WEIL-PETERSSON CURVES AND FINITE TOTAL CURVATURE

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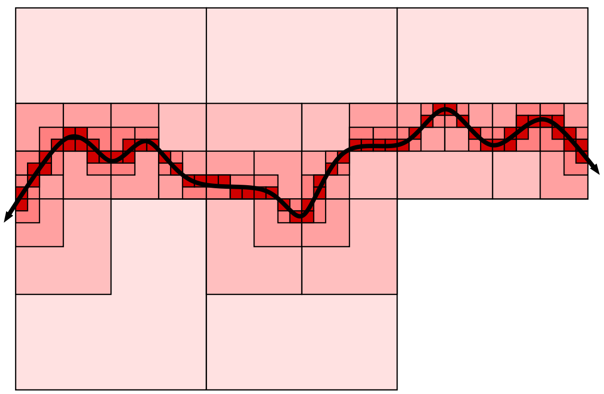
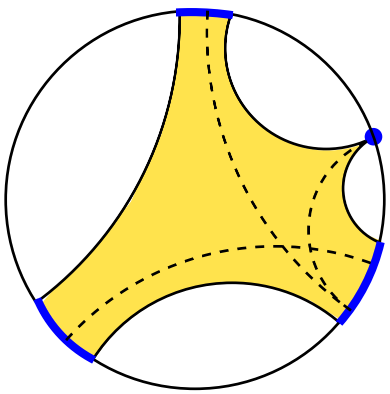
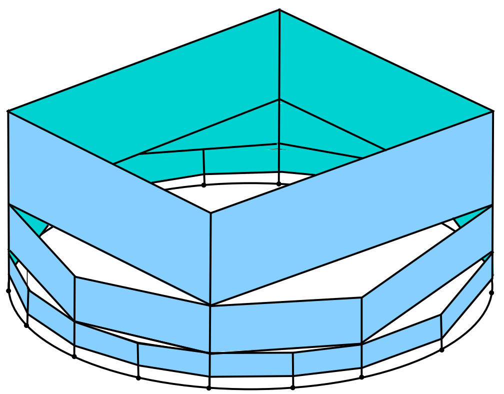
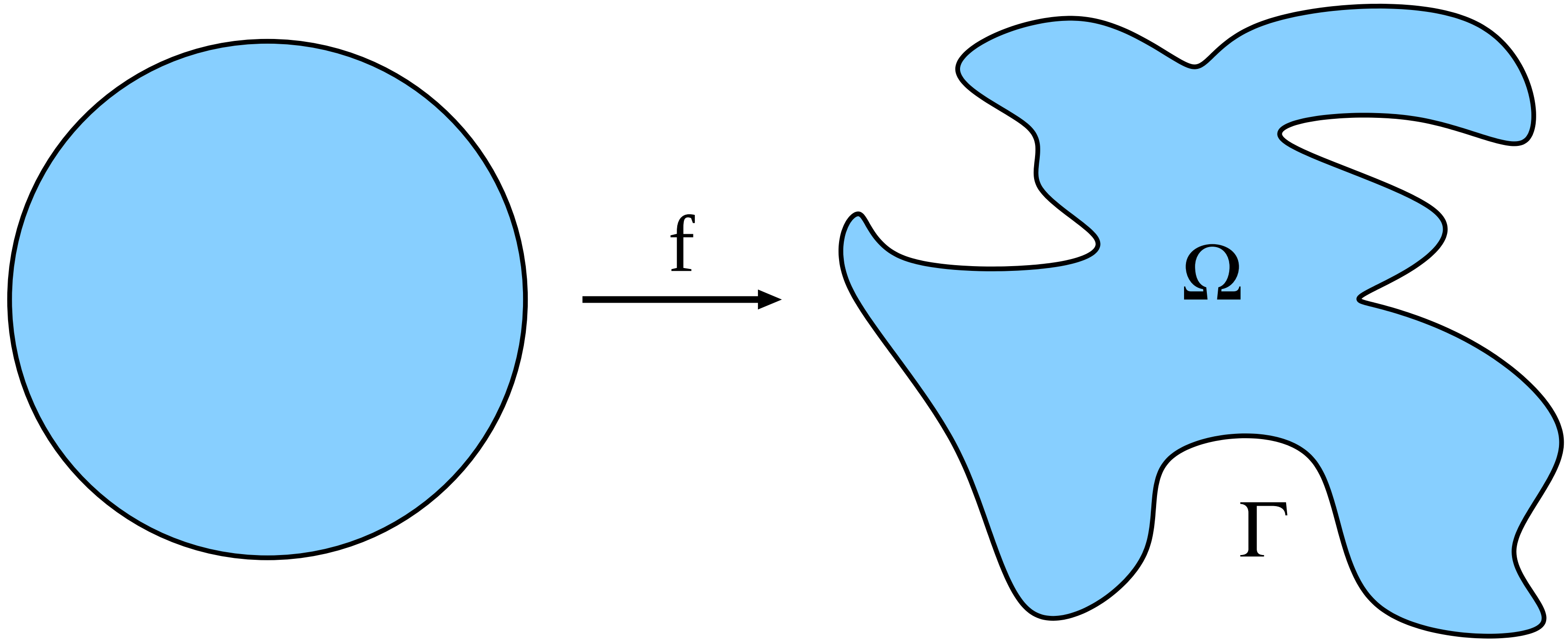
`www.math.sunysb.edu/~bishop/lectures`



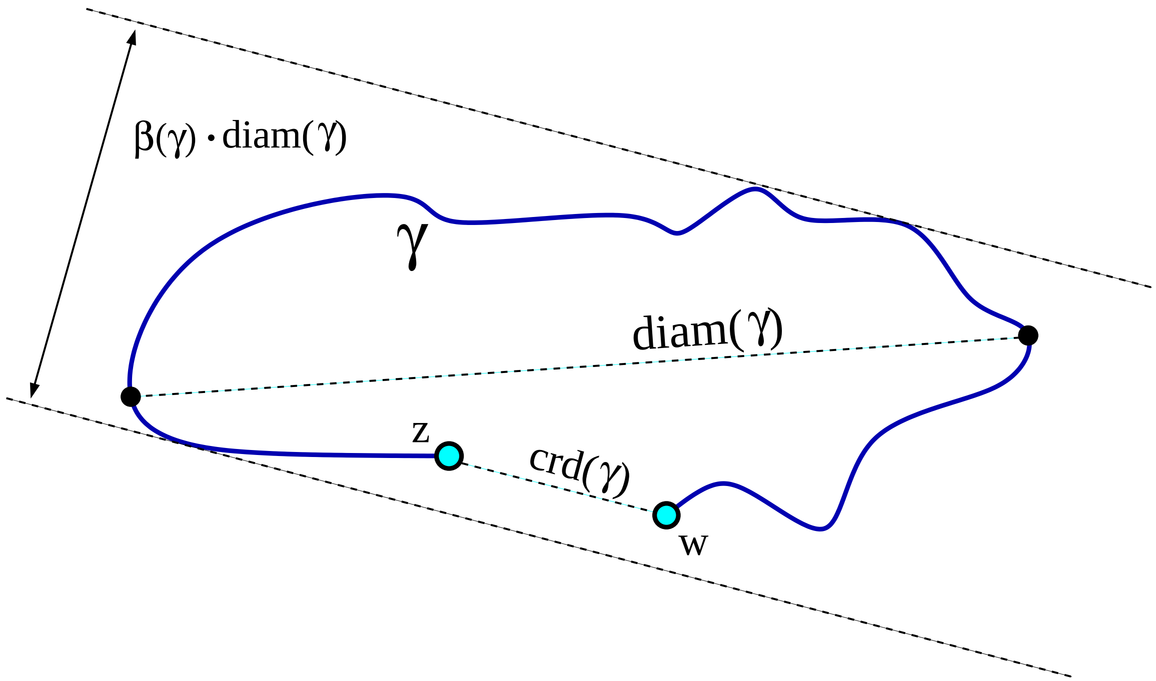






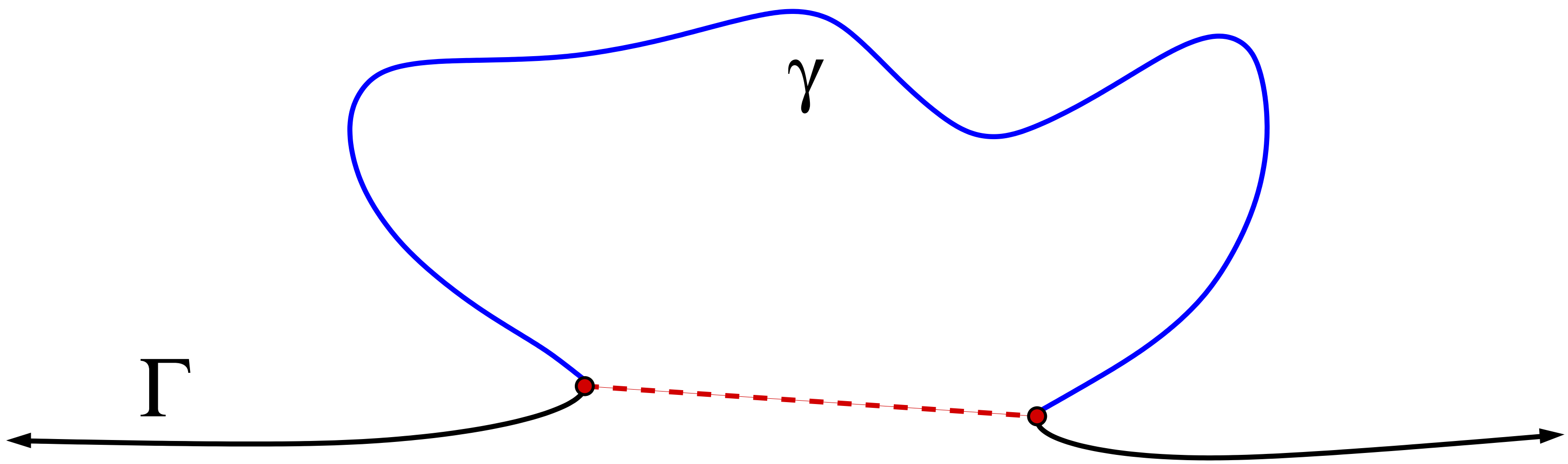
Suppose $\Gamma = \partial\Omega$ is Jordan curve, $f : \mathbb{D} \to \Omega$ is conformal.

Basic problem: how is geometry of Γ related to properties of f?



If γ is a planar Jordan arc with endpoints z, w, we set:

- $\mathrm{diam}(\gamma)$ = diameter of γ
- $\mathrm{crd}(\gamma) = z - w$ = chord length of γ
- $\ell(\gamma)$ = length of γ,
- $\Delta(\gamma) = \ell(\gamma) - \mathrm{crd}(\gamma)$ = excess length
- $\beta(\gamma) = \sup\{z \in \gamma : \mathrm{dist}(z, L)/\mathrm{diam}(\gamma)\}$, L = line through z, w



Γ is a **quasicircle** iff $\text{diam}(\gamma) = O(\text{crd}(\gamma))$ for $\gamma \subset \Gamma$.

