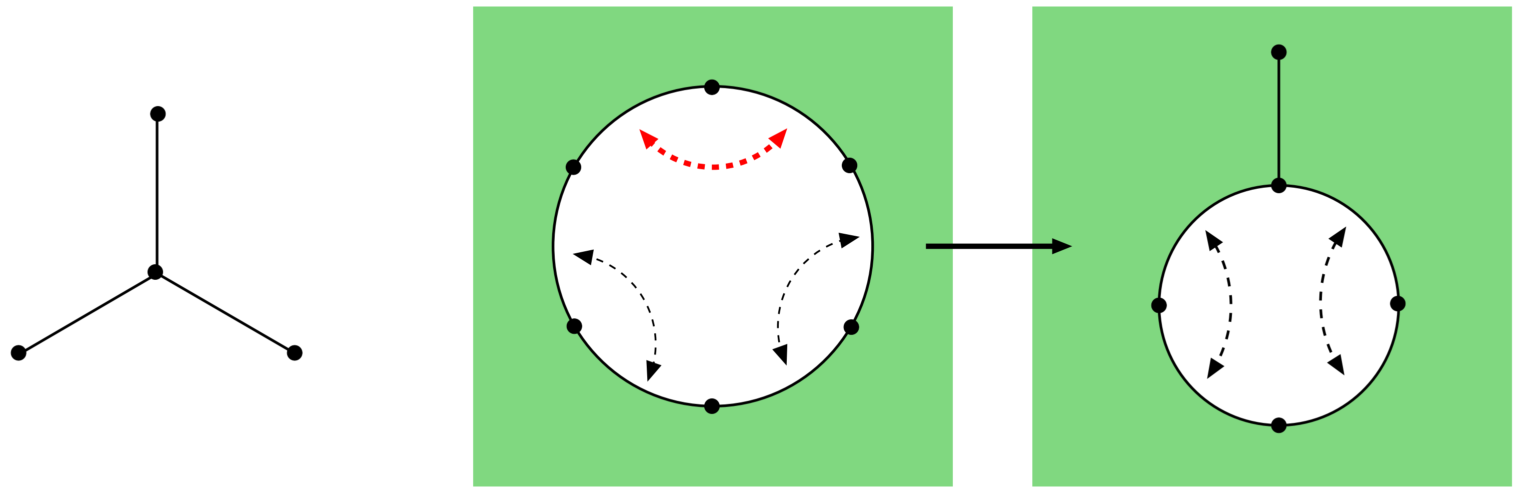
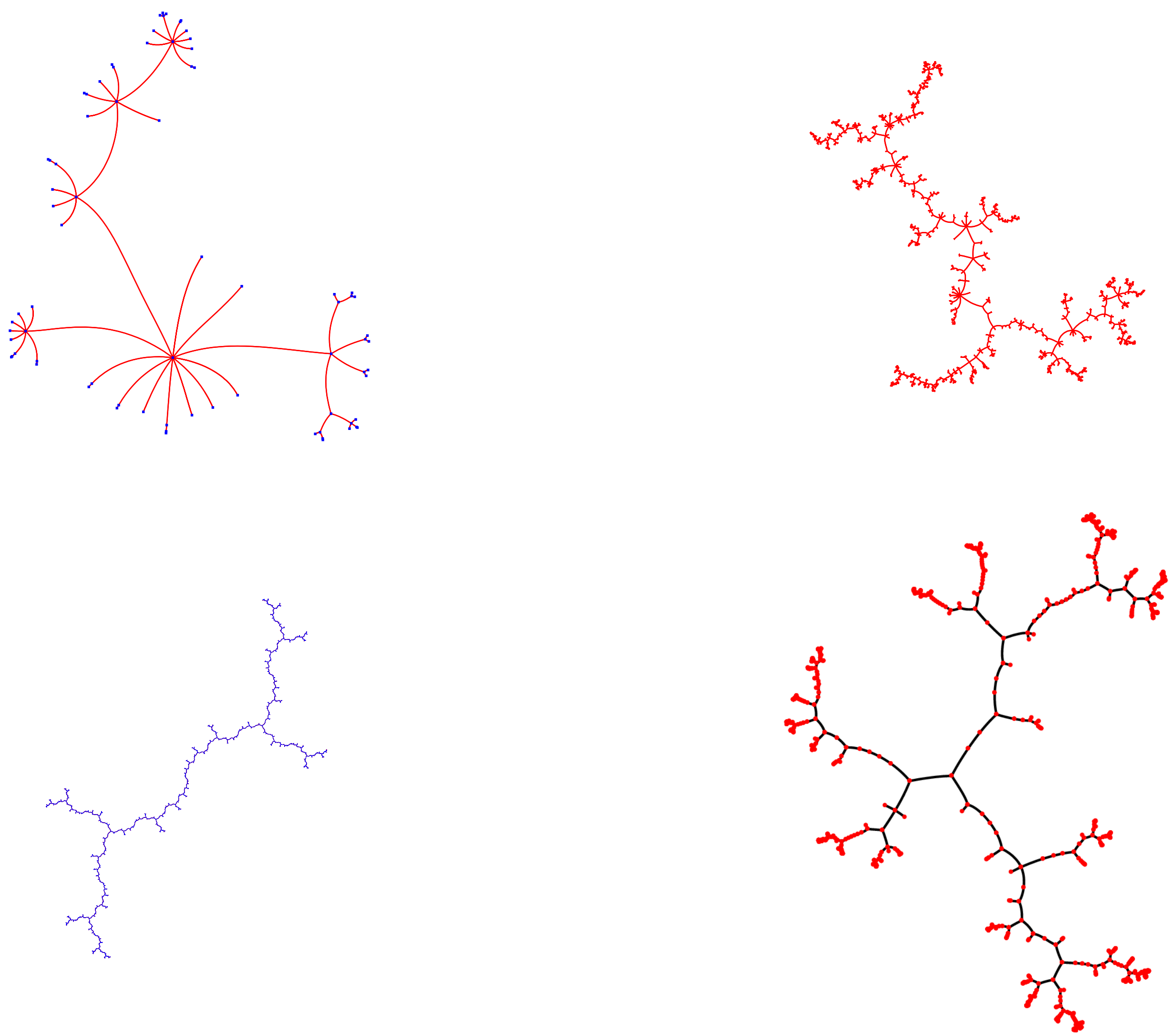


Marshall and Rohde approximated all true trees with ≤ 14 edges.

They can compute vertices to 1000's of digits of accuracy.

Test if $\alpha \in \mathbb{C}$ is algebraic by seeking integer relationships between $1, \alpha, \alpha^2, \ldots$ using lattice reduction or Helaman Ferguson's PSLQ algorithm.

See Rohde's excellent talk Thursday.



Some true trees, courtesy of Marshall and Rohde

Do true trees approximate all possible shapes?

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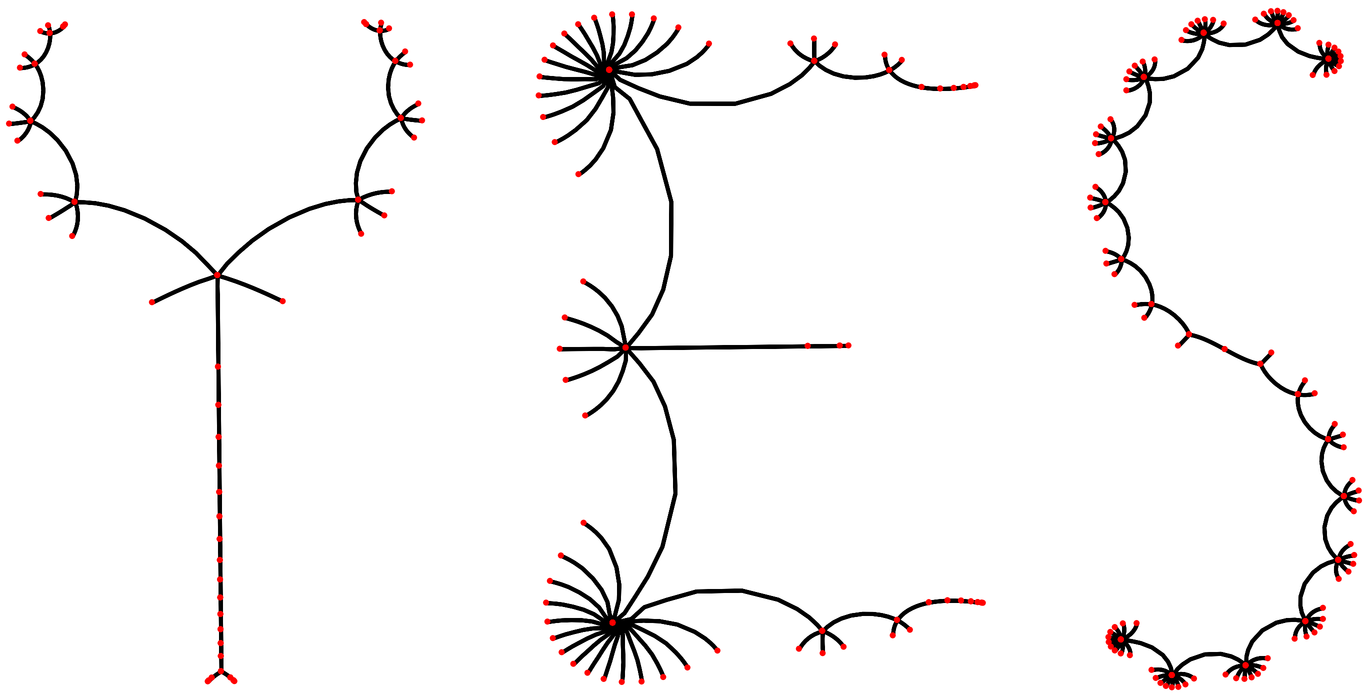
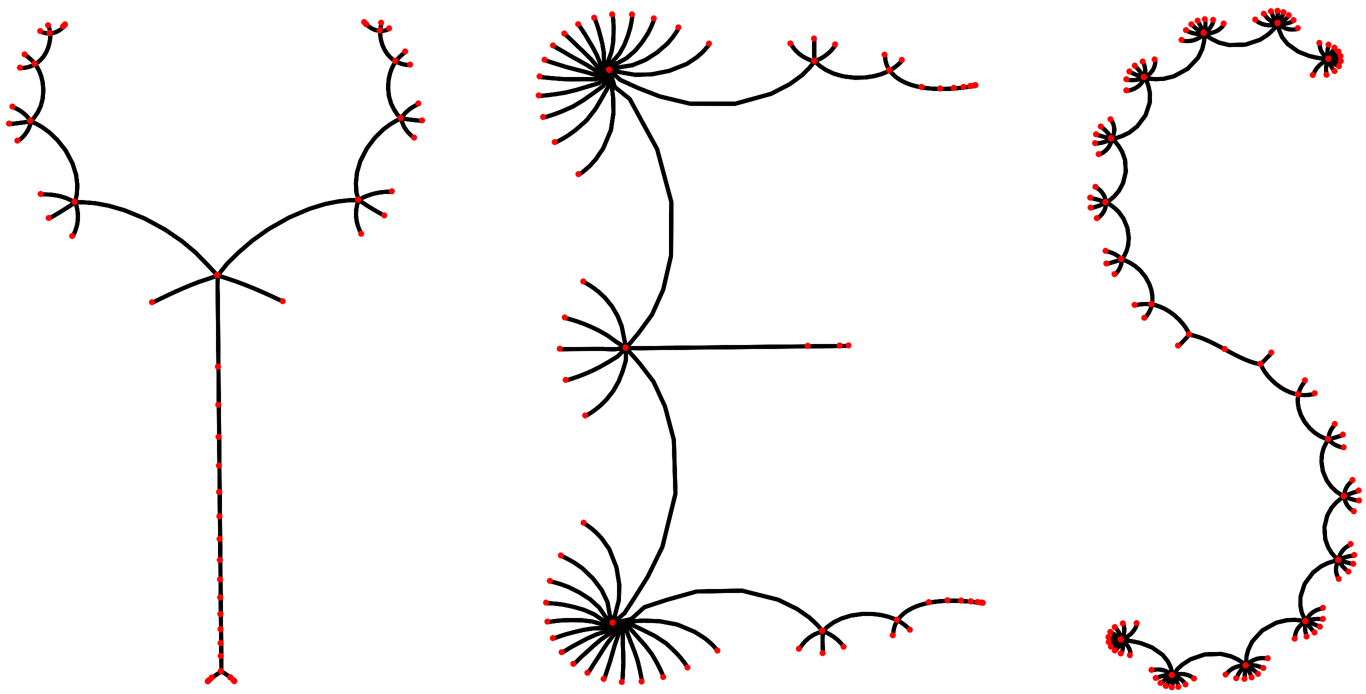
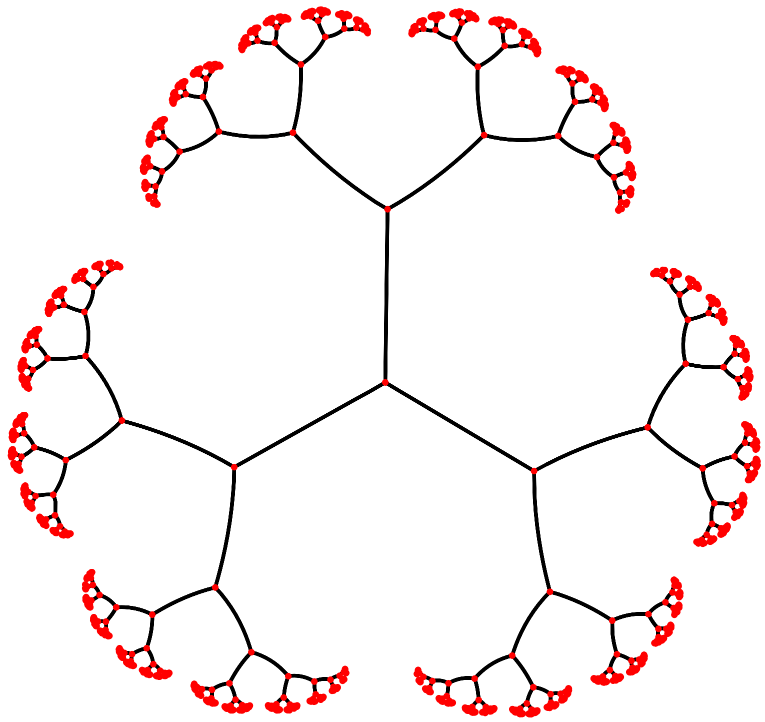
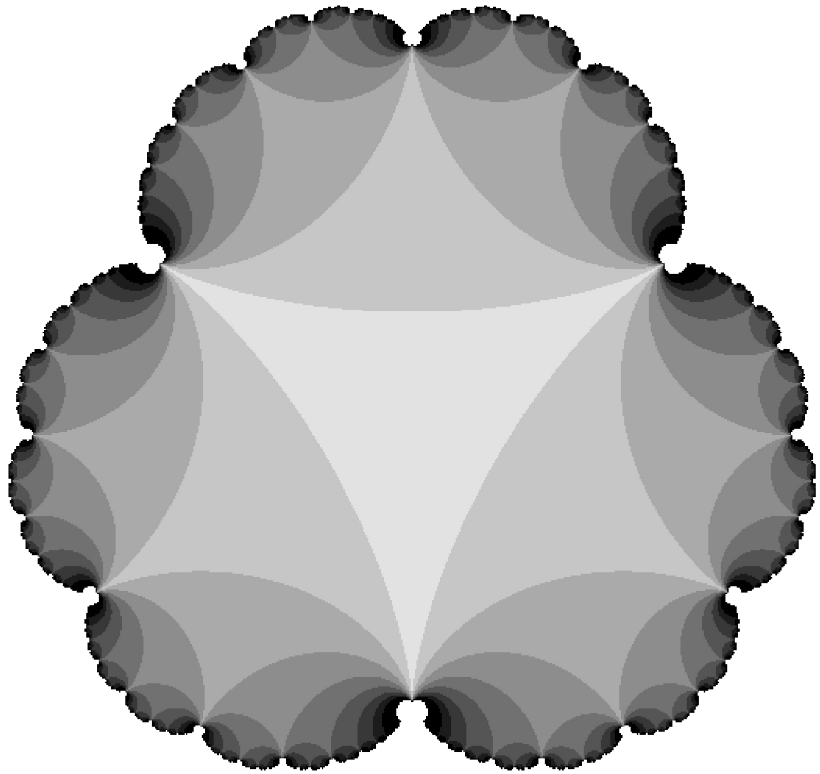


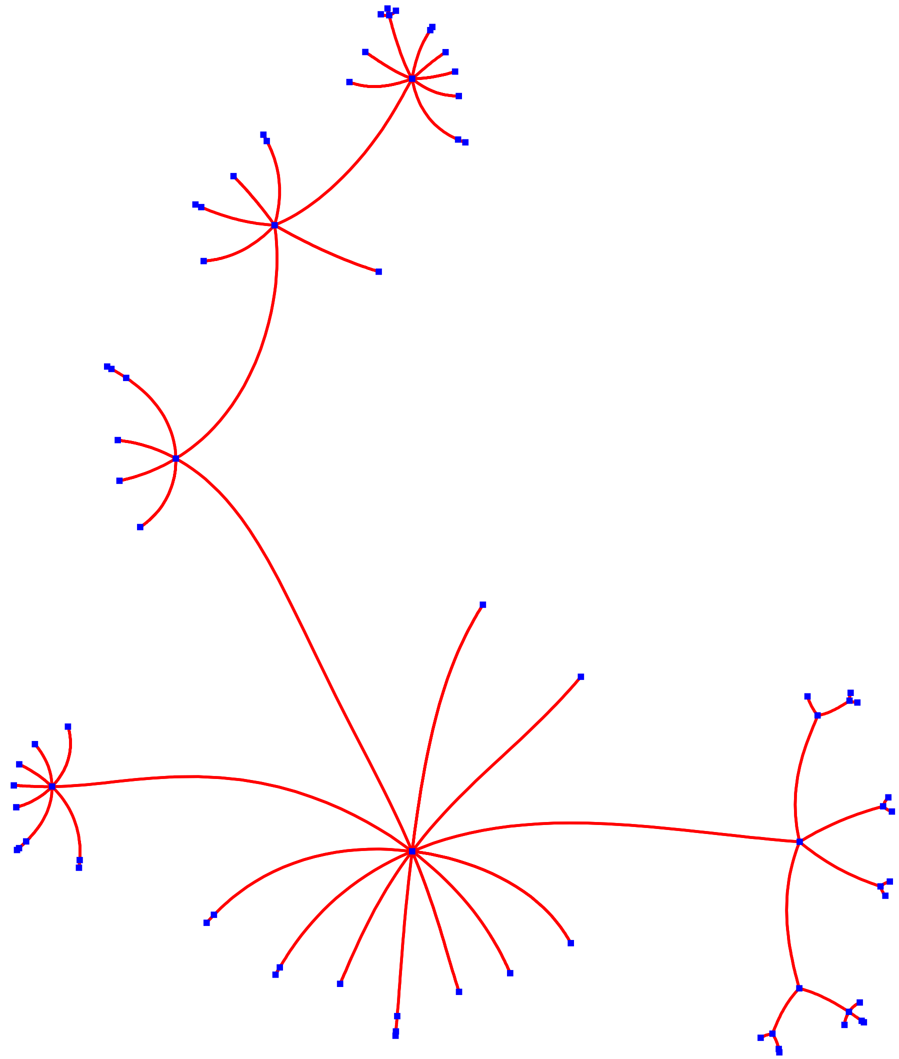
Do true trees approximate all possible shapes?

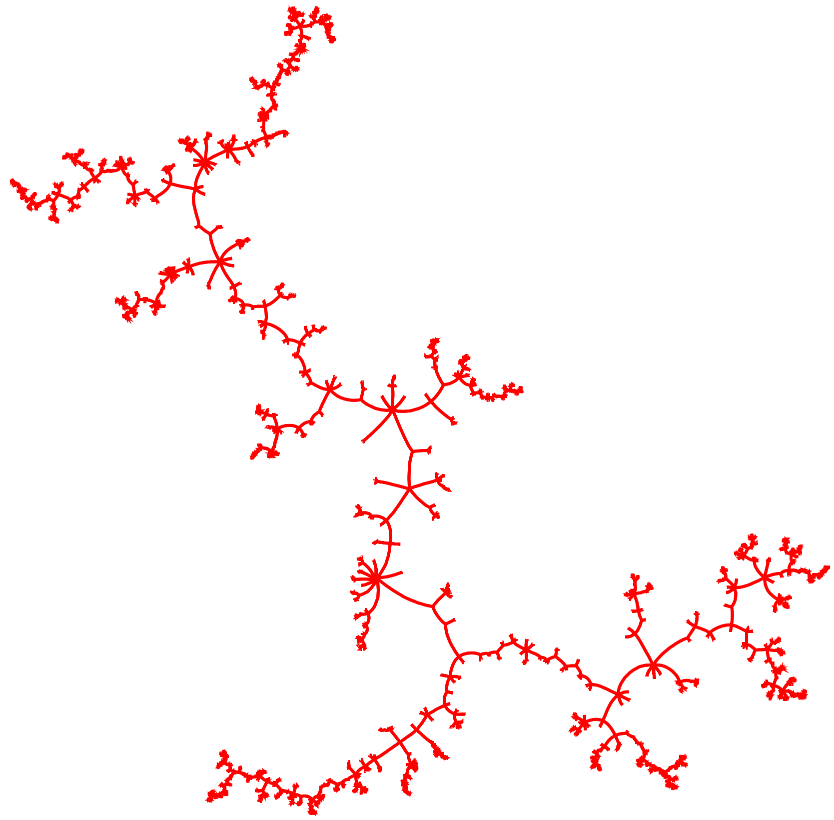


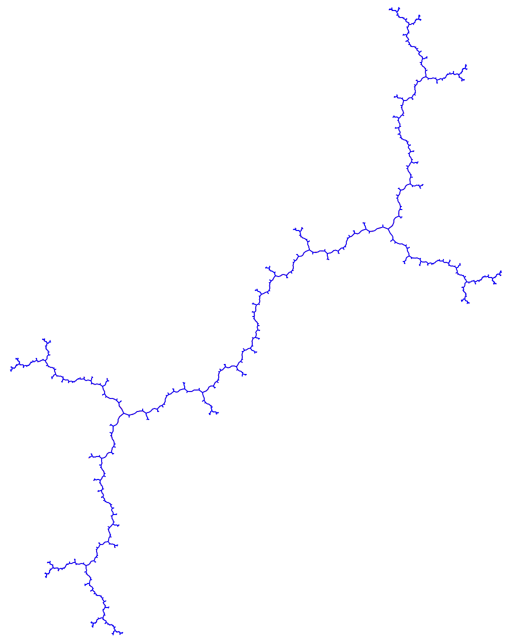
Thm: (B. 2013) Every planar continuum is Hausdorff limit of true trees.



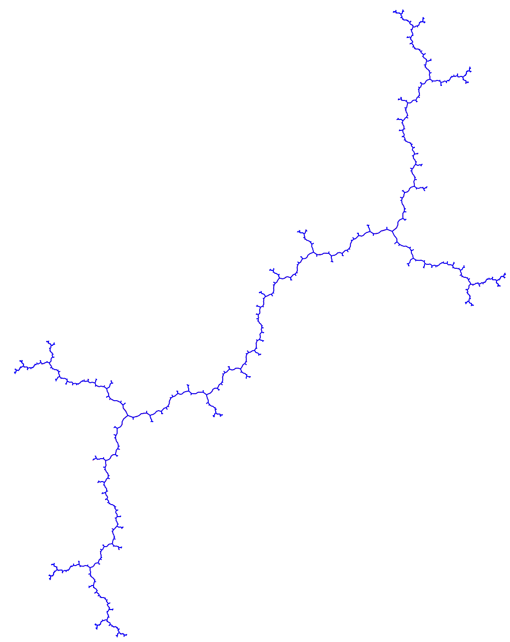




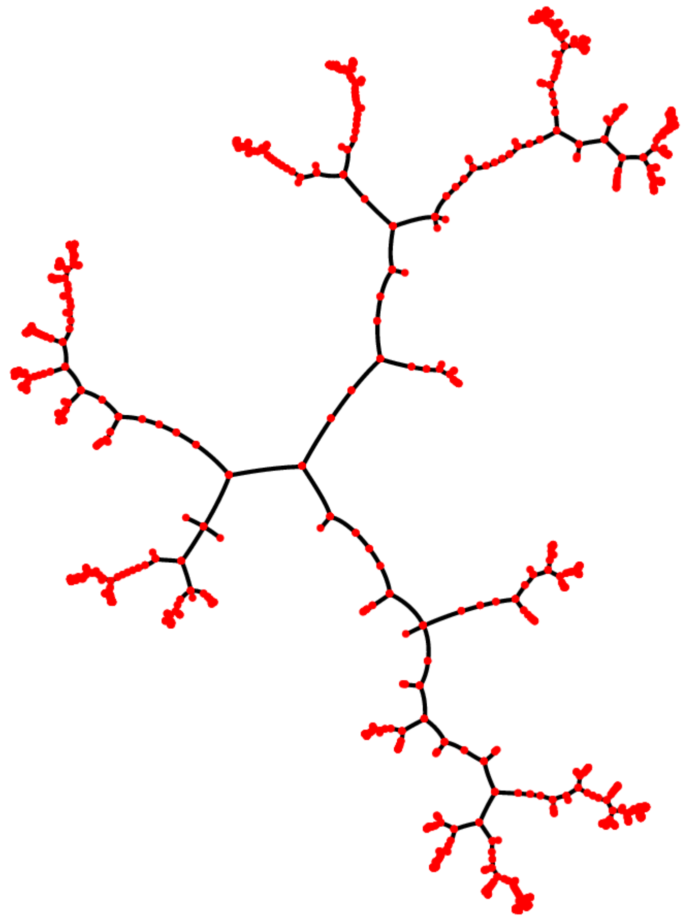




Ceci n'est pas un ensemble de Julia.

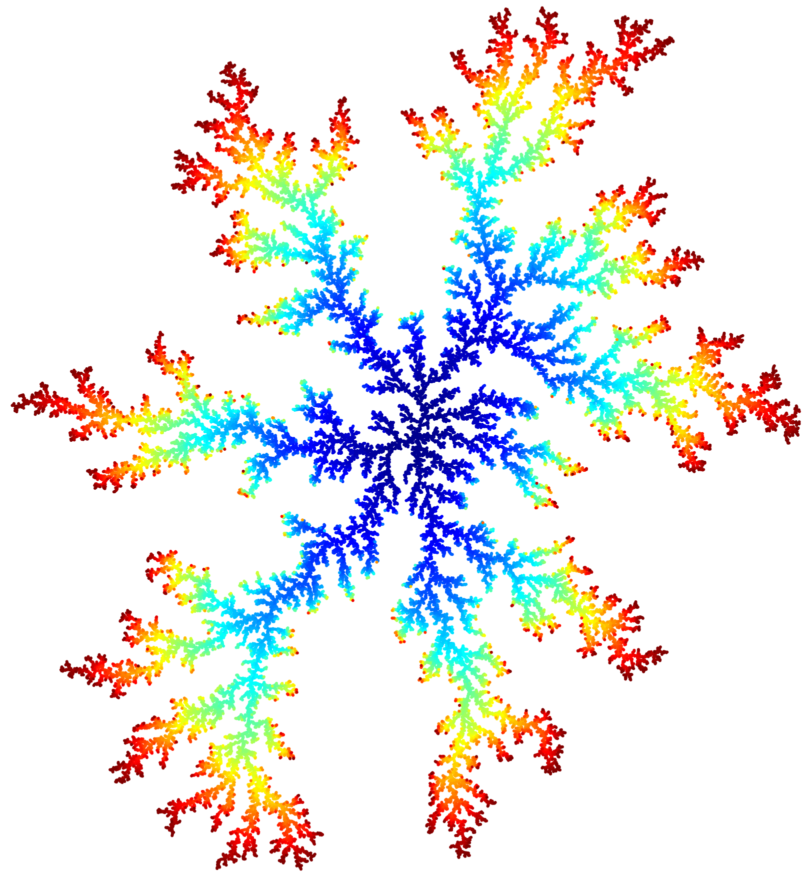


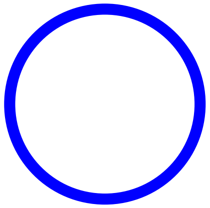
True tree based on combinatorics of Julia set of $z^2 + i$.
Example of "rigidity": combinatorics determines geometry.

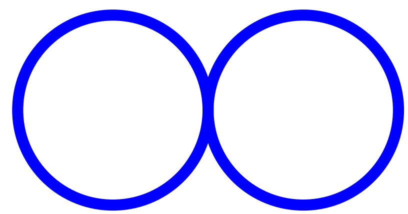


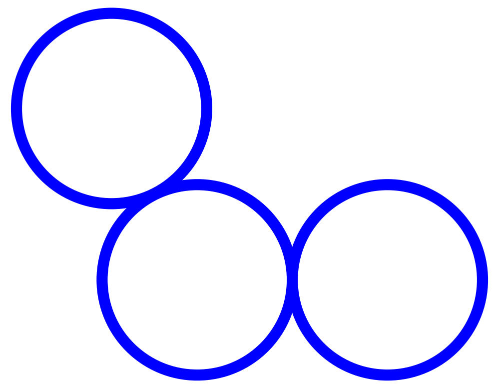
True DLA

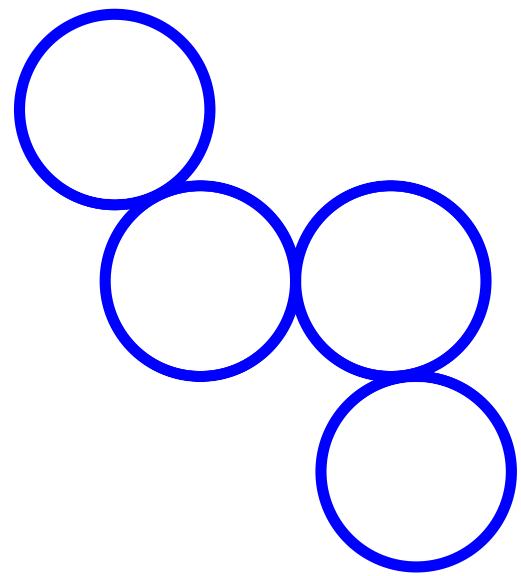
DIFFUSION LIMITED AGGREGATION (DLA)