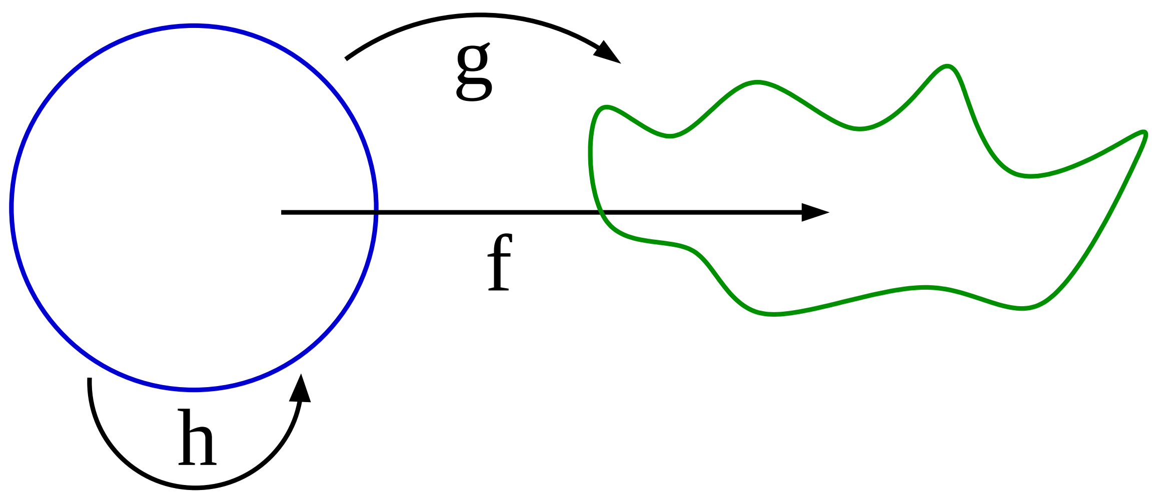
Γ is **chord-arc** iff $\ell(\gamma) = O(\text{crd}(\gamma))$ for $\gamma \subset \Gamma$.

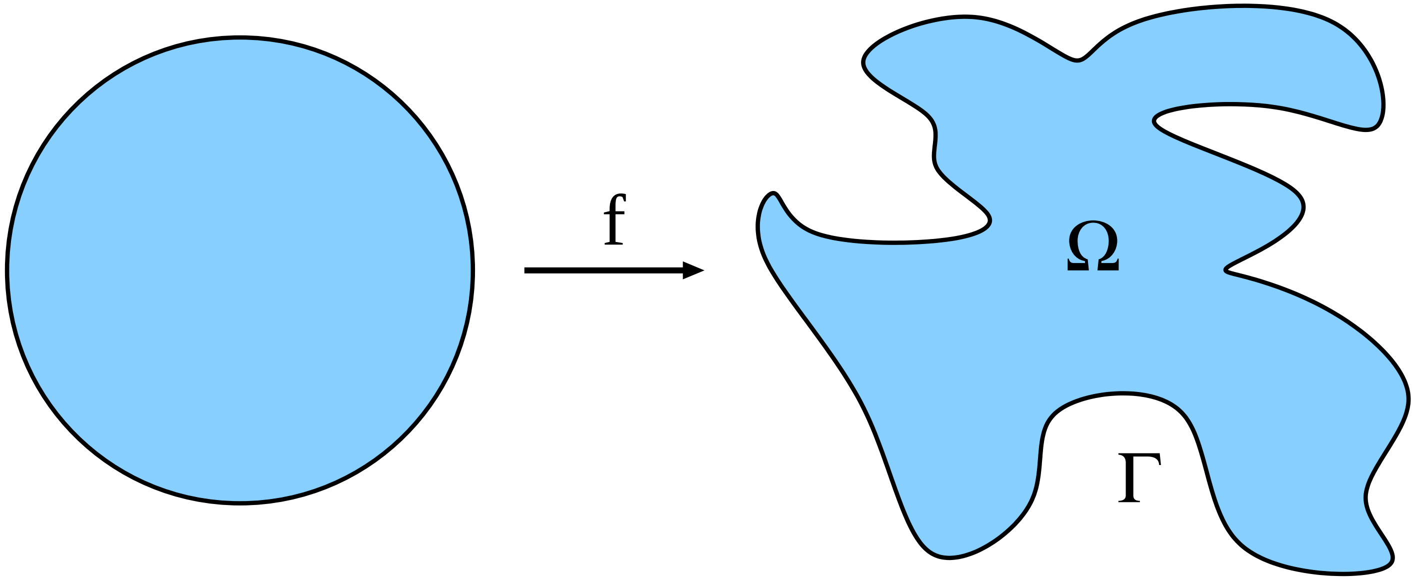
- For a conformal map, f' is never zero, so $\log f'$ makes sense.

- Also interested in Schwarzian derivative: $S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$.

- If Γ is a quasicircle, f has a quasiconformal extension to plane, with dilation μ defined on $\mathbb{D}^* = \{|z| > 1\}$.

- $\Gamma = f(\mathbb{T})$ is associated to a conformal welding $h = g^{-1} \circ f : \mathbb{T} \to \mathbb{T}$:



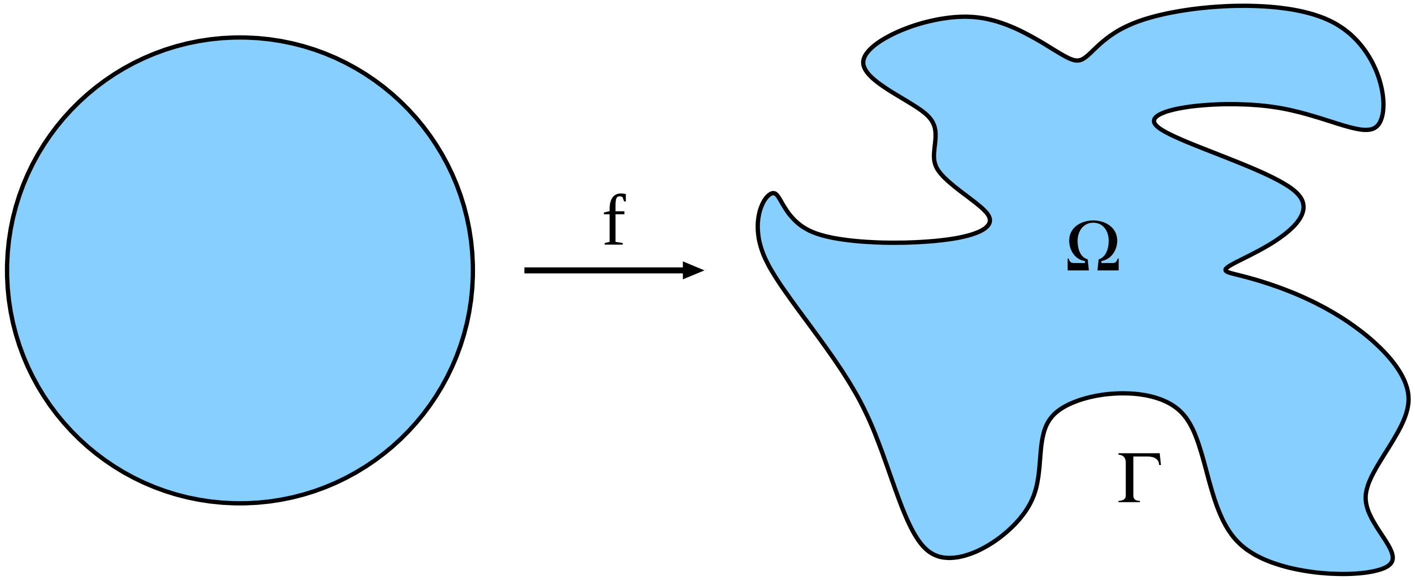


Theorem (Pommerenke, 1978): Γ is asymptotically conformal, i.e.,

$$\beta(\gamma) \to 0, \text{ as } \operatorname{diam}(\gamma) \to 0,$$

iff $\log f'$ is in little Bloch class

$$\mathcal{B}_0 = \left\{ g \text{ holomorphic on } \mathbb{D} : \frac{|g'(z)|}{1-|z|} \to 0 \right\}.$$



Theorem (Pommerenke, 1978): Γ is asymptotically smooth, i.e.,

$$\frac{\Delta(\gamma)}{\mathrm{crd}(\gamma)} = \frac{\ell(\gamma) - \mathrm{crd}(\gamma)}{\mathrm{crd}(\gamma)} \to 0, \text{ as } \mathrm{diam}(\gamma) \to 0,$$

iff $\log f' \in \mathrm{VMO}$.

$\log f'$	$\mathcal{B}_0$	BMOA	VMOA	Dirichlet
$(\log f')'(1-\|z\|^2)$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$S(z)(1-\|z\|^2)^2$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
μ	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$h = g^{-1} \circ f$	symmetric	strongly quasisymmetric	$\log h' \in \mathrm{VMO}$	$\log h' \in H^{1/2}$
$\Gamma = f(\mathbb{T})$	asymptotically conformal	Bishop-Jones condition	asymptotically smooth	

Each of four columns is a theorem giving 5 equivalent conditions.

Conditions become more restrictive moving left to right.

$\log f'$	$\mathcal{B}_0$	BMOA	VMOA	Dirichlet
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Pommerenke's theorem: $\log f' \in \mathcal{B}_0 \Leftrightarrow \Gamma$ asymptotically conformal.

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Astala-Zinsmeister theorem:

$$\log f' \in \mathrm{BMO} \Leftrightarrow |S(f)|^2(1-|z|^2)^3 dzxdy \text{ is Carleson.}$$

μ is a Carleson measure if $\mu(D(x,r)) = O(r)$.

BMOA are homomorphic functions such that

$$|g'(z)|^2(1-|z|^2)dxdy \text{ is Carleson}$$

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$$\mathrm{C}_0(\mathbb{D}) = \{f : \text{ continuous, tending to } 0 \},$$
$$\mathrm{CM}(\mathbb{D}) = \{f : |f|^2(1-|z|^2)^{-1}dxdy \text{ is a Carleson measure } \},$$
$$\mathrm{CM}_0(\mathbb{D}) = \{f : |f|^2(1-|z|^2)^{-1}dxdy \text{ is vanishing Carleson}\},$$
$$L^2(dA_\rho) = \{f : \int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^{-2}dxdy < \infty\},$$

$$A_\rho = \text{ hyperbolic area}$$

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Today's talk is about Dirichlet space $= \{F : \int |F'|^2 dxdy < \infty\}$.

Takhtajan and Teo defined metric on universal Teichmüller space ($\approx$ space of quasicircles) that makes it a disconnected Hilbert manifold.

One connected component is closure of the smooth curves. Elements are called **Weil-Petersson** curves; they are rectifiable quasicircles.

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- Γ is WP $\Leftrightarrow$ Γ has finite Loewner energy (Rohde and Wang)
- Large deviations of SLE(κ) as $\kappa \searrow 0$ (Wang).
- Connections to Brownian loop soup of Lawler and Werner (Wang).
- Computer vision, morphing (Sharon and Mumford, Feiszli and Narayan)
- "WP ... correct analytic choice for the formulation of CFT." (Radnell, Schippers and Staubach)
- Connections to renormalized area, quantum entanglement, Ads/CFT.

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$$\begin{aligned}
\Gamma \text{ is WP} \quad &\Leftrightarrow \ \log f' \text{ is in Dirichlet class (Takhtajan and Teo)} \\
&\Leftrightarrow \ \int_{|z|<1} |S(f)(z)|^2(1-|z|^2)^2 dxdy < \infty \text{ (Cui)} \\
&\Leftrightarrow \ \int_{|z|>1} |\mu(z)|^2 dA_\rho < \infty \text{ (Cui)} \\
&\Leftrightarrow \ \log h' \in H^{1/2} \text{ (Shen)}
\end{aligned}$$

$$H^{1/2}(\mathbb{T}) = \{g : \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|g(z)-g(w)|^2}{|z-w|^2}|dw||dz| < \infty\}.$$

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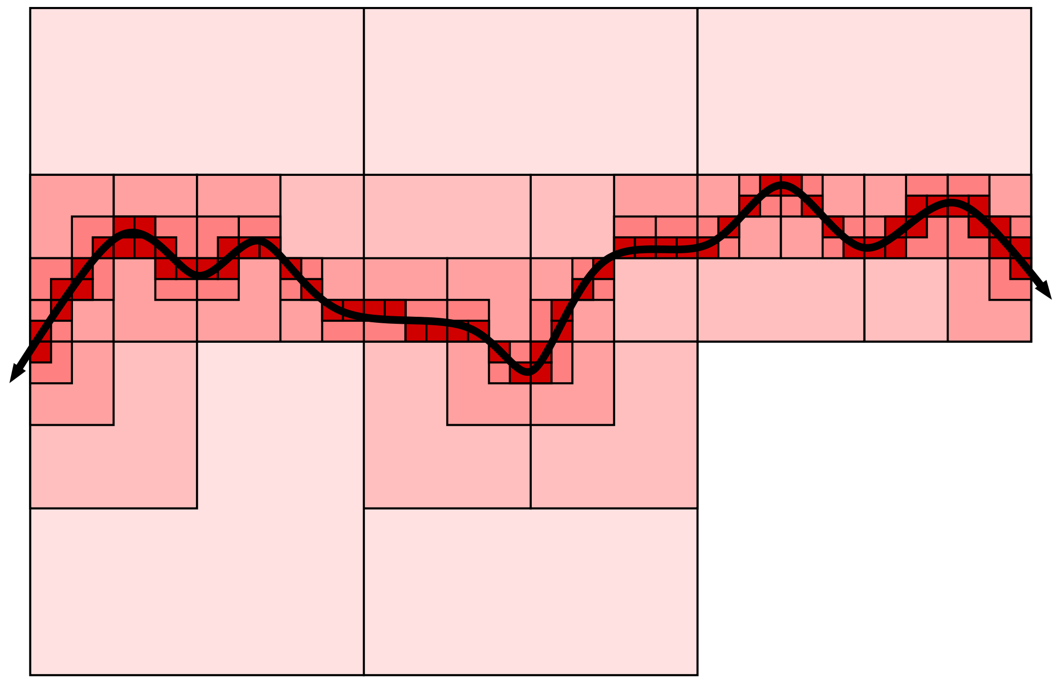
What geometric condition goes in the empty box?

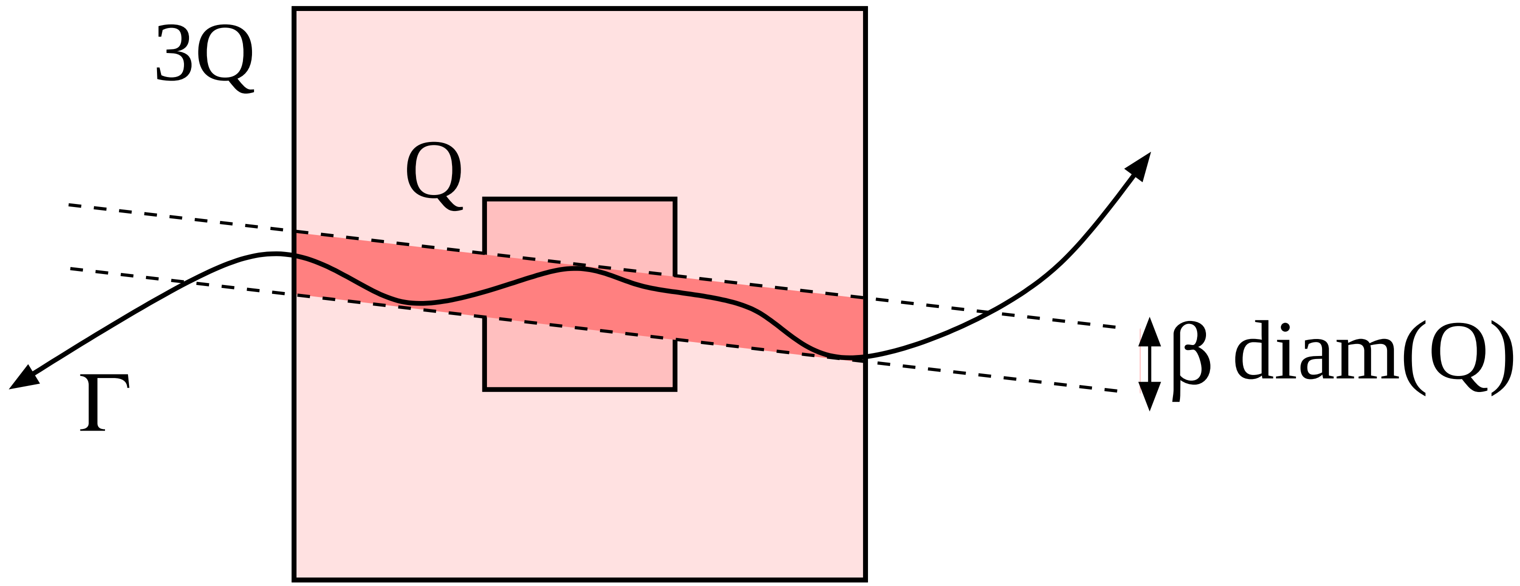
BMO case + Peter Jones' traveling salesman theorem (TST) gives a hint.

Traveling Salesman Theorem (Jones): Γ is rectifiable iff

$$\ell(\Gamma) \simeq \mathrm{diam}(\Gamma) + \sum_{\text{dyadic } Q} \beta_\Gamma^2(Q)\mathrm{diam}(Q) < \infty.$$

Sum is over dyadic squares in plane.





$$\beta_\Gamma(Q) = \frac{1}{\mathrm{diam}(Q)} \inf_L \sup\{z \in \Gamma \cap 3Q : \mathrm{dist}(z, L)\}$$

β measures local deviation from a line.

Peter Jones and I proved (1994) that a quasicircle Γ is rectifiable iff

$$\iint_\Omega |f'(z)||S(f)(z)|^2(1-|z|^2)^3 dxdy < \infty$$

By the TST this is equivalent to

$$\sum_Q \beta^2(Q)\mathrm{diam}(Q) < \infty.$$

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"Canceling" gives

$$\iint |S(f)(z)|^2(1-|z|^2)^2 dxdy < \infty \quad \Leftrightarrow \quad \sum_{Q} \beta^2(Q) < \infty.$$

We already know that LHS characterizes Weil-Petersson class.

Theorem: Γ is Weil-Petersson iff it is a quasicircle and

$$\sum_{Q} \beta^2(Q) < \infty.$$

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This is "finite total curvature".

Deviation from flatness is square integrable over all scales and locations.

Contains all C^2 curves. Critical smoothness near $\beta(Q) \simeq \sqrt{\text{diam}(Q)}$.

Allows infinite spirals to occur, so not C^1.

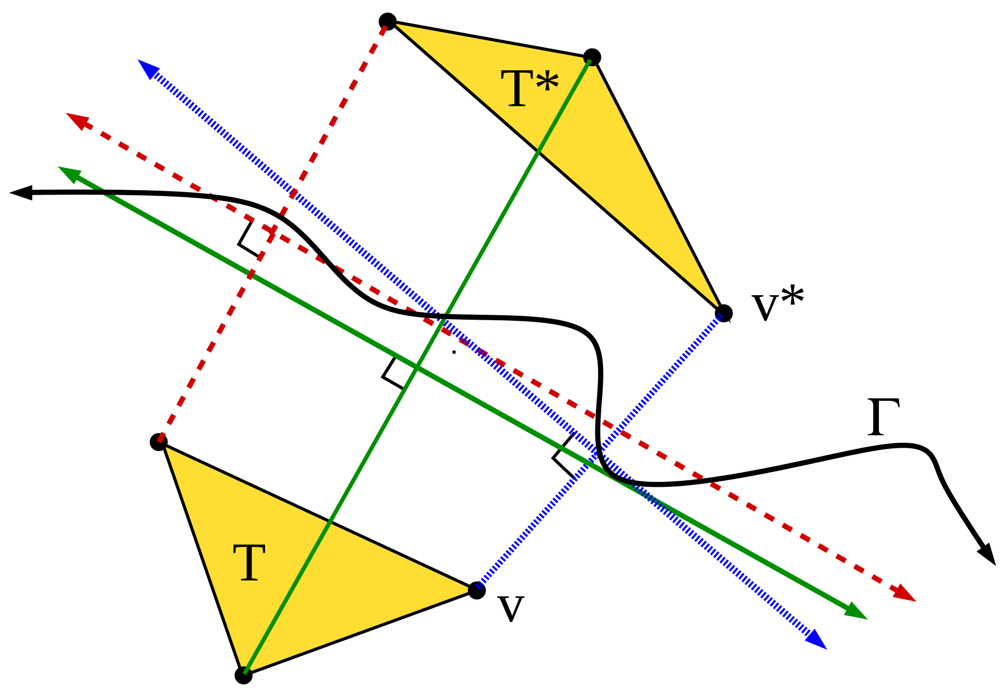
There are many equivalent characterizations.

$=$ boundaries of finite total curvature surfaces in hyperbolic 3-space.

Idea of proof: $\log f'$ in Dirichlet class implies f' is "mostly" constant near boundary. Follow Pommerenke's estimates to show images of short boundary arcs are almost line segments and $\sum \beta^2 < \infty$.