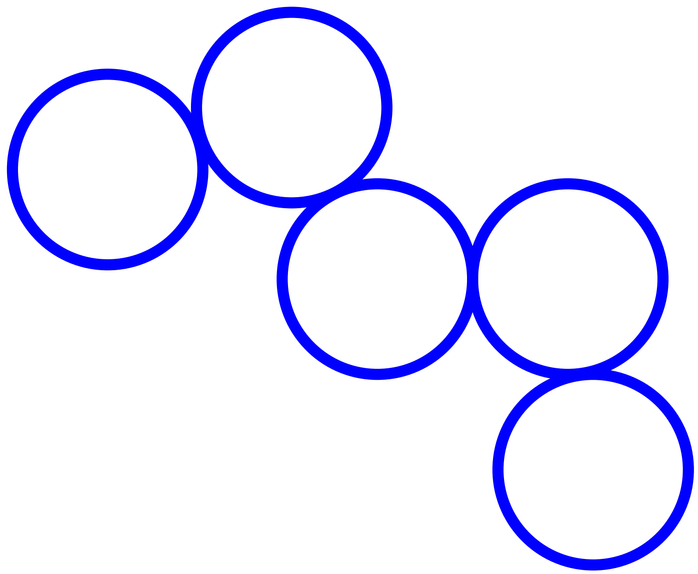


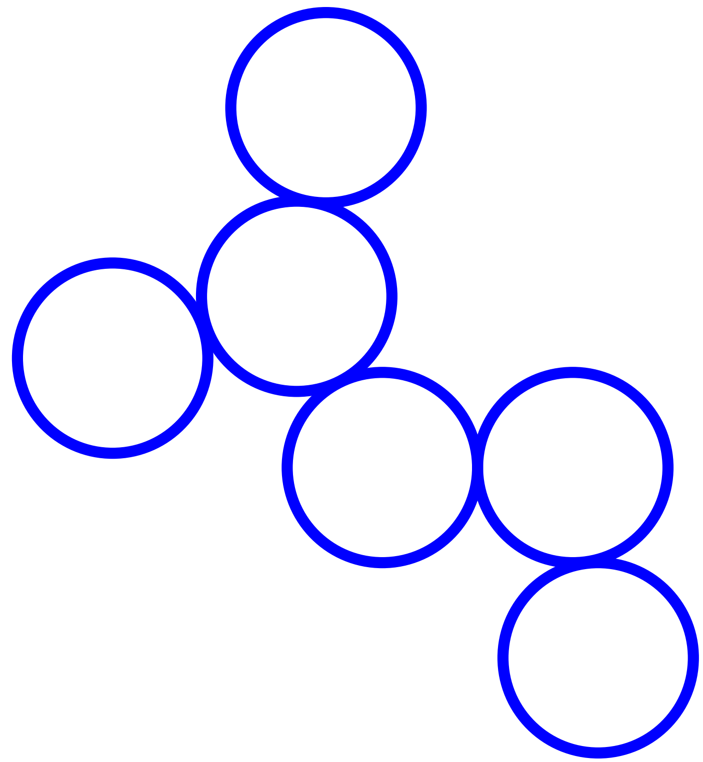


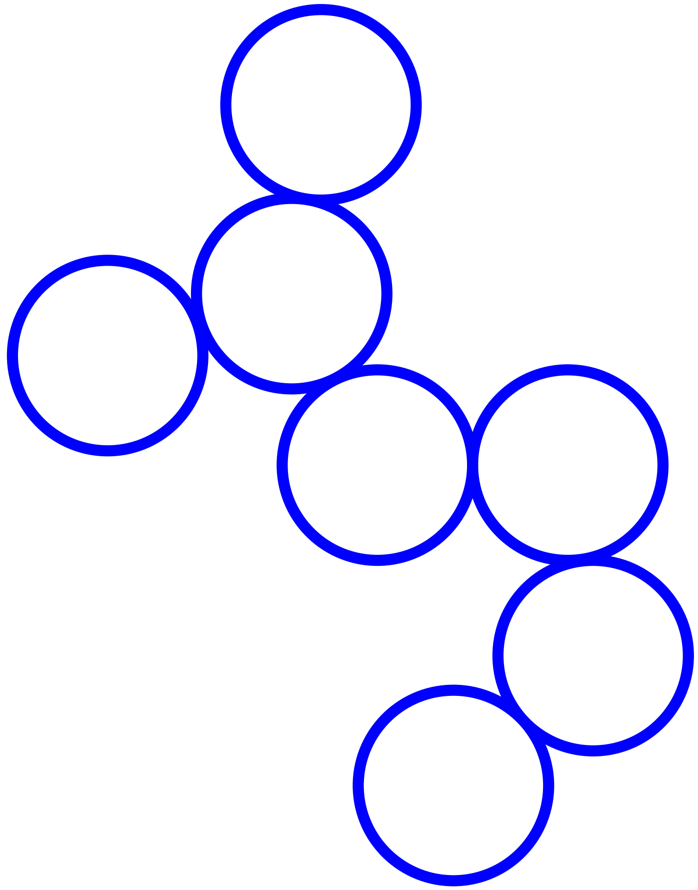


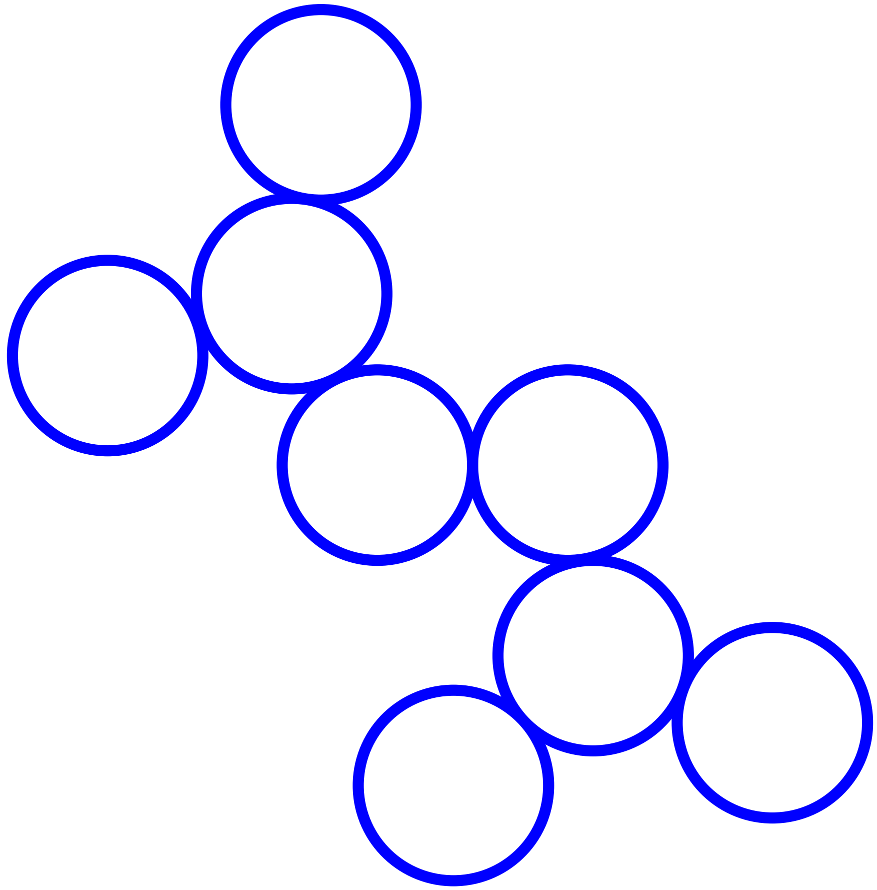


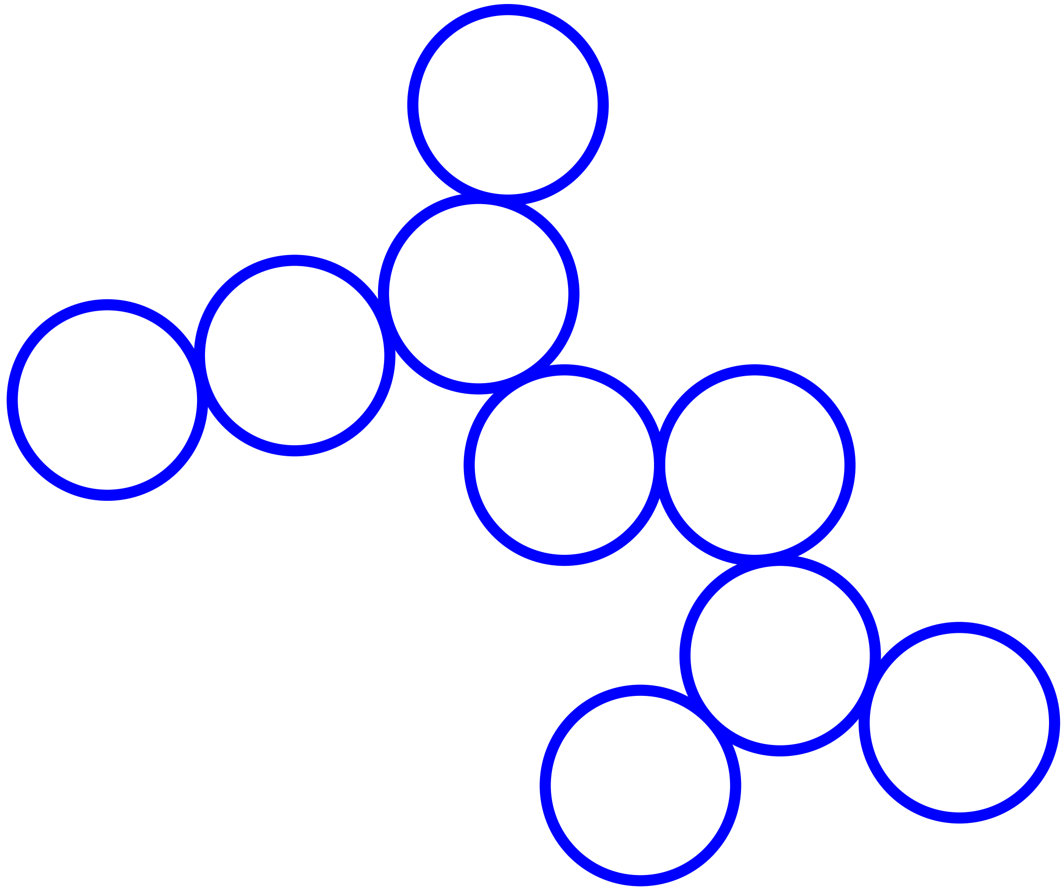


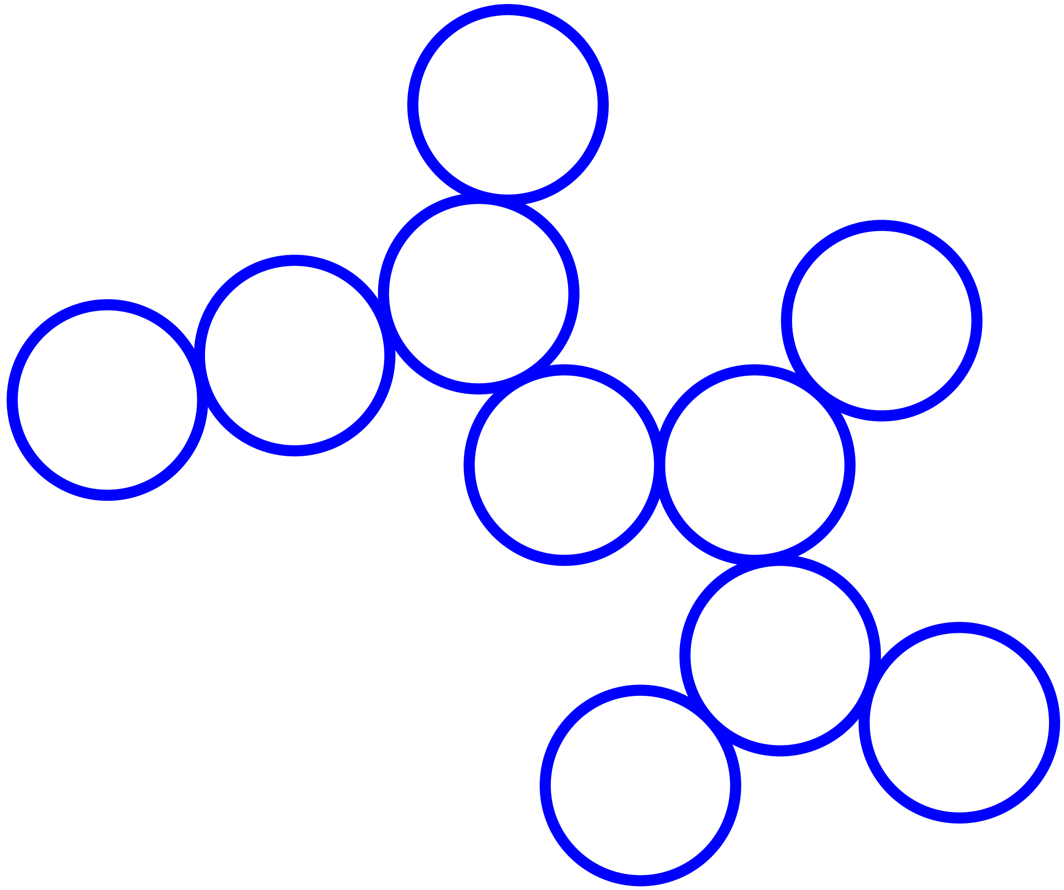


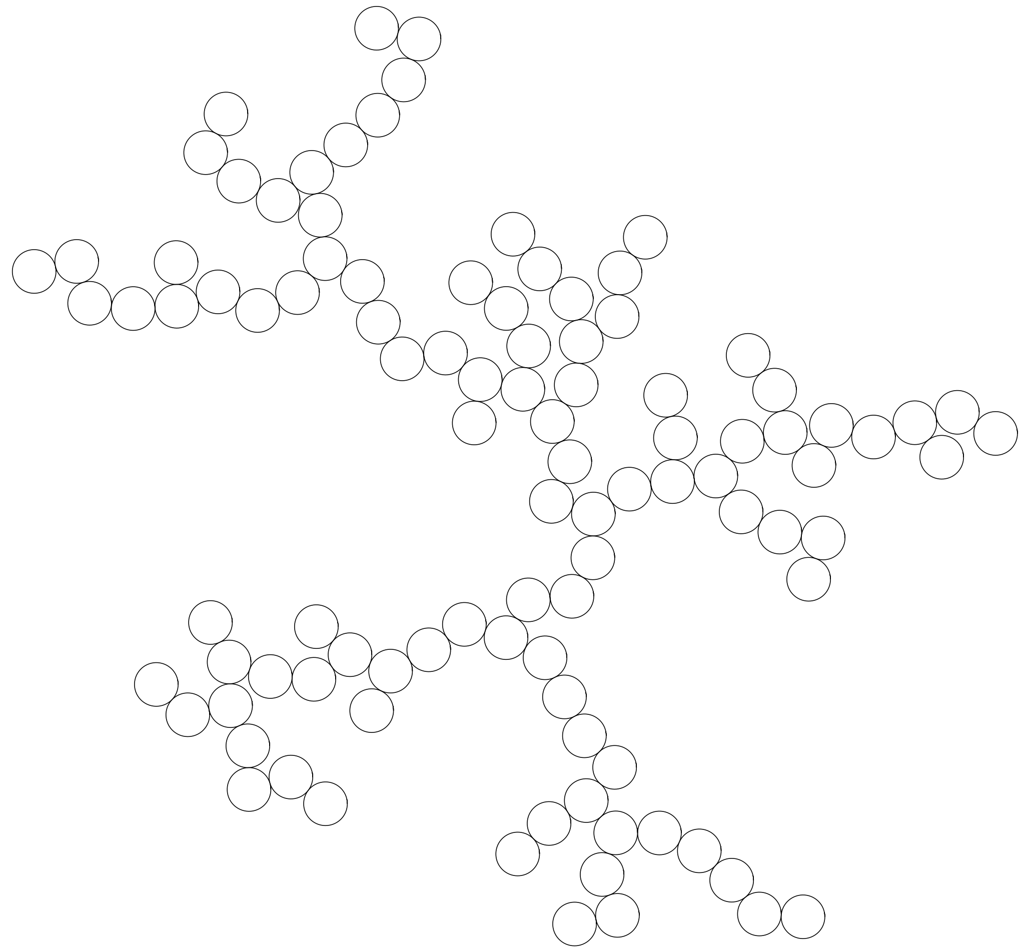


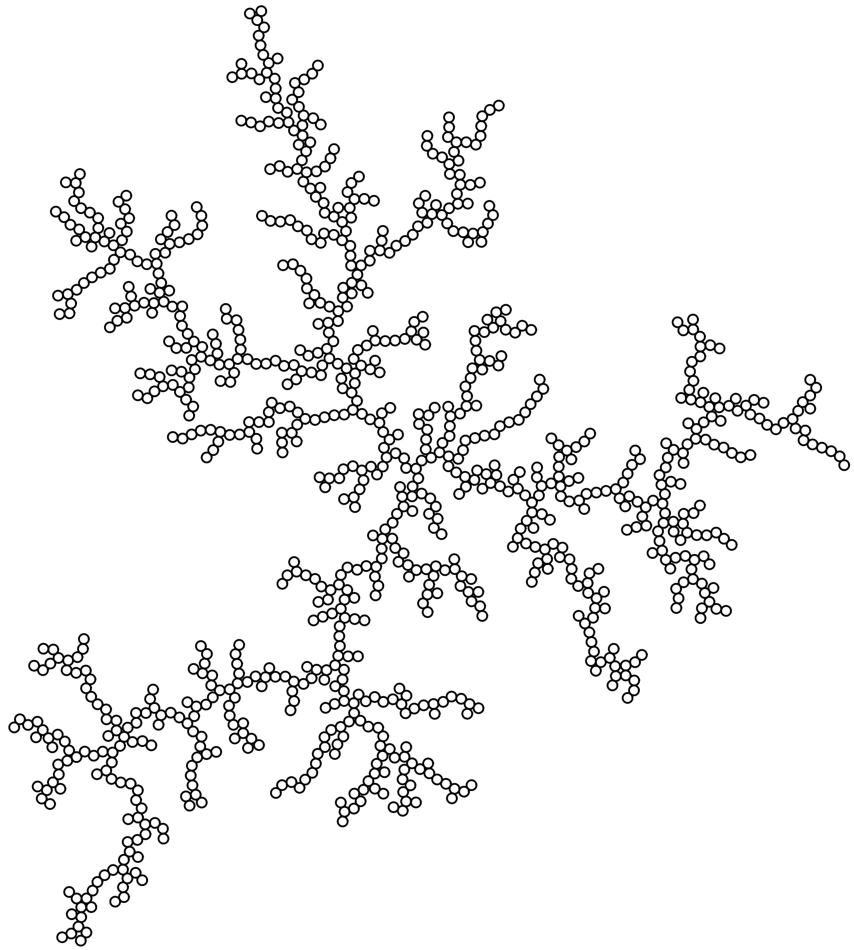


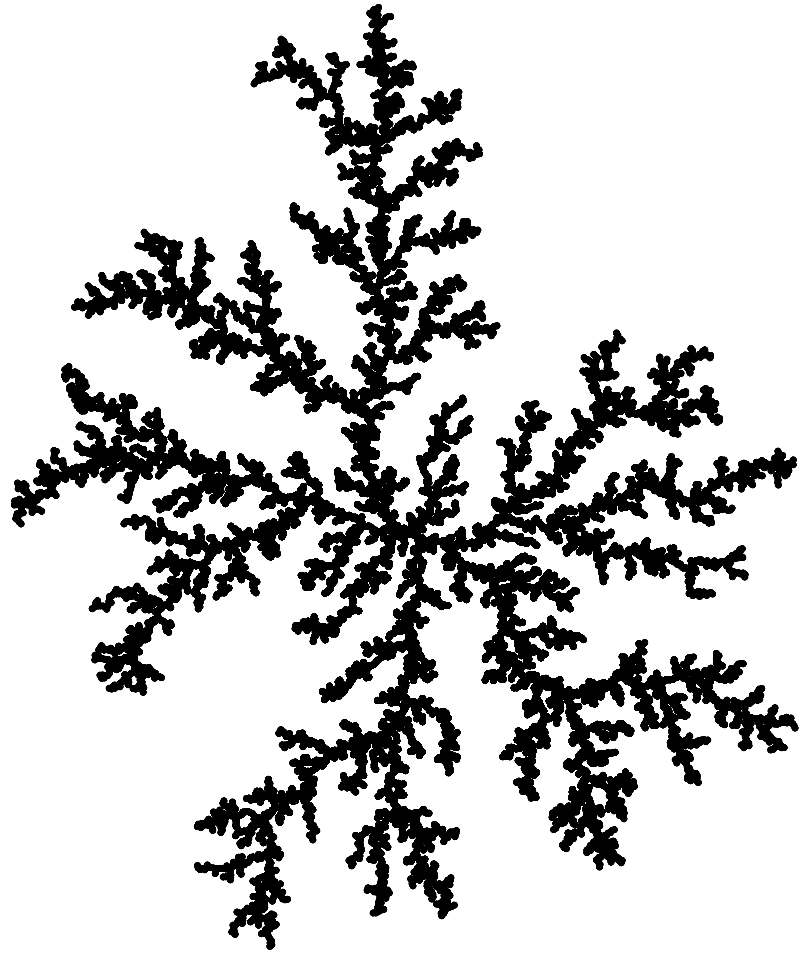












DLA, n = 10000

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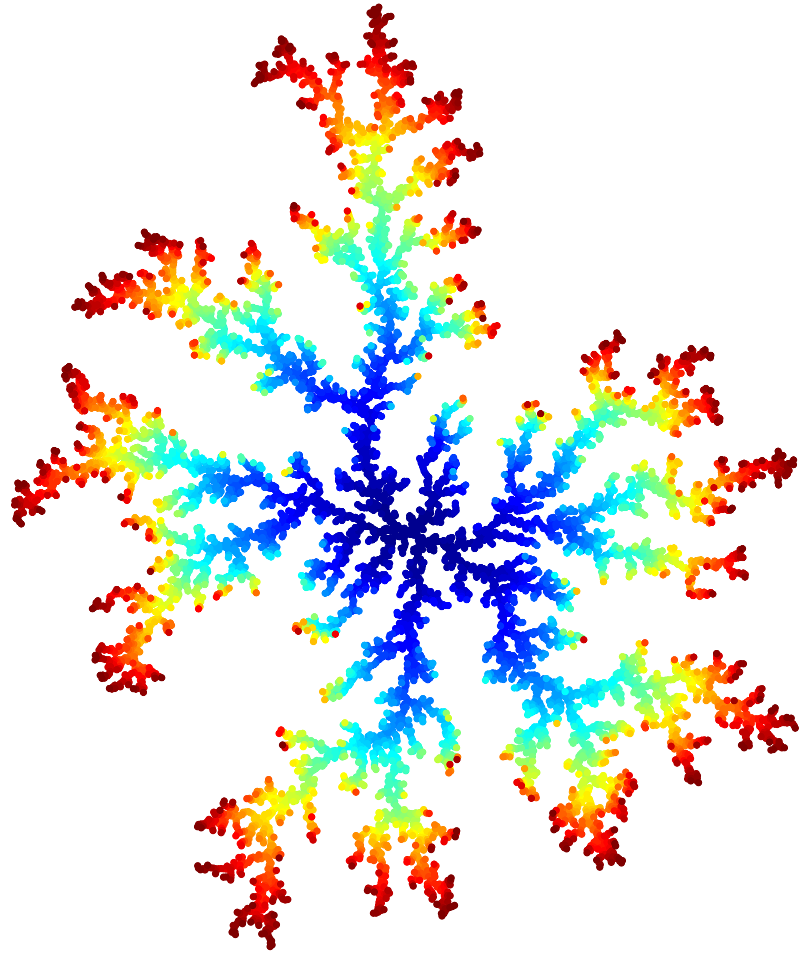
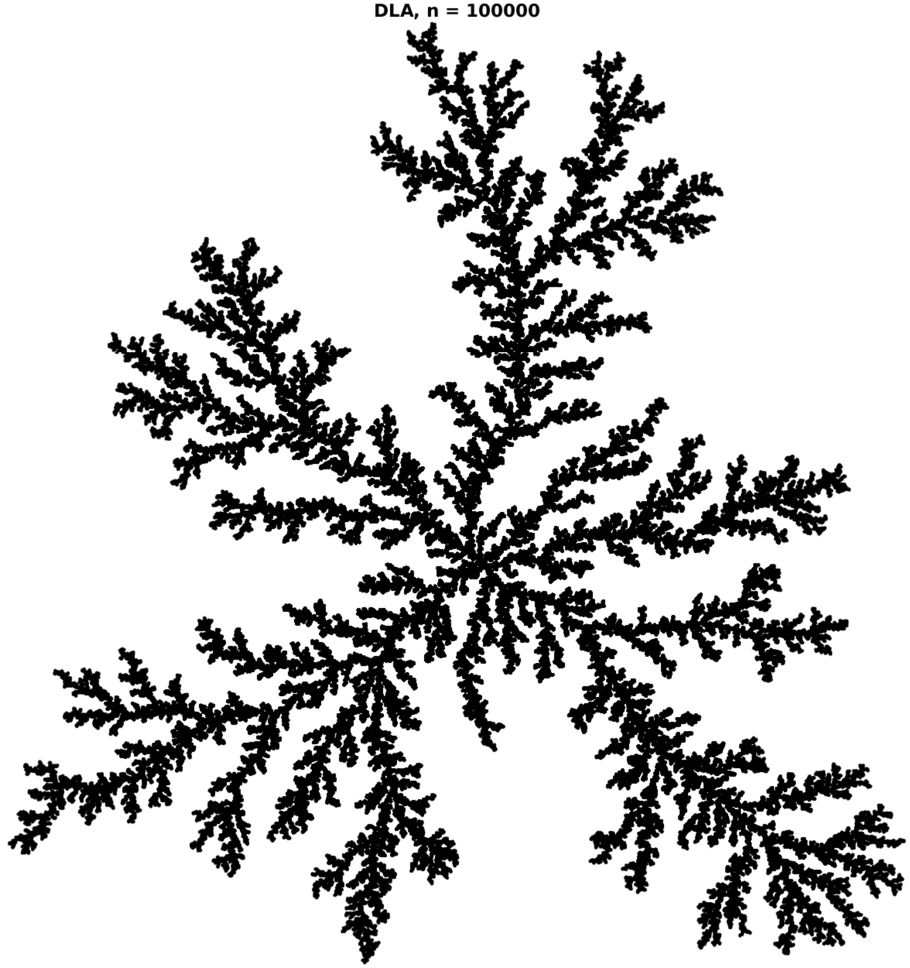
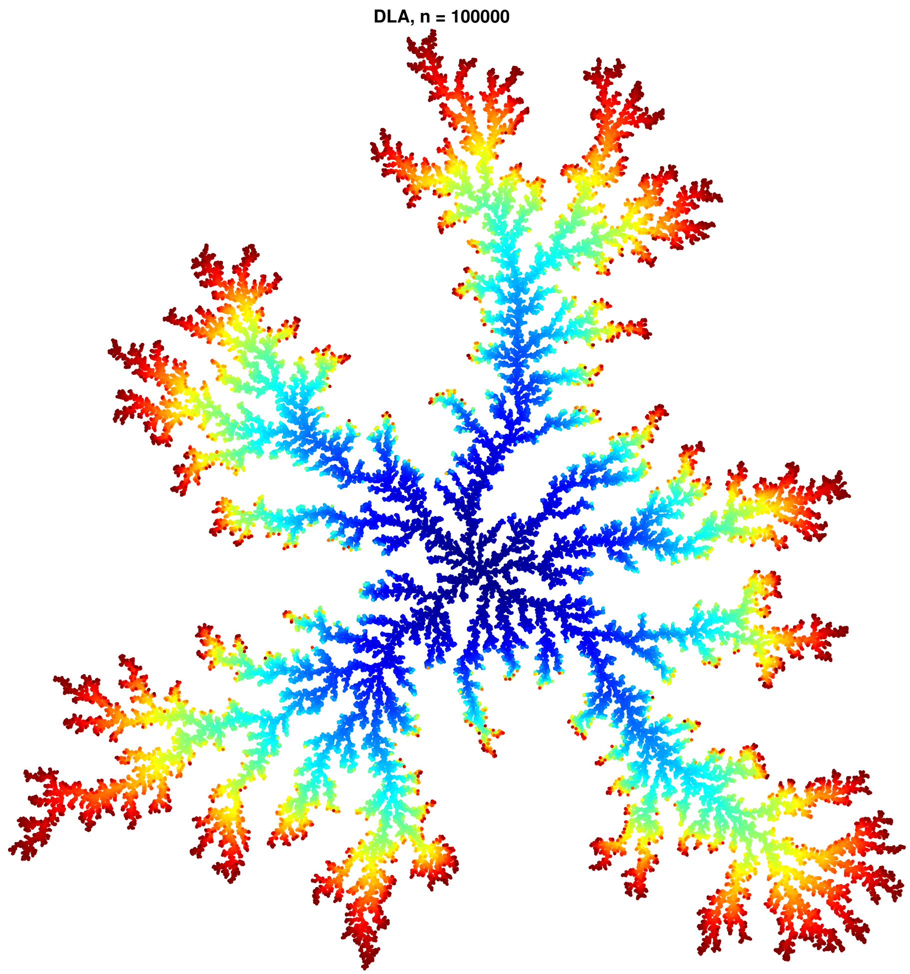
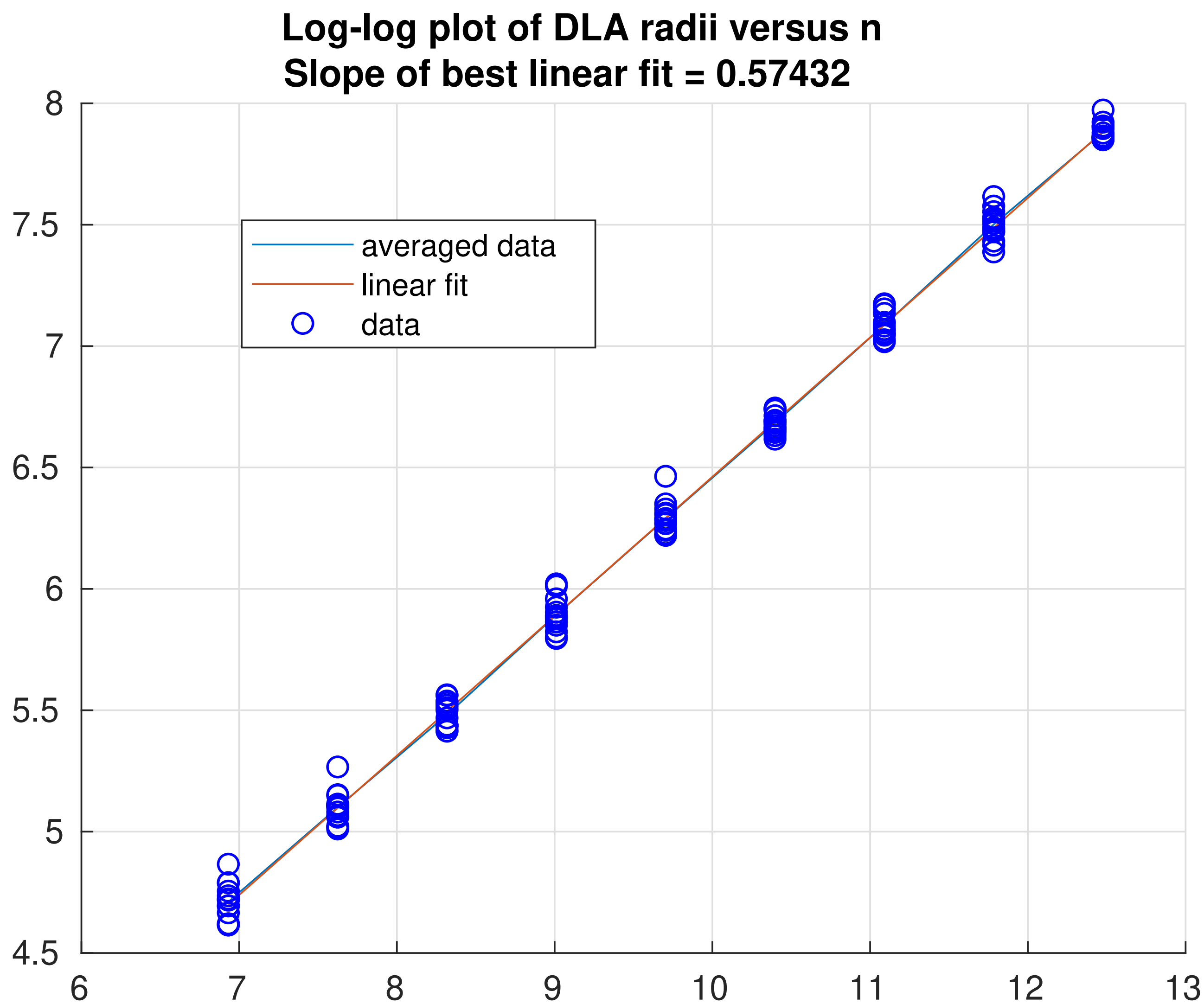


DLA, n = 100000

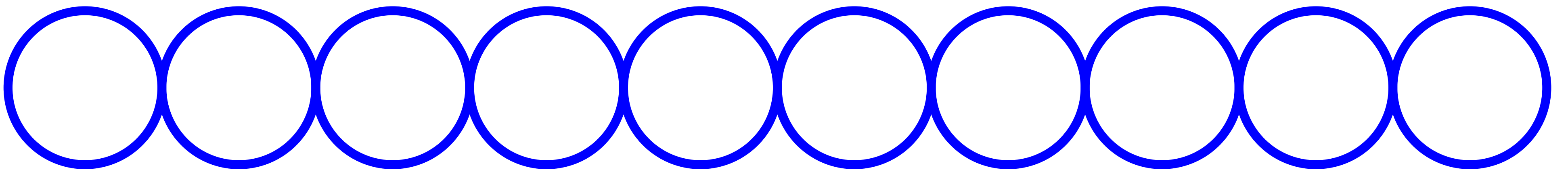




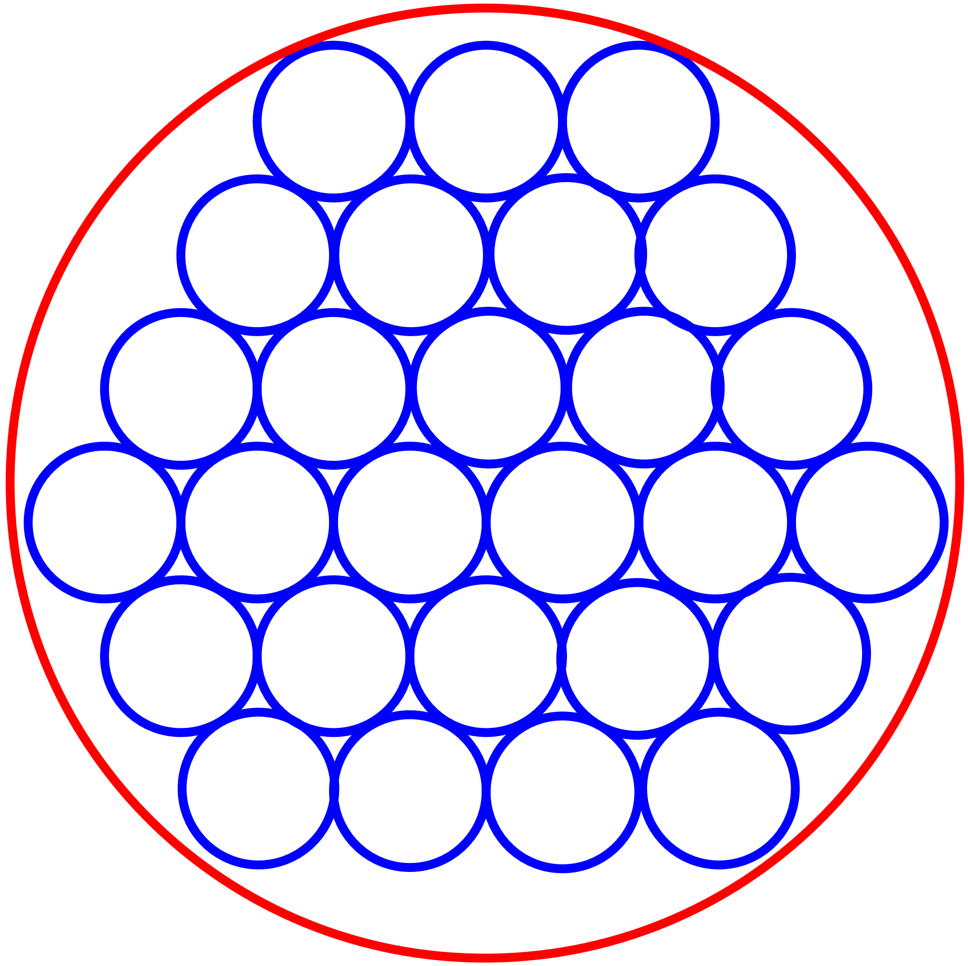
How fast does the diameter grow?



Numerical experiment for growth rate.



Trivial upper bound is $O(n)$.



Trivial lower bound is $\Omega(\sqrt{n})$.

Theorem (Kesten): $\operatorname{diam}(\mathrm{DLA}(n)) = O(n^{2/3})$.

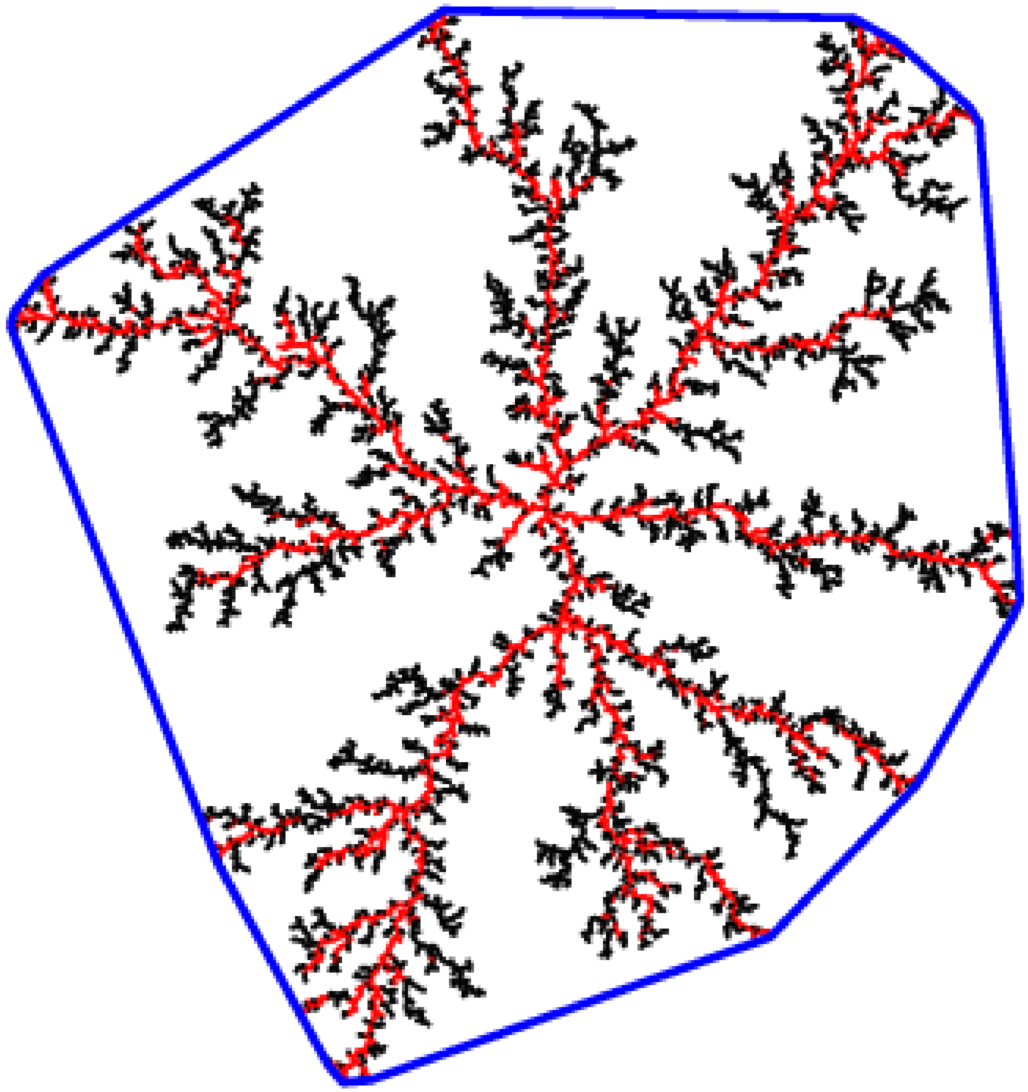
Amazingly, there is no known better lower bound than the trivial $\sqrt{n}$.

Conjecture: Almost surely,

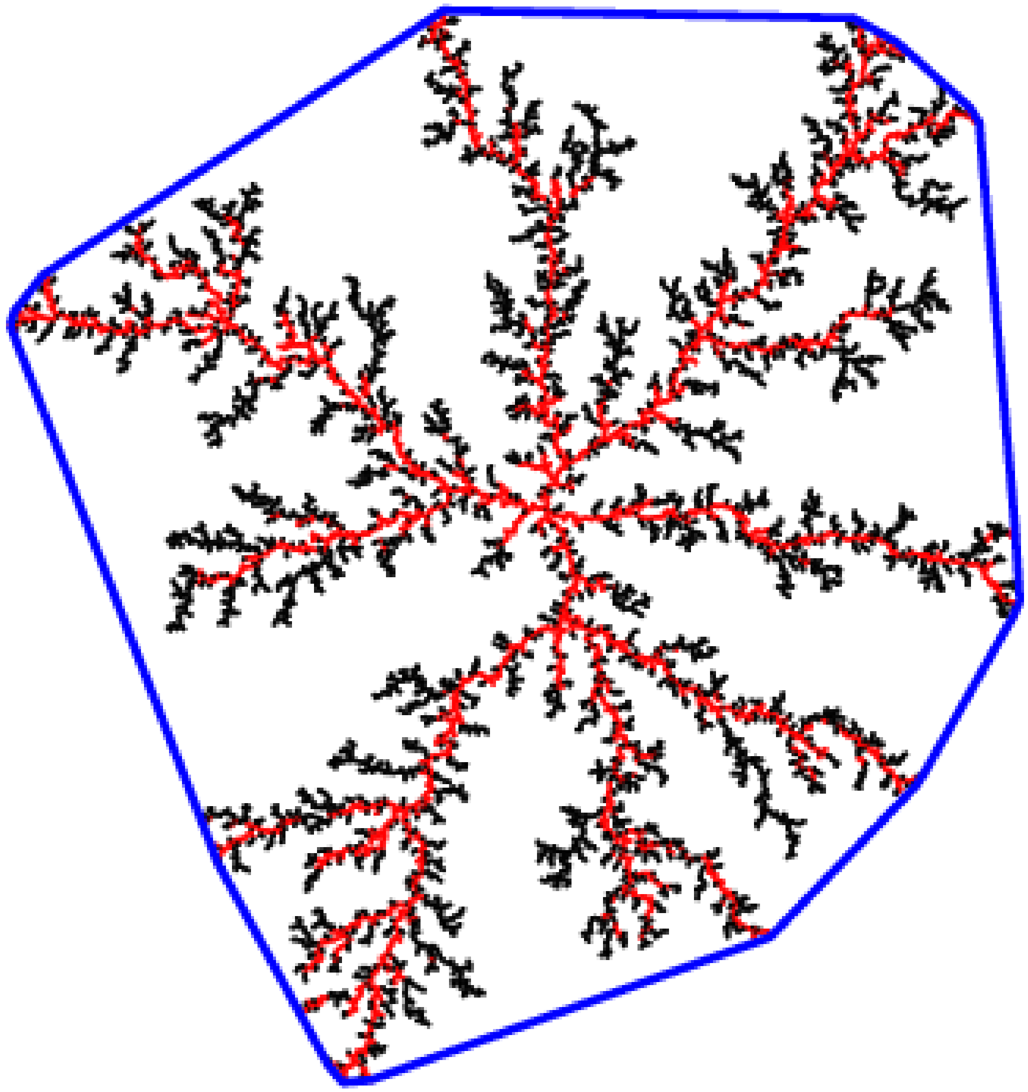
$$\lim_{n\to\infty} \frac{\operatorname{diam}(\mathrm{DLA}(n))}{\sqrt{n}} = \infty.$$

If $\mathrm{DLA}(n)$ is roughly a disk of radius $\sqrt{n}$ then any boundary disk is hit with probability $\simeq 1/\sqrt{n}$, which gives which gives the trivial lower bound. For non-trivial lower bound, we need to show there are points that get hit with probability $\gg n^{-1/2}$.

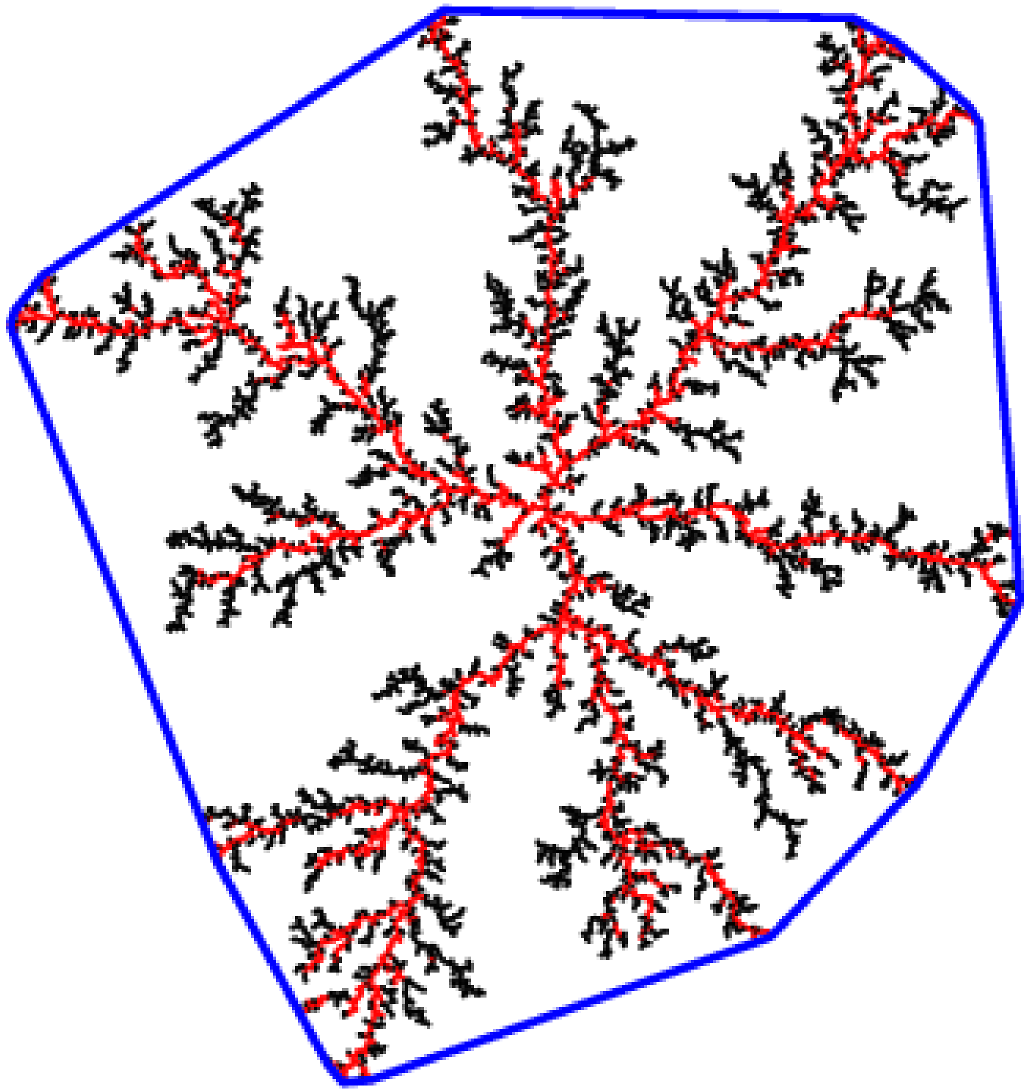
Consider convex hull of the DLA cluster. What is the harmonic measure of the disks that touch the convex hull boundary?

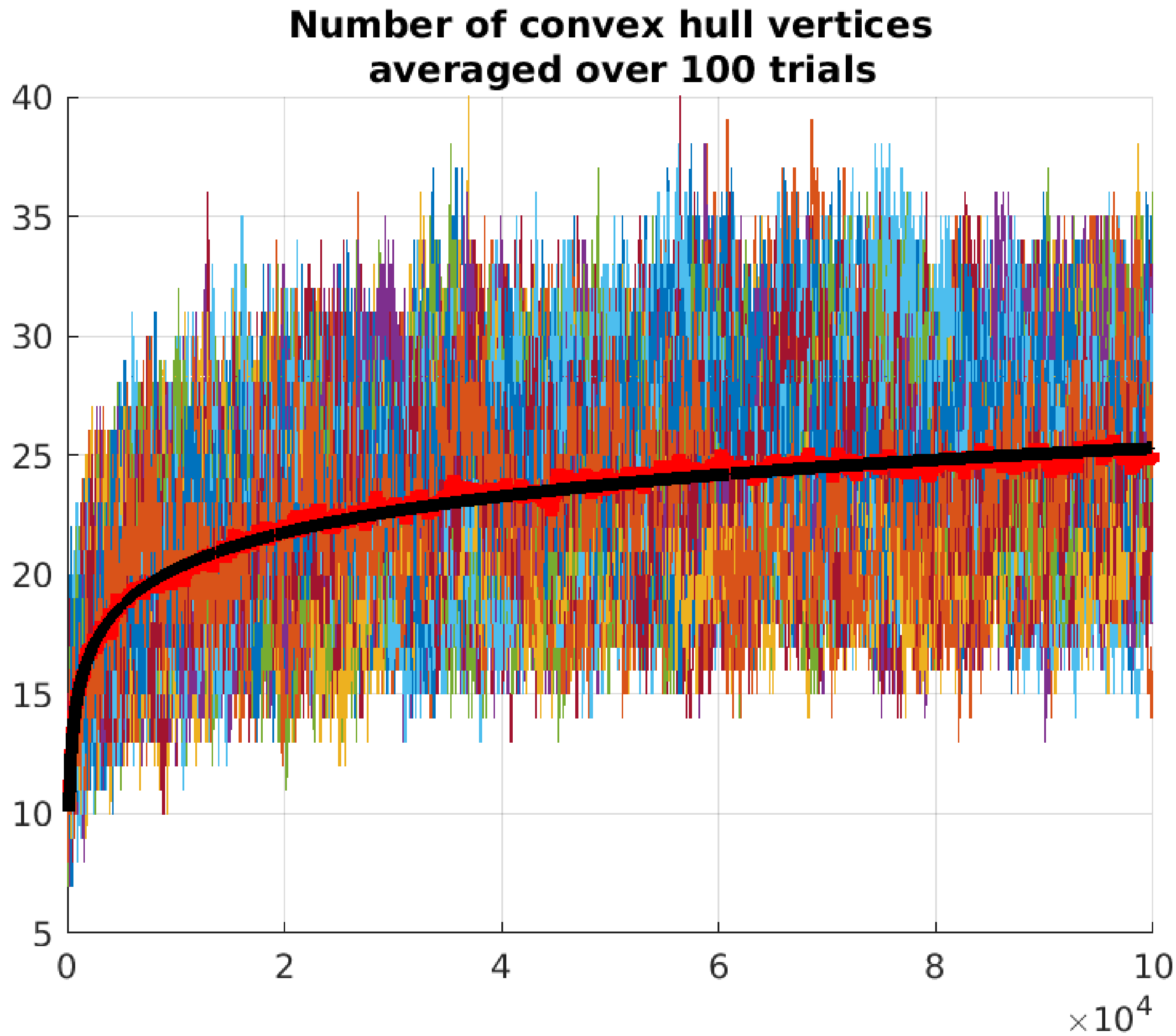


If convex hull has "sharp angles" some vertices have larger than average harmonic measure, implies faster than trivial growth.