Conversely, triangulate one side of Γ and reflect vertices using lines approximating Γ. PL extension gives a QC reflection with $|\mu(z)| \simeq \beta$.Then

$$\sum \beta^2 < \infty \quad \Rightarrow \quad \int |\mu|^2 dA_\rho < \infty \quad \Rightarrow \quad \text{WP}.$$



Recall the TST: if $\Gamma \subset \mathbb{R}^2$ is a Jordan arc, then

$$\ell(\Gamma) \simeq \operatorname{diam}(\Gamma) + \sum_Q \beta_\Gamma^2(Q)\operatorname{diam}(Q).$$

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Here is a slight improvement:

$$\ell(\Gamma) = \operatorname{crd}(\Gamma) + O\left(\sum_Q \beta_\Gamma^2(Q)\operatorname{diam}(Q)\right).$$

or

$$\Delta(\Gamma) \equiv \ell(\Gamma) - \operatorname{crd}(\Gamma) \simeq \sum_Q \beta_\Gamma^2(Q)\operatorname{diam}(Q).$$

Modification of a known proof (e.g., Chapter 10, Bishop-Peres).

Theorem: Γ is Weil-Petersson iff it is chord-arc and

$$\sum \frac{\Delta(\gamma_j)}{\ell(\gamma_j)} < \infty$$

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Sum is over a multi-resolution family of subarcs, e.g., take a dyadic decomposition and triple length of each arc.

A dyadic decomposition of Γ consists of sets $Z_1 \subset Z_2 \subset \cdots \subset \Gamma$ so that Z_n divides Γ into 2^n arcs of length $2^{-n}\ell(\Gamma)$.

For a closed curve, this depends on the initial division into two pieces.

Theorem: Γ is Weil-Petersson iff it is chord-arc and

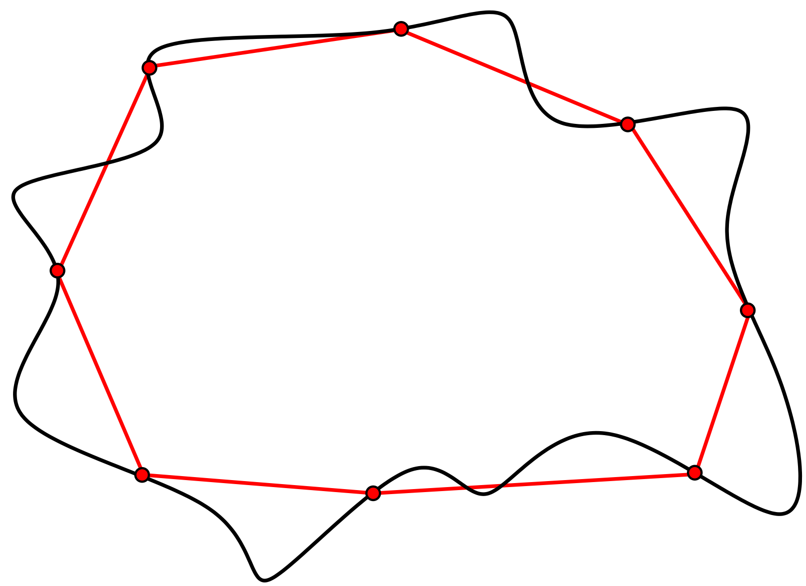
$$\sum \frac{\Delta(\gamma_j)}{\ell(\gamma_j)} < \infty$$

Continuous version: Γ is WP iff it is a quasicircle and

$$\int_{\Gamma\times\Gamma} \frac{\Delta(z,w)}{|z-w|^3}|dw||dz| < \infty$$

$\Delta(z,w) = \Delta(\gamma)$ for shorter subarc γ connecting z, w.

Γ_n is polygon given by nth generation dyadic points. Then $\ell(\Gamma_n) \nearrow \ell(\Gamma)$.



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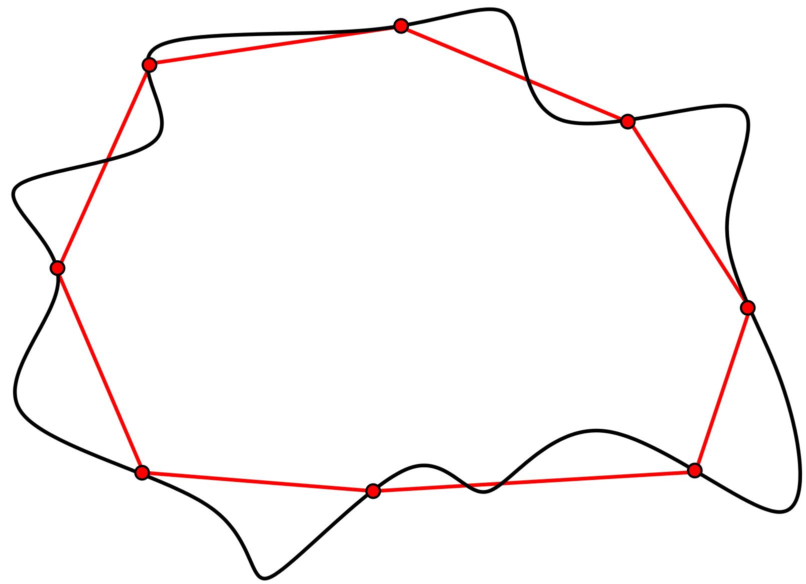
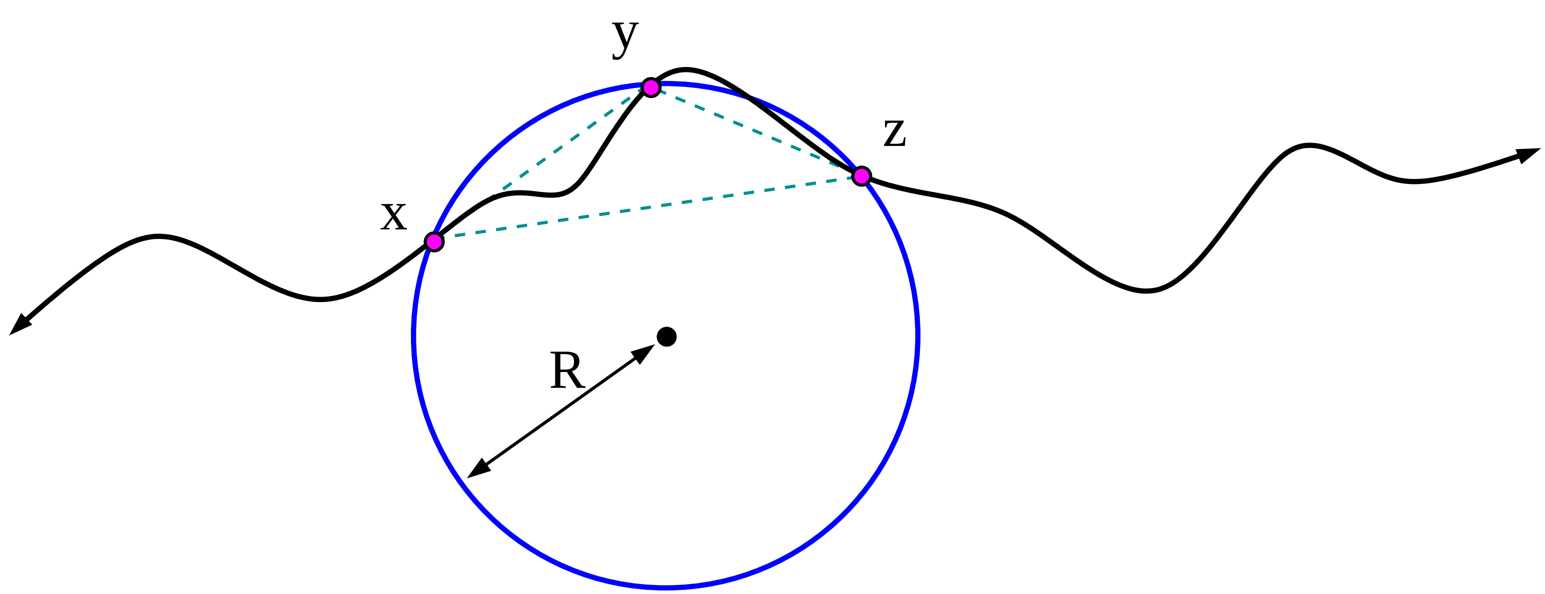


Theorem: Γ is Weil-Petersson iff it is a quasicircle and

$$\sum_{n=1}^{\infty} 2^n \left[\ell(\Gamma) - \ell(\Gamma_n)\right] < \infty$$

The **Menger curvature** of three points $x, y, z \in \mathbb{R}^2$ is $c(x, y, z) = 1/R$ where R is the radius of the circle passing through these points.

Let $\ell(x, y, x) = |x - y| + |y - z| + |z - x| =$ perimeter of triangle (x, y, z).



Theorem: Γ is WP iff it is chord-arc and satisfies

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It is known that

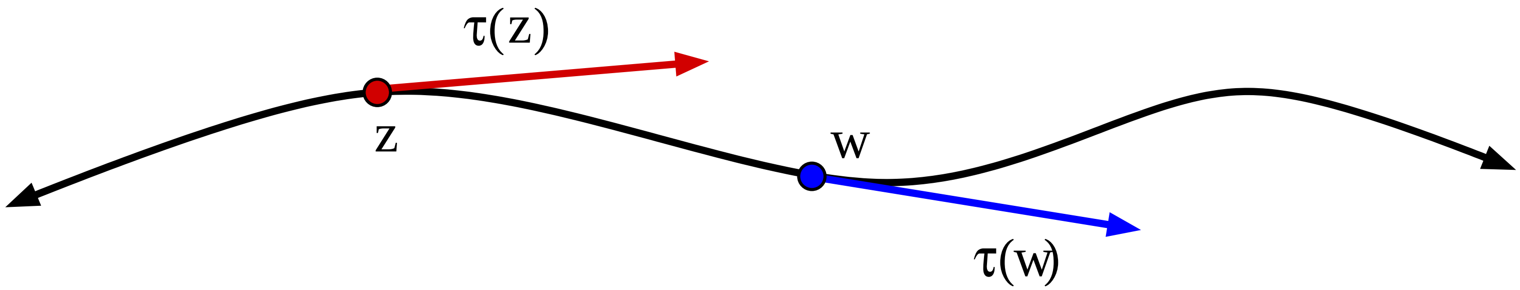
$$\int_\Gamma \int_\Gamma \int_\Gamma c(x,y,z)^2 |dx||dy||dz| < \infty.$$

iff

$$\sum_Q \beta^2(Q) \mathrm{diam}(Q) < \infty.$$

To get above result, divide both sides by a scale factor.

If Γ is rectifiable, it has a tangent direction $\tau(z) \in \mathbb{T}$ a.e. on Γ.



Theorem: Γ is Weil-Petersson iff it is chord-arc and

$$\int_\Gamma \int_\Gamma \left| \frac{\tau(z) - \tau(w)}{z - w} \right|^2 |dz||dw| < \infty.$$

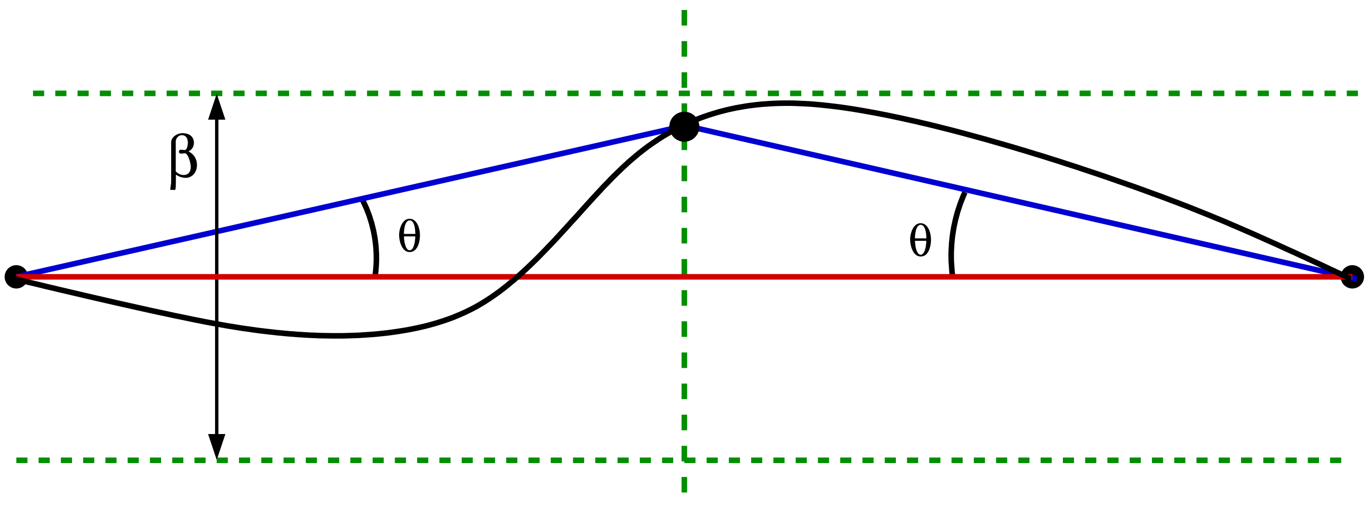
In other words, $\tau \in H^{1/2}(\Gamma)$ (Sobolev space).

Proof uses discrete analog summed over multi-resolution family:

$$\sum_j \frac{1}{\ell(\Gamma_j)} \int_{\Gamma_j} |\tau(z) - \tau(\Gamma_j)|^2 |dz| < \infty$$

Fairly easy: Discrete version $\Rightarrow$ continuous version $\Rightarrow \sum \beta^2 < \infty$

"$\sum \beta^2 < \infty \Rightarrow$ discrete version" uses martingales.



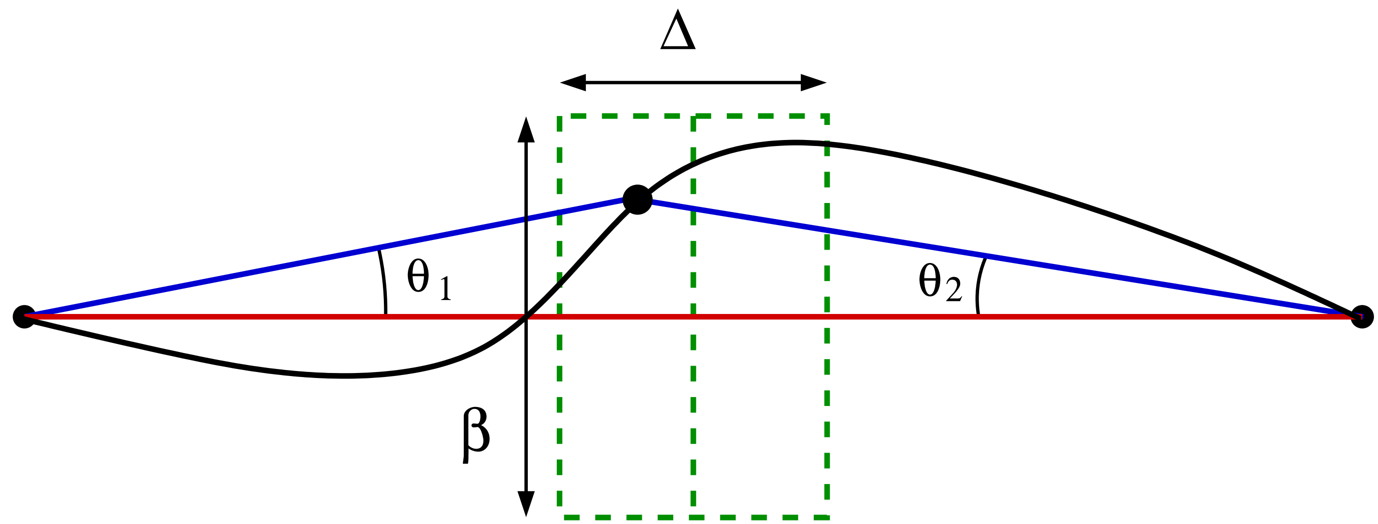
Suppose that midpoint of a dyadic arc is on bisector of its endpoints.

Then new tangent directions differ by $\pm\theta = O(\beta)$ from old one.

$\Rightarrow$ polygonal tangents define a martingale on Γ .

$\sum \beta^2 < \infty \Rightarrow L^2$ bounds on martingale.

Maximal functions and martingale convergence imply result.

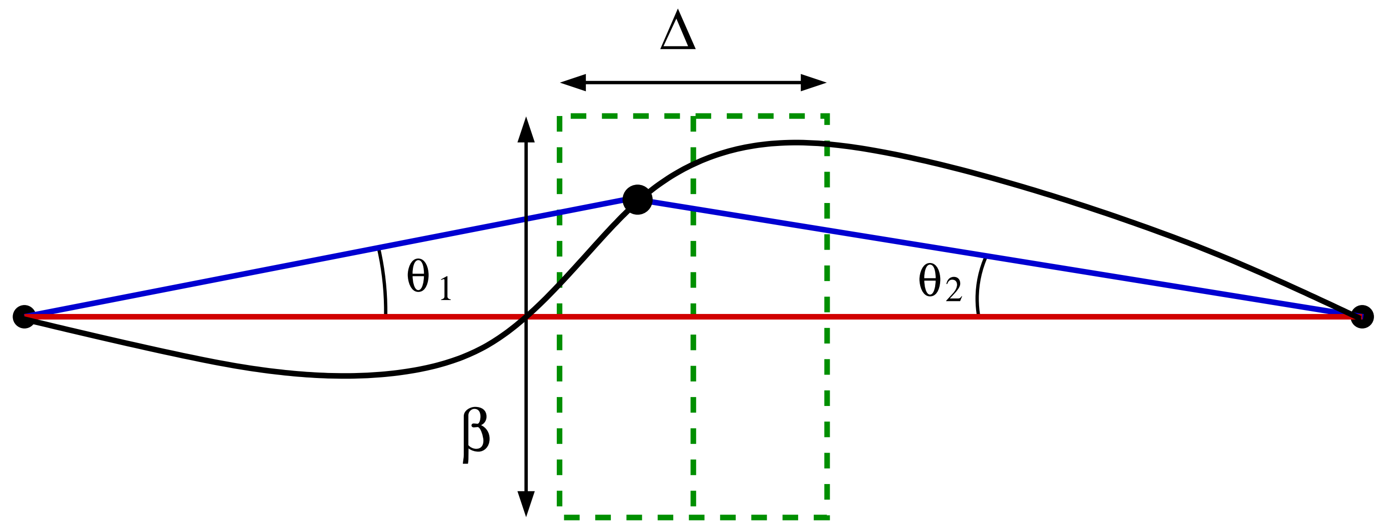


In reality, midpoint is not on bisector, but is within $\Delta(\Gamma_j)$ of bisector.

Change in angles are not equal, but satisfy

$$|\theta_1 - \theta_2| \leq \beta(\Gamma_j) \cdot \Delta(\Gamma_j)/\ell(\Gamma_j).$$

So polygon angles are a small perturbation of a martingale. This suffices.



When we bisect arc by length, harmonic measure (for either side) is also bisected up to a factor controlled by sums of β's and Δ's.

Thus bisecting harmonic measure on one side almost bisects for other side.

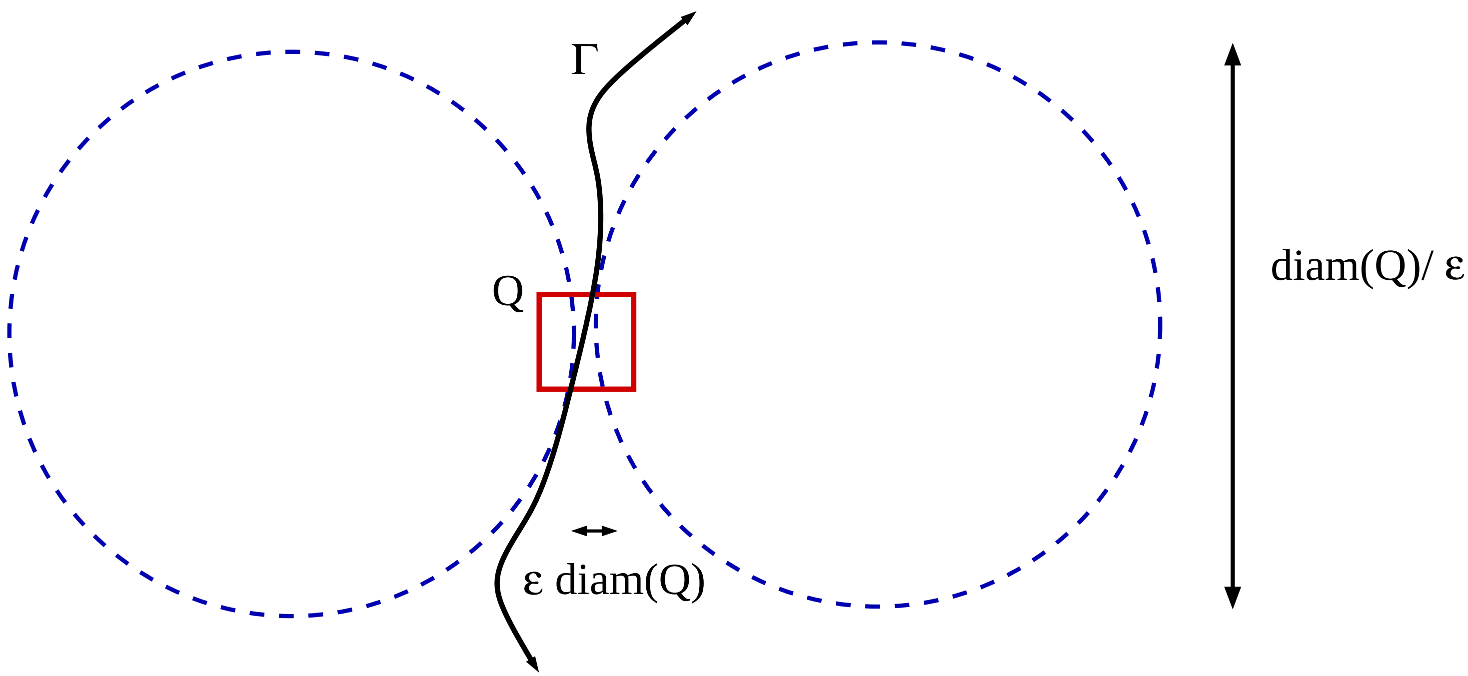
This proves Shen's theorem:

$$\mathrm{WP} \quad \Leftrightarrow \quad \log h' \in H^{1/2}.$$

Is there a higher dimensional analog?