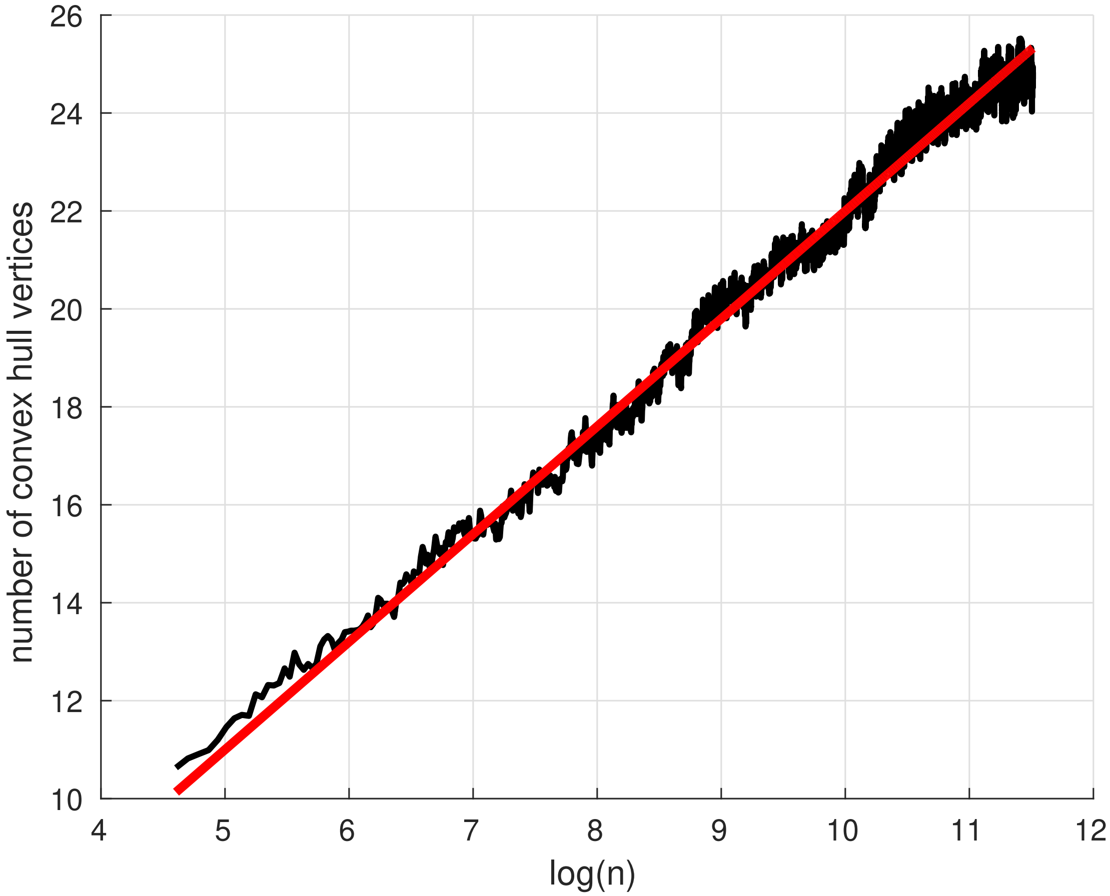


One way to have "sharp angles" is to have few vertices: if the convex hull boundary has few vertices, some of the angles should be large.

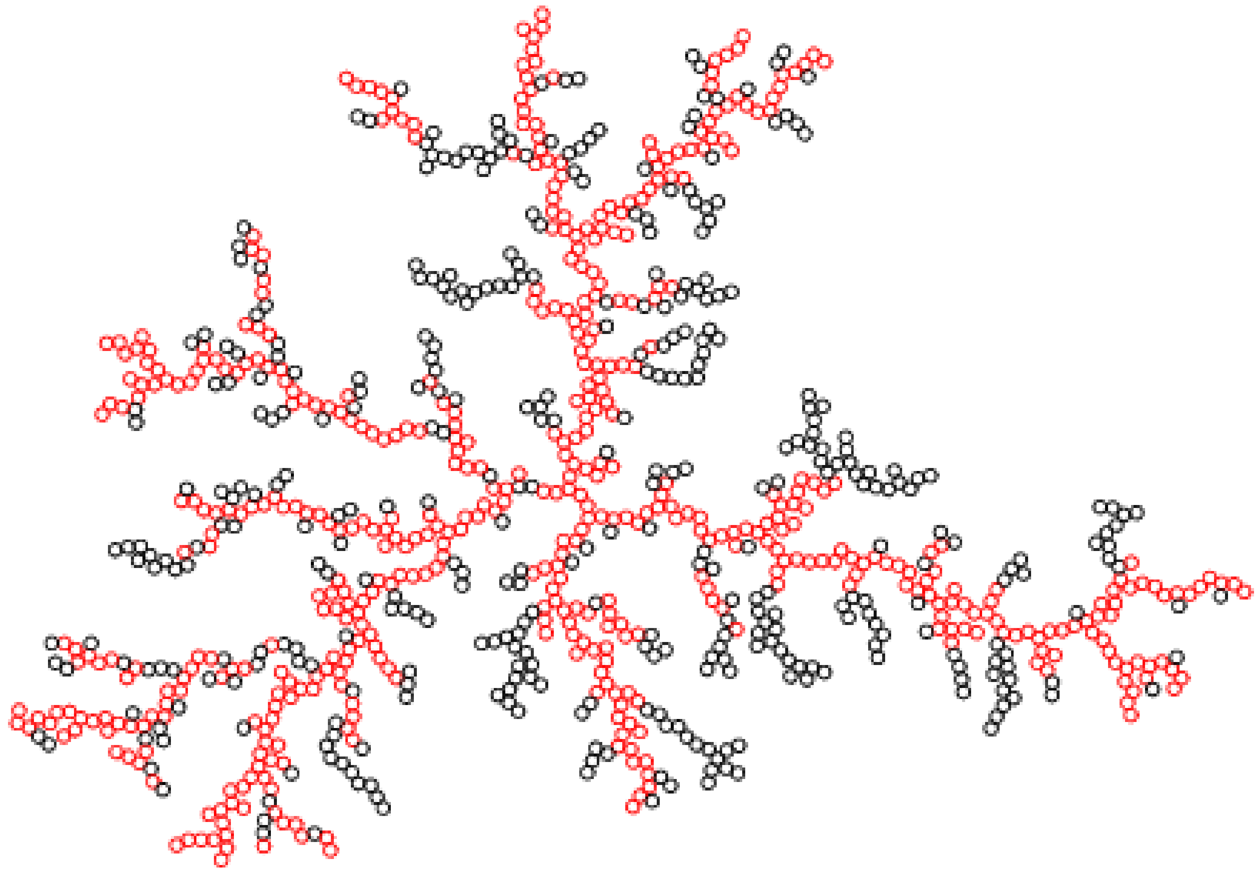




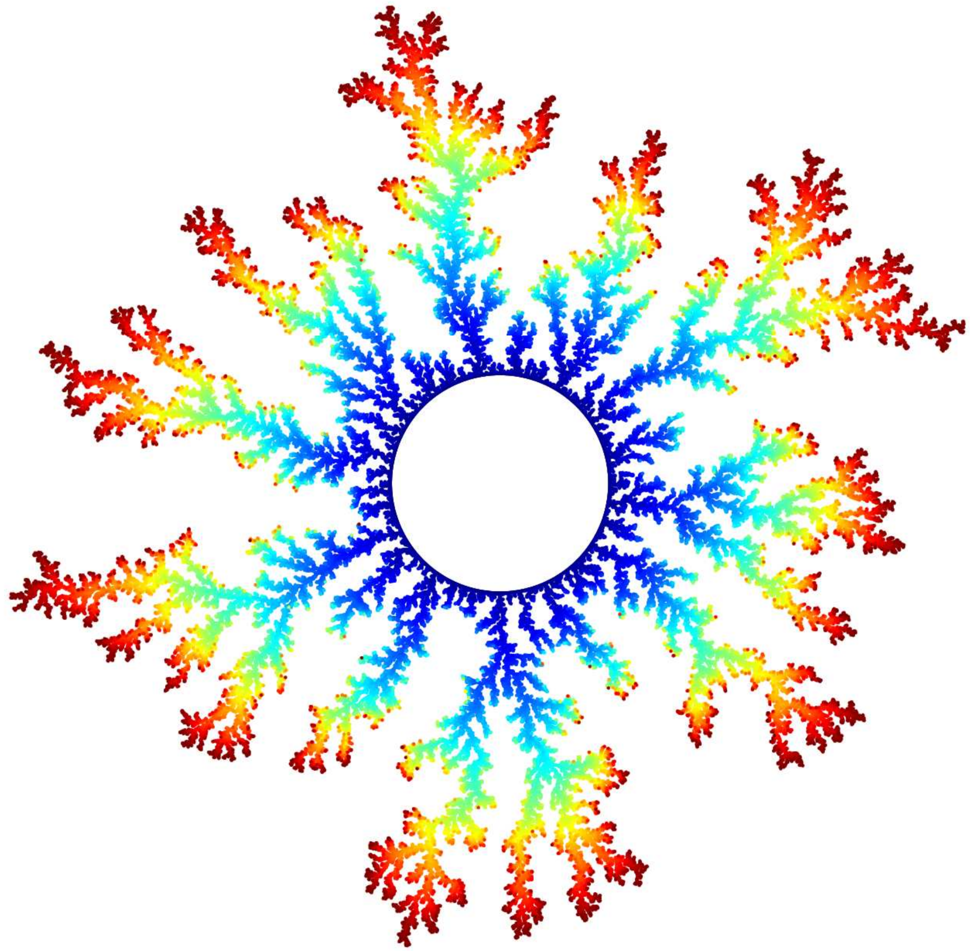
How many convex hull vertices are there at time n?

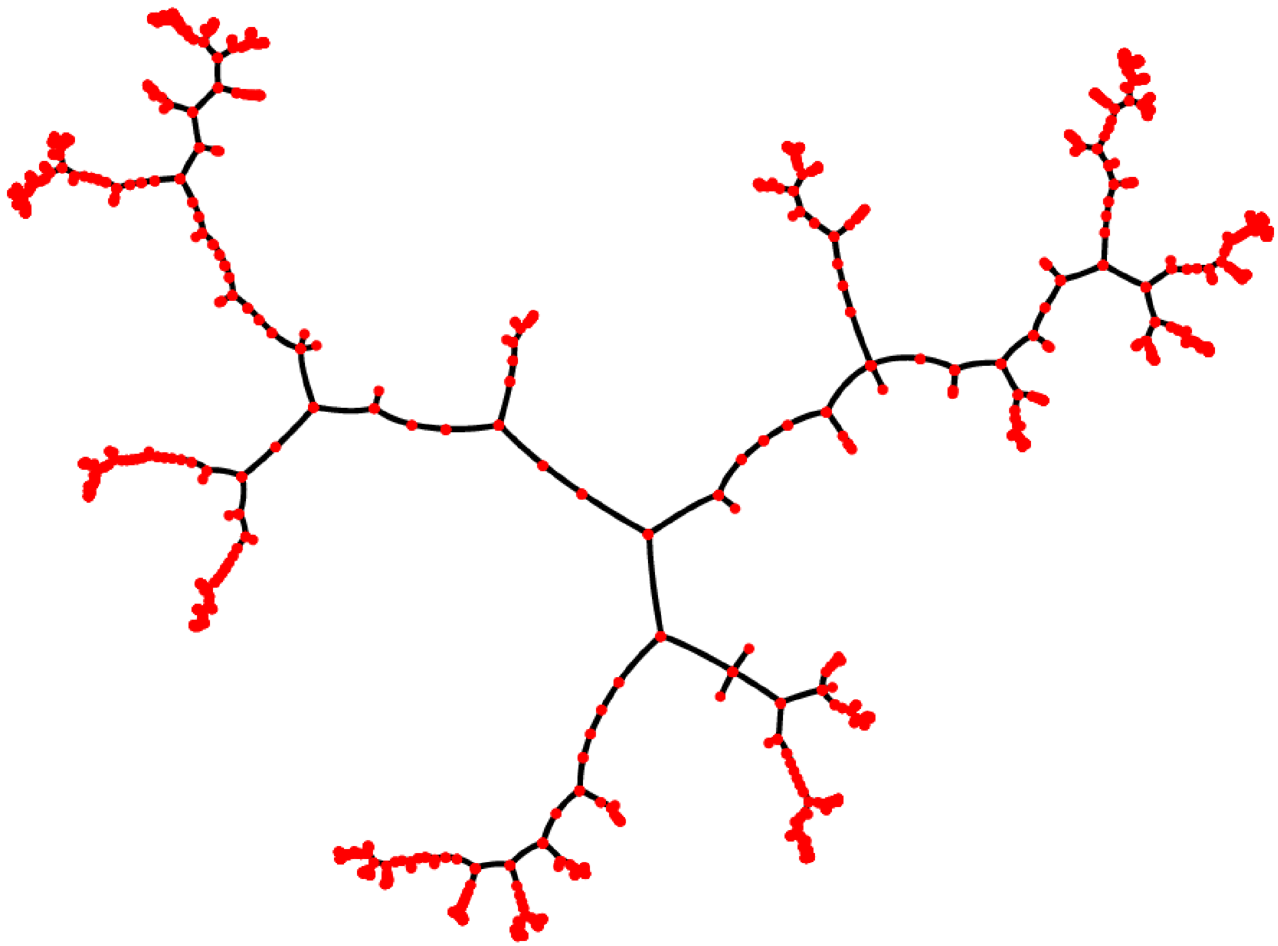


Number of convex hull vertices, averaged over 100 trials.
Plotted versus $\log(n)$, looks linear.
Numerically, $\approx (2.2)\log n$.

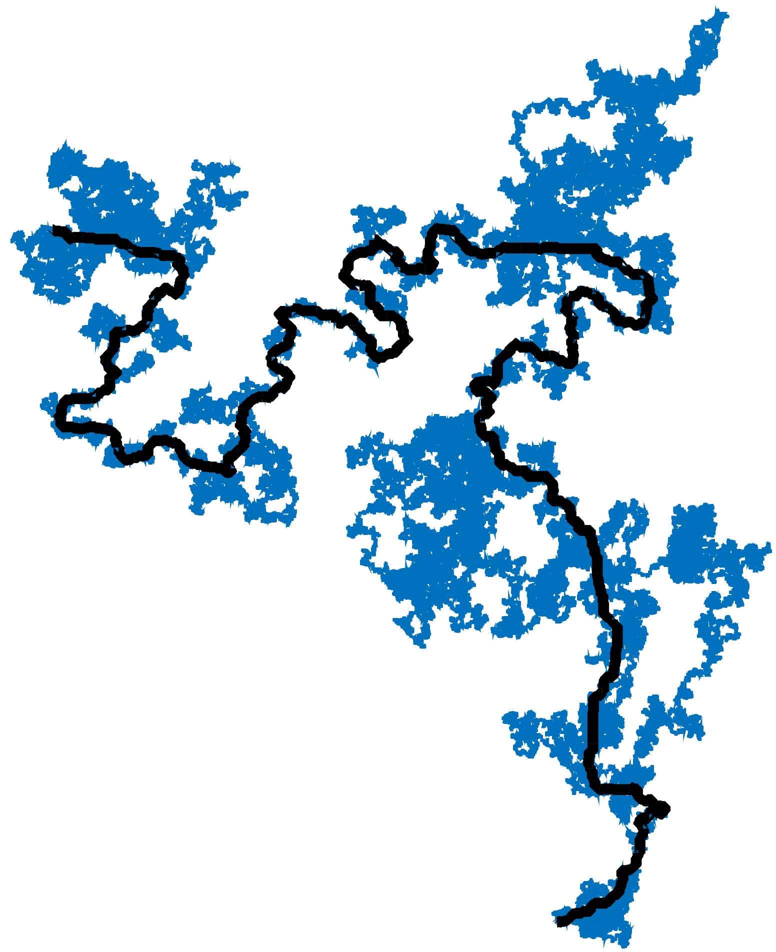


Red disks where on convex hull boundary when added.
Percentage probably tends to zero, but how fast?





SHORT PATHS IN THE BROWNIAN TRACE



Robin Pemantle proved Brownian motion does not cover a line segment.

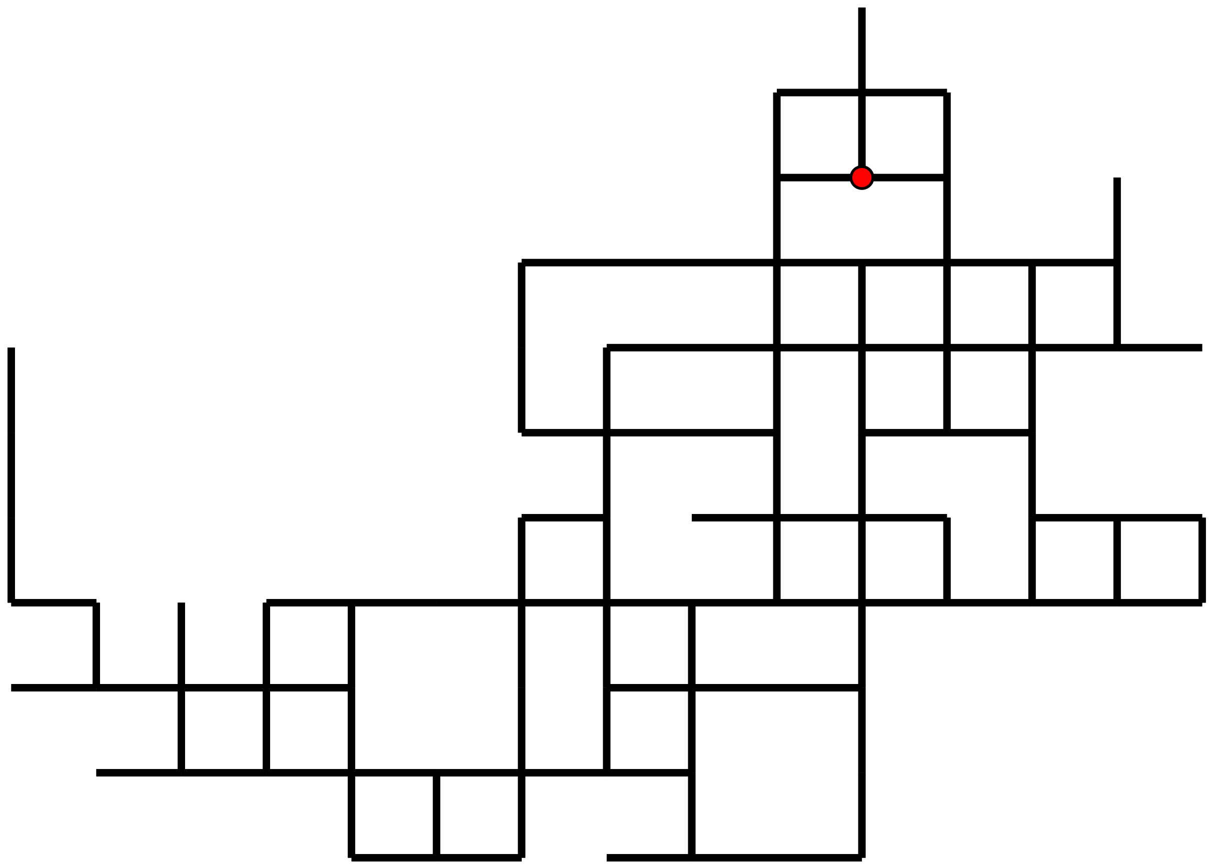
Question: Does it cover a rectifiable curve?

Question: Does it cover a curve of dimension $1+\epsilon$, any $\epsilon > 0$?

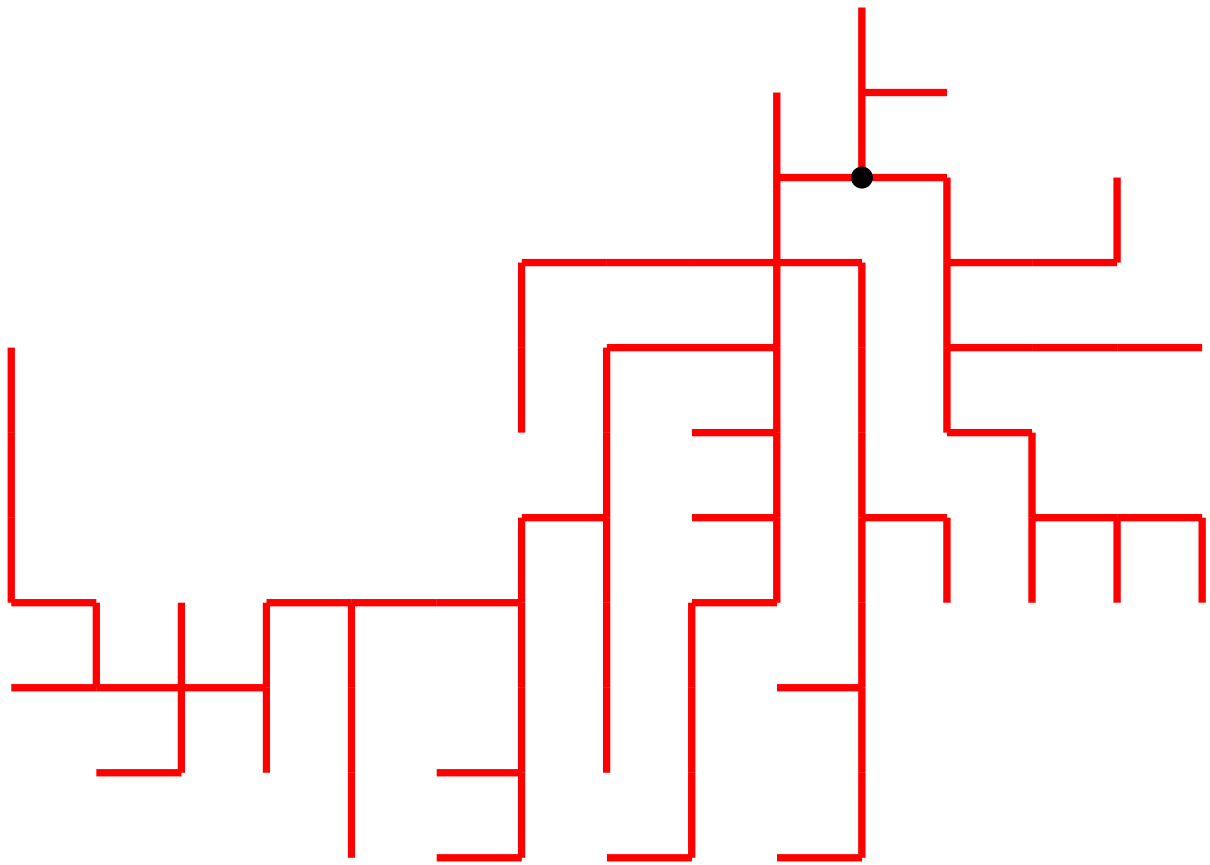
Percolation dimension = minimal dimension of Jordan arc inside the set.

Frontiers have dimension $4/3$ (Lawler et. al.).

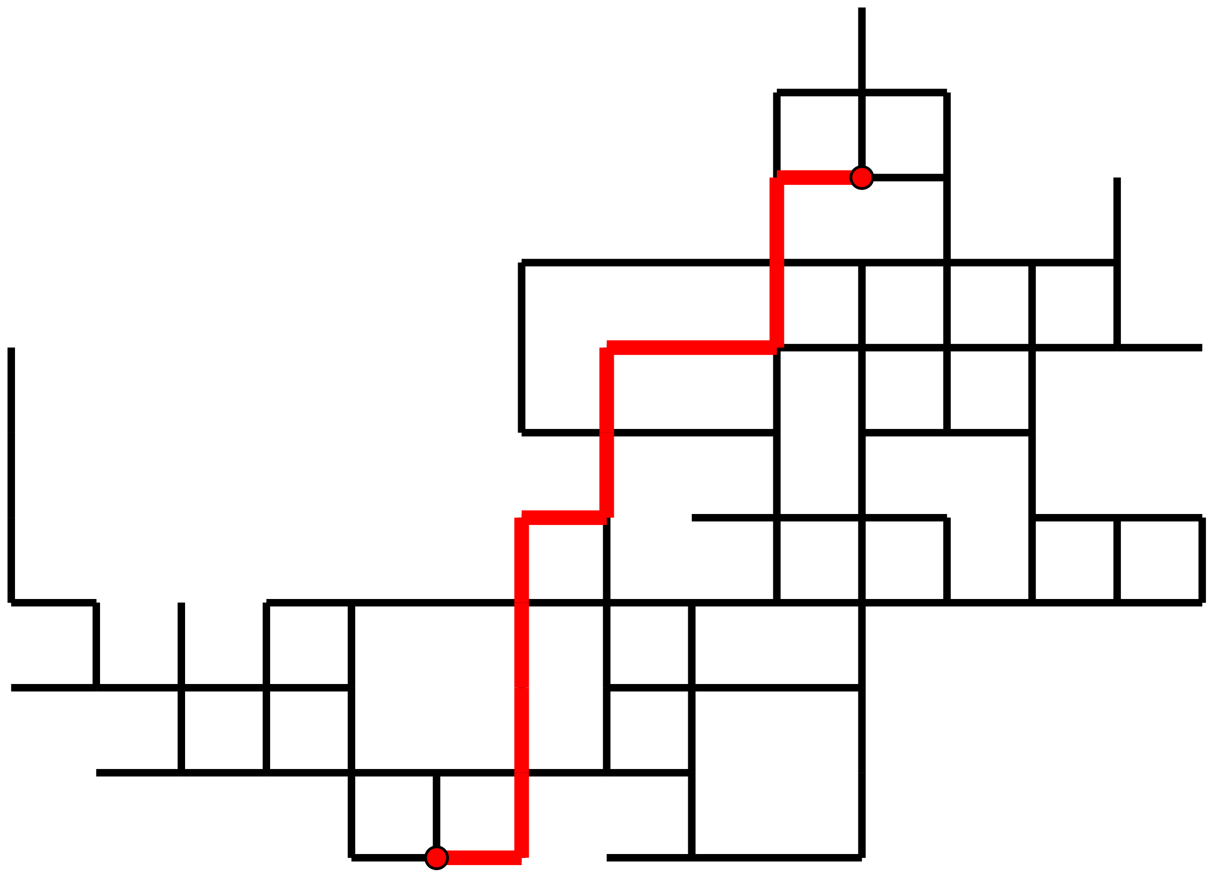
Brownian trace contains curves of dimension $5/4$ (Dapeng Zhan).



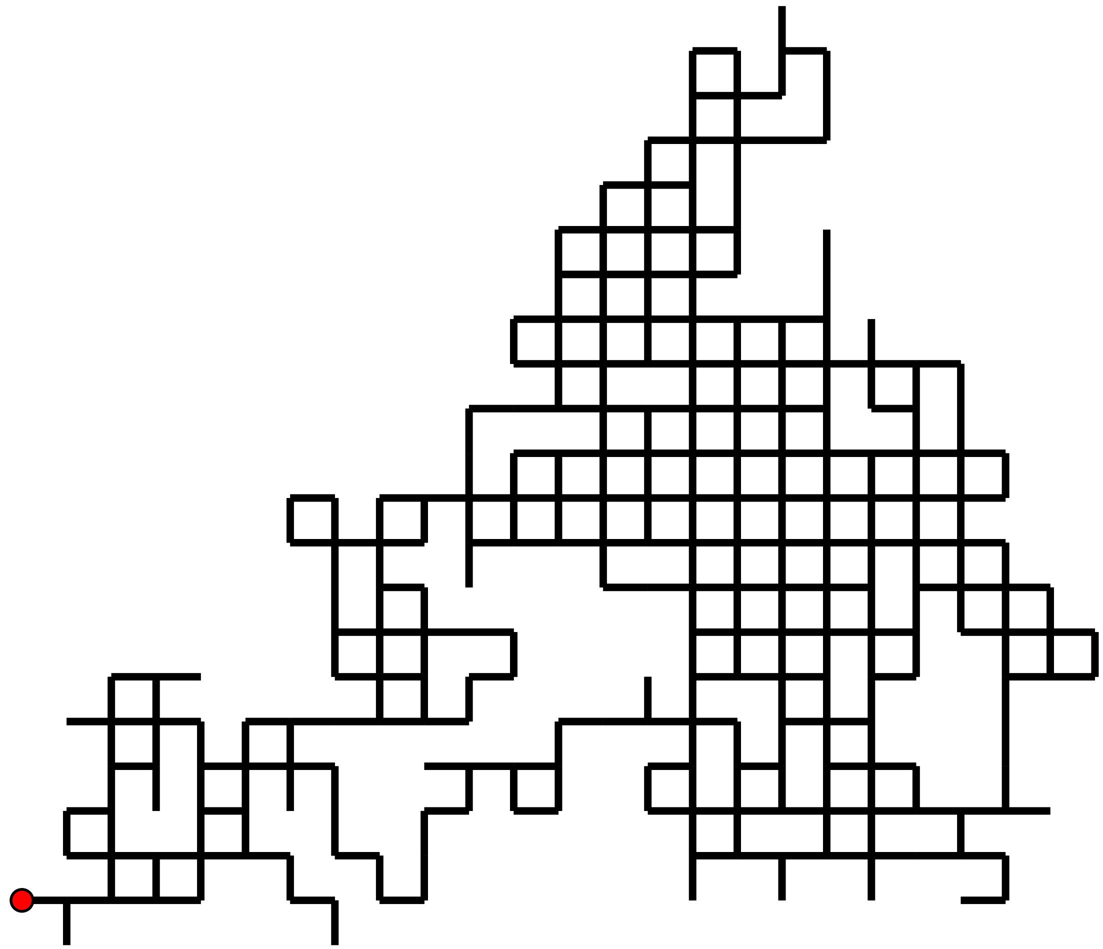
200 step random walk.



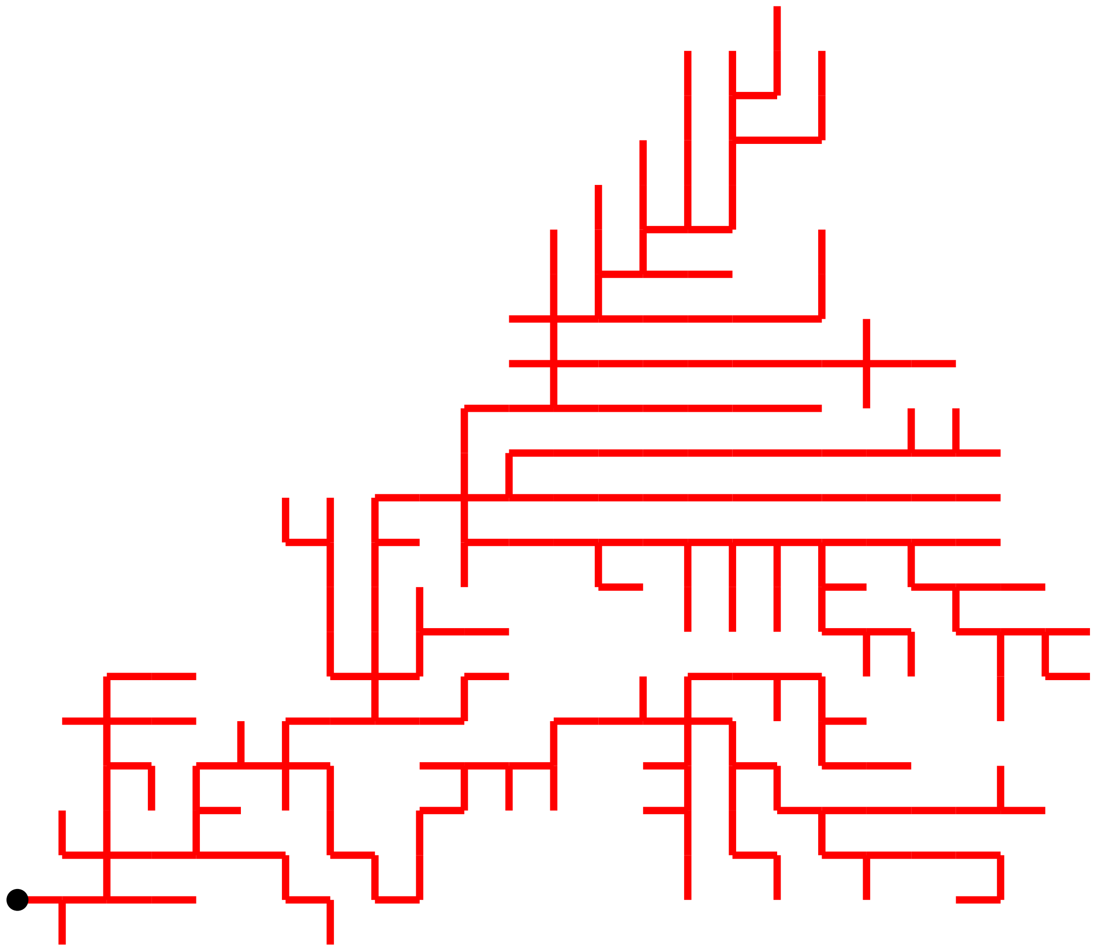
Minimal distance rooted spanning tree.



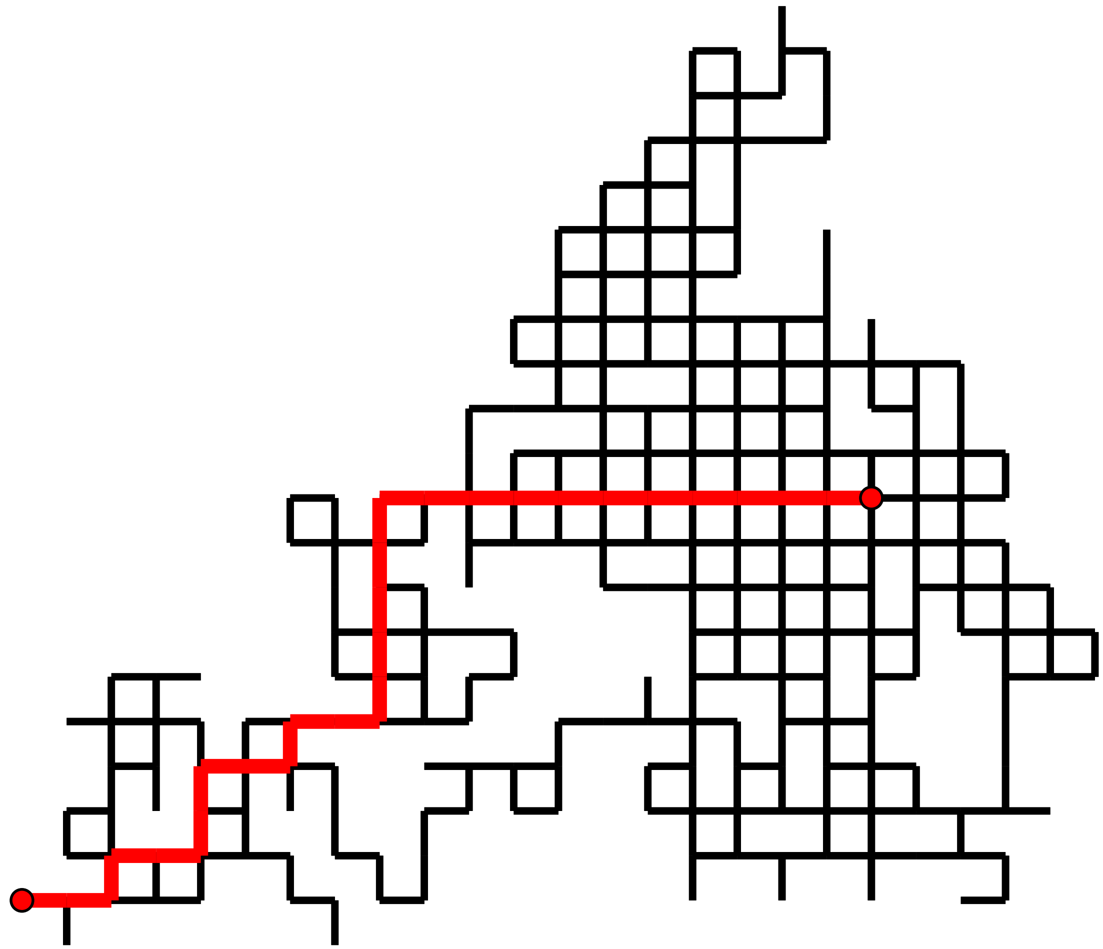
A shortest path from 0 to $\sqrt{n}/2$.



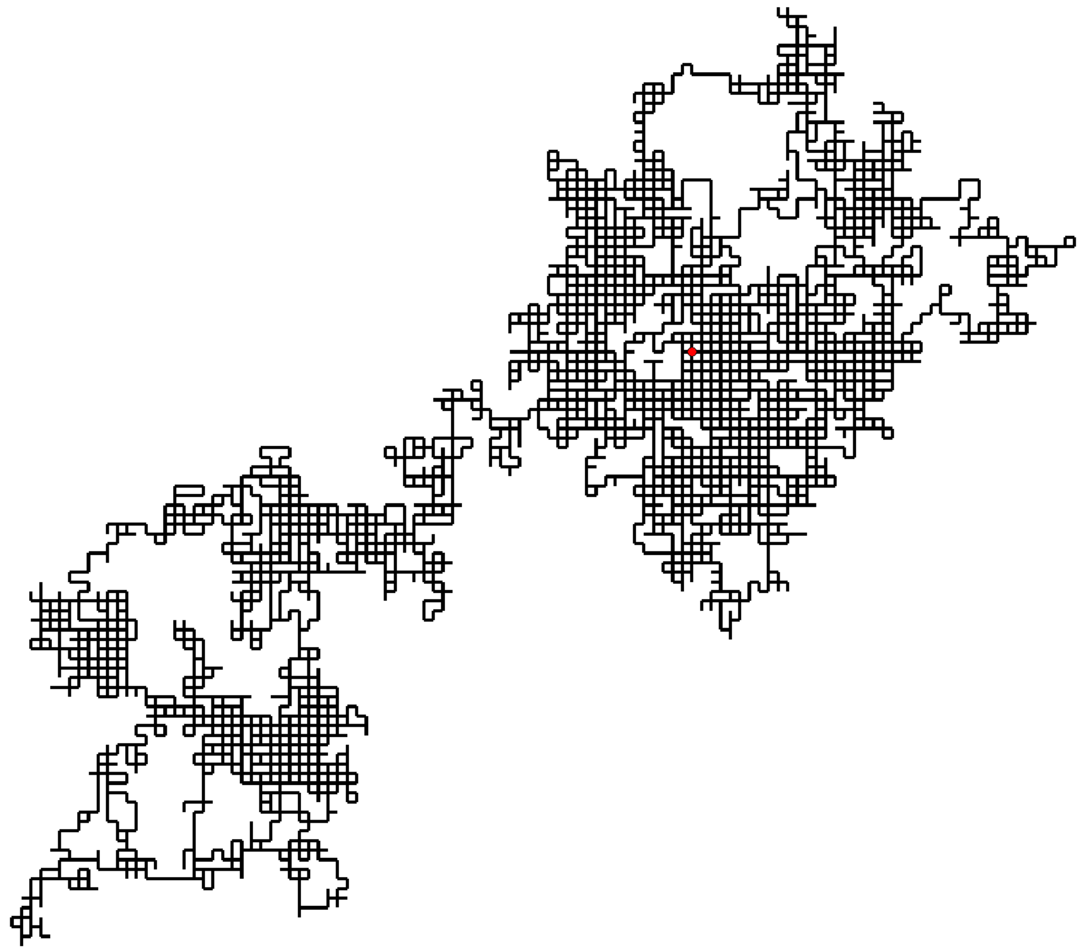
1000 step random walk.



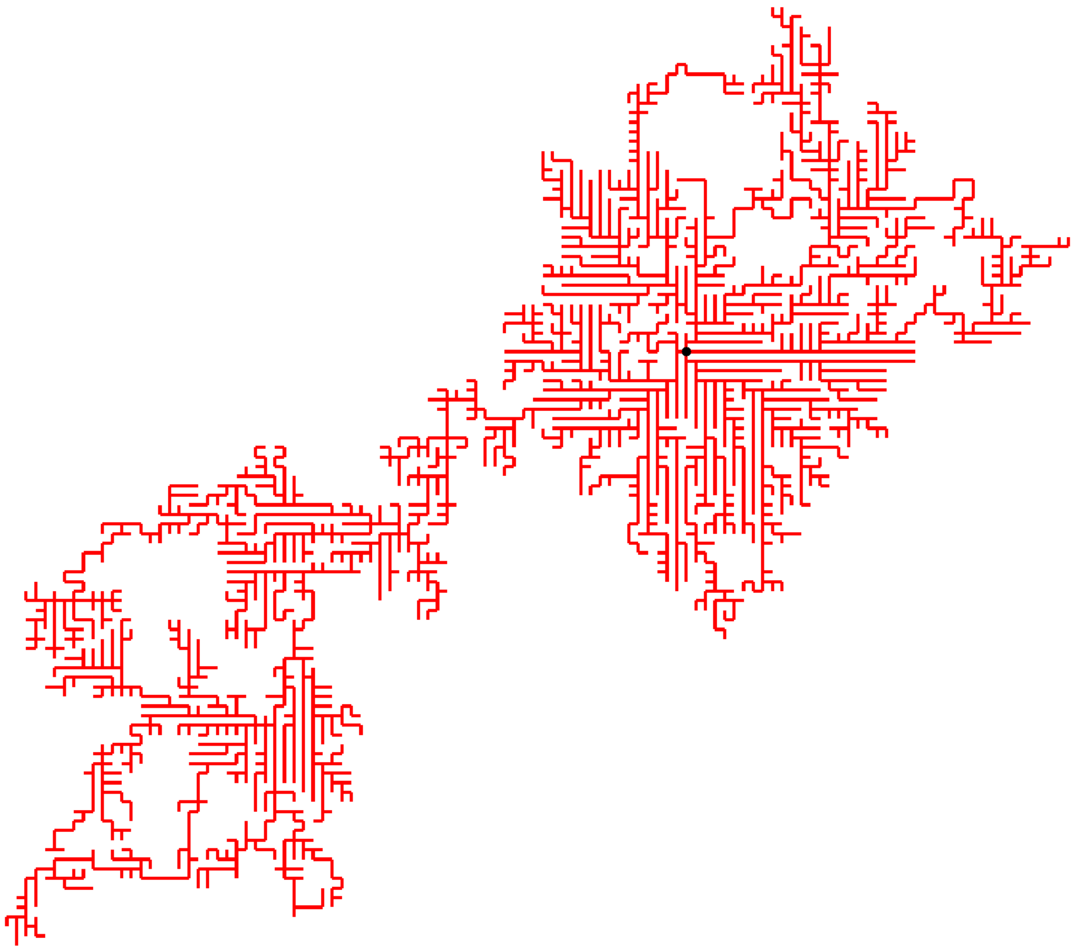
Minimal distance rooted spanning tree.



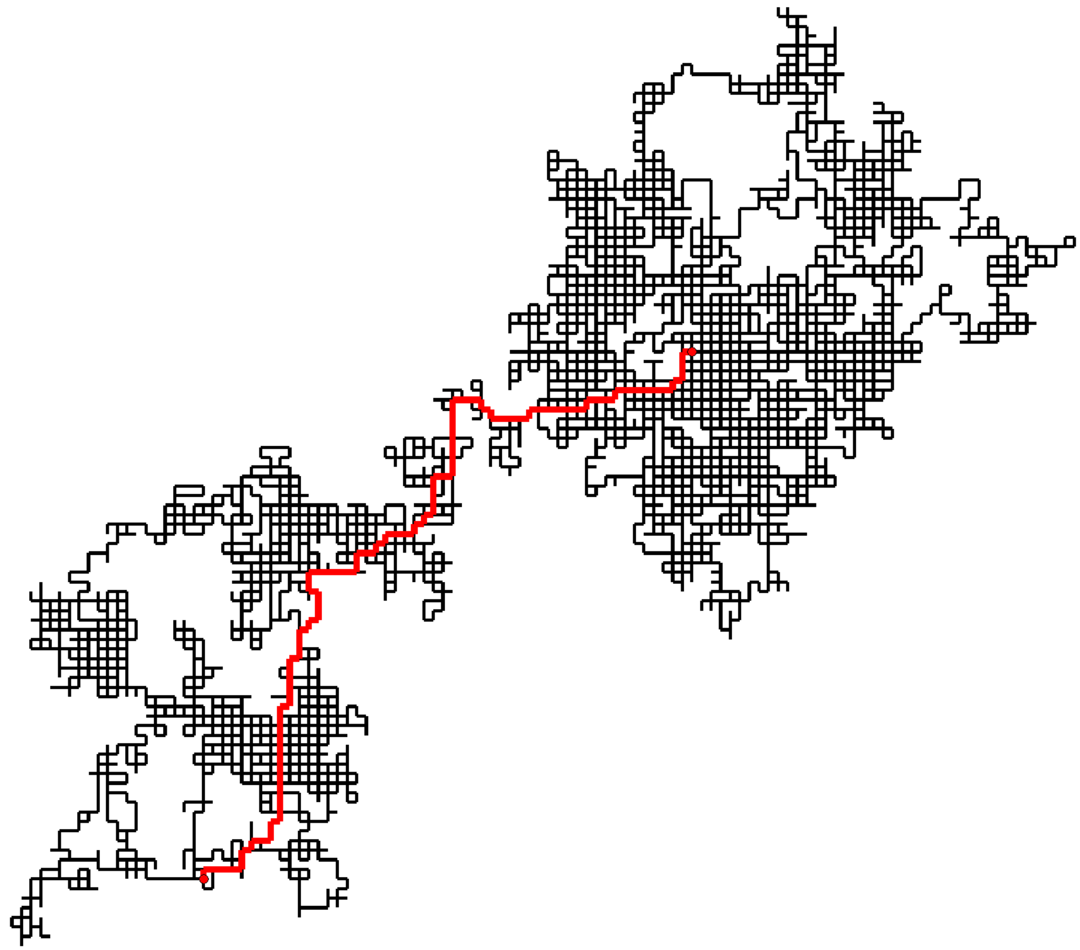
A shortest path from 0 to $\sqrt{n}/2$.



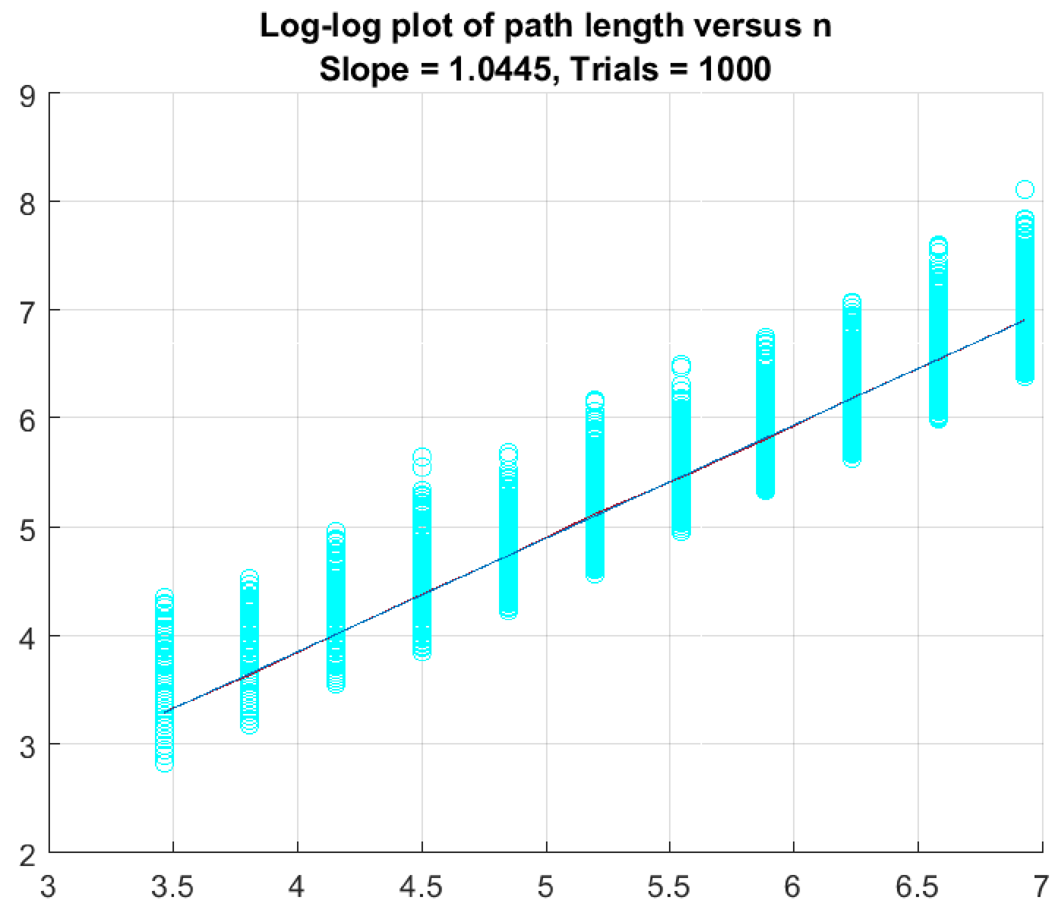
10000 step random walk.



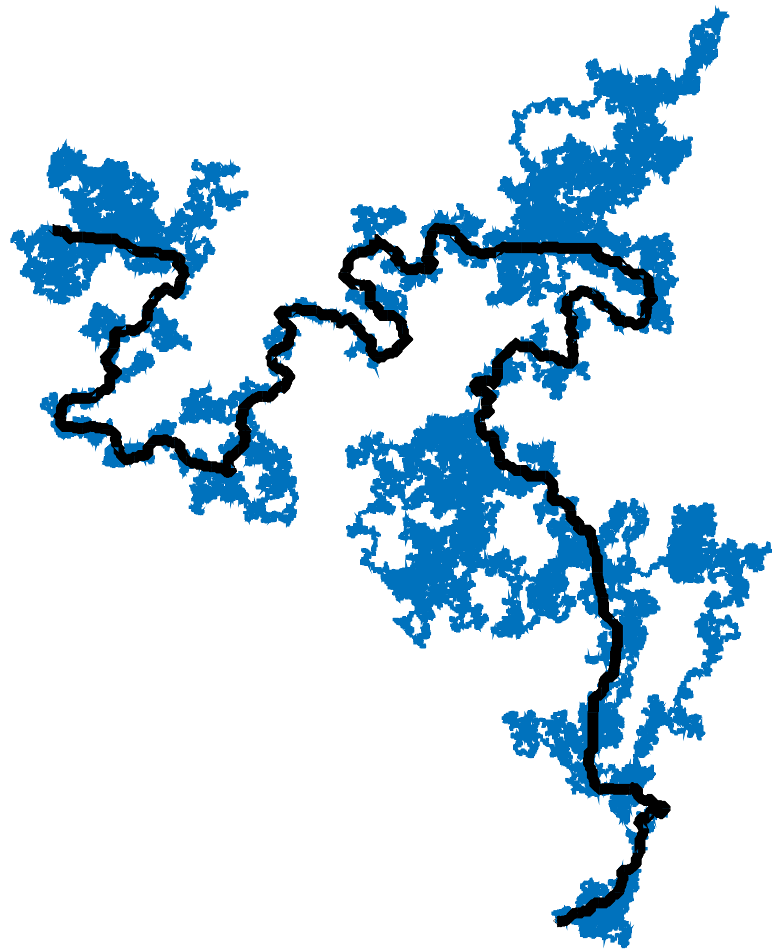
Minimal distance spanning tree, wrt to origin.

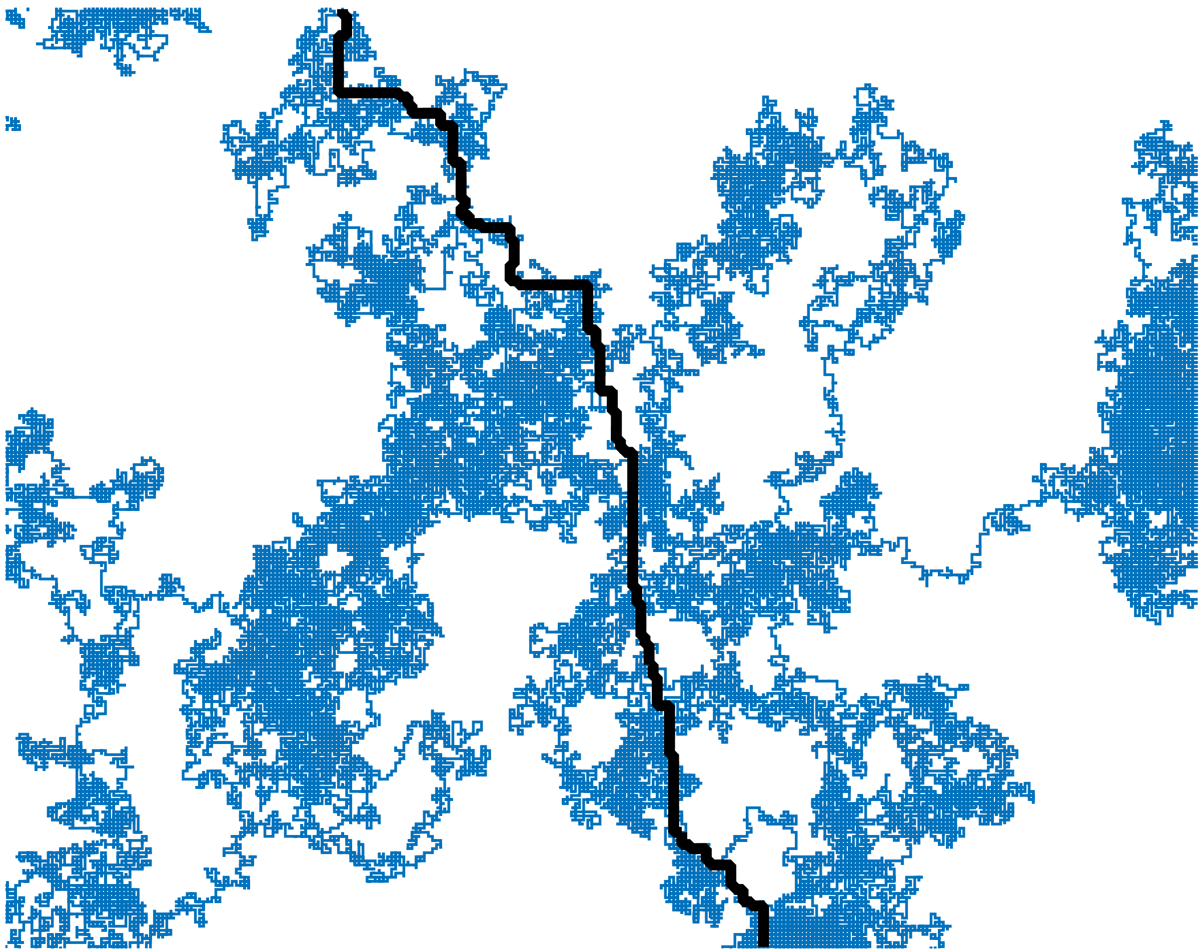


Minimal length path to distance $\sqrt{n}$.

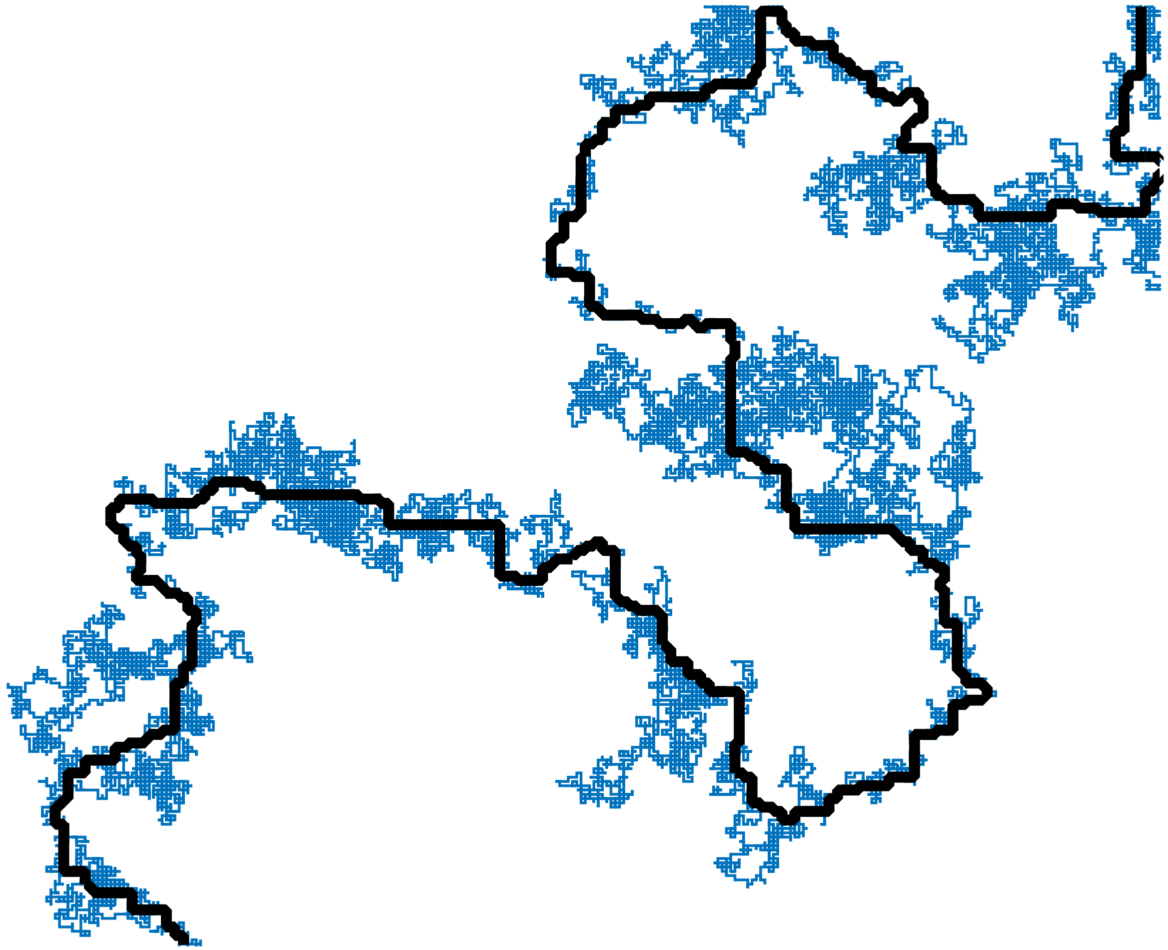


Log-log plot of graph distance 0 to $\{|z| = \frac{1}{2}\sqrt{n}\}$.

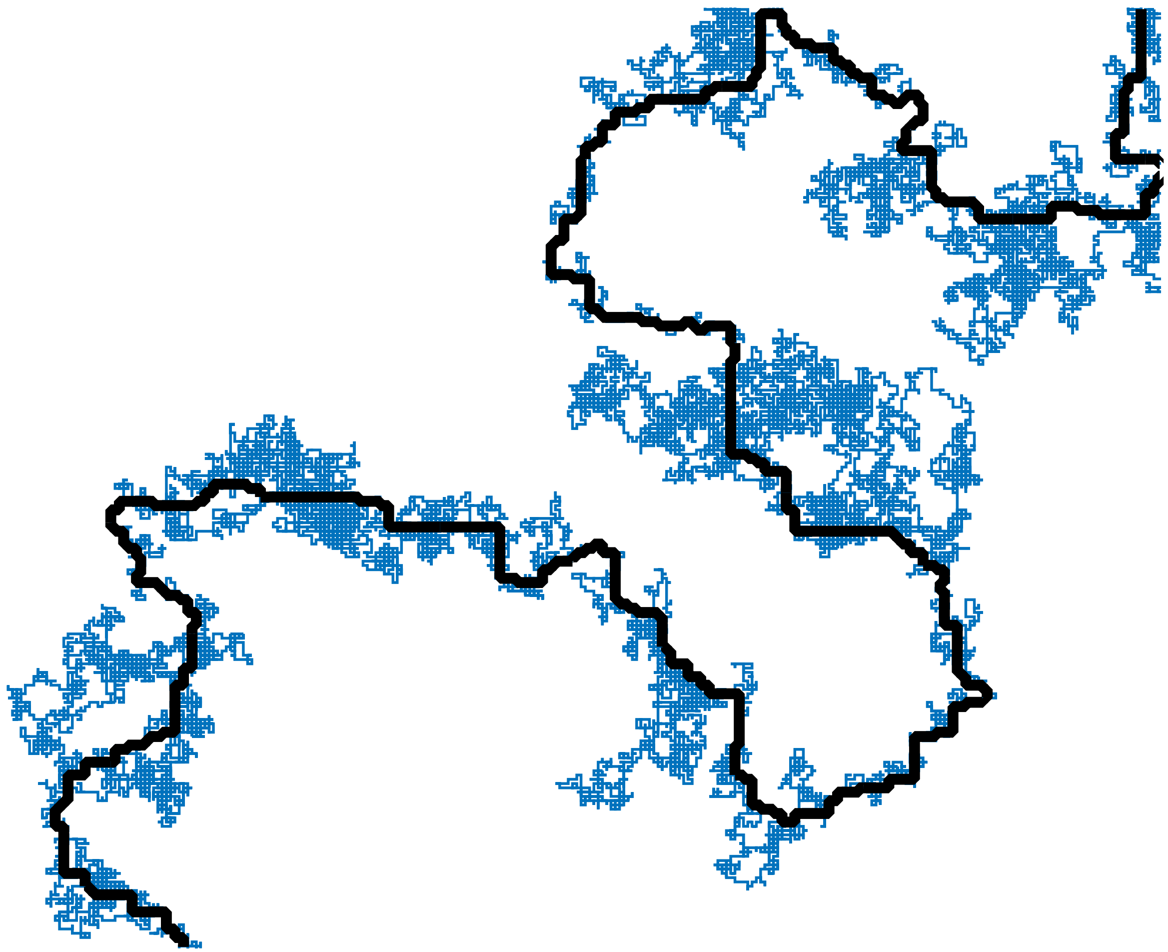




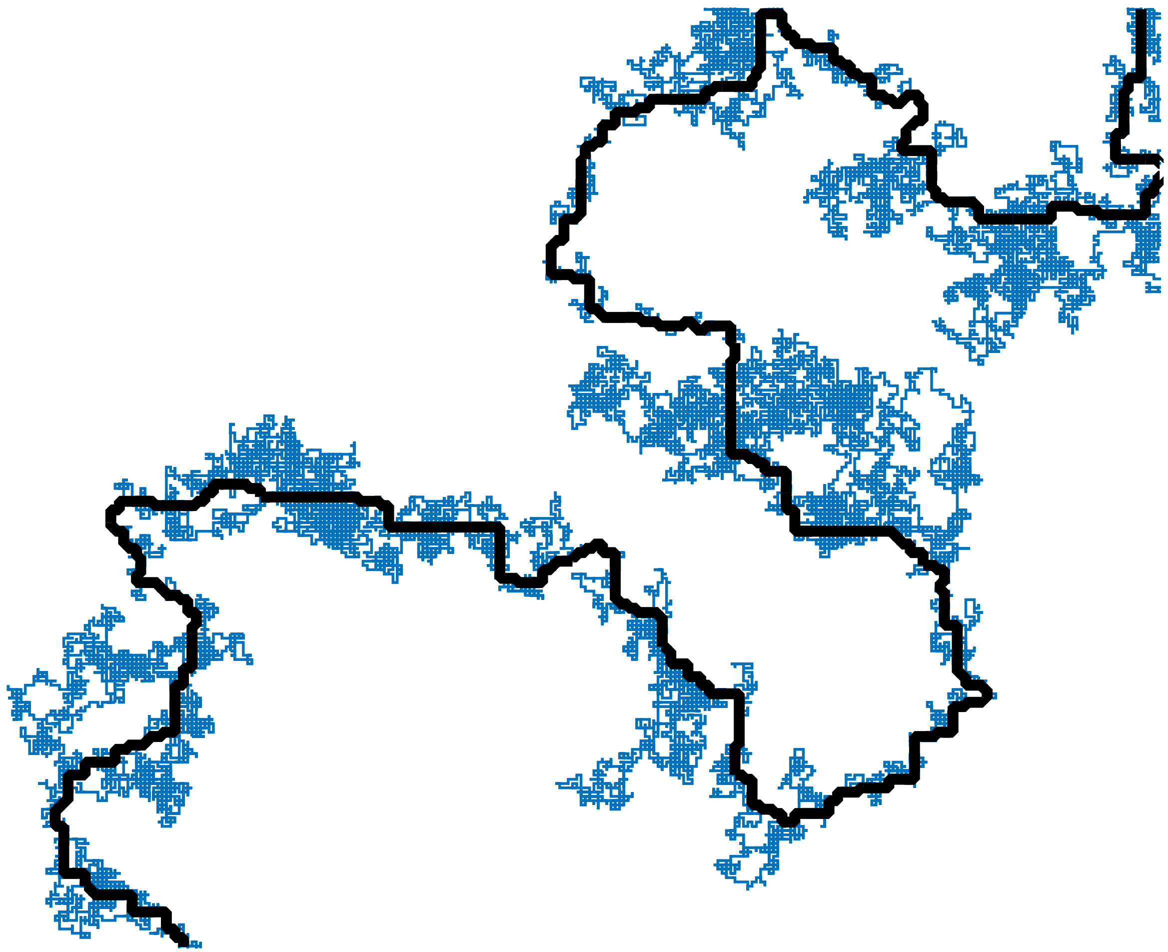
Paths can be straight where trace is dense.



Paths wiggle more when trapped between components.

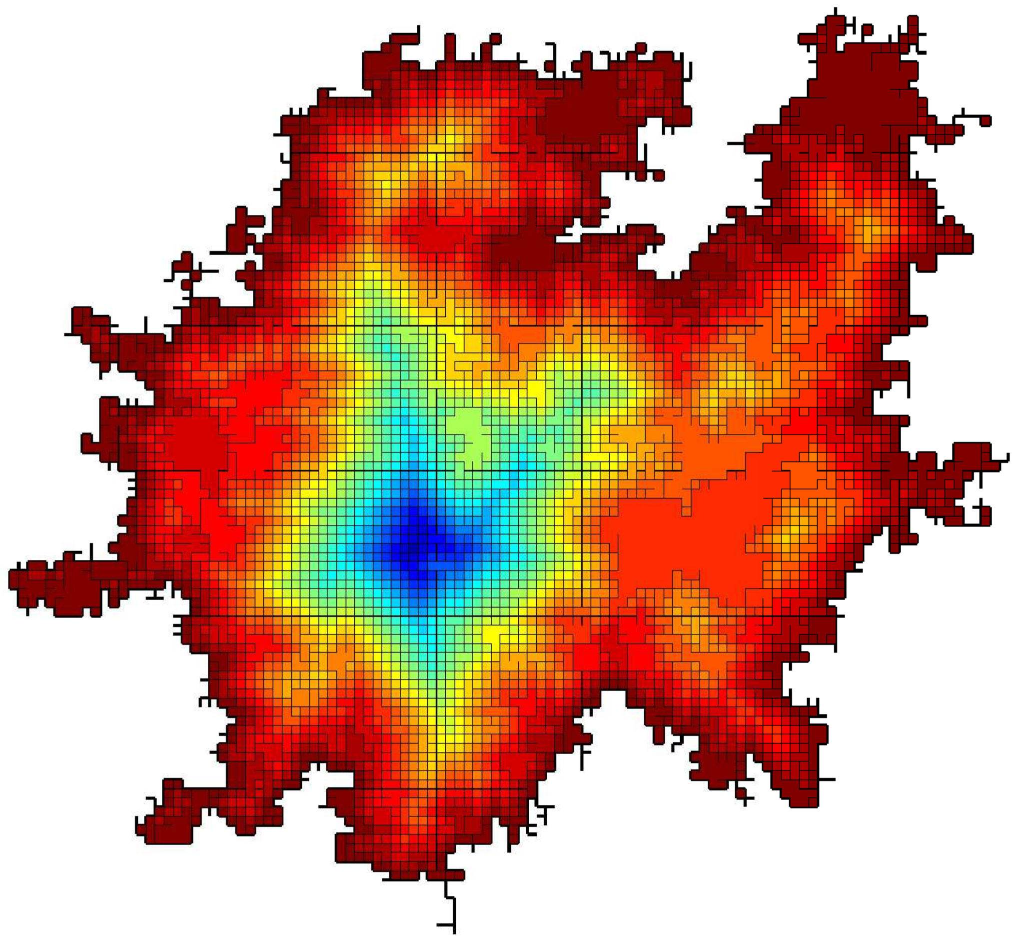


Can any two components be separated by some rectifiable curve?



Is the intersection of two boundaries on a rectifiable curve?

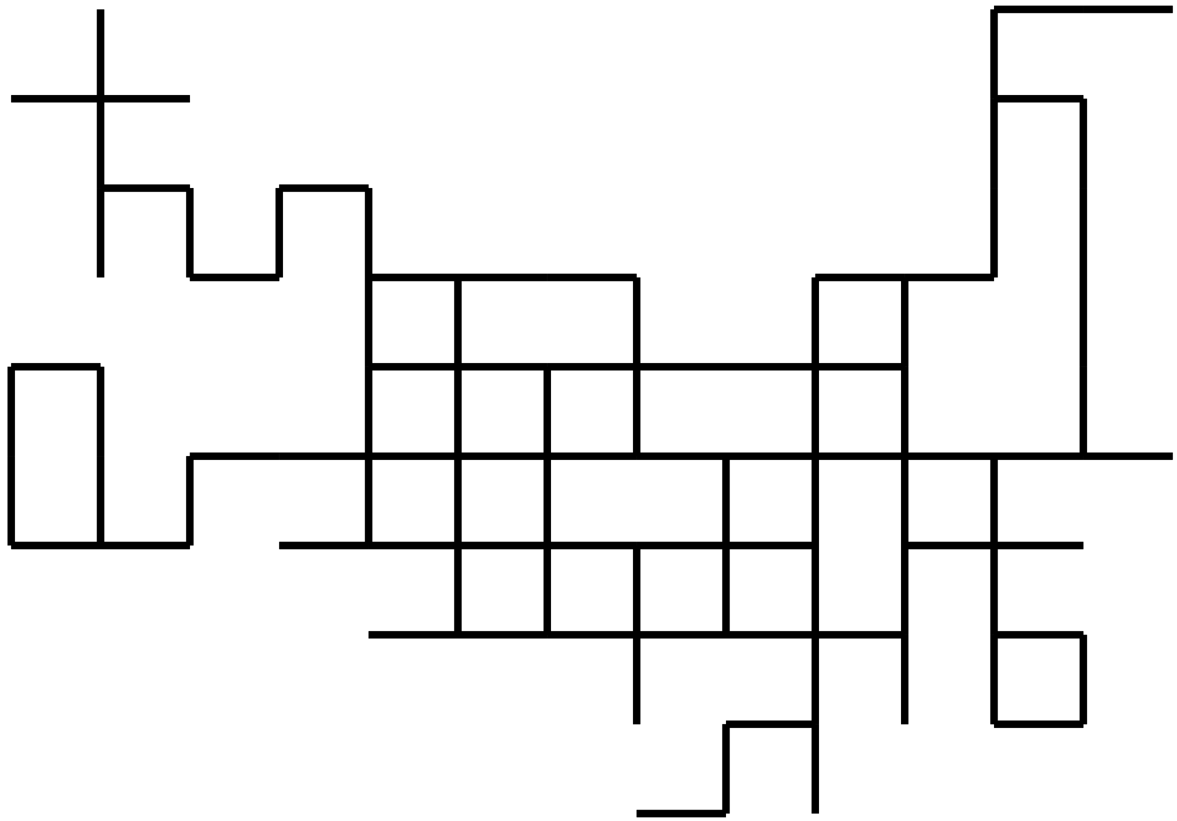
WERNER'S CONJECTURE



Consider the complementary components of the Brownian trace as vertices of a graph, with two being adjacent if their boundaries overlap.

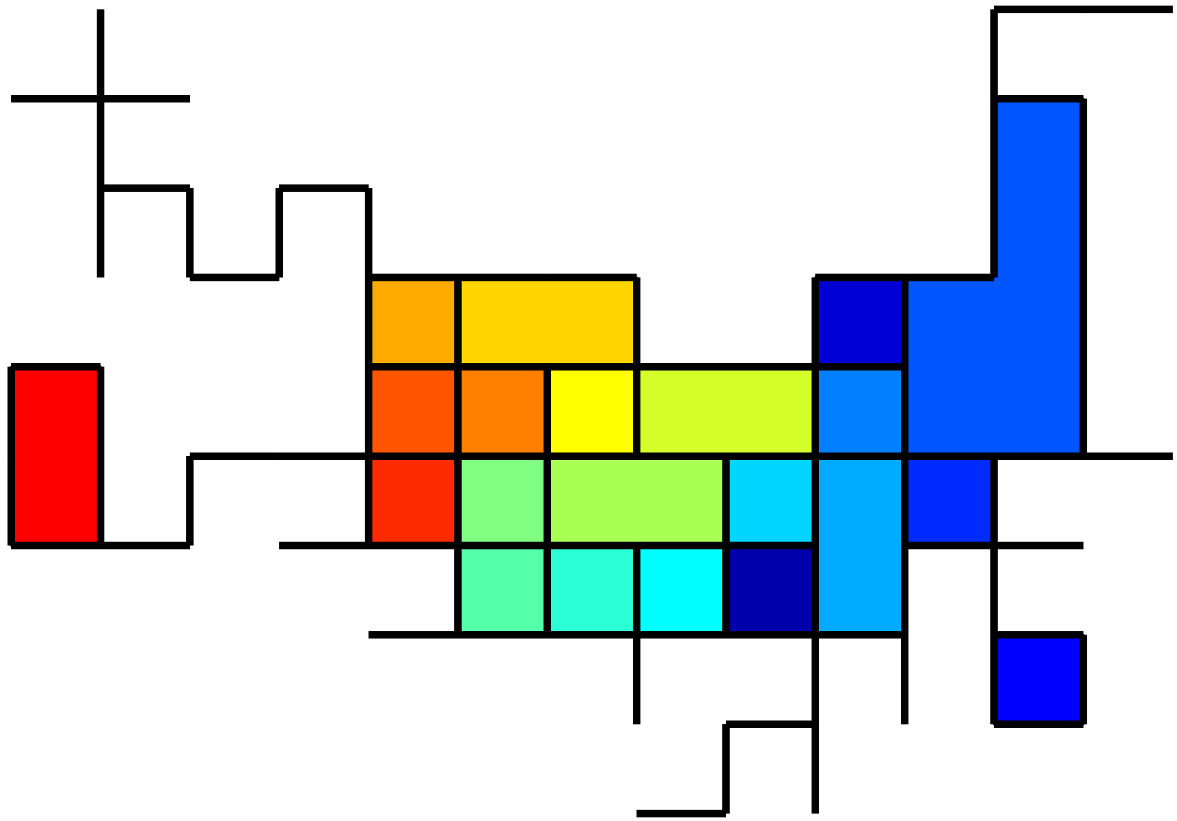
Wendelin Werner conjectured this graph is connected, i.e., any two components are connected by a path hitting the trace only finitely often.

N = 200, The trace



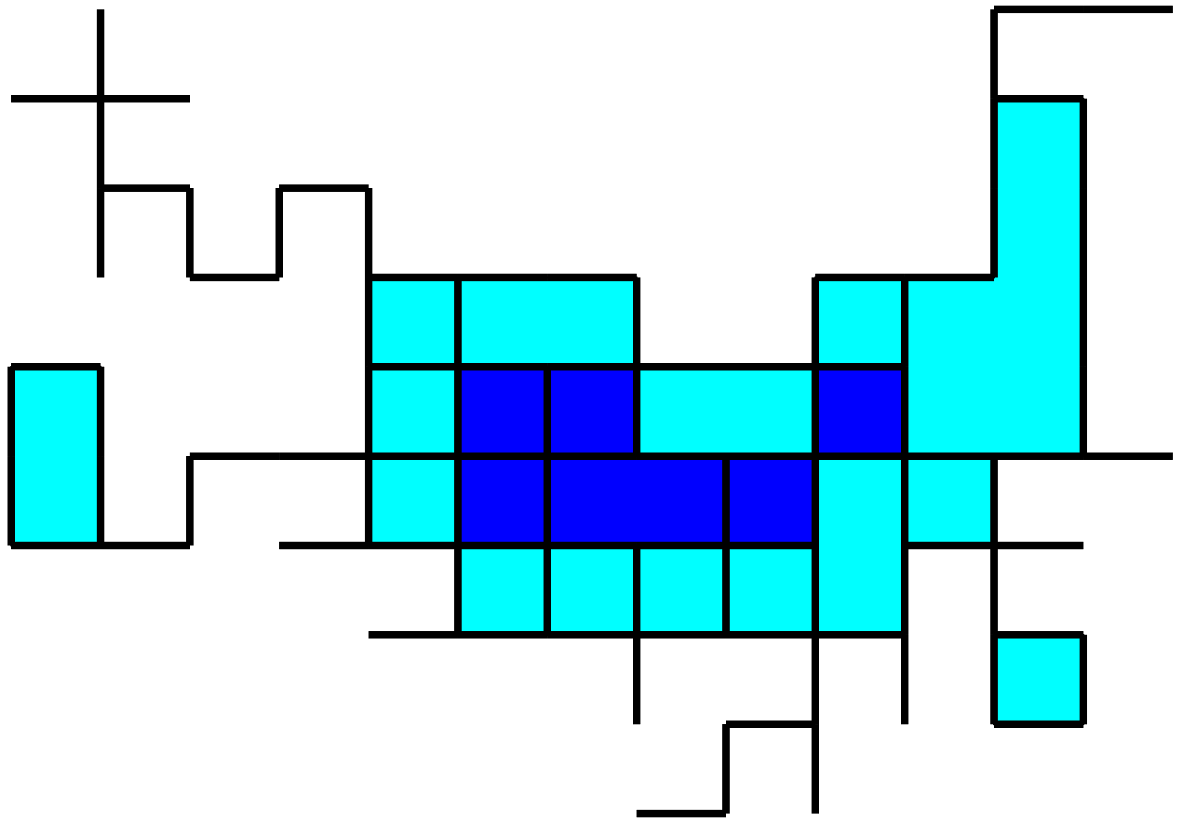
200 random steps on square grid.