“Round” βs:

Let $\varepsilon_\Gamma(Q)$ be the smallest $\varepsilon > 0$ so that $3Q$ intersects disks D_1, D_2 separated by Γ, both with radius $\mathrm{diam}(Q)/\varepsilon$ and $\mathrm{dist}(D_1, D_2) \leq \varepsilon \cdot \mathrm{diam}(Q)$.



For single Q, we have $\beta(Q) \lesssim \varepsilon(Q)$, but not conversely.

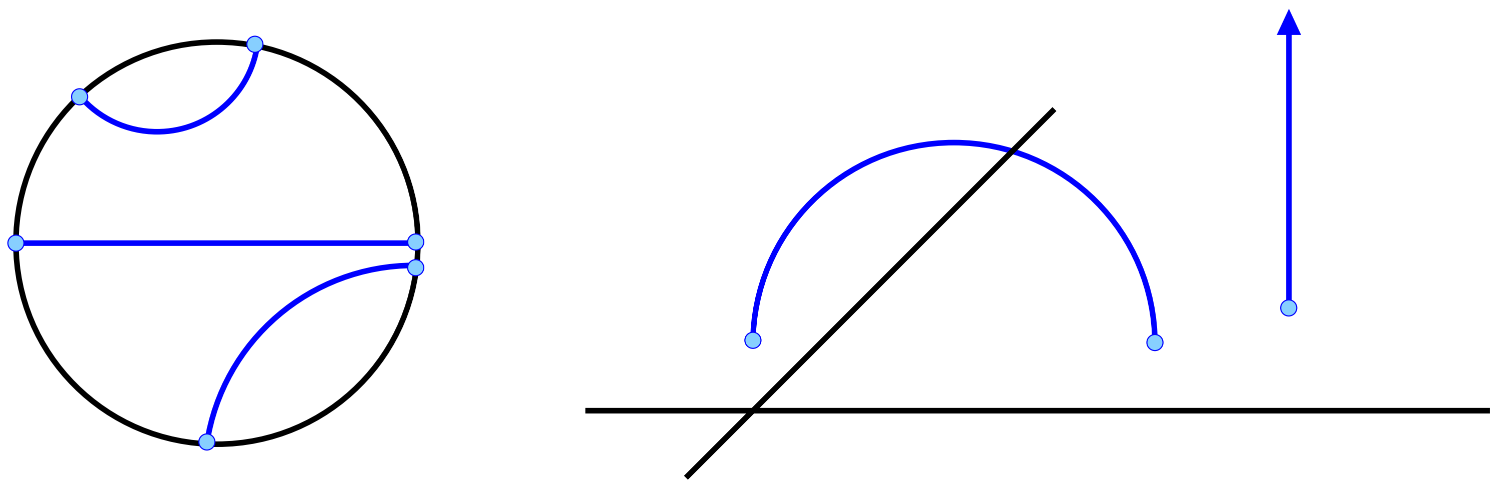
Theorem: Γ is WP iff it is chord-arc and satisfies

$$\sum_{Q} \varepsilon_{\Gamma}^{2}(Q) < \infty$$

$\varepsilon(Q)$ provides a link to hyperbolic geometry.

Hyperbolic metric the unit ball $\mathbb{B}$ is $d\rho = \frac{ds}{1-|z|^2}$.

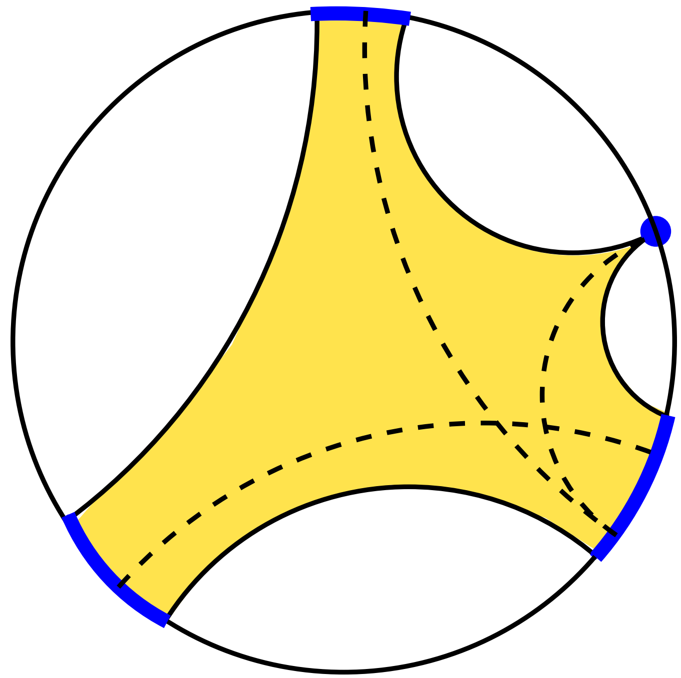
In the upper half-space $\mathbb{R}^3_+ = \{(x, y, t) : t > 0\}$, metric is $d\rho = ds/2t$.



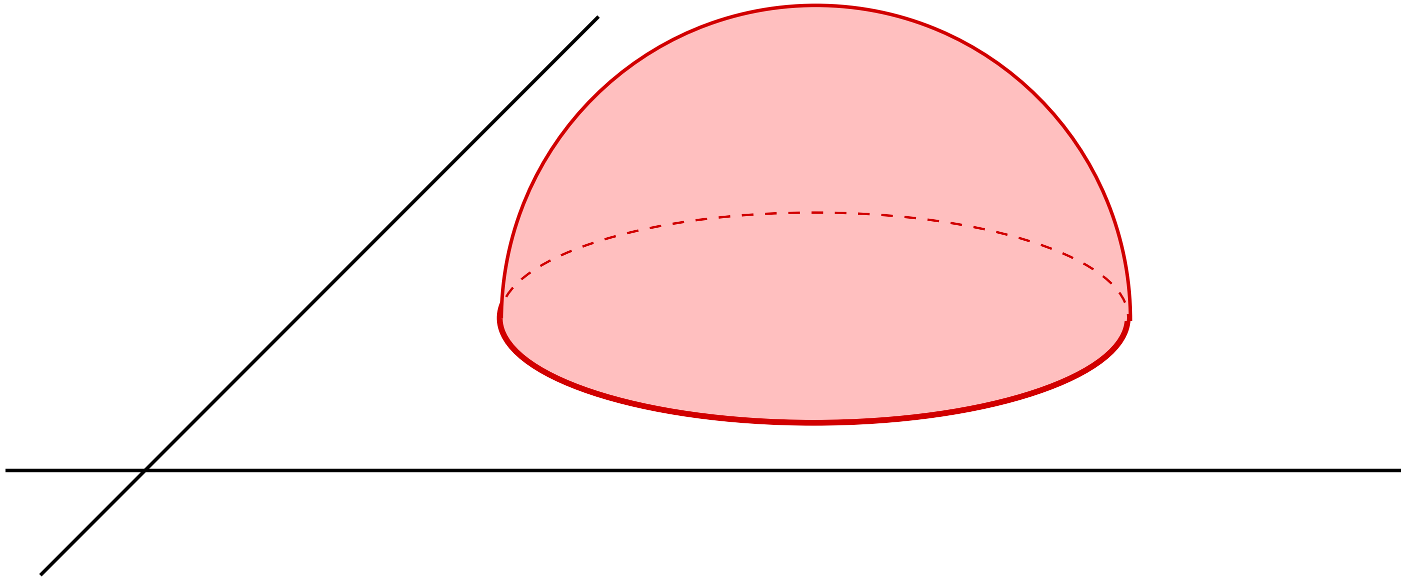
Geodesics in $\mathbb{B}$ are diameters or circular arcs perpendicular to $\partial\mathbb{B}$

Geodesics in $\mathbb{R}^3_+$ are vertical rays or semi-circles perpendicular to $\mathbb{R}^2$.

CH(Γ) denotes the hyperbolic convex hull of Γ



For a curve, CH(Γ) is the union of all geodesics with endpoints in Γ.



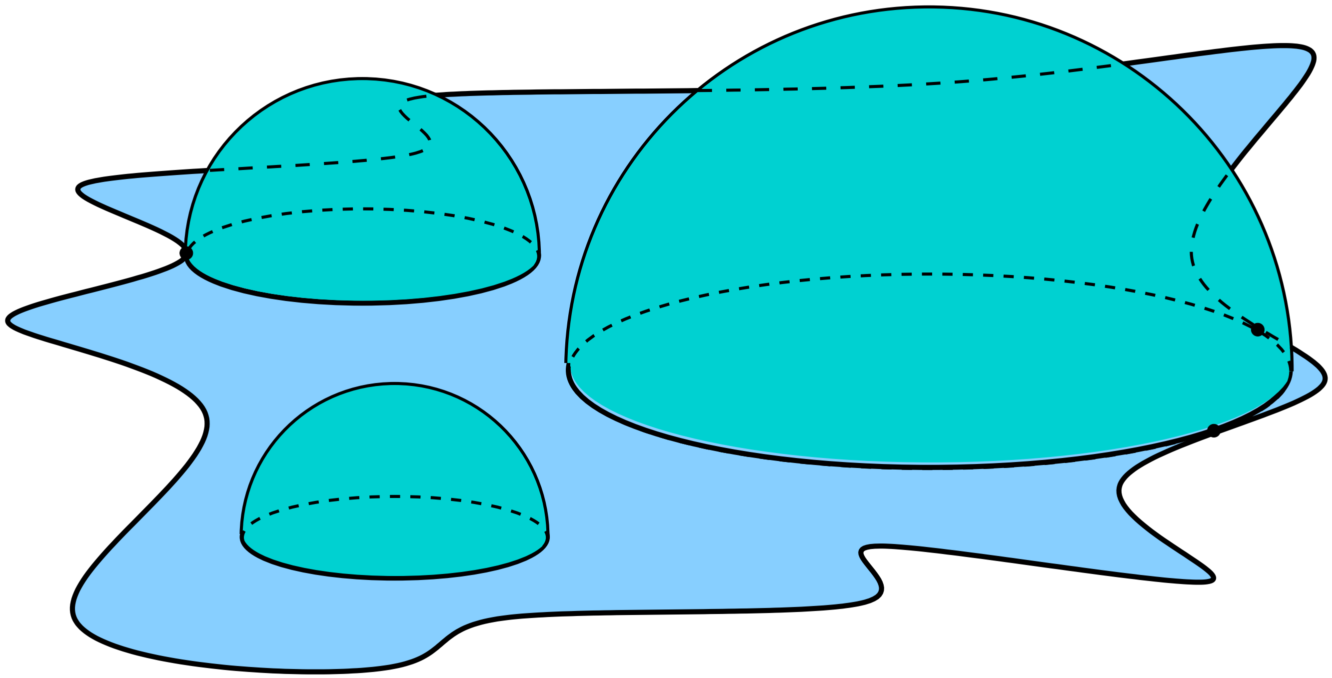
For a circle, this is a hemisphere.