N = 200, Number of components = 22

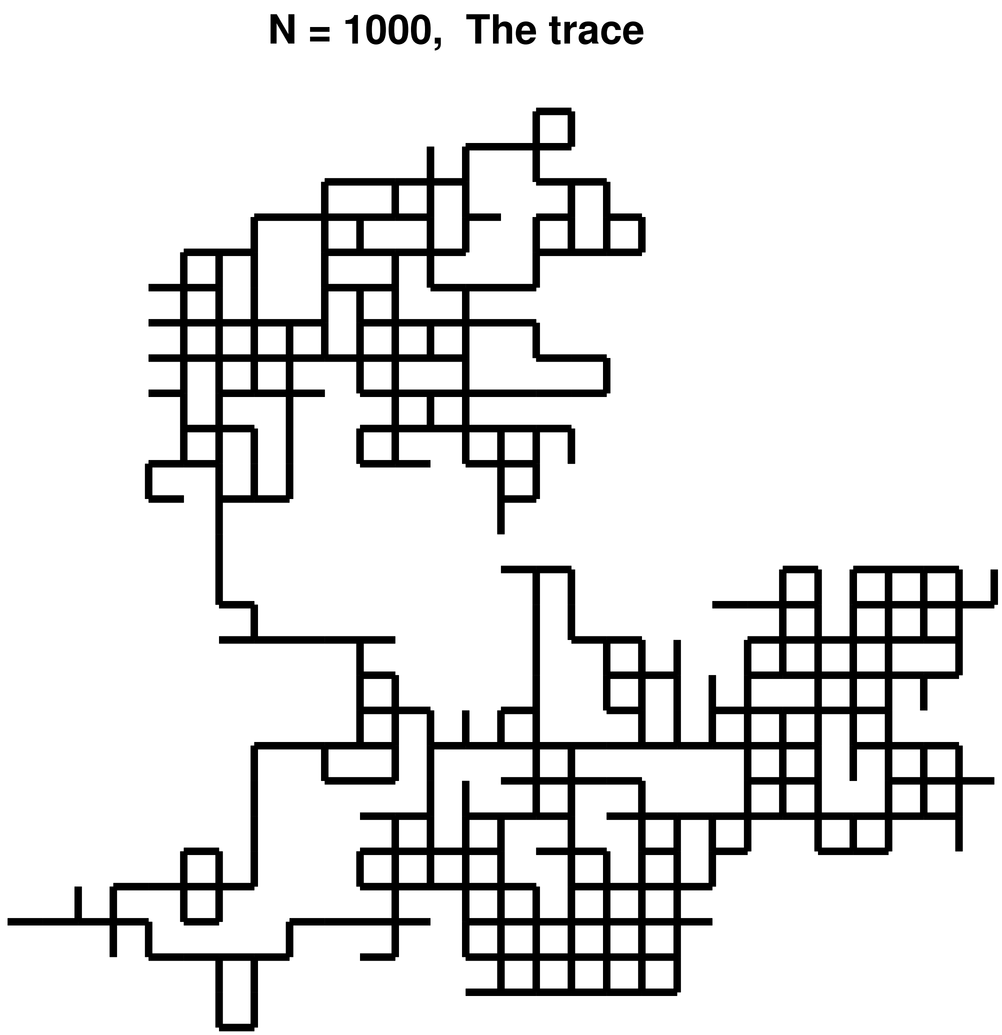


Components form a graph under edge adjacency.

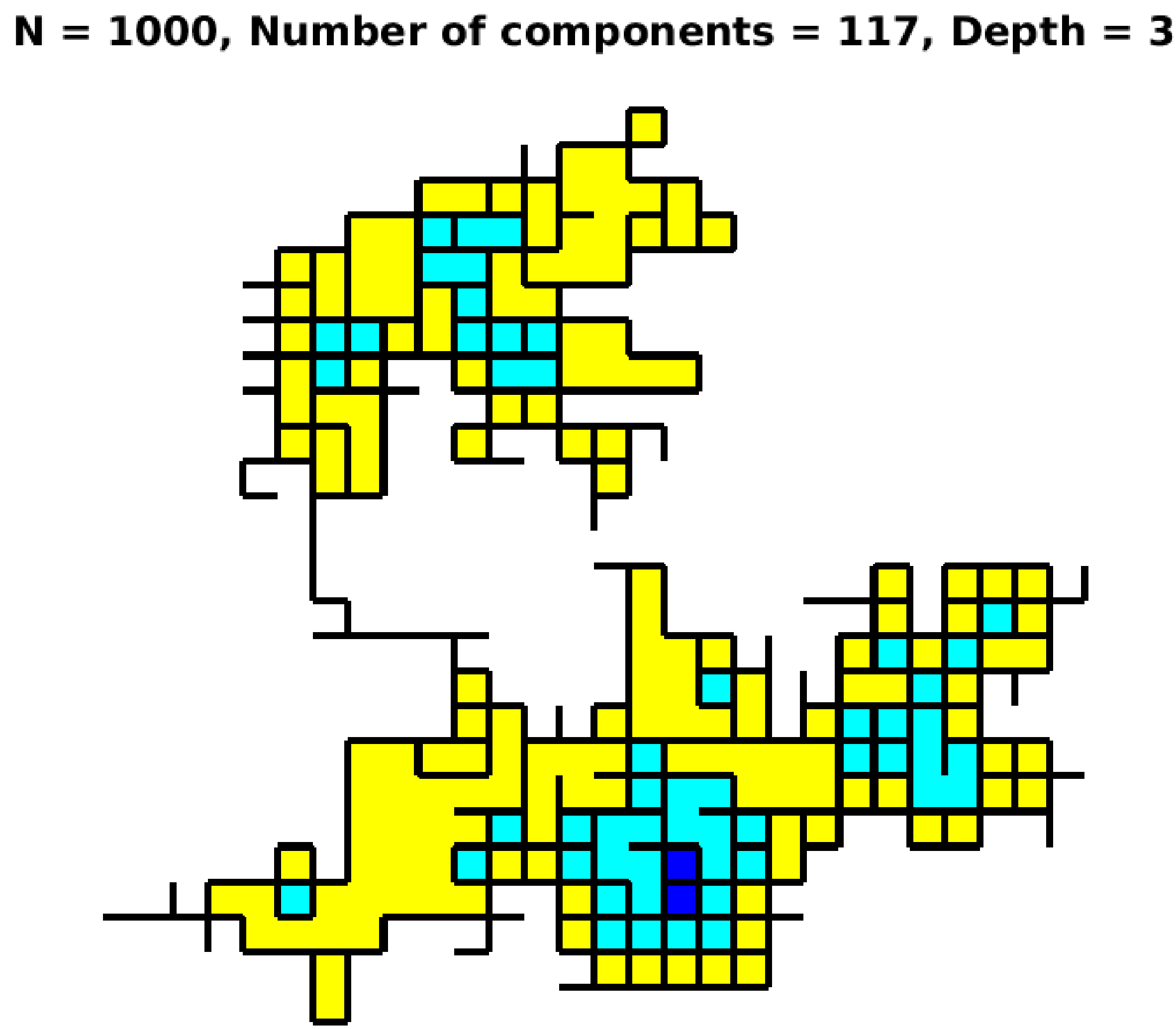
N = 200, Number of components = 22, Depth = 2



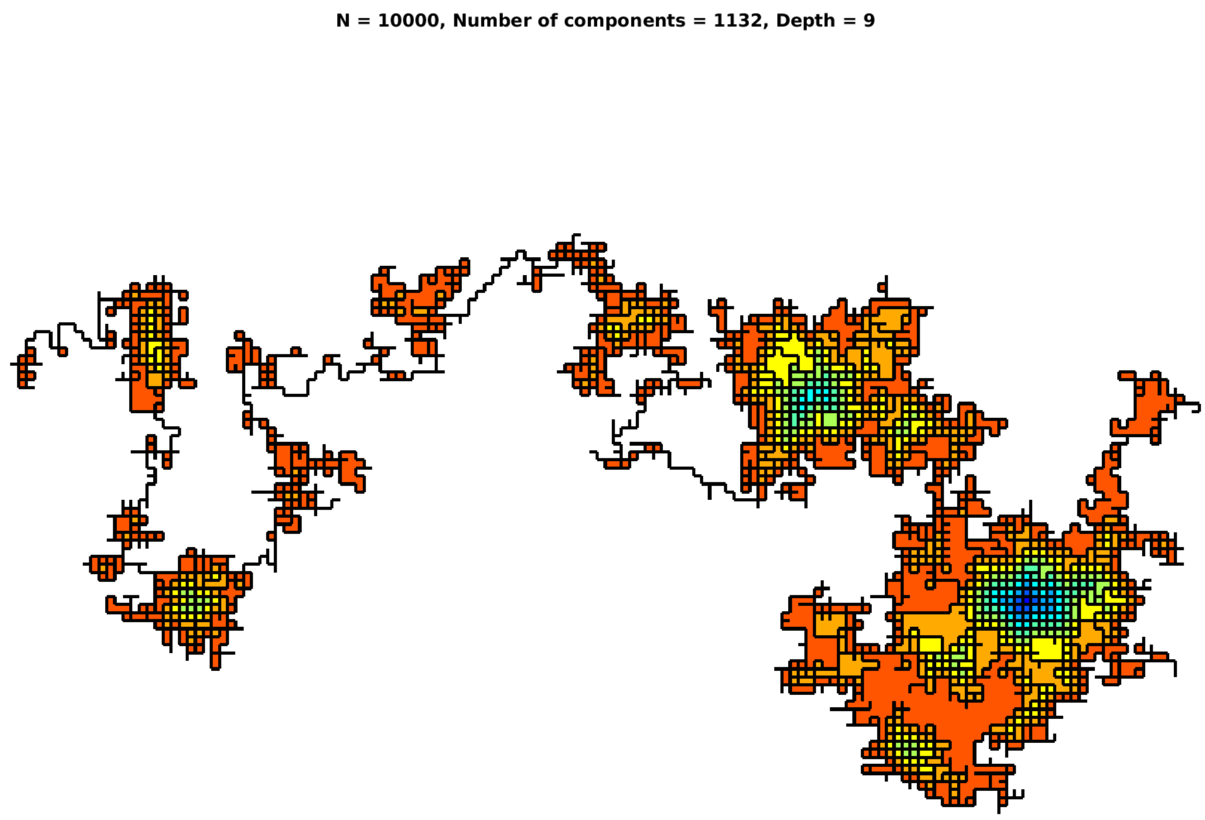
The bounded components colored by graph distance to outer component.



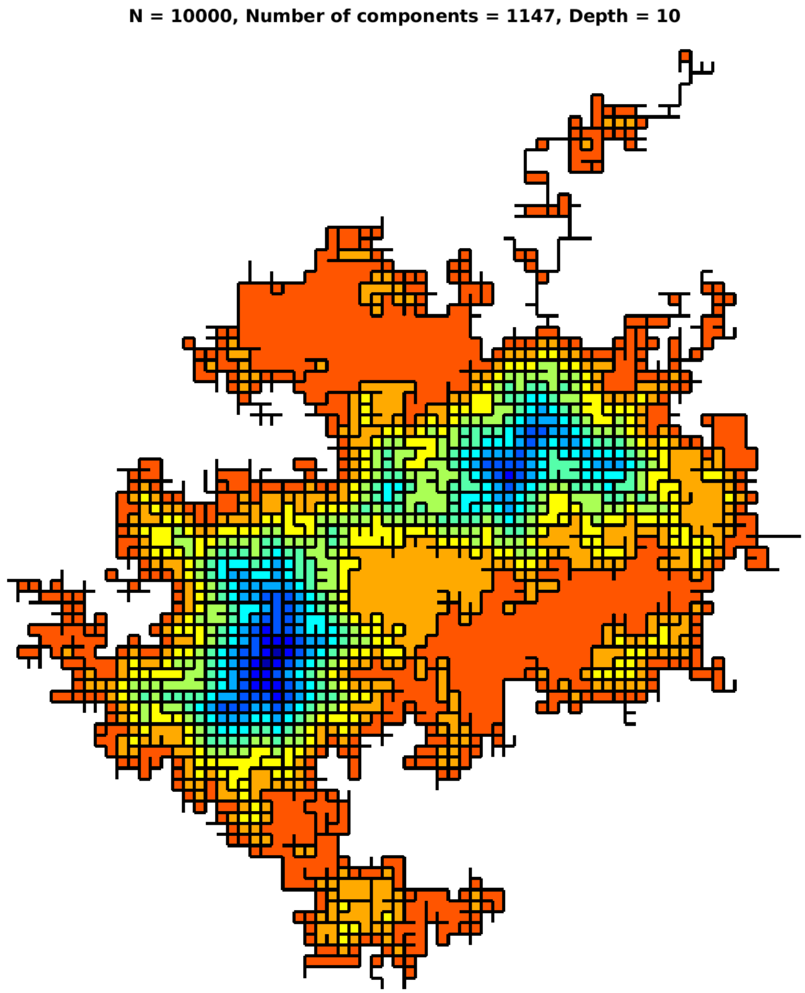
1000 random steps on square grid.



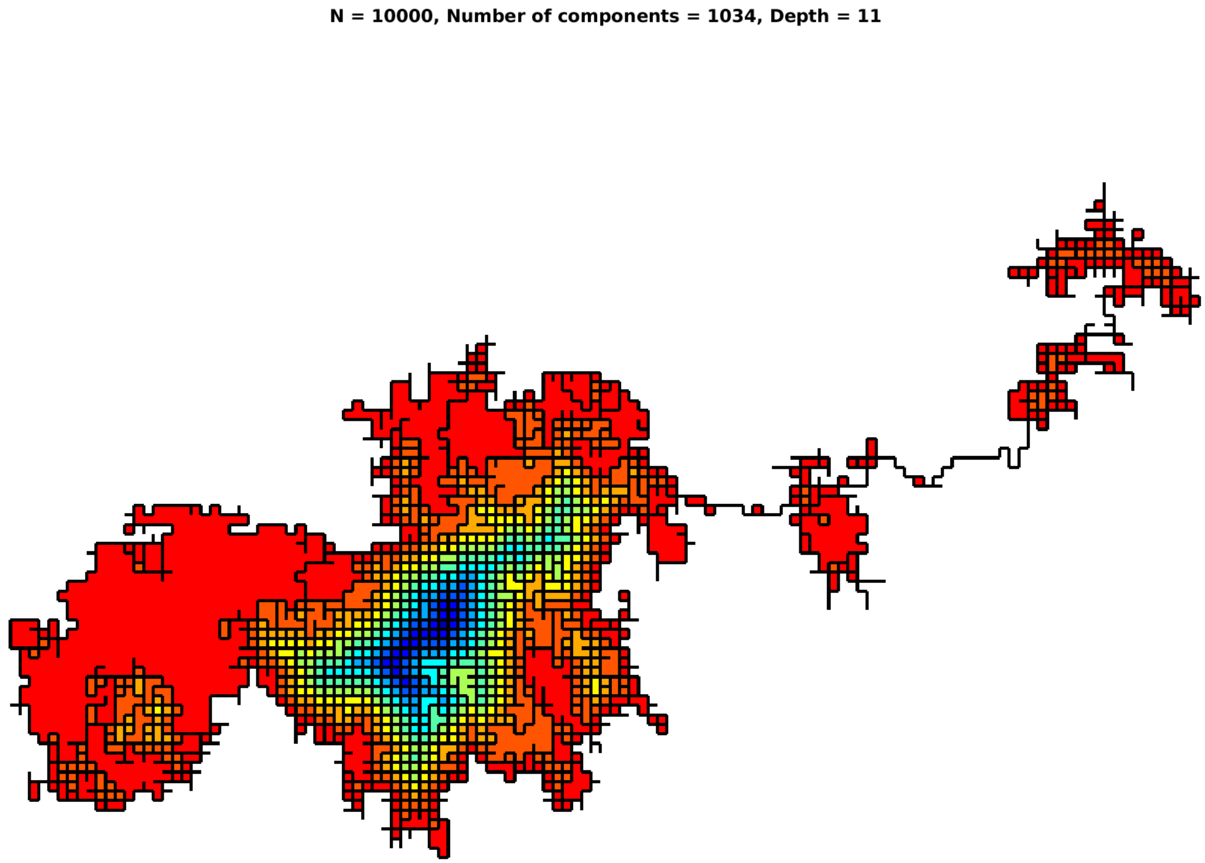
Components colored by graph distance to outer component.



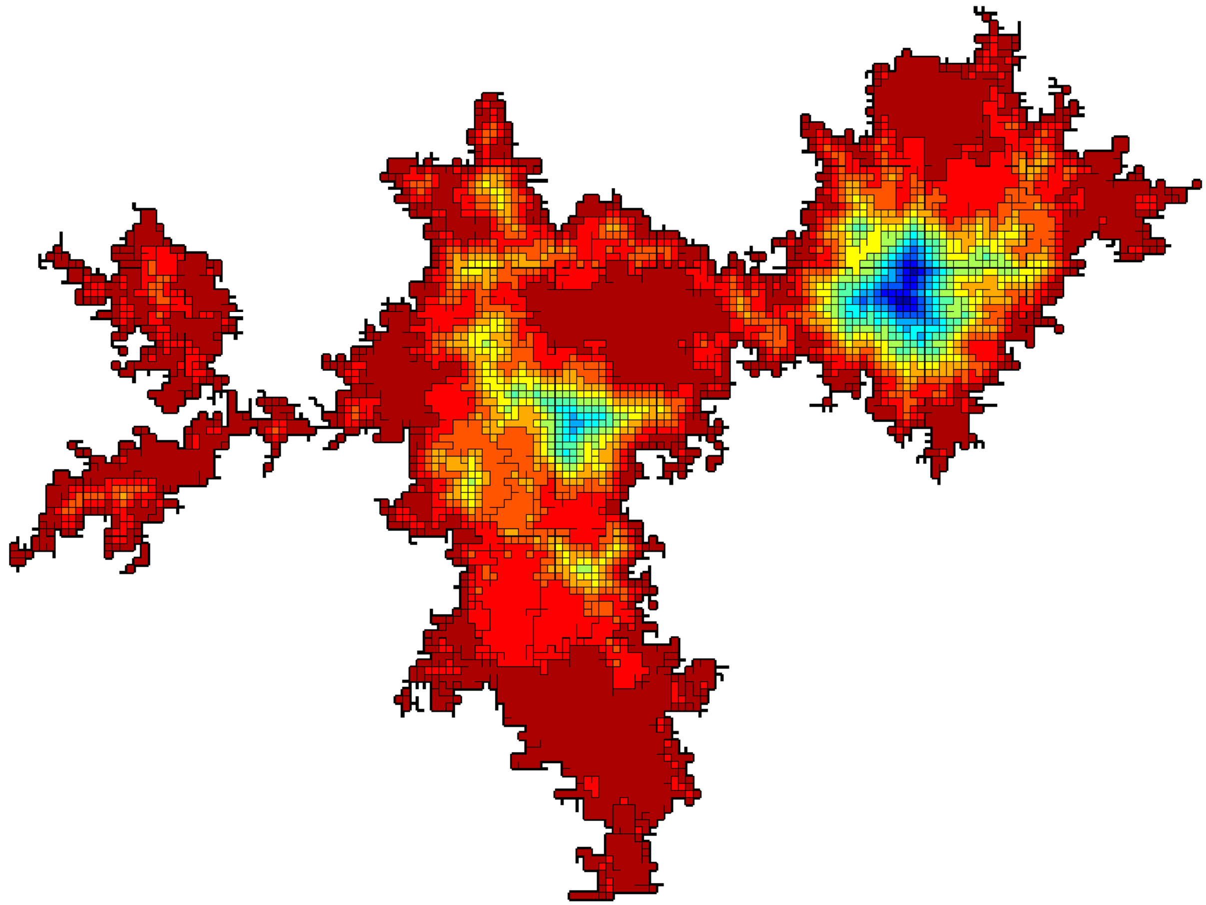
10,000 steps



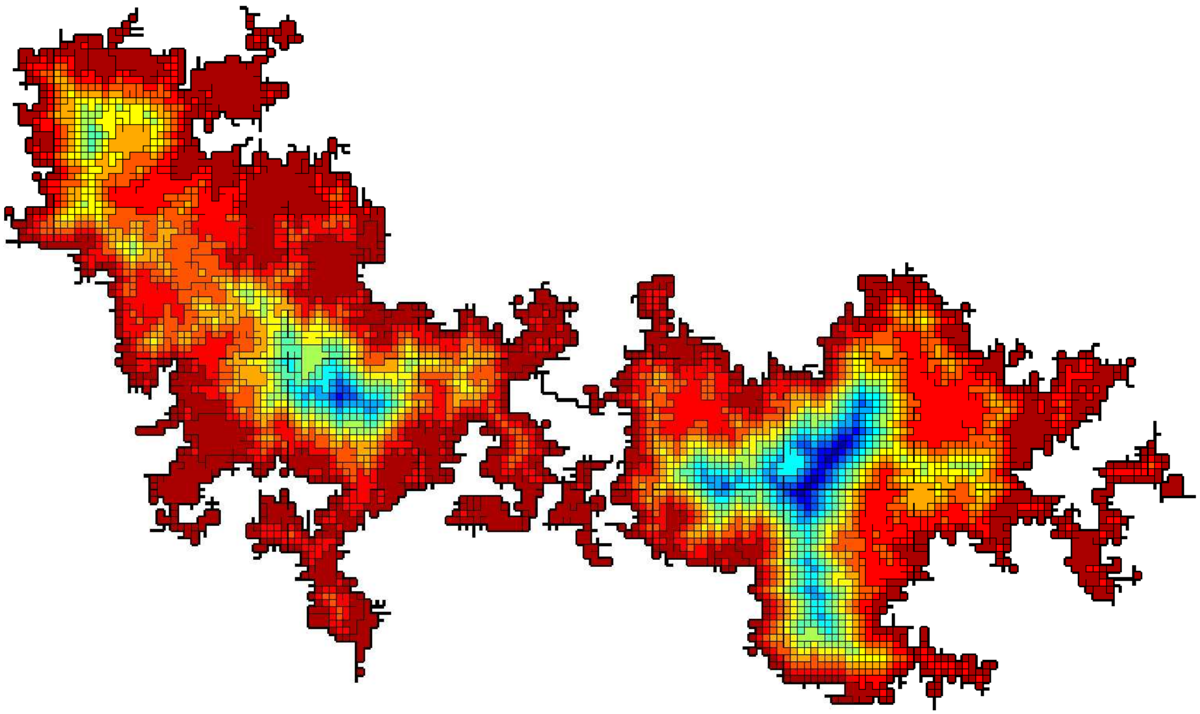
10,000 steps



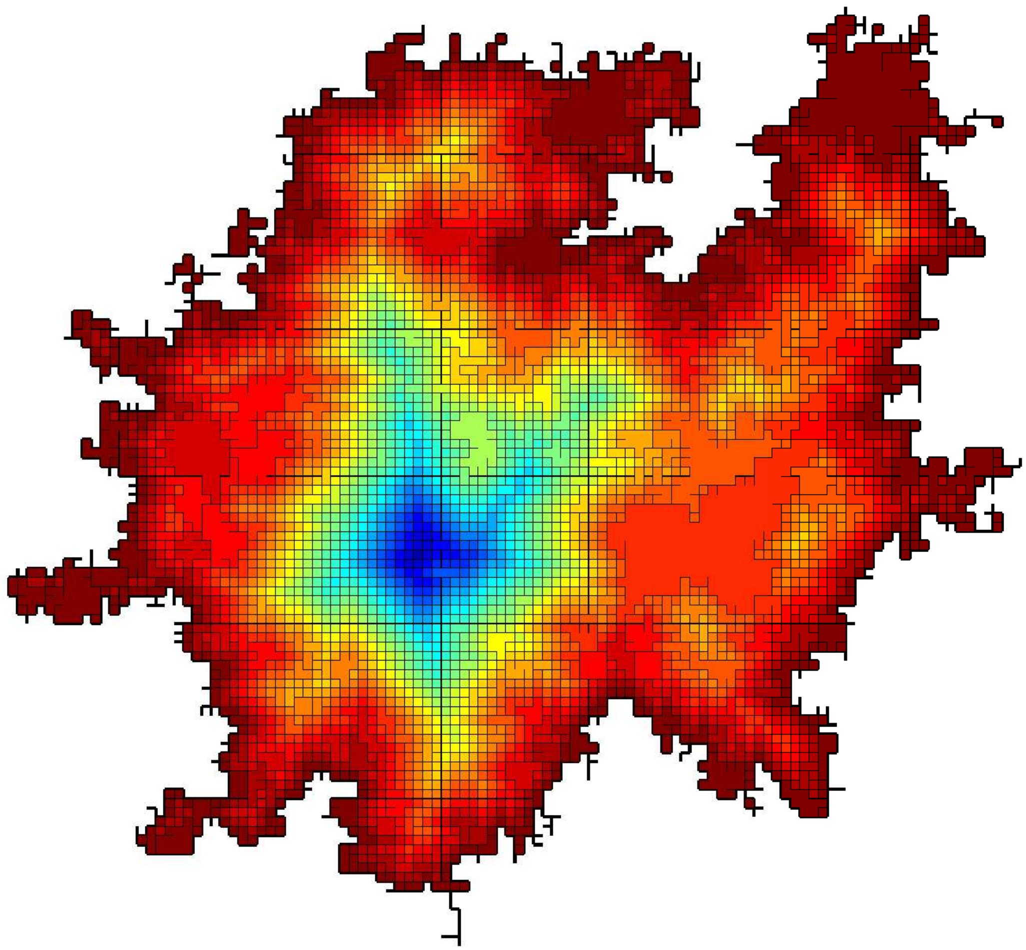
10,000 steps



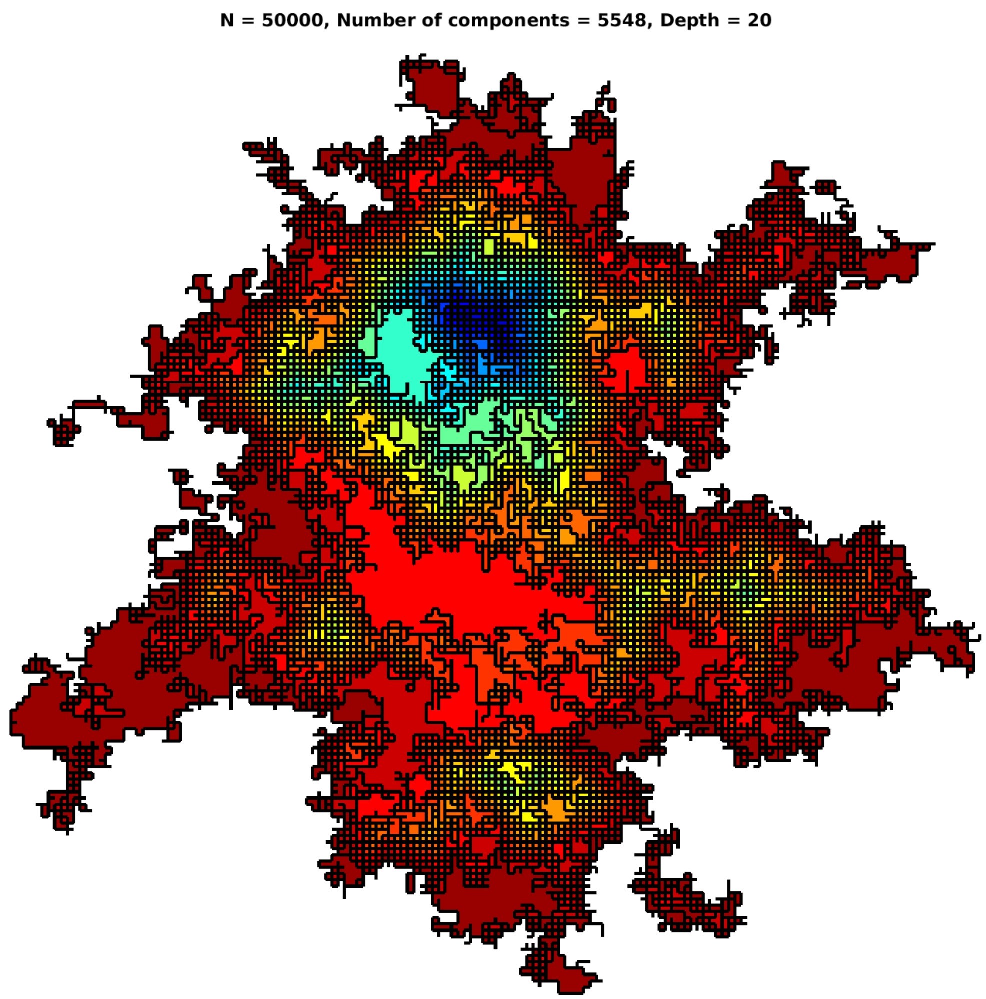
20,000 steps



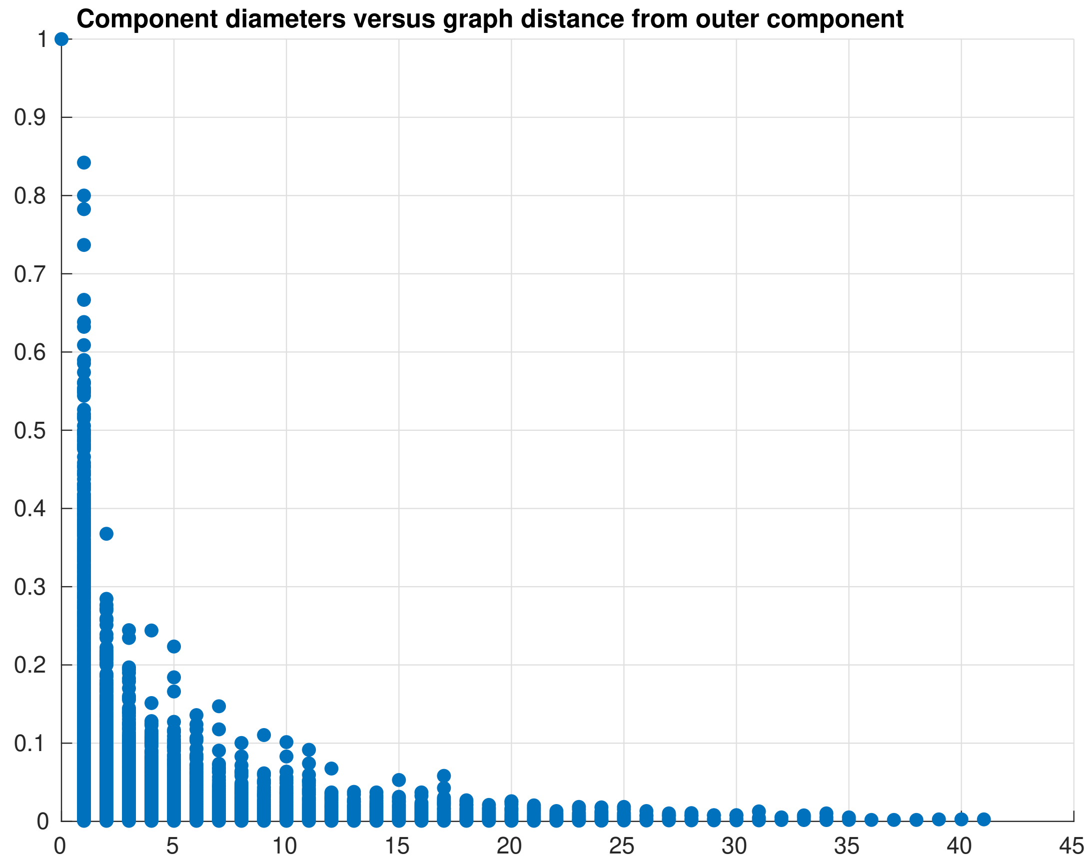
30,000 steps



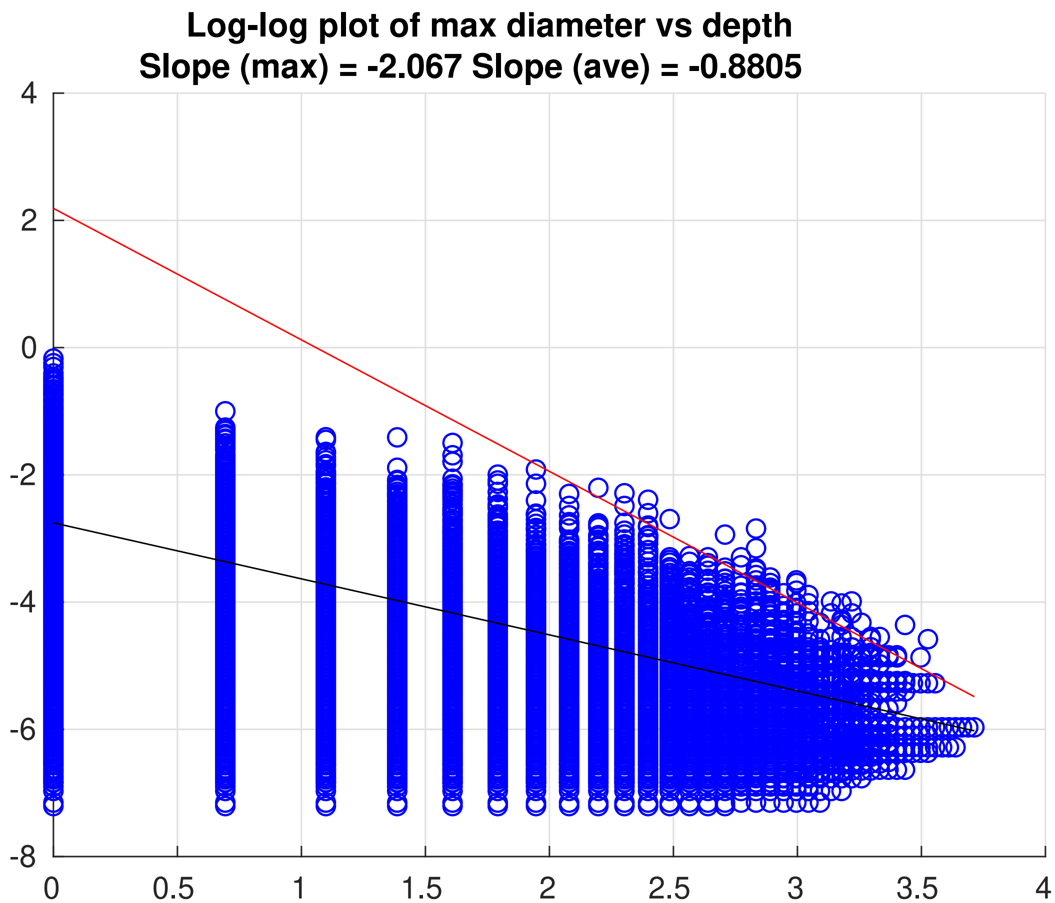
40,000 steps



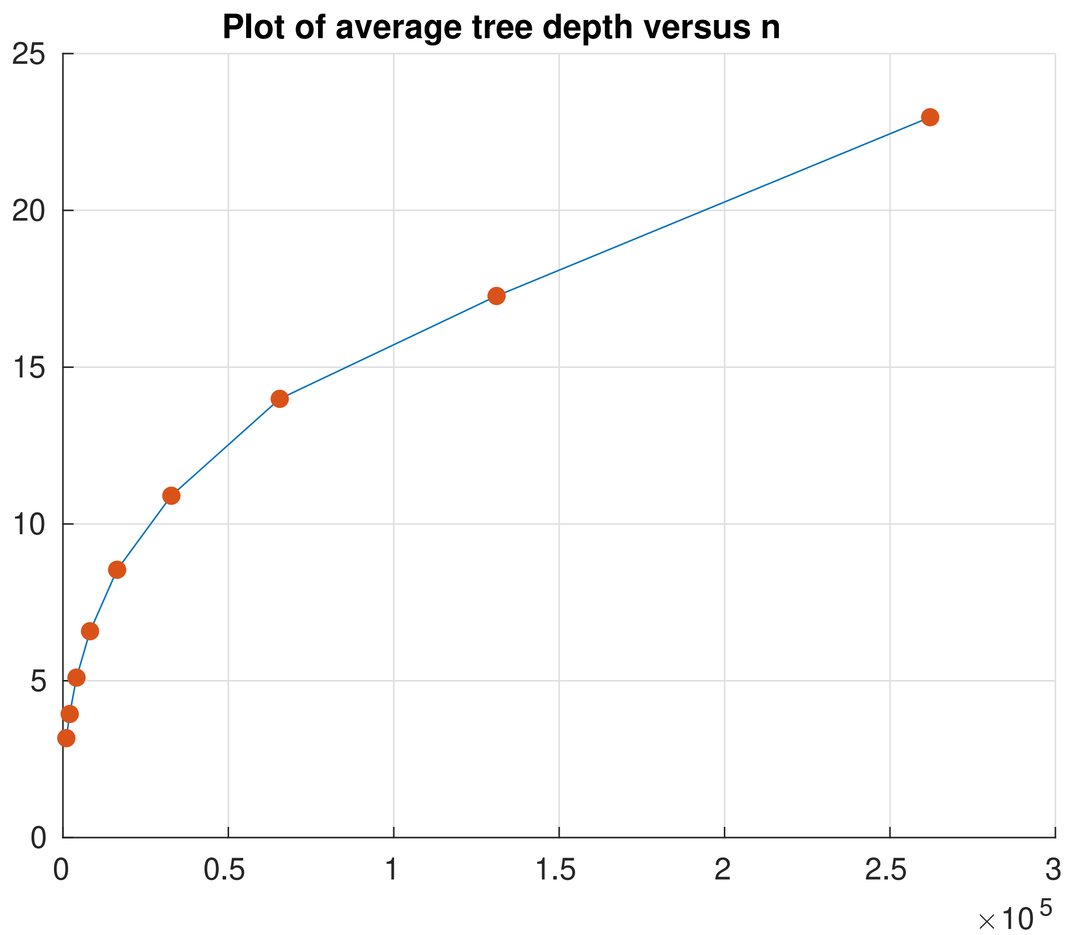
50,000 steps, 10%-component at depth 12



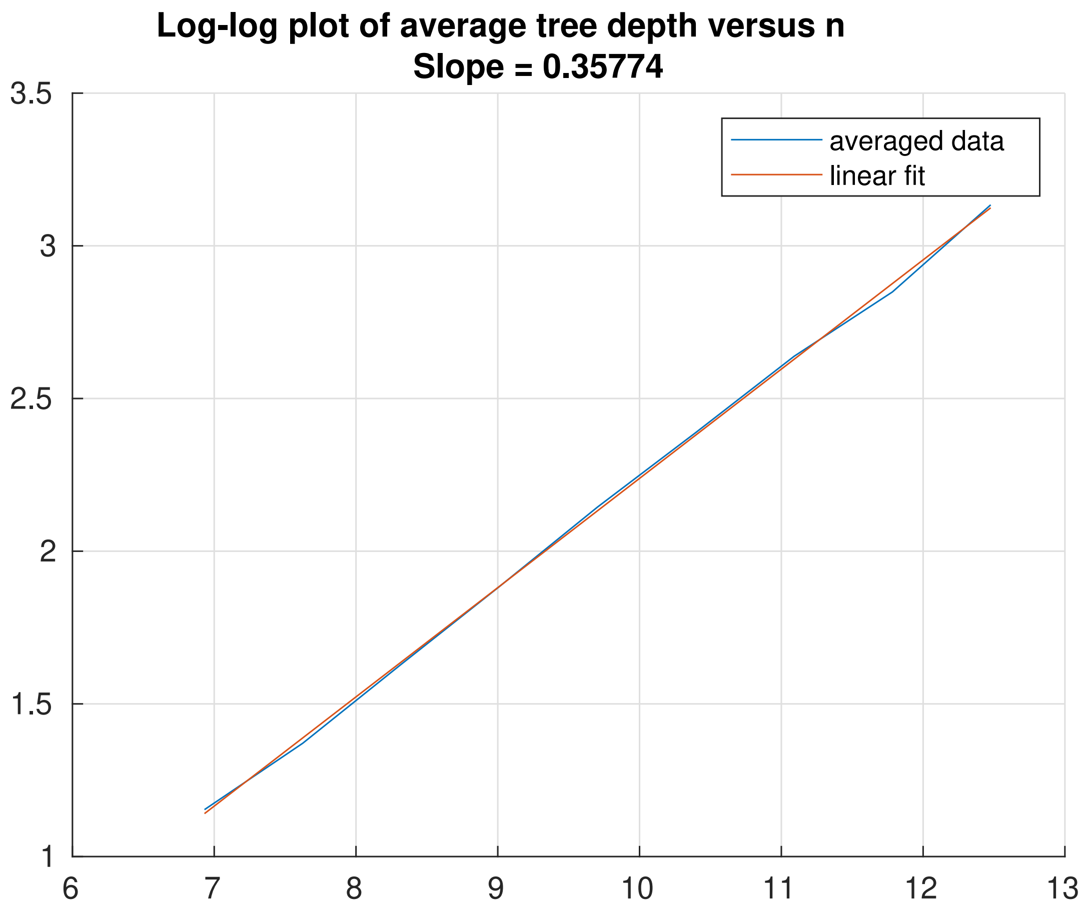
Component diameter versus graph distance from outer component.



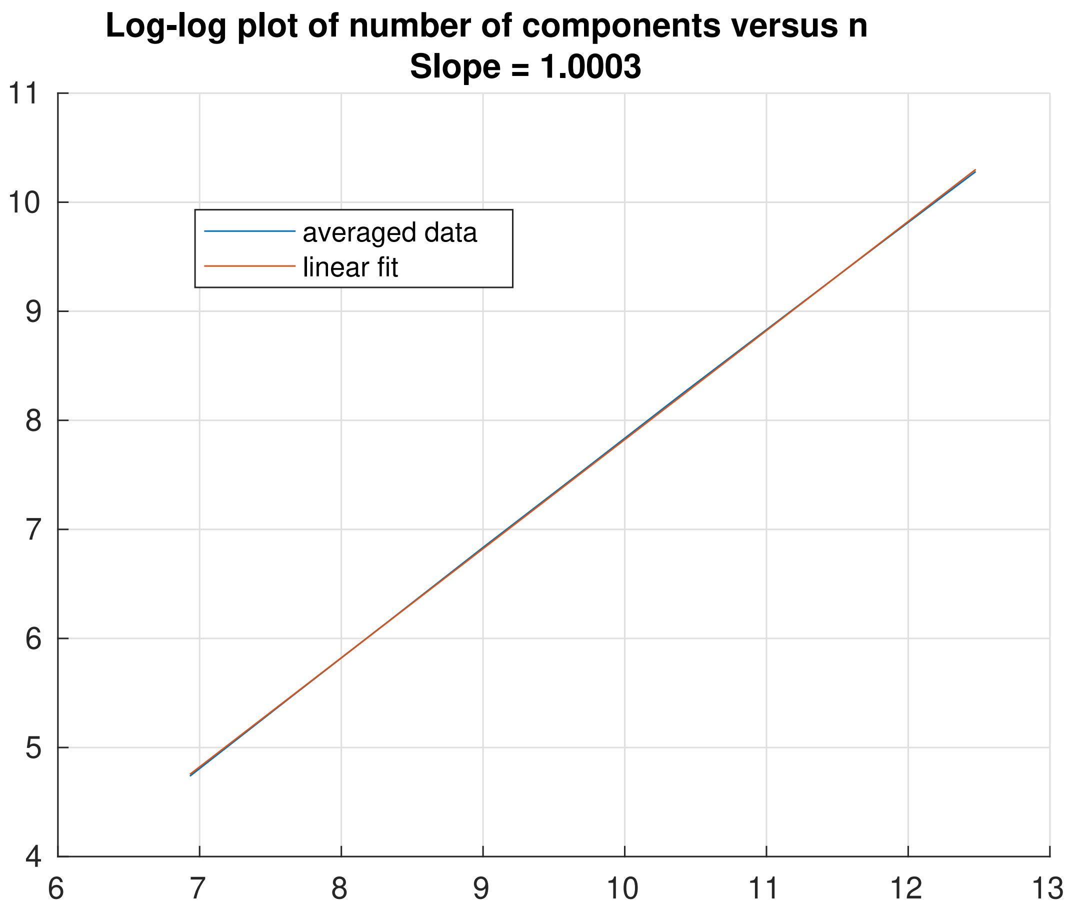
Same plot in log-log coordinates.



Growth of maximal graph distance (depth) to outer component.



Maximum distance from outer component looks like $n^{.36}$.



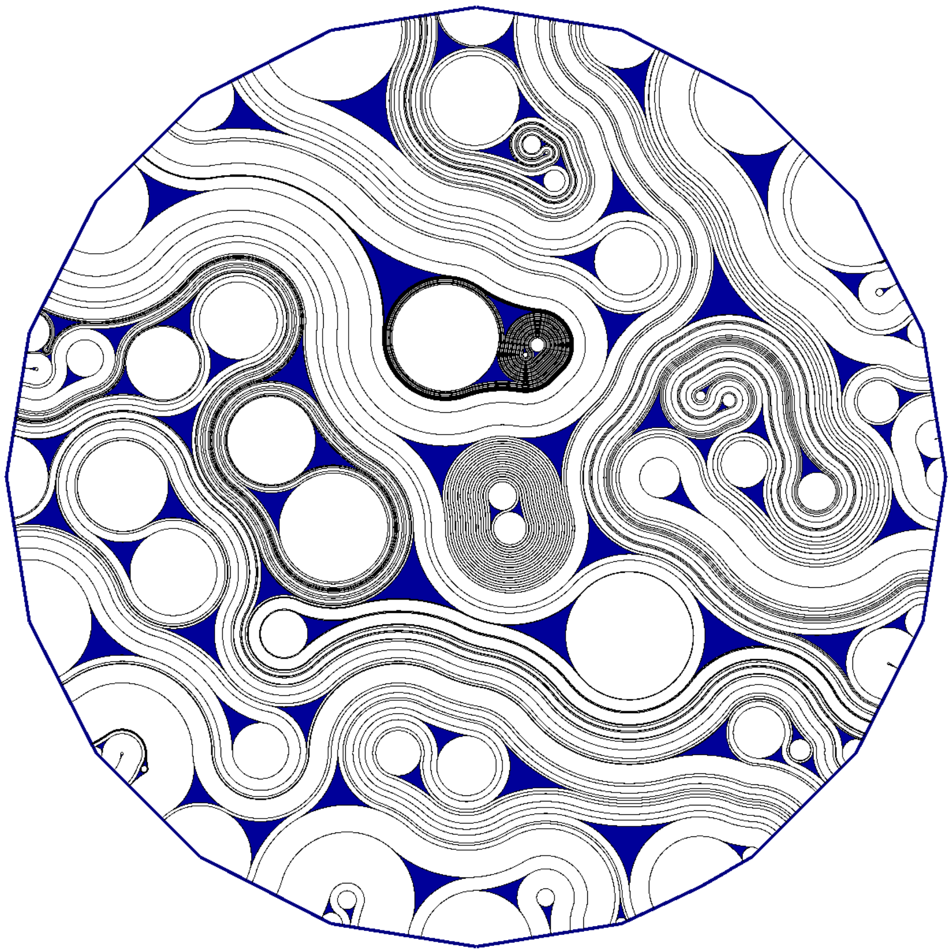
Number of components appears linear in n

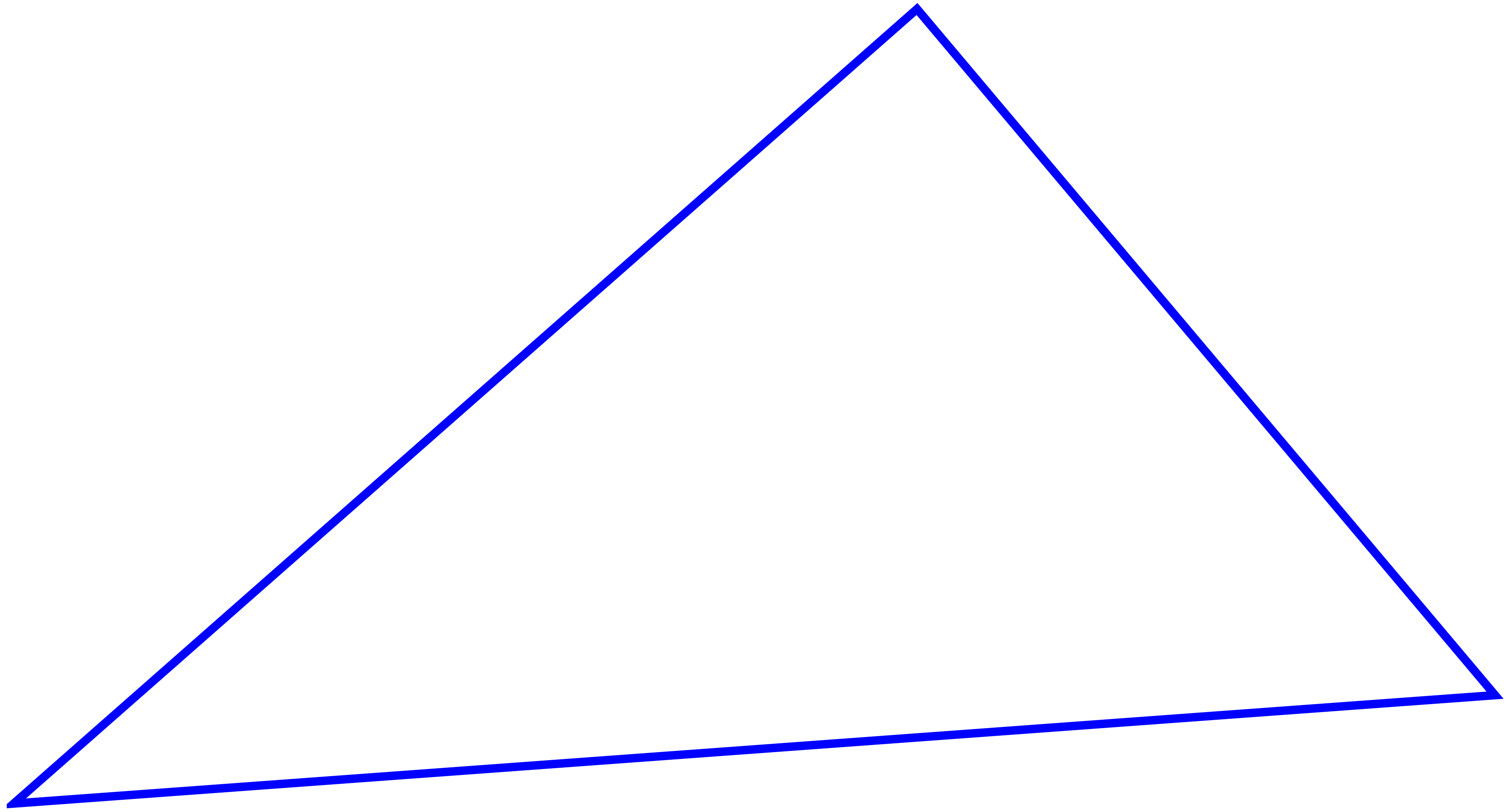
How could Werner's conjecture fail?

Need large component surrounded by much smaller components. Would happen if it was surrounded by ϵ-dense squares for every ϵ.

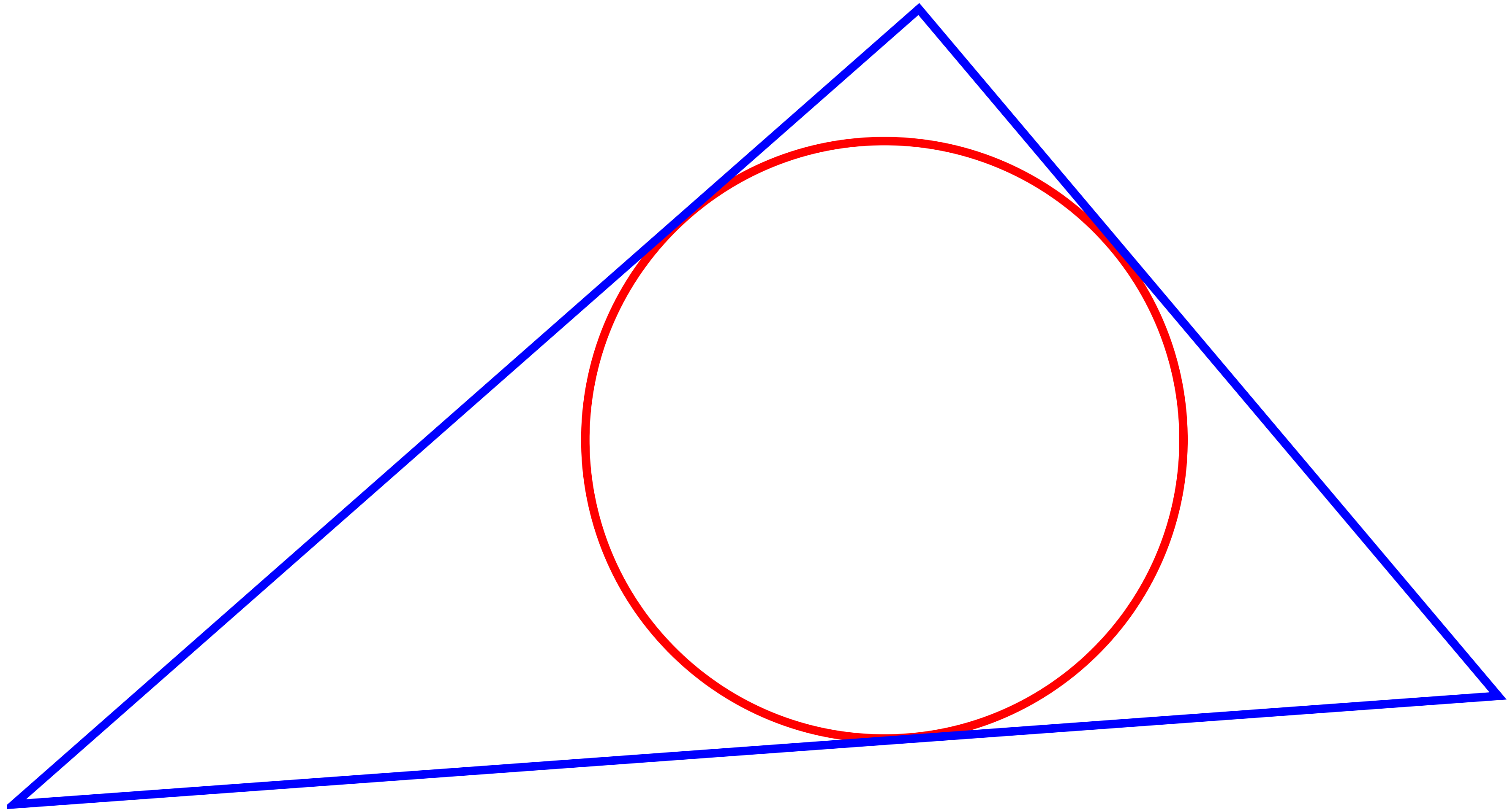
If ϵ-dense squares "percolate", they might form macroscopic loops that cause Werner's conjecture to fail.

TRIANGULATION FLOWS

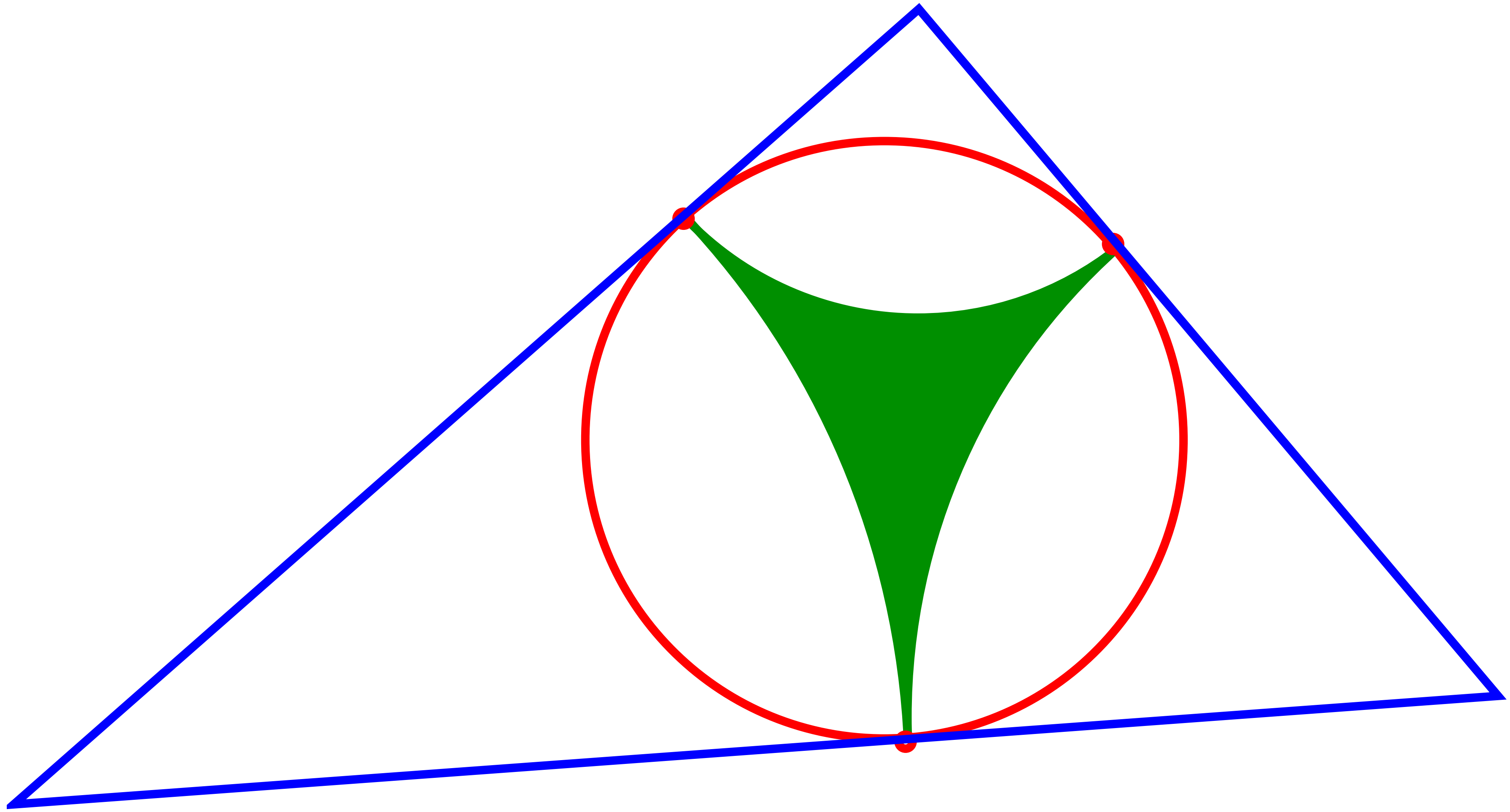




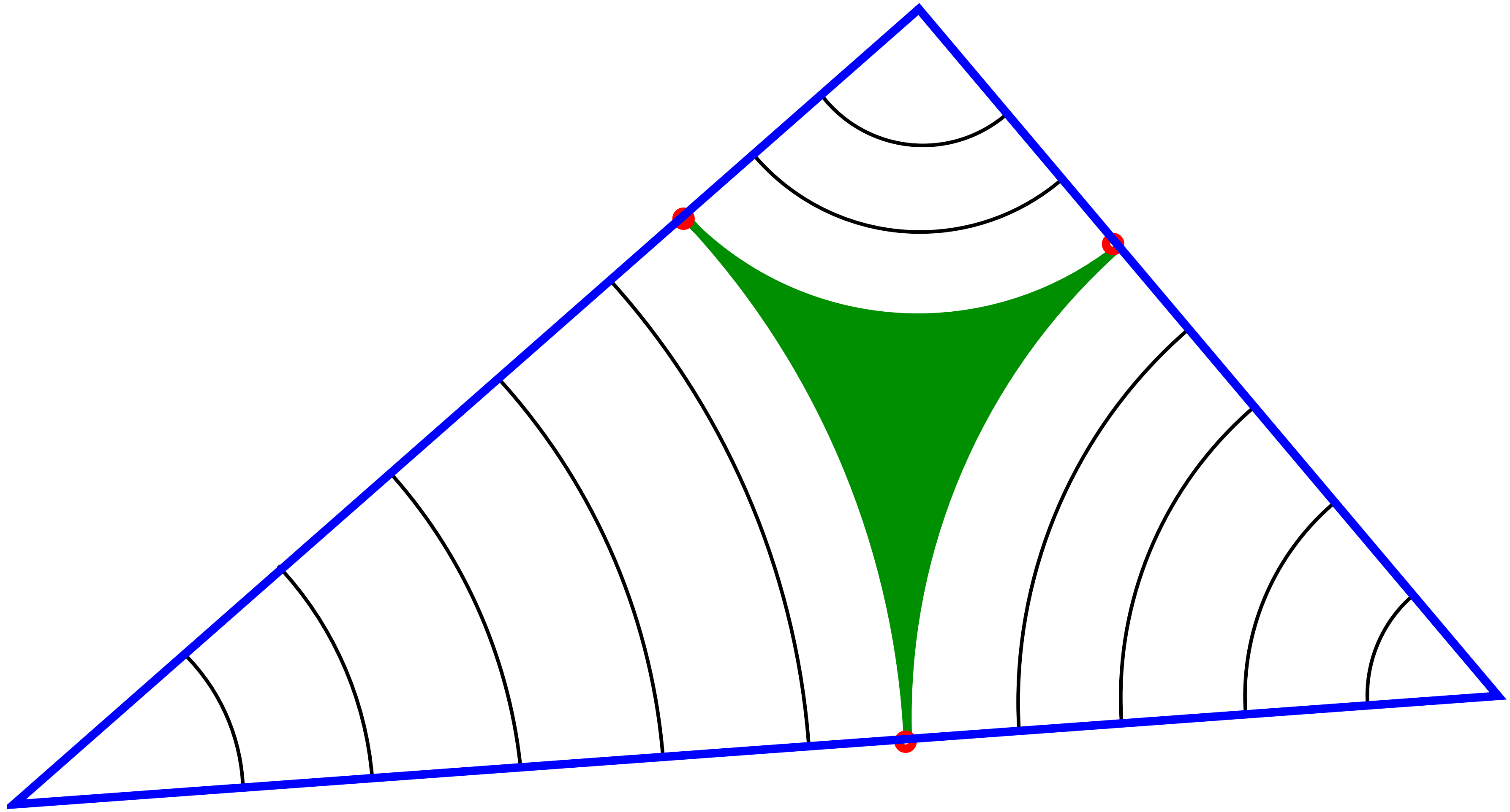
A triangle.



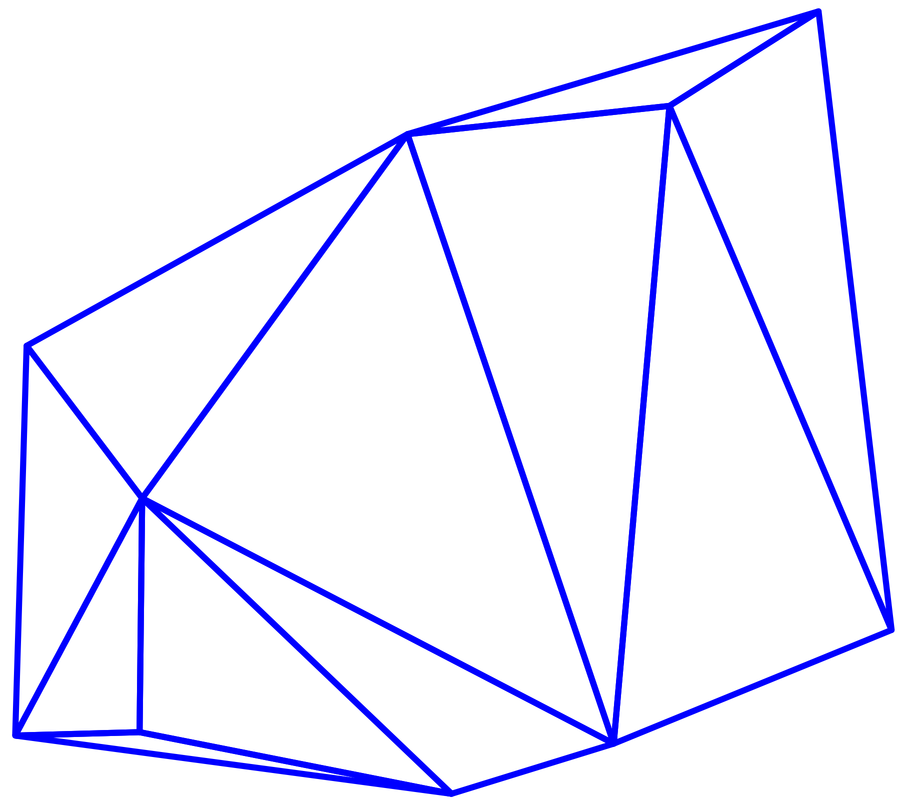
Its in-circle.



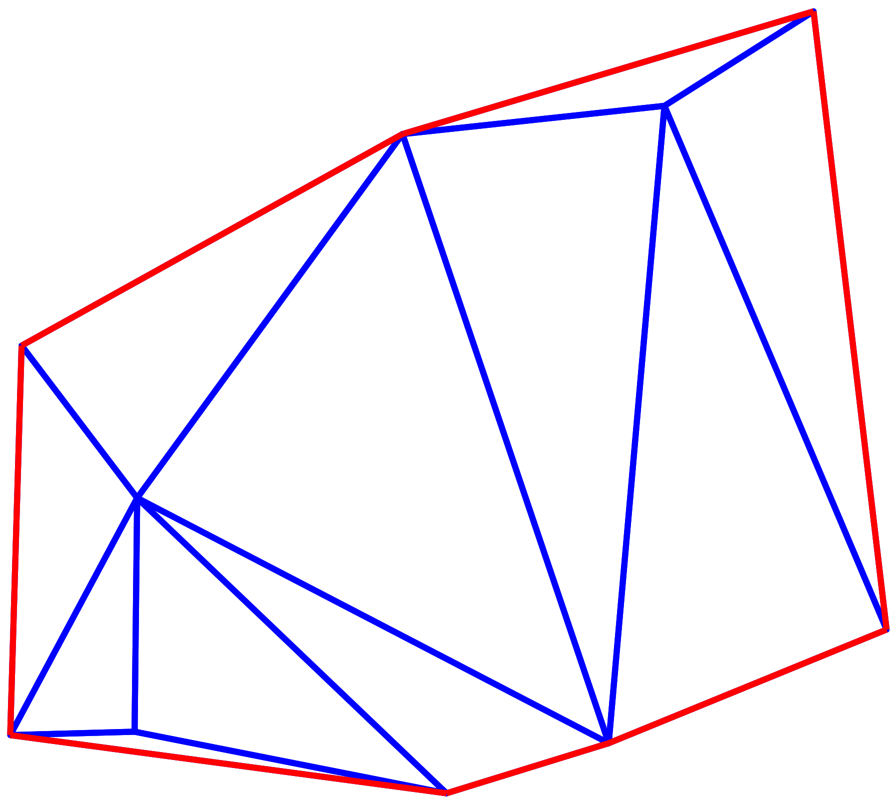
The central region and three sectors (thin parts).



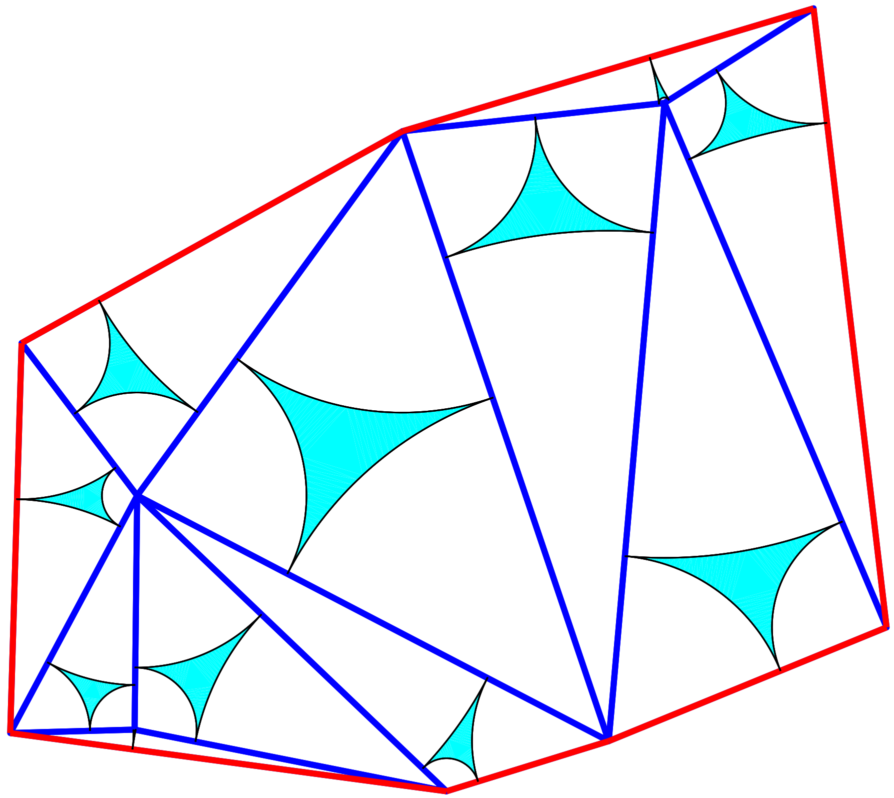
The three sectors are foliated by circular arcs.
Defines flow on a triangulation that stops at boundary or cusp point.



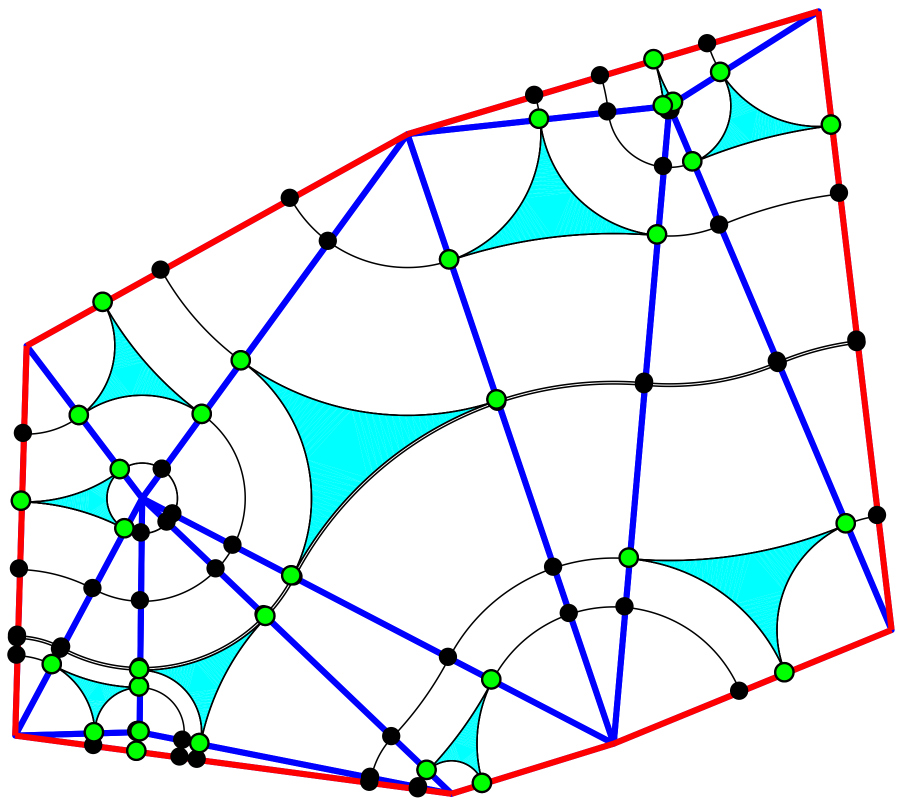
Delaunay triangulation of 10 random points,



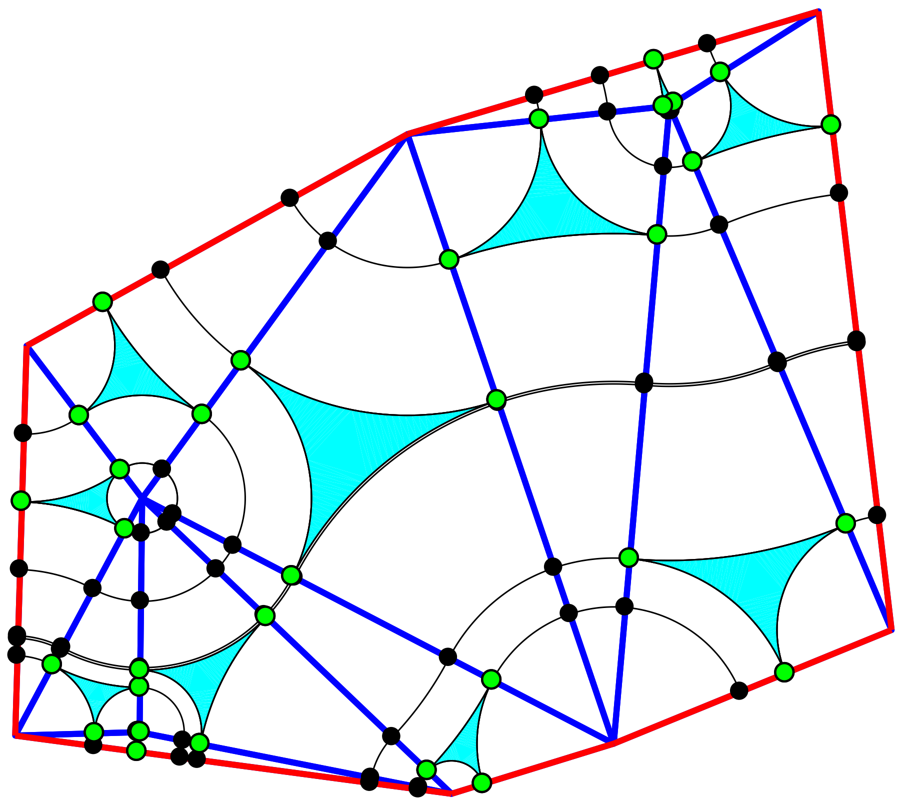
The boundary of the triangulation



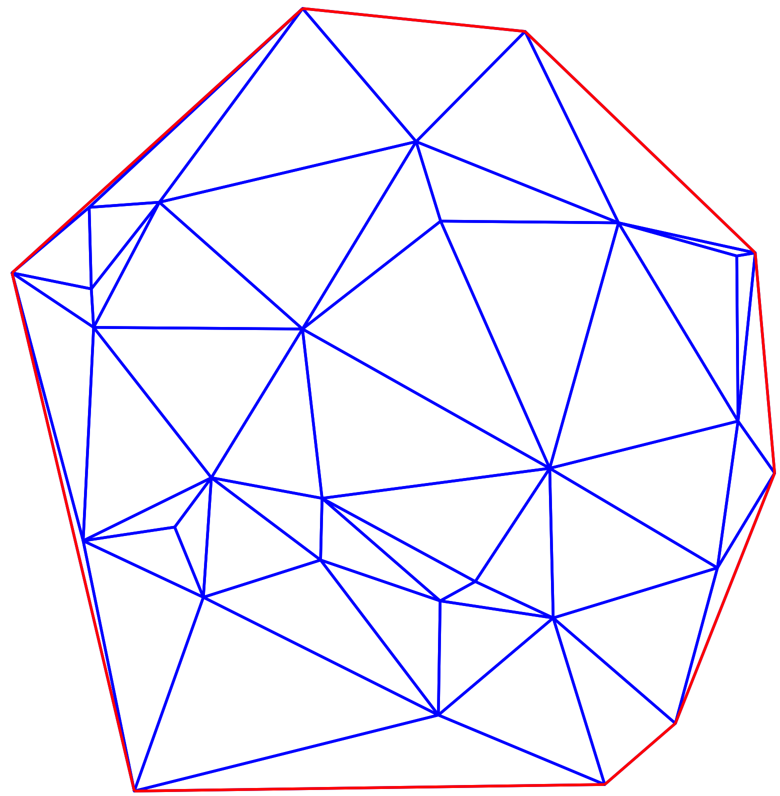
The central regions.



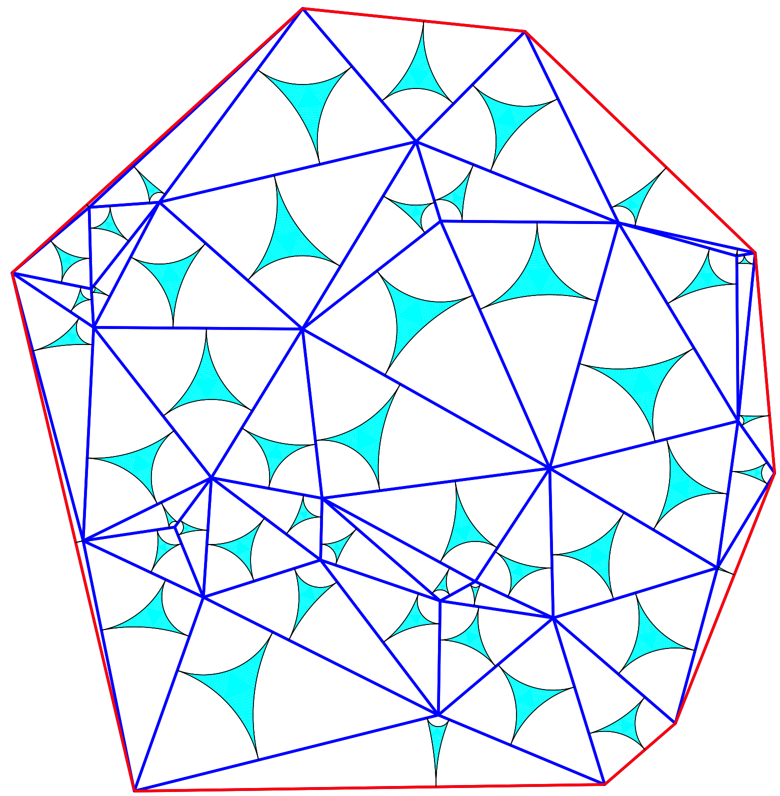
Propagation lines starting at all cusp points.



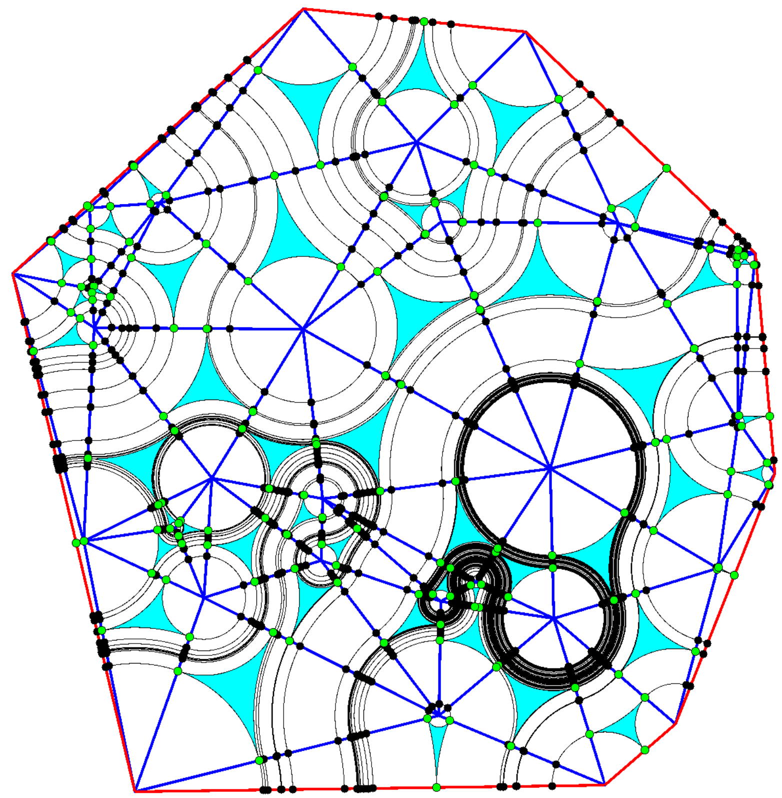
Propagation lines identify boundary points; induces tree.
Discontinuous, but piecewise length preserving.



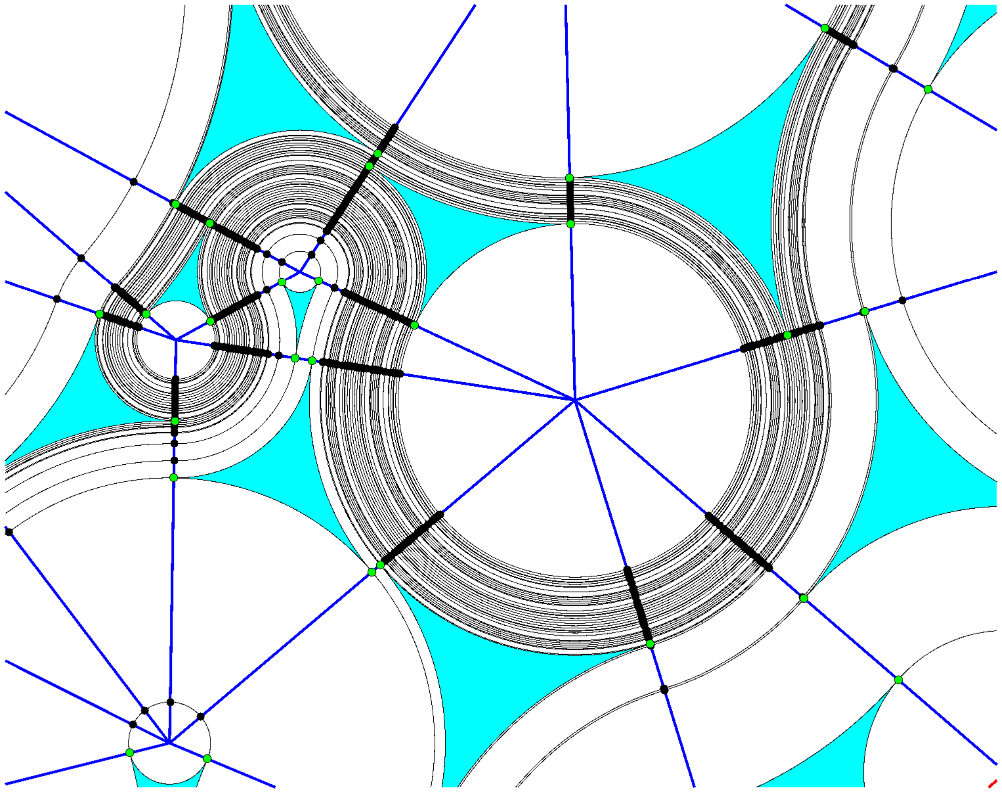
Delaunay triangulation of 30 random points in disk.