In general, $\mathrm{CH}(\Gamma)$ has non-empty interior.

There are two boundary surfaces, each asymptotic to Γ.

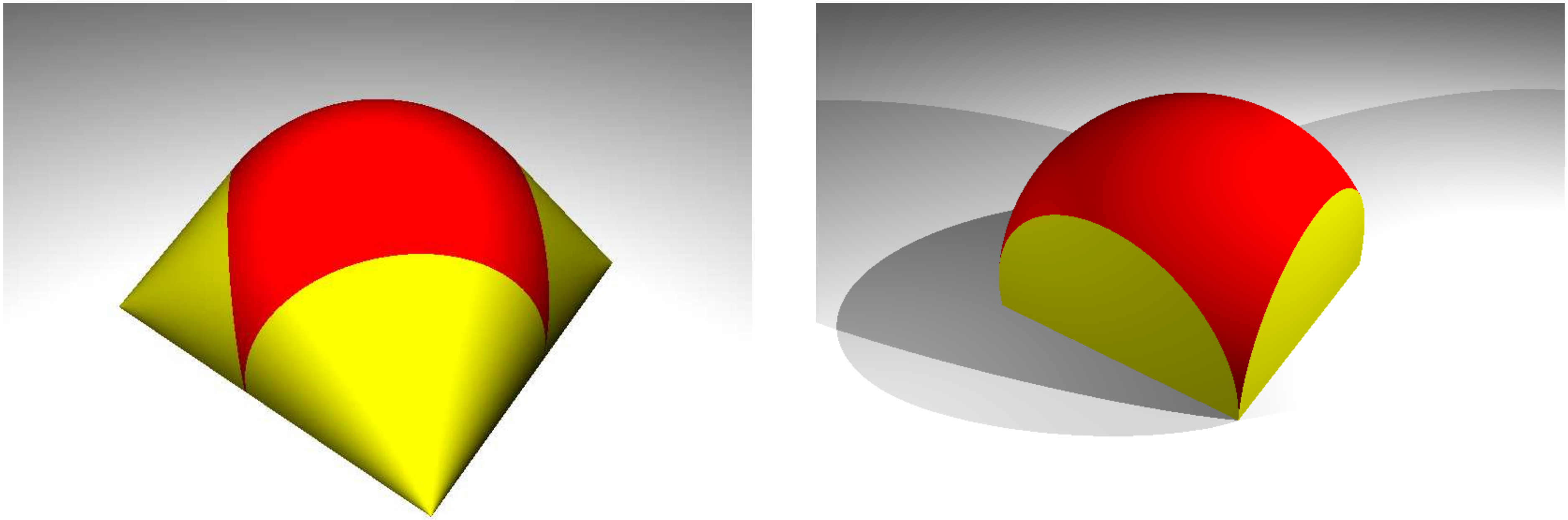
Suppose Ω is Jordan domain with boundary Γ.

The dome of Ω is upper envelope of all hemispheres with base disk in Ω.



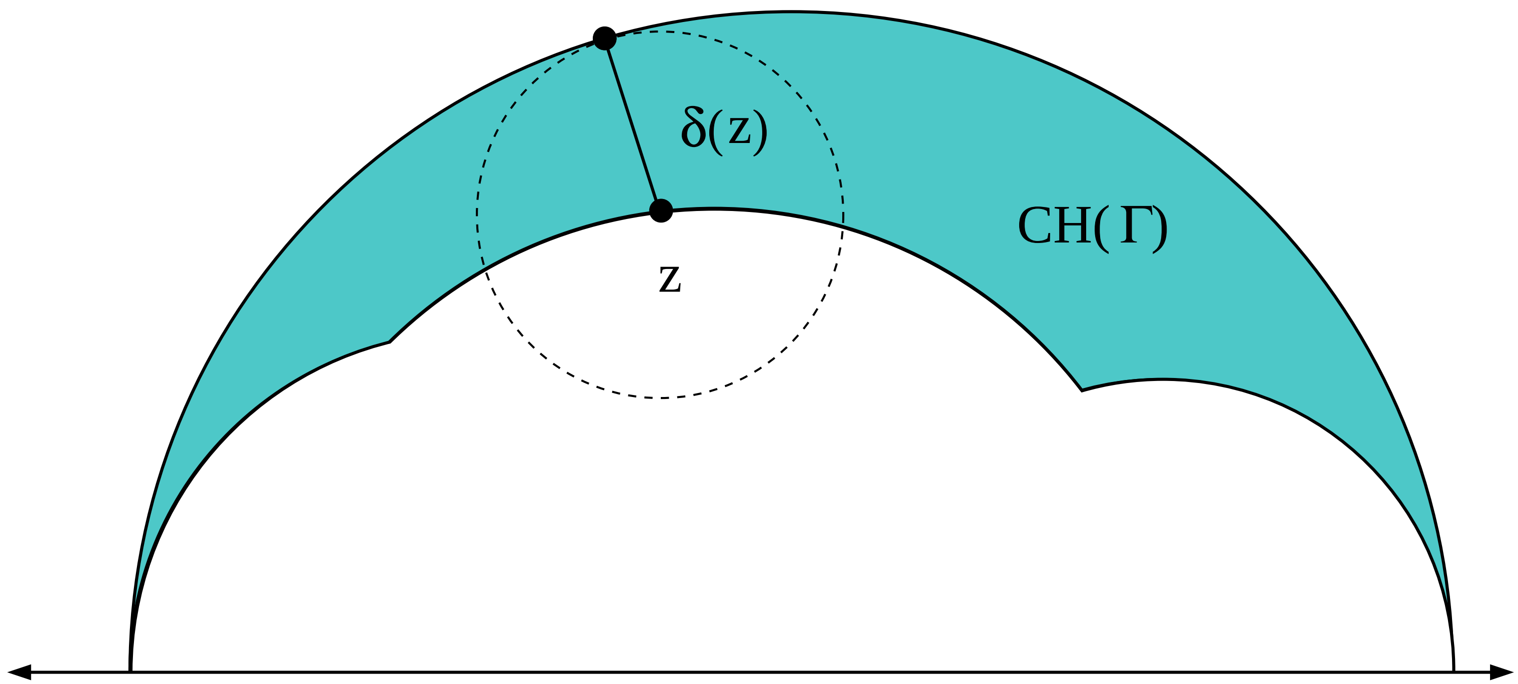
Region above dome is intersection of half-spaces, hence convex.

$\mathrm{CH}(\Gamma)$ is region between domes of "inside" and "outside" of Γ.



Upper and lower boundaries of hyperbolic convex hull of a square.

Let $\delta(z)$ be the hyperbolic distance to farther boundary component.



For $z \in \mathrm{CH}(\Gamma)$, $\delta(z)$ measures "width" of convex hull near z.

$\delta(z) = 0$ iff Γ is a circle (hull has no interior).

For quasicircles, $\delta(z) \in L^\infty$ (not conversely).

Theorem: A quasicircle Γ is WP iff $\delta(z) \in L^2$, i.e.,

$$\int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z) d\mathrm{A}_\rho(z) < \infty.$$

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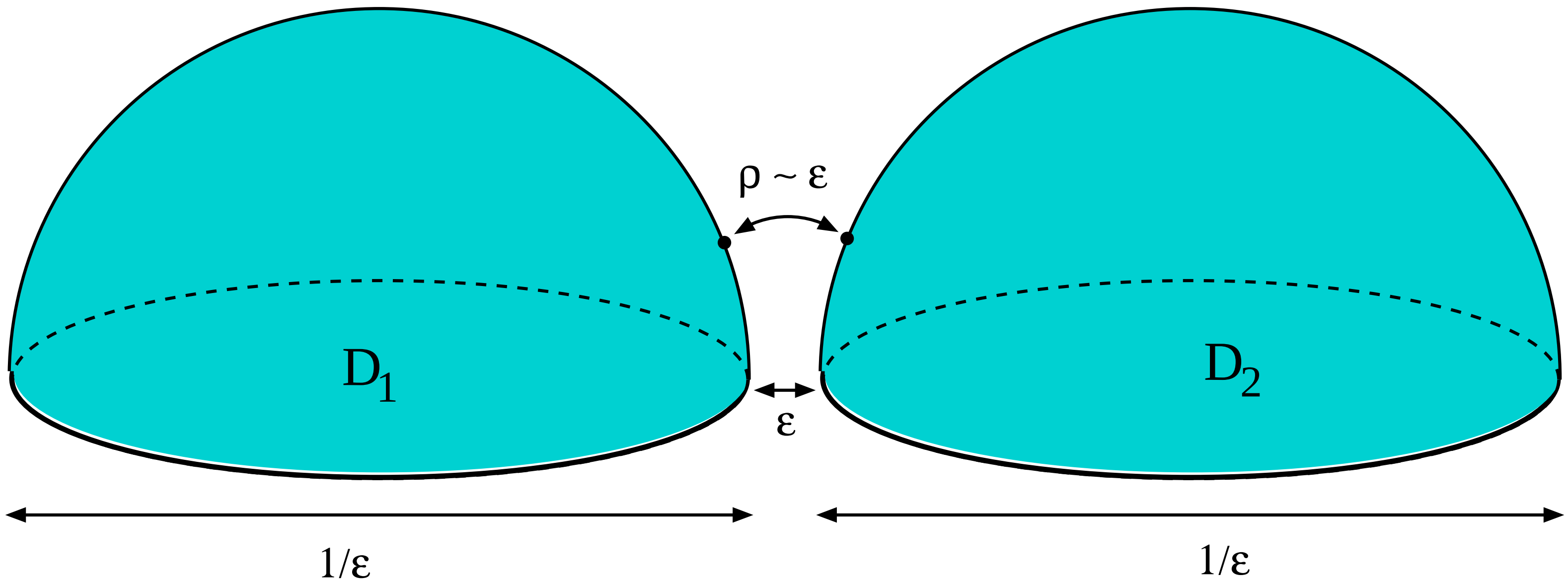
Thus Γ is WP iff its hyperbolic convex hull is asymptotically "thin".