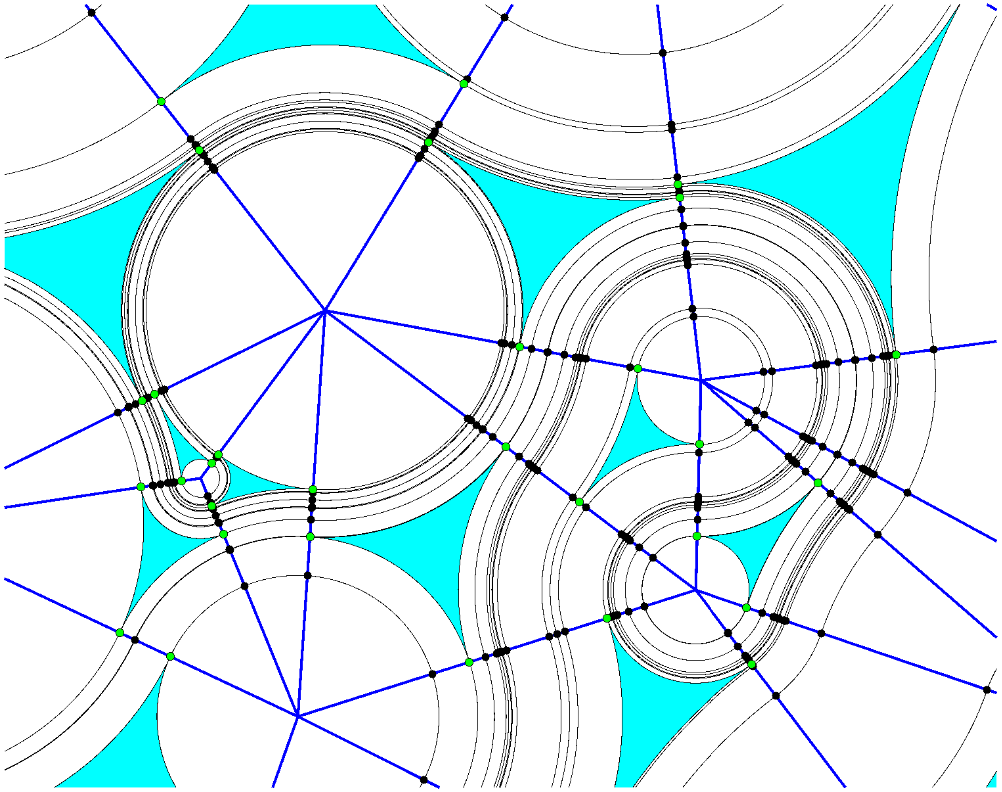
The central regions.



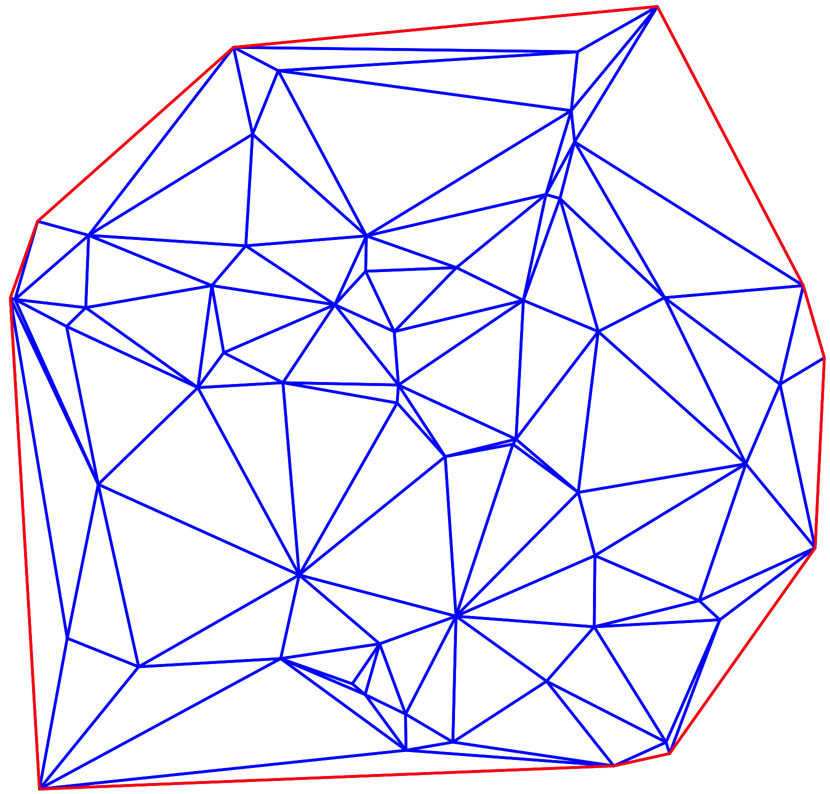
Propagation lines starting at all cusp points.



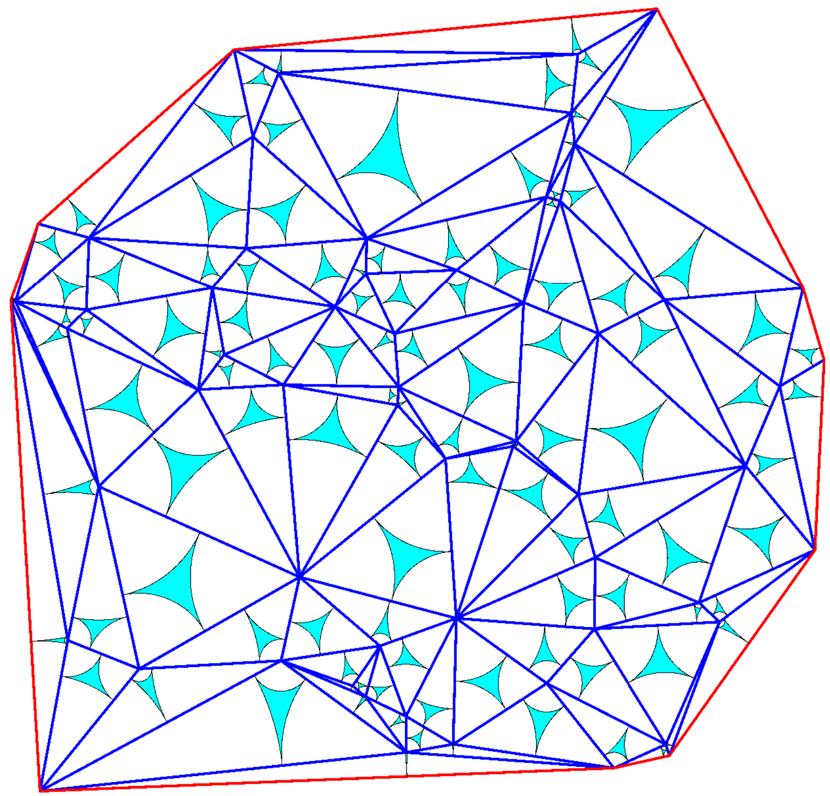
Enlargement 1.



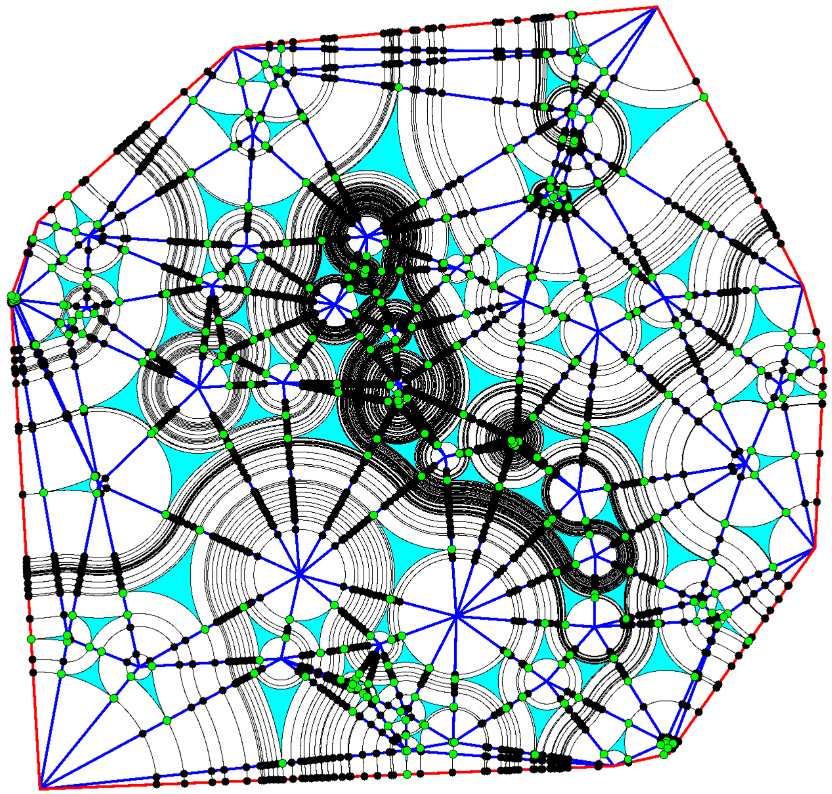
Enlargement 2.



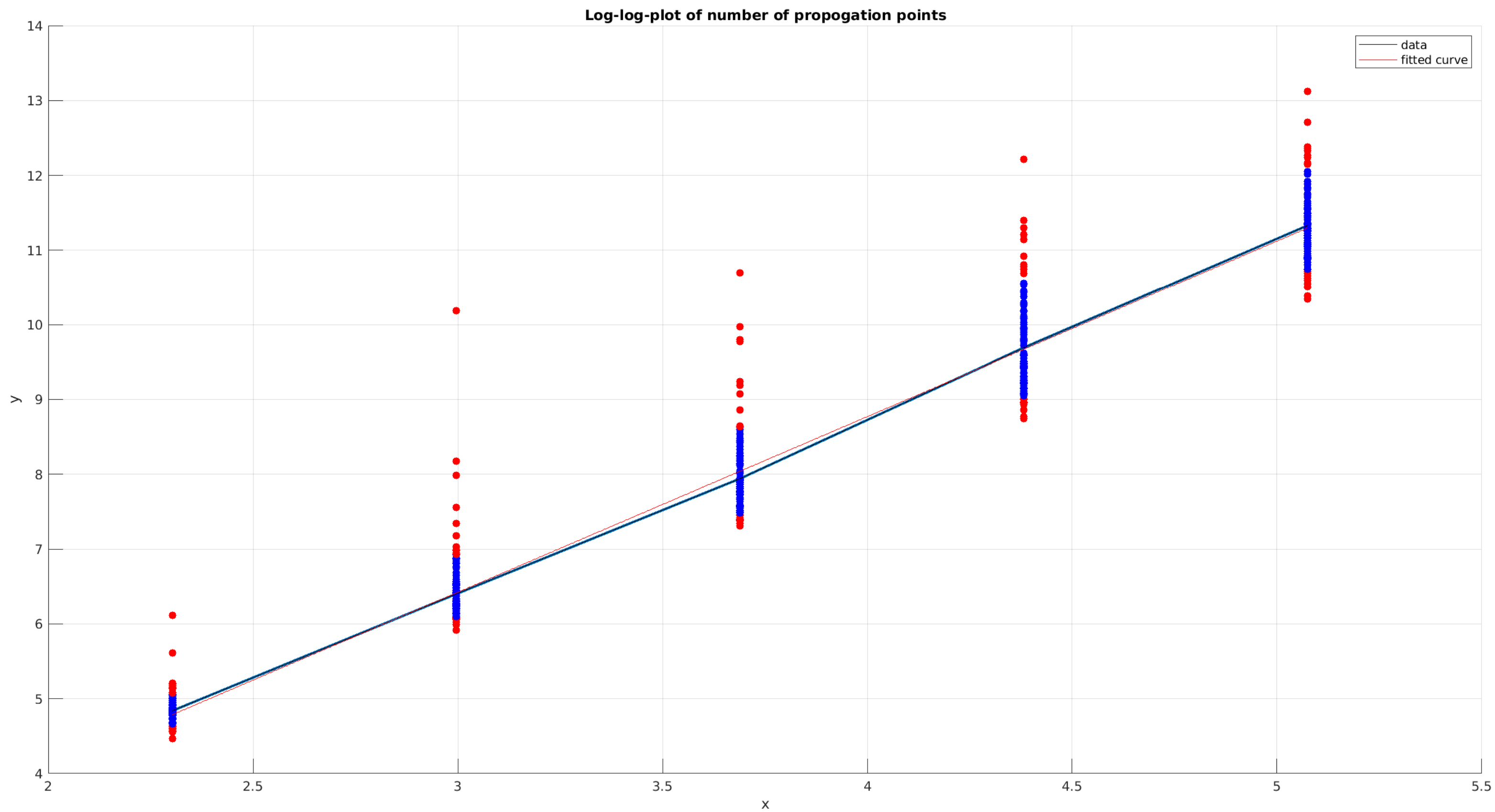
60 points



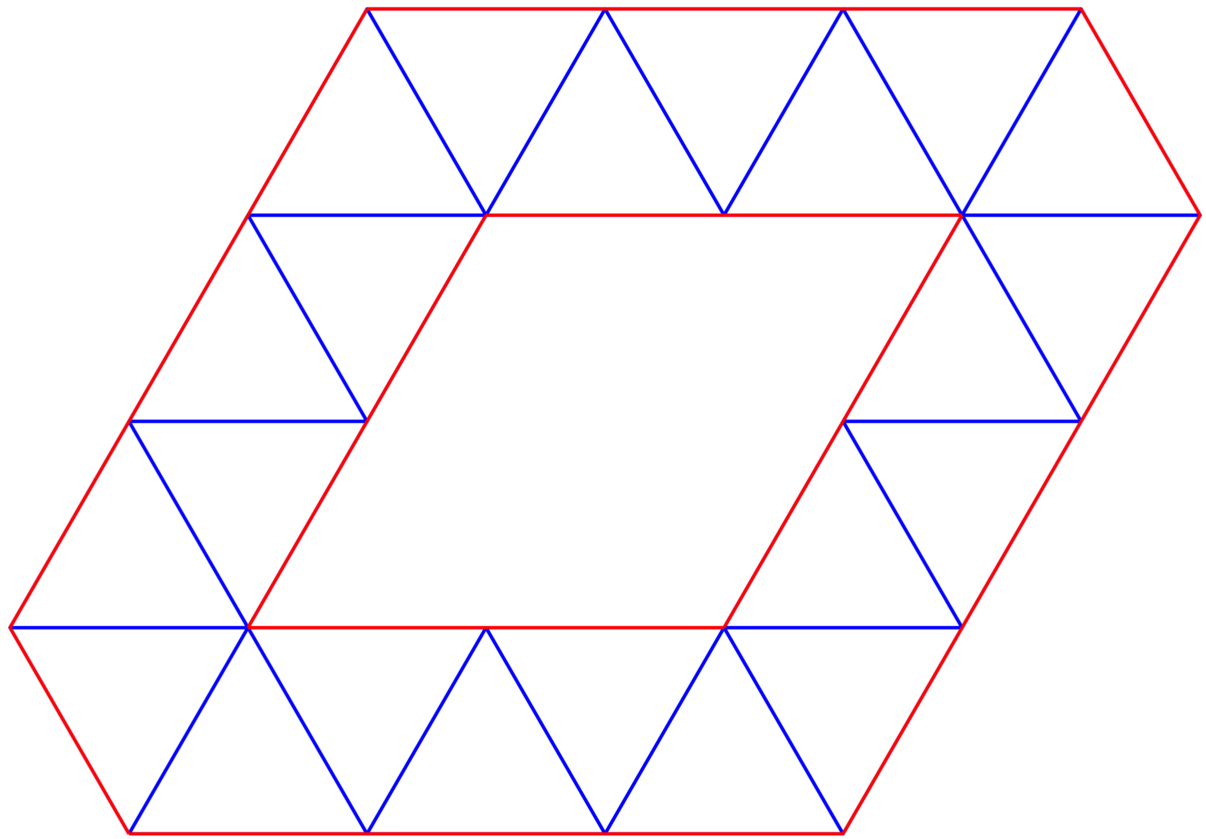
The central regions.



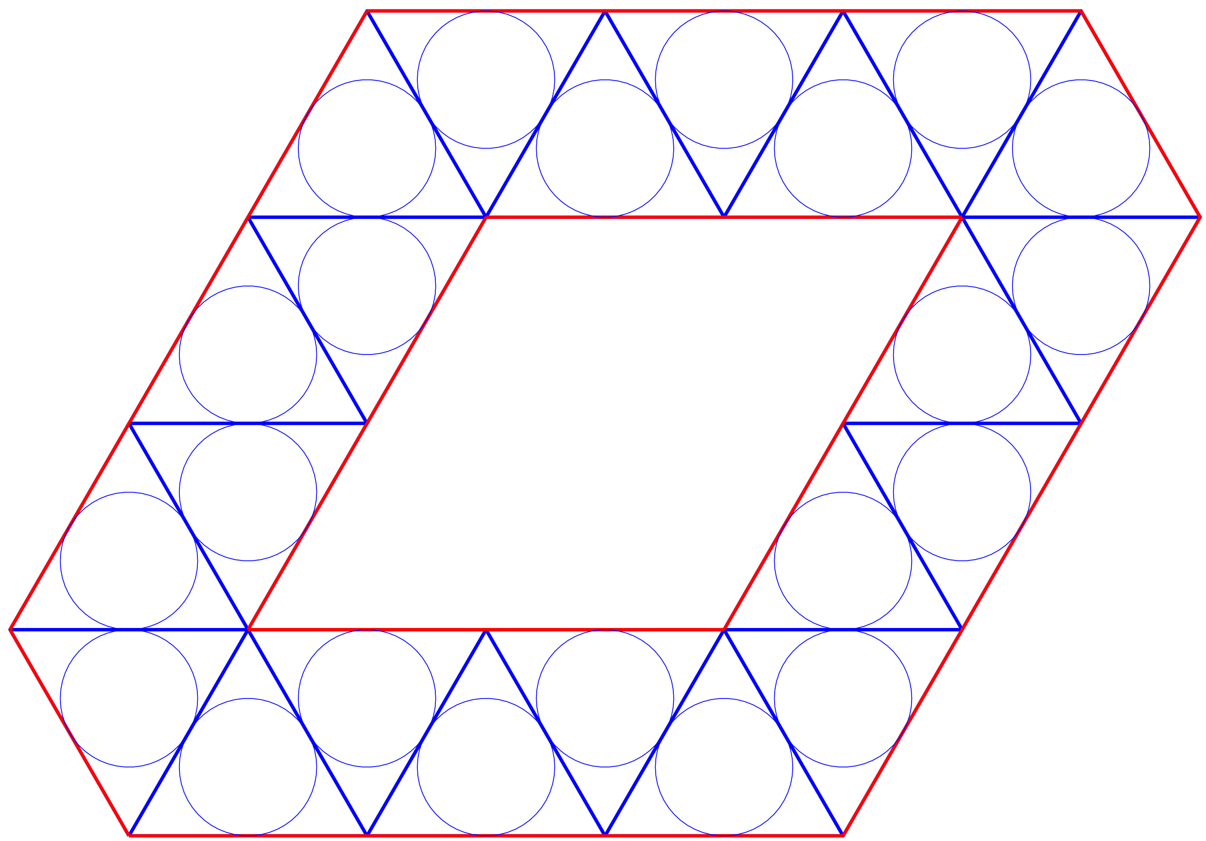
Propagation lines starting at all cusp points.



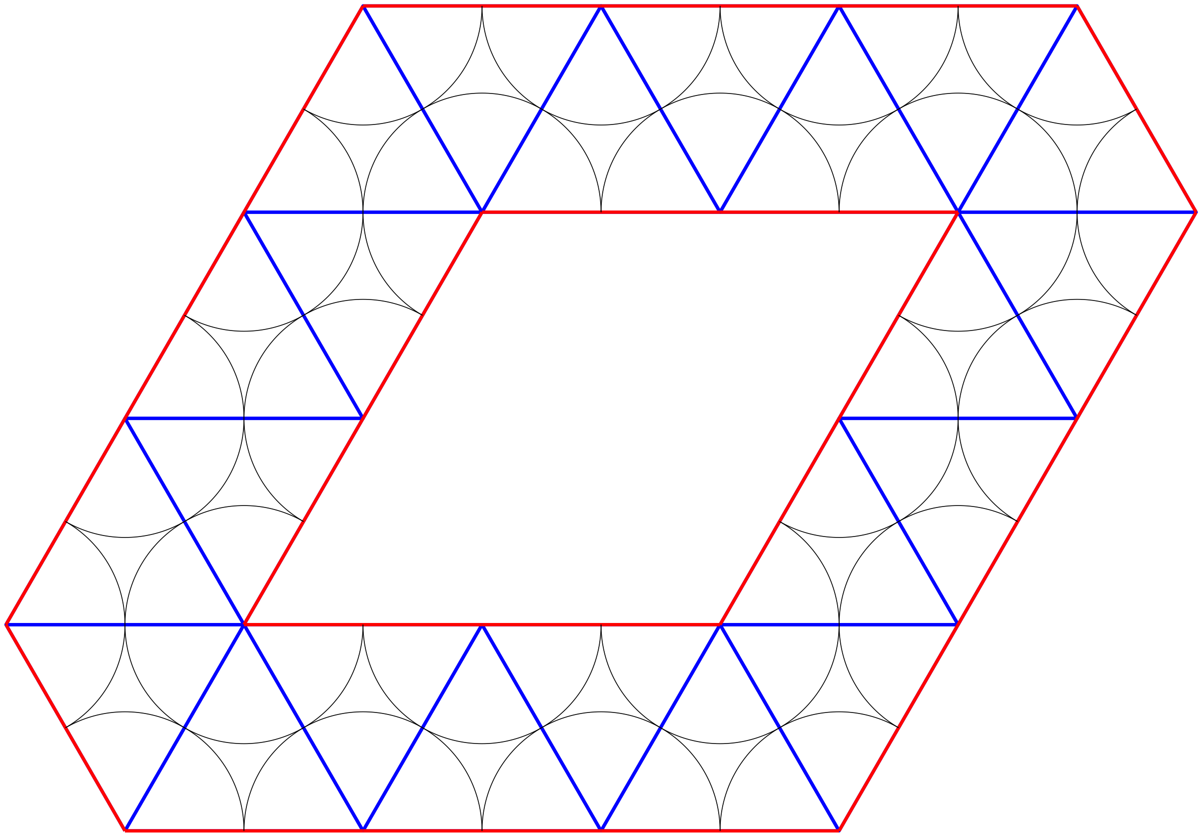
Log-log plot of number points created versus n.



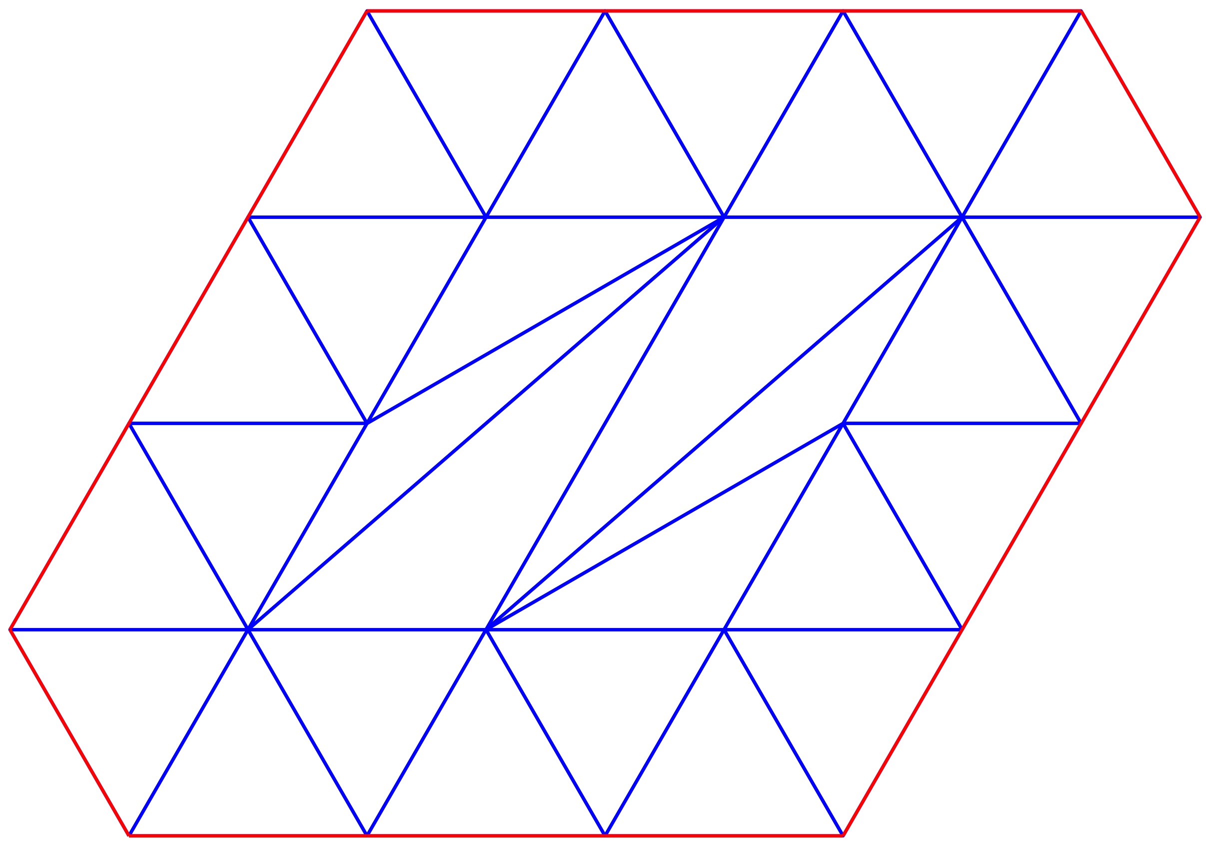
A ring of equilateral triangles.



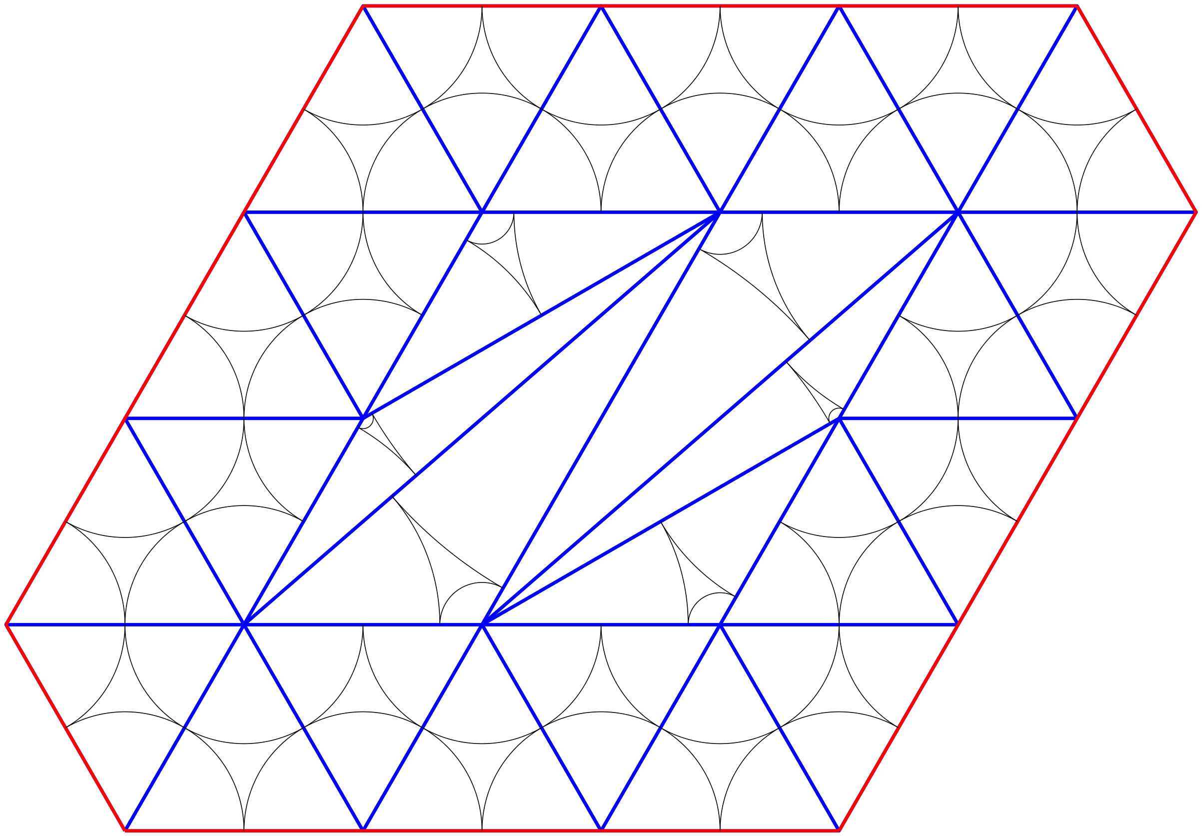
The in-circles are tangent.



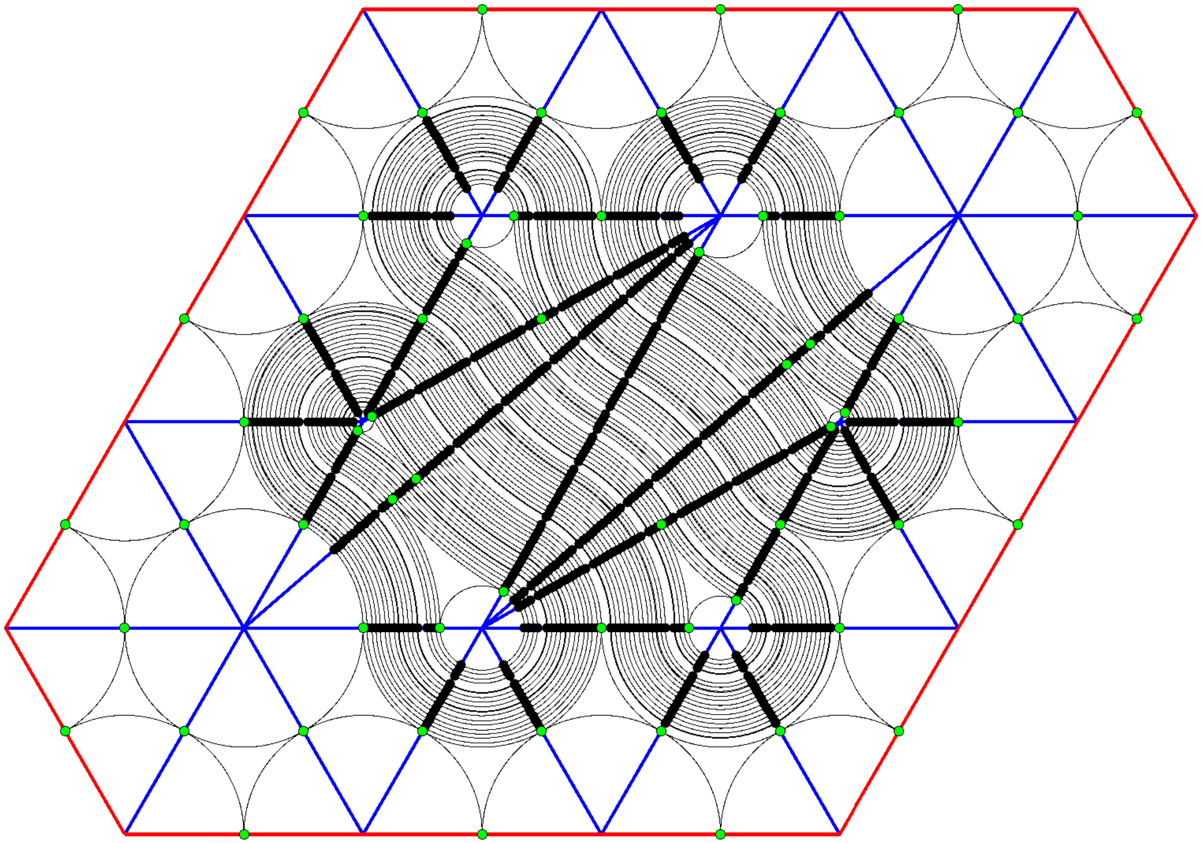
The the cusps touch; for a closed flow line.



No matter how we triangulate interior, flow lines never exit.



No matter how we triangulate interior, flow lines never exit.



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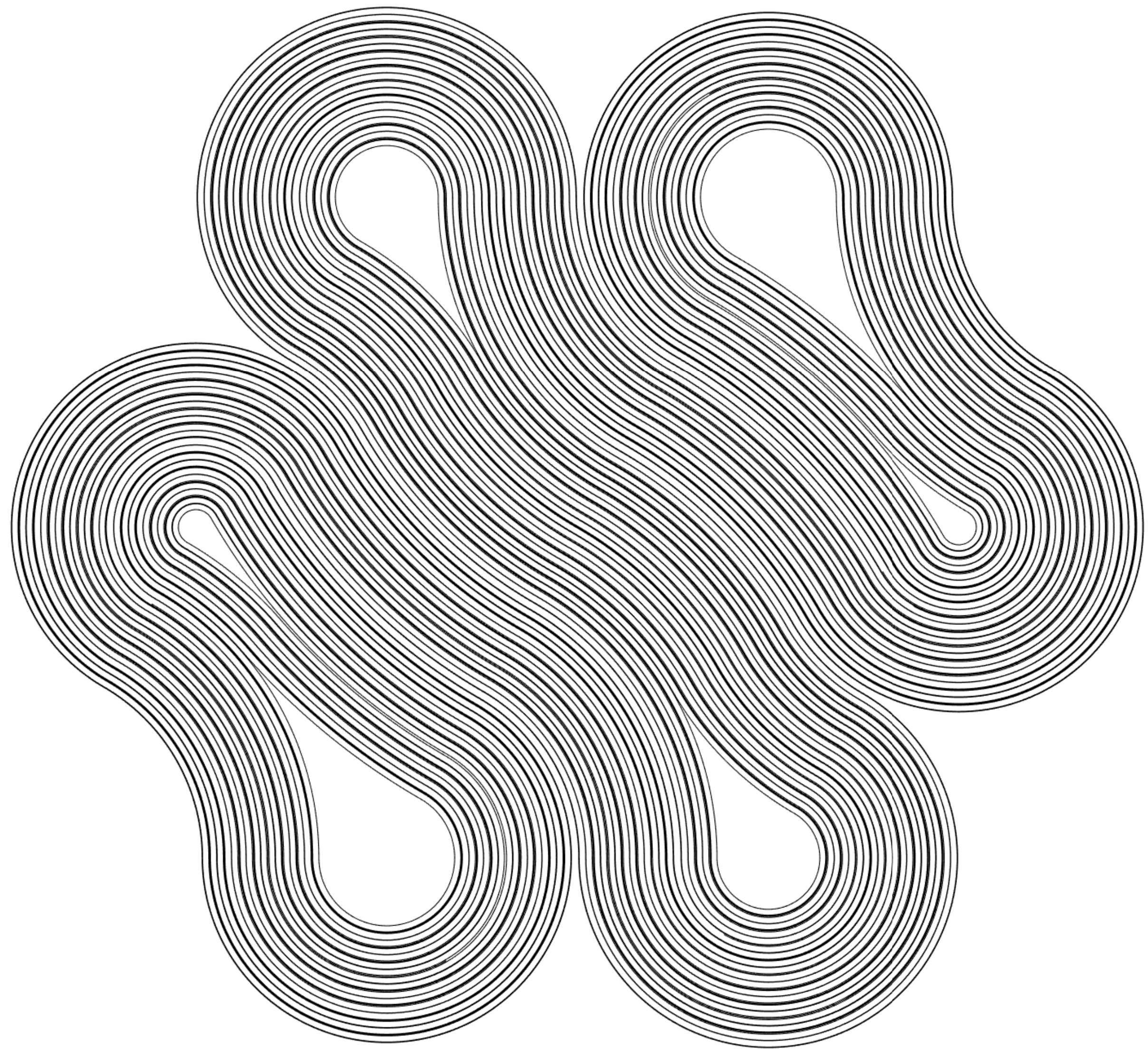
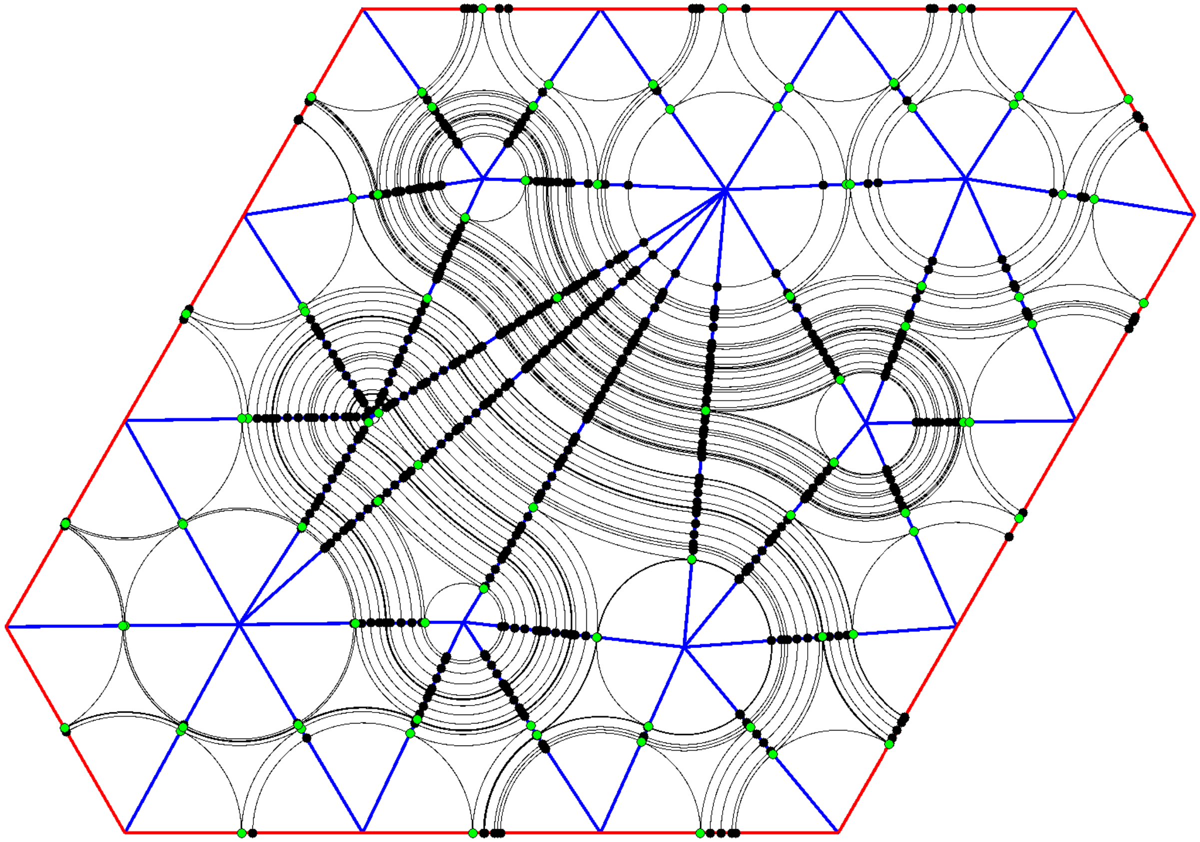




Randomly perturb some vertices; flow lines “leak” to the boundary.

The triangulation flow arises from applications to optimal meshing.

A triangulation is **non-obtuse** if every angle is $\leq 90^\circ$.

Non-obtuse triangulations are important in various numerical methods.

For example, Vavasis showed that matrices arising from finite element method for a certain PDE have conditions numbers that grow exponentially (in number of triangles) for general triangulations, but only linearly for non-obtuse triangulations.

Other numerical methods are faster, simpler to implement, or provably correct when using non-obtuse triangulations.

Fact: every triangulation has a non-obtuse refinement (possibly huge).

Fact: some n-triangulations require n^2 elements to refine non-obtusely.

Fact: No polynomial bound is possible with angle bound $\theta < 90^\circ$.

Conj: every triangulation has a polynomial sized non-obtuse refinement.

Fact: every triangulation has a non-obtuse refinement (possibly huge).

Fact: some n-triangulations require n^2 elements to refine non-obtusely.

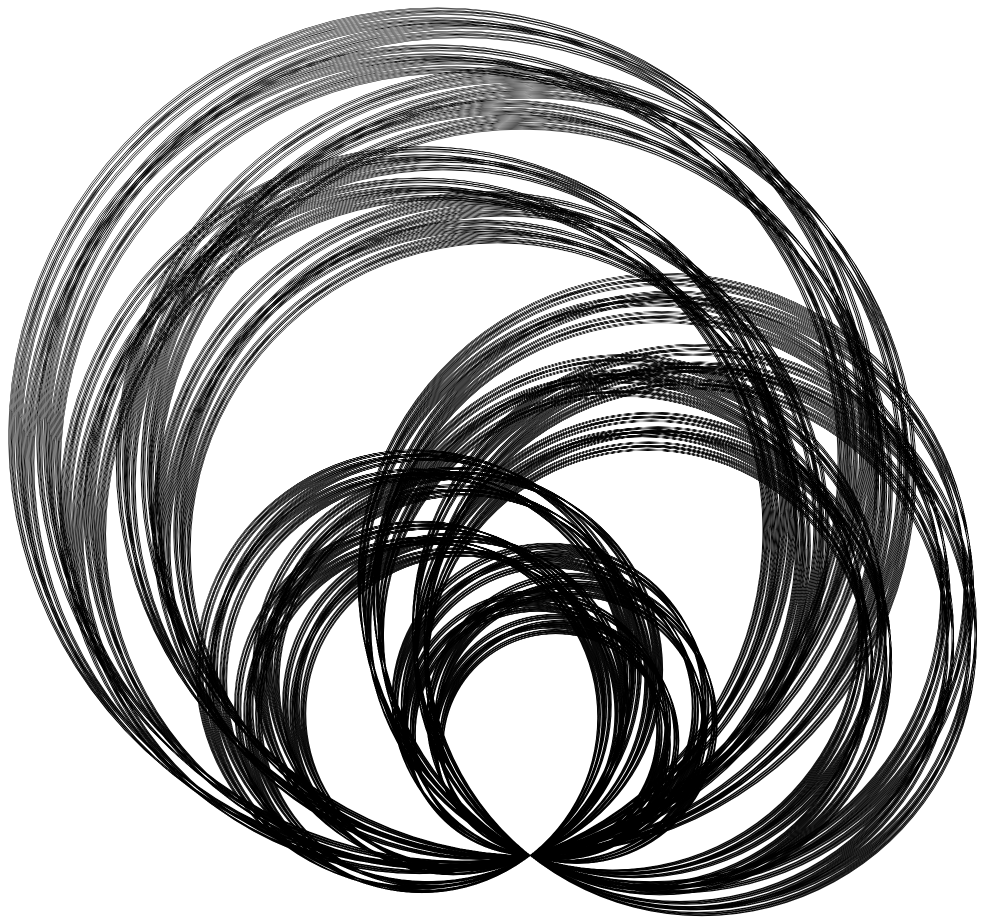
Fact: No polynomial bound is possible with angle bound $\theta < 90°$.

Conj: every triangulation has a polynomial sized non-obtuse refinement.

Theorem (B. 2016): There is always a non-obtuse refinement with $O(n^{2.5})$ elements.

Proof "bends" flow lines to make them collide (discrete closing lemma).

Besicovitch-Kakeya sets

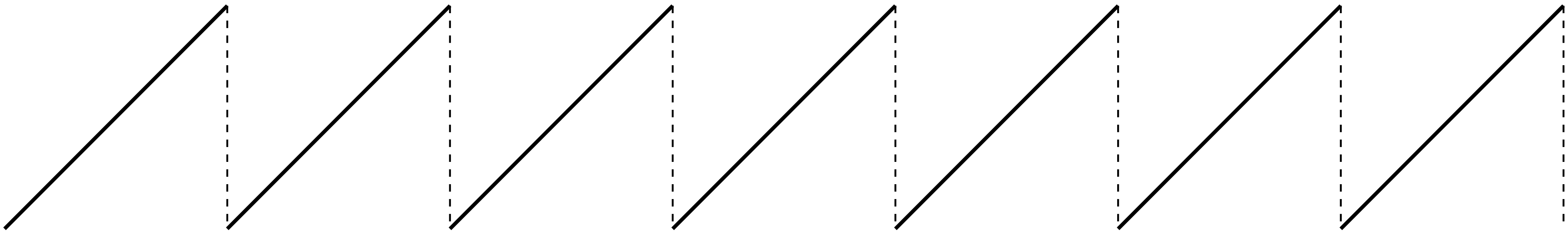


A Besicovitch-Kakeya set is a compact set in the plane that has zero area, but contains a unit line segment in every direction.

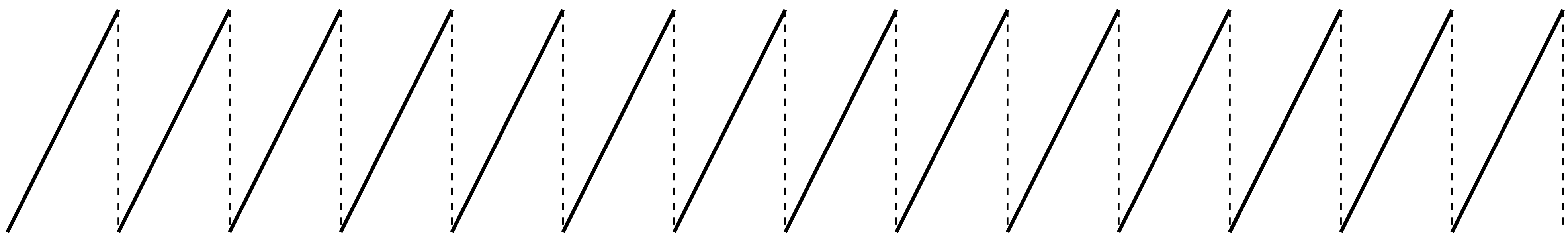
Such sets were constructed by Besicovitch in 1919, while investigating the differences between Riemann and Lebesgue integration.

Around same time Kakeya and Fujiwara were investigating minimal area needed to continuously turn a needle around.

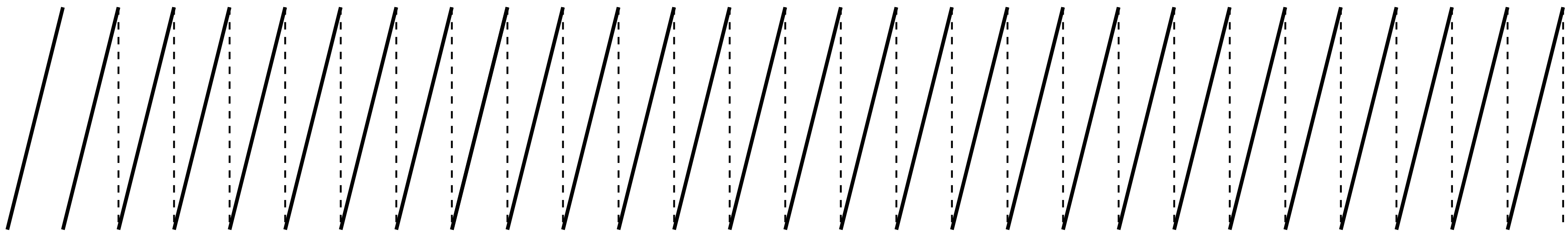
There are now many constructions. Will sketch one based on probabilistic construction of Babichenko, Peres, Peretz, Sousi and Winkler (talking later today). Similar to a construction of Sawyer.



$$g(t) = t - \lfloor t \rfloor.$$



$g(2t)$



$g(4t)$

Let $\{a_k\}_0^\infty$ be dense in $[0,1]$, $|a_{k+1}-a_k| \le \epsilon(k) \searrow 0$, set

$$f_k(t) = \sum_{m=1}^{k} (a_{m-1}-a_m)\frac{g(2^m t)}{2^m}, \qquad f(t) = \lim_{k\to\infty} f_k(t).$$

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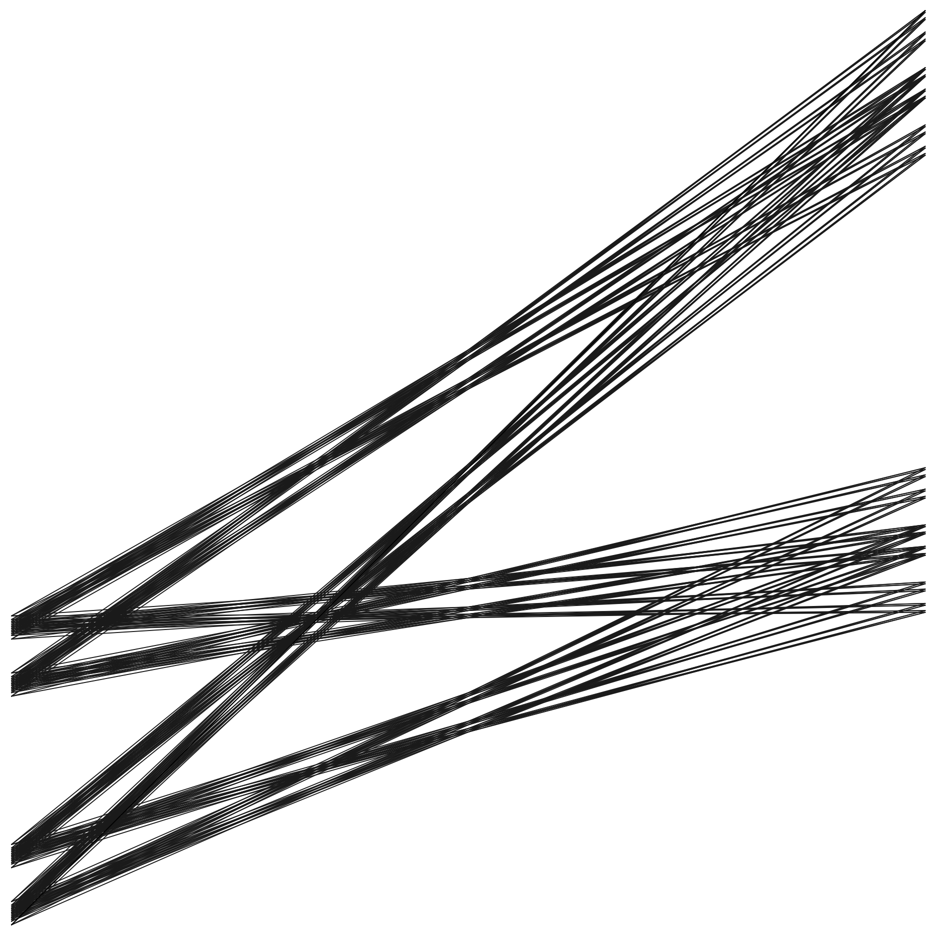
If $a_0 = 0$, then by telescoping series $f_k'(t) = -a_k$ on each component I of $U = [0,1] \setminus 2^{-k}\mathbb{Z}$. Thus $t \to f_k(t) + a_k t$ maps each such I to a point.



Define $K = \{(a, f(t) + at) : a, t \in [0, 1]\}$.
Fixing t and varying a gives a segment of slope t inside K.
Fixing a and varying t gives a (disconnected) vertical slice.

Fix $a \in [0,1]$ and choose k so that $|a - a_k| < \epsilon(k)$. Since

$$f(t) + at = (f(t) - f_k(t)) + (f_k(t) + a_k t) + (a - a_k)t,$$

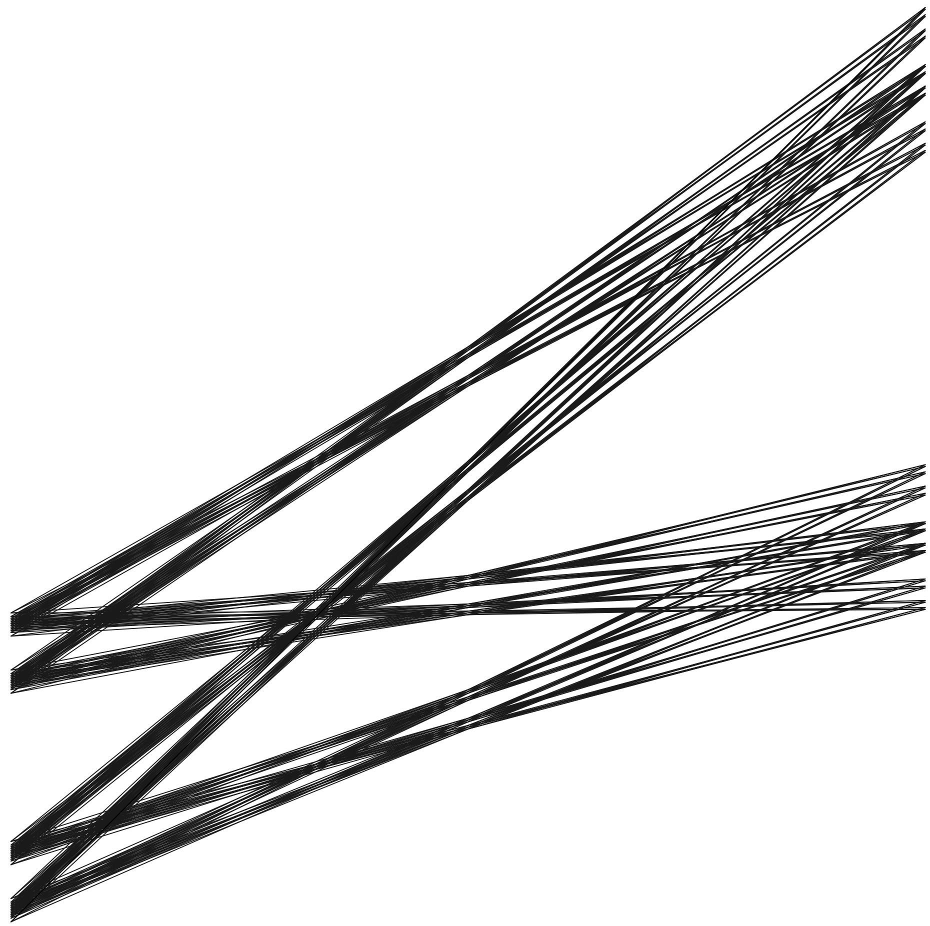
and

$$|f(t) - f_k(t)| \leq \sum_{m=k+1}^{\infty} \frac{|a_{m-1} - a_m|}{2^m} g(2^m t) = \epsilon(k) 2^{-k},$$

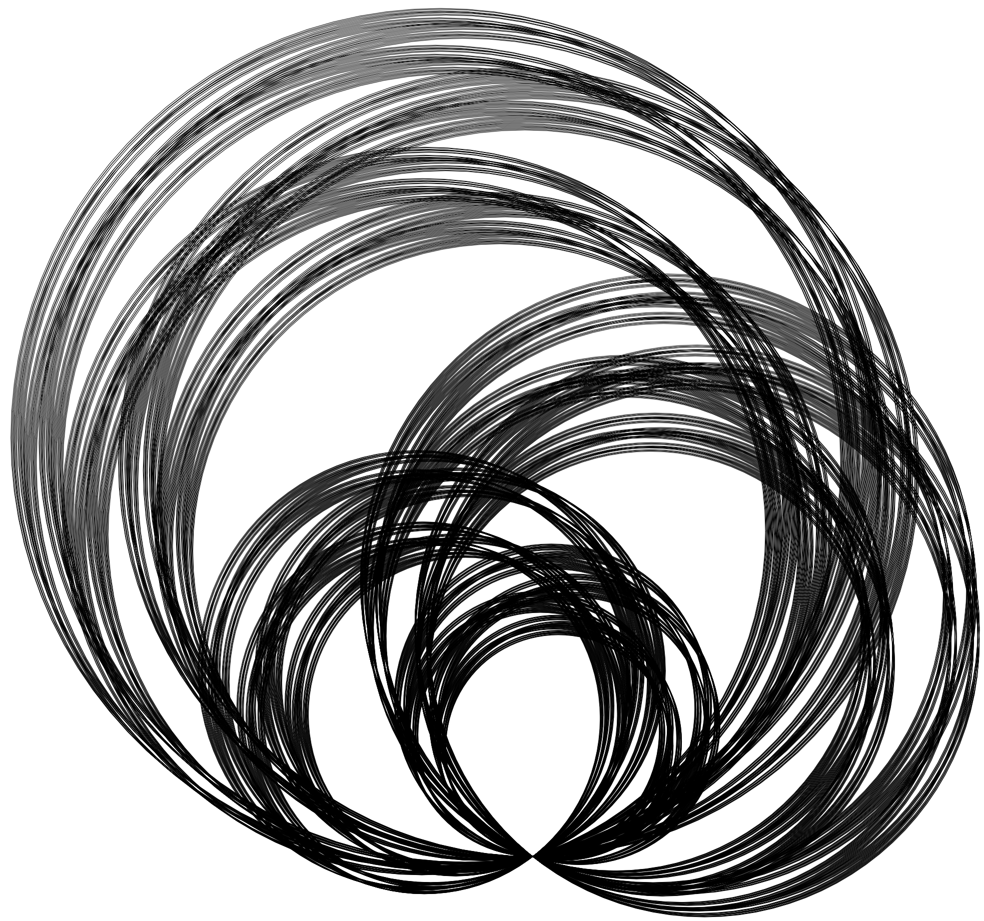
each of the 2^k intervals I maps to a set of diameter

$$\leq \epsilon(k)|I| + 0 + \epsilon(k)|I| \leq \epsilon(k) 2^{-k+1} = 2\epsilon(k)|I|$$

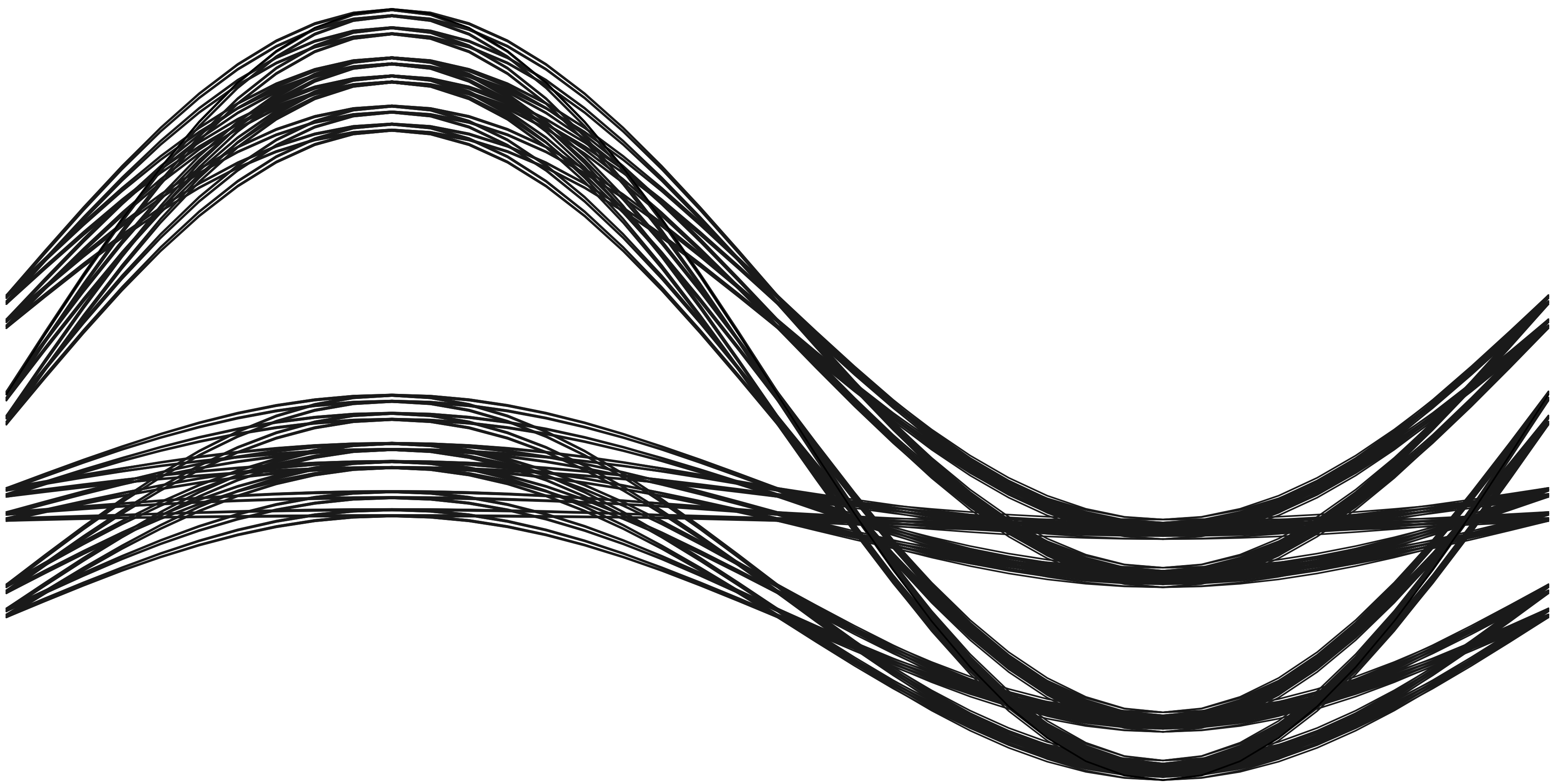
under $t \to f(t) + at$.



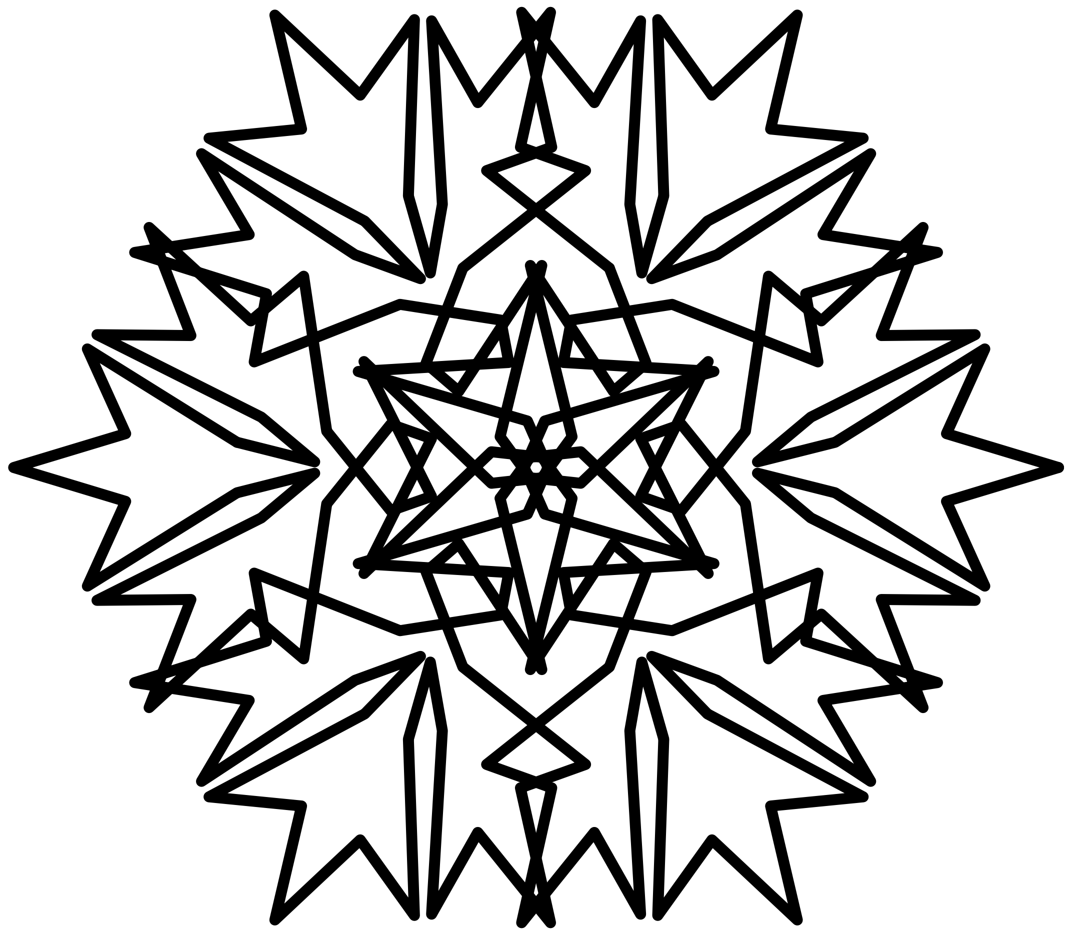
For every a, the slice $\{t : (a, t) \in \overline{K}\}$ has length zero, so $\text{area}(\overline{K}) = 0$.



Null set containing circles of every radius



Null set containing graph of $a\sin(t) + b$ for every a.



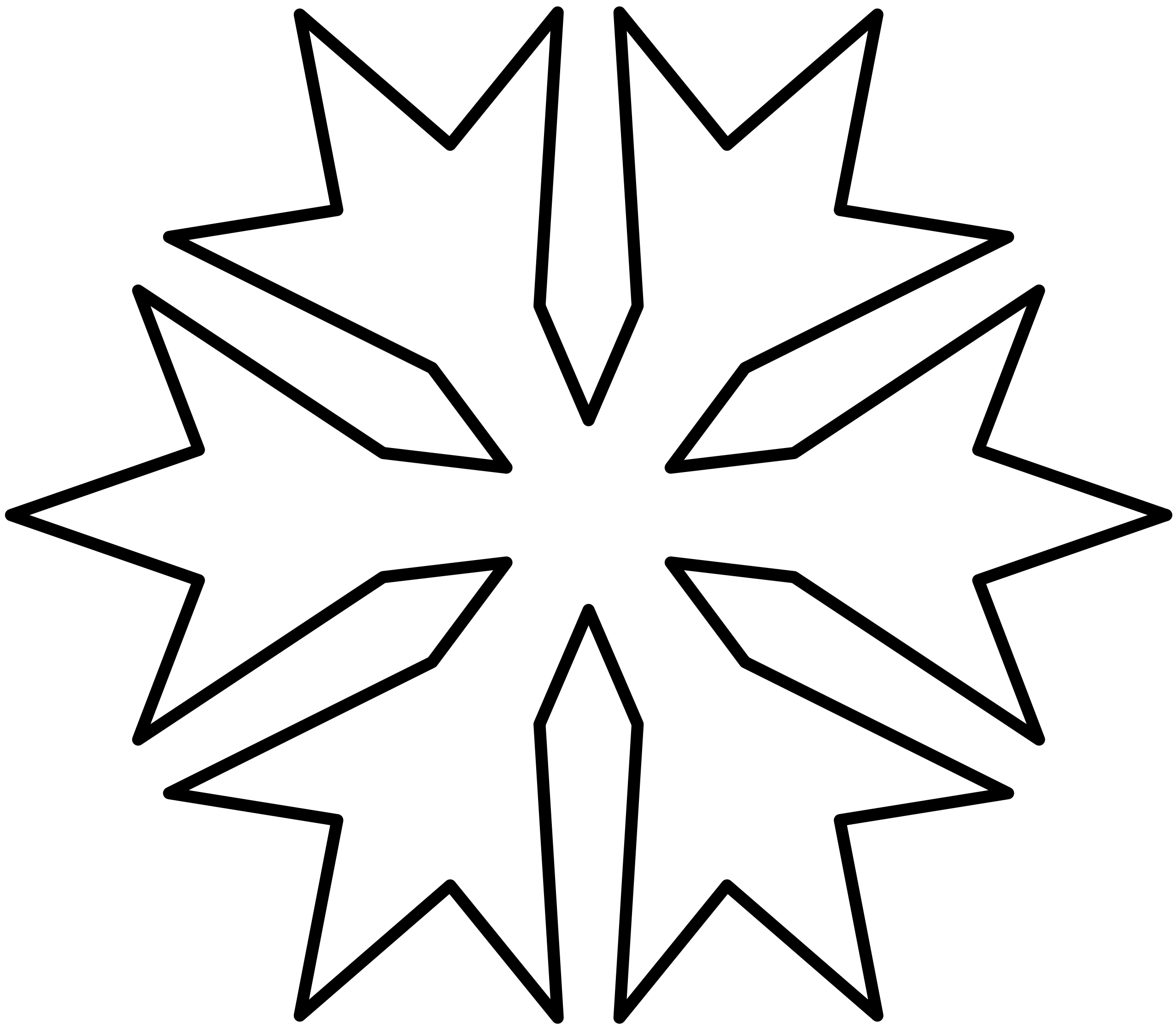
A Peano curve is a continuous map from an arc into the plane whose image has non-empty interior.

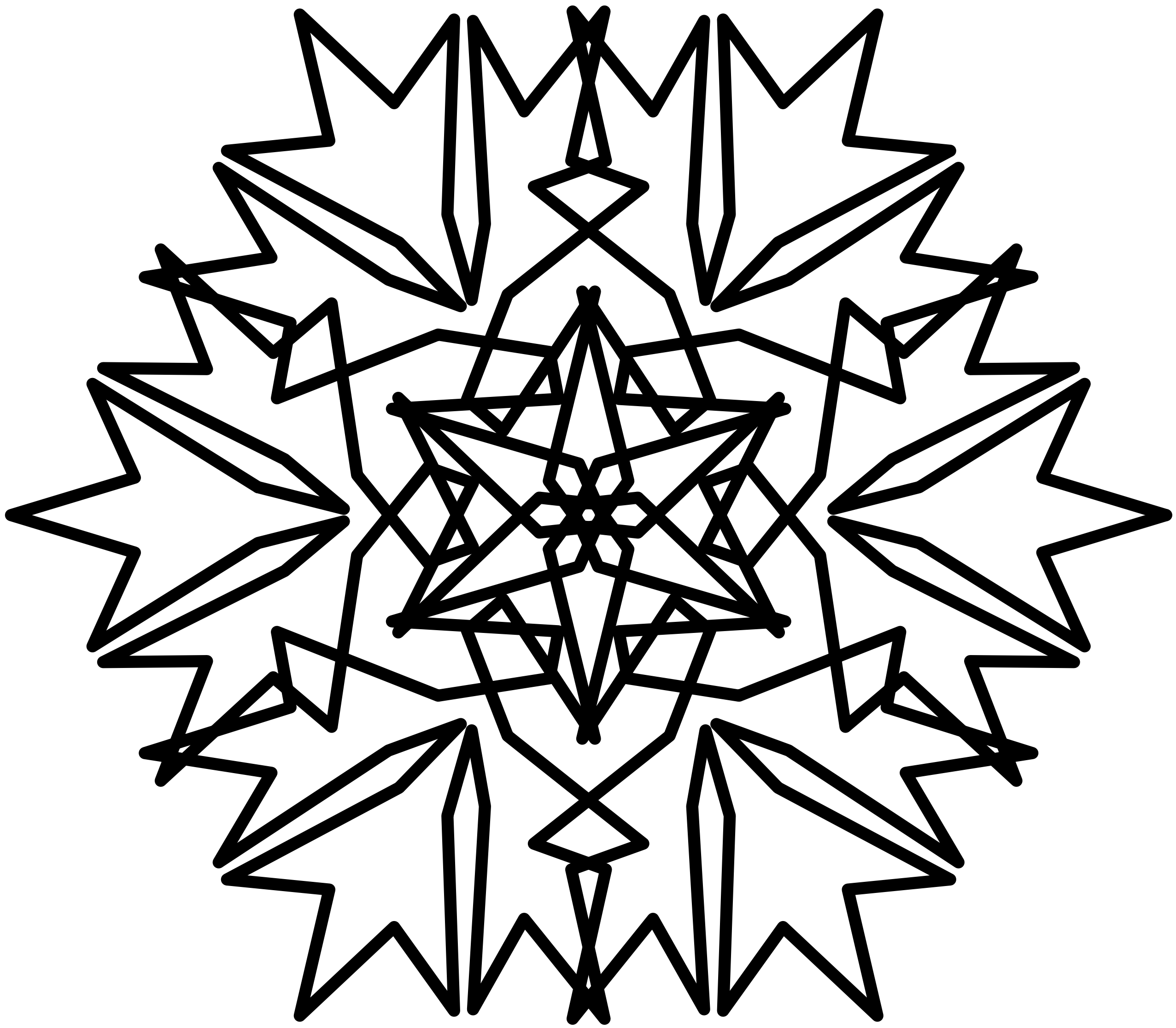
A Peano curve is a continuous map from an arc into the plane whose image has non-empty interior.

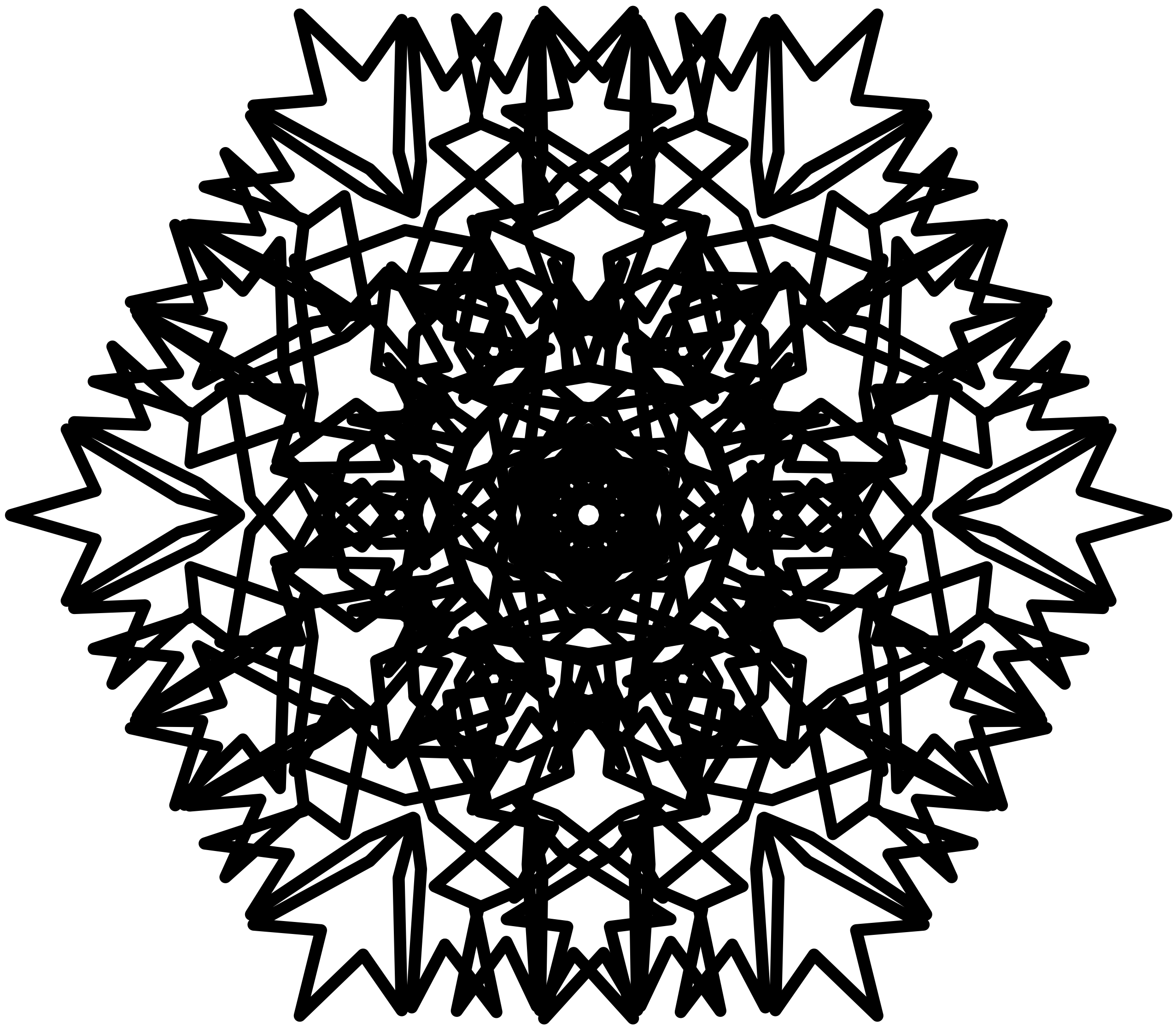
Suppose Γ is closed curve and F is continuous on sphere and holomorphic off Γ. Argument principle implies $F(\mathbb{C}) = F(\Gamma)$. If F is not constant, then $F(\Gamma)$ has interior.

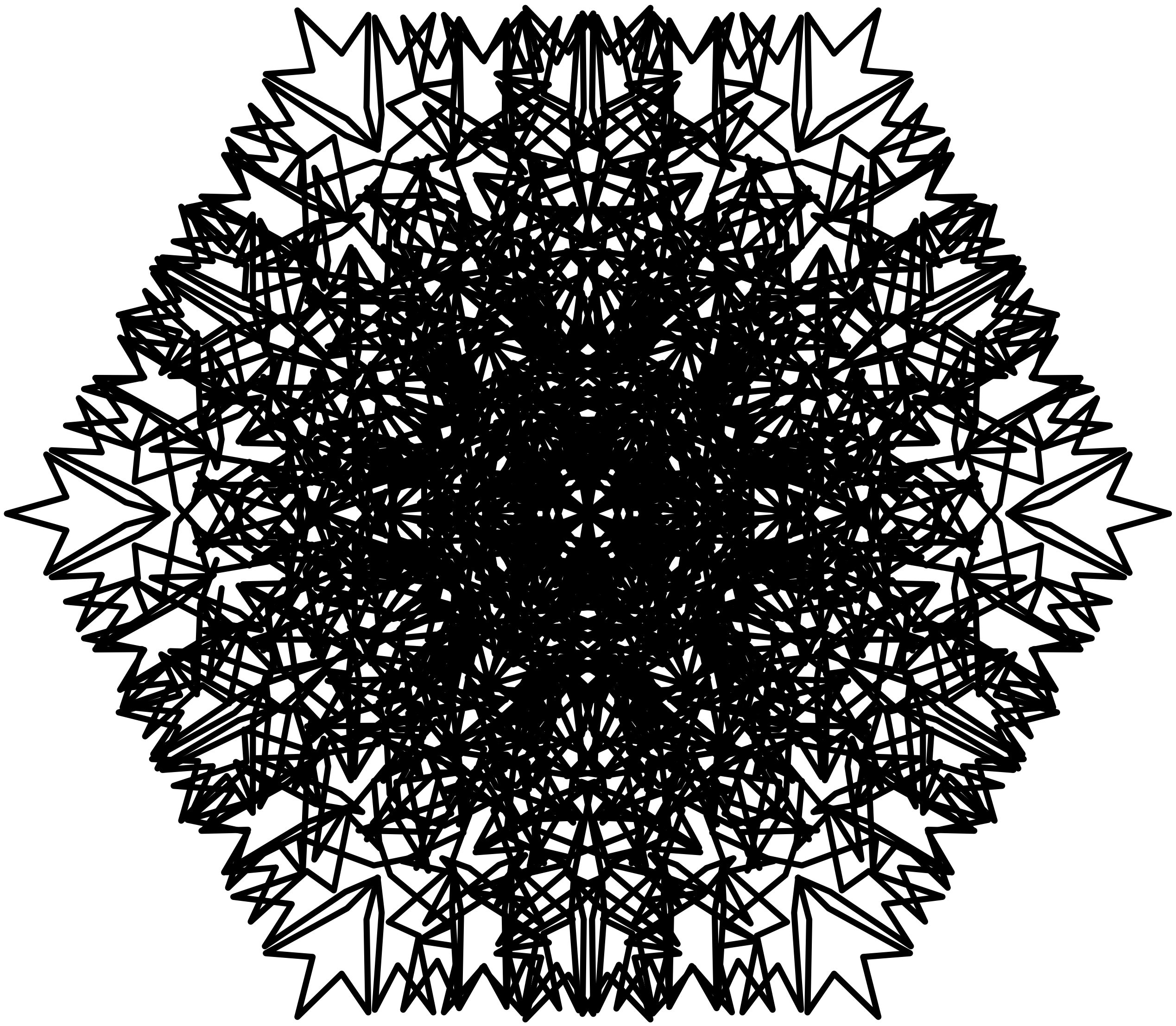
Such Peano functions exist whenever inside and outside harmonic measures are mutually singular. Theorem of Andrew Browder and John Wermer (reproved in my PhD thesis).

The von Koch snowflake S has dimension $\alpha = \log 4/\log 3$. Convolving $1/z$ against α-measure on S gives a continuous function F that is non-constant and holomorphic off S. Thus is Peano. Following pictures show convolution with point masses approximating Hausdorff measure.









Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, and it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine." [Nowadays you can think of it as a computer!] ... the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning.

Sir Michael Atiyah, "Mathematics in the 20th Century",
Fields lecture, Toronto, June


ICM 2018
Christopher Bishop
United States of America

Mathematical research ... is a process that Charles Fefferman ... likens to "playing chess with the devil." ... The devil is vastly superior at chess, but, Fefferman explained, you may take back as many moves as you like, and the devil may not. You play a first game, and, of course, "he crushes you." So you take back moves and try something different, and he crushes you again, "in much the same way." If you are sufficiently wily, you will eventually discover a move that forces the devil to shift strategy; you still lose, but – aha! – you have your first clue.

Gareth Cook, "The singular mind of Terry Tao", July 24, 2015, *NY Times Magazine*

“The Chess Players” by Morits Retzch

Thanks for playing along!