First geometric condition that is obviously conformally invariant.

"Thickness" of convex hull also important in Teichmüller theory and Kleinian groups. Measures "complexity" of the quotient 3-manifold.

Idea of proof: Use $\sum \varepsilon^2 < \infty$ and

Lemma: If two disks of diameter $1/\epsilon$ are distance ϵ apart in plane, then their domes are hyperbolic distance $\simeq \epsilon$ apart.



Corollary: $\delta(z) \lesssim \varepsilon(Q)$ if z is height $\mathrm{diam}(Q)$ above Q.

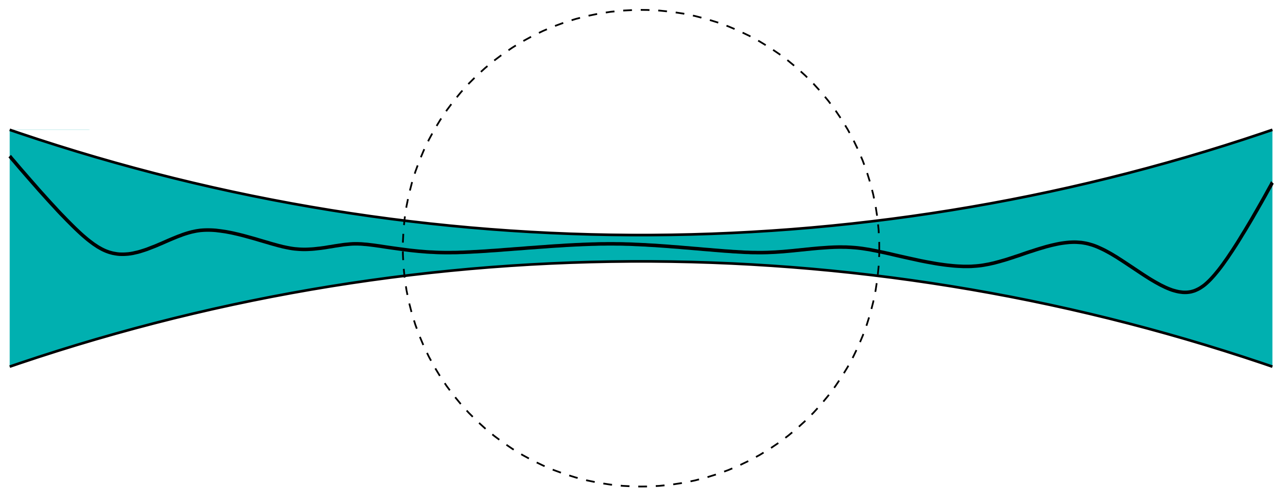
Theorem (Mike Anderson, 1983): Every Jordan $\Gamma \subset \mathbb{R}^2$ is boundary of at least one minimal surface S in $\mathbb{R}^3_+$.

The minimal surface S has infinite hyperbolic area: minimal means minimal under compact variations.

Important Fact: Any such minimal surface is contained in $\mathrm{CH}(\Gamma)$.

2nd fundamental form $K(z)$ measures curvature of an embedded surface. Roughly $K^2 = a^2 + b^2$ if surface is graph of $ax^2 + by^2$ locally.

Theorem (Andrea Seppi, 2016): For a minimal surface S trapped between non-intersecting planes distance δ apart near z, $K(z) = O(\delta)$.



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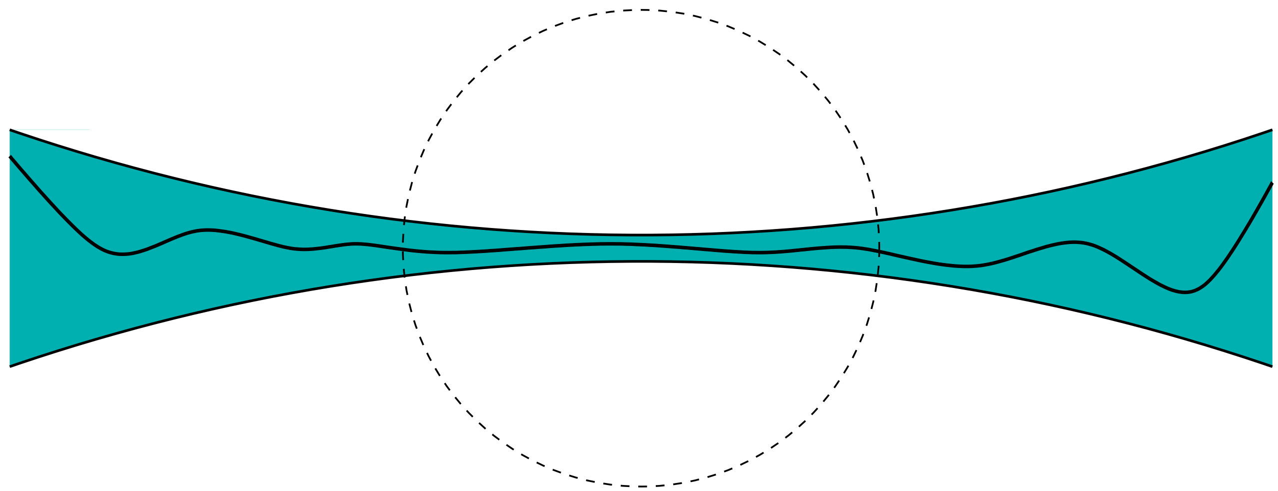
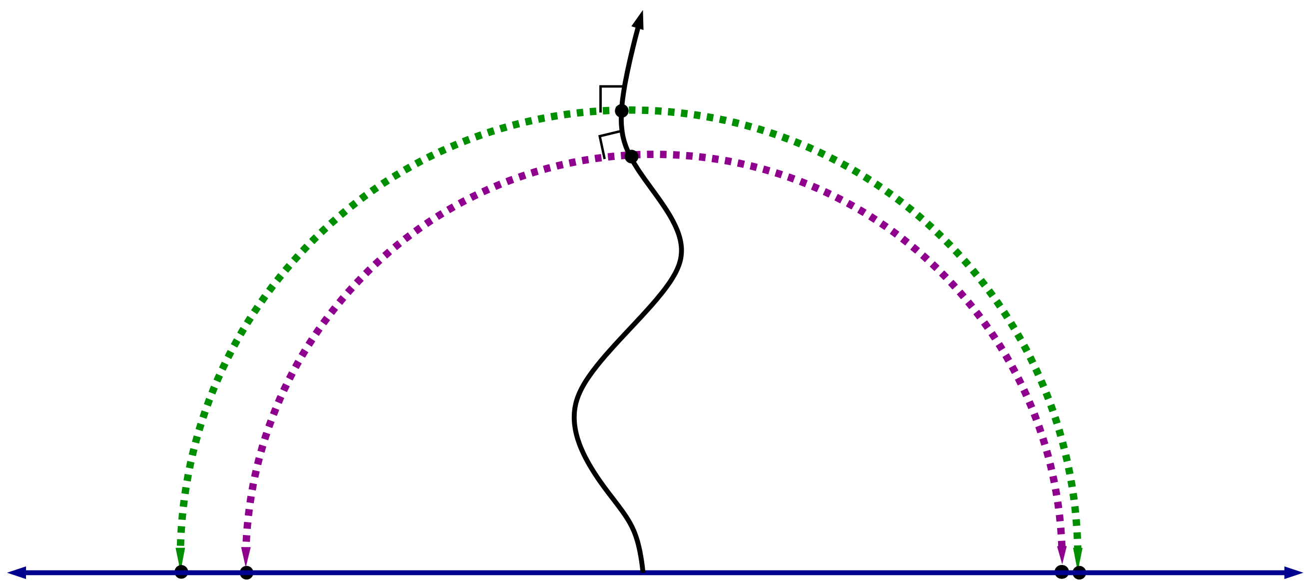


Cor: If Γ is WP then it bounds a surface with finite total curvature, i.e.,

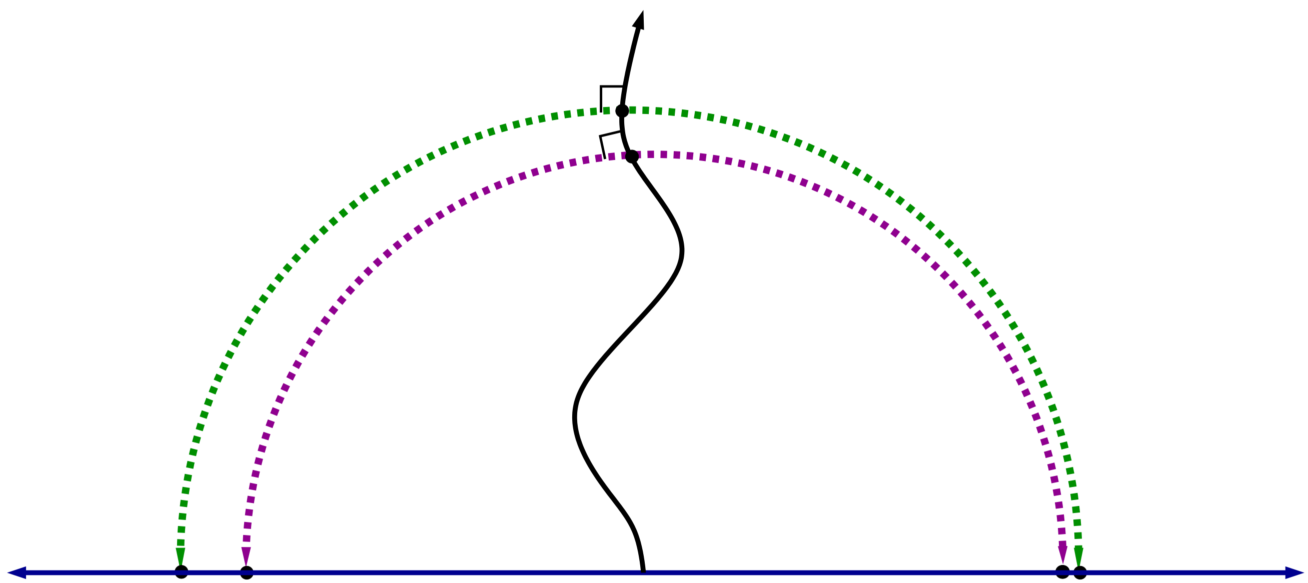
$$\int_S K^2(z)\, dA_\rho(z) < \infty.$$

Theorem (Charles Epstein, 1986): If S is a surface with $|K| < 1$, then the Gauss maps G_j, $j = 1, 2$ define a quasiconformal reflection across Γ with dilatation $|\mu(G(z))| = O(K(z))$.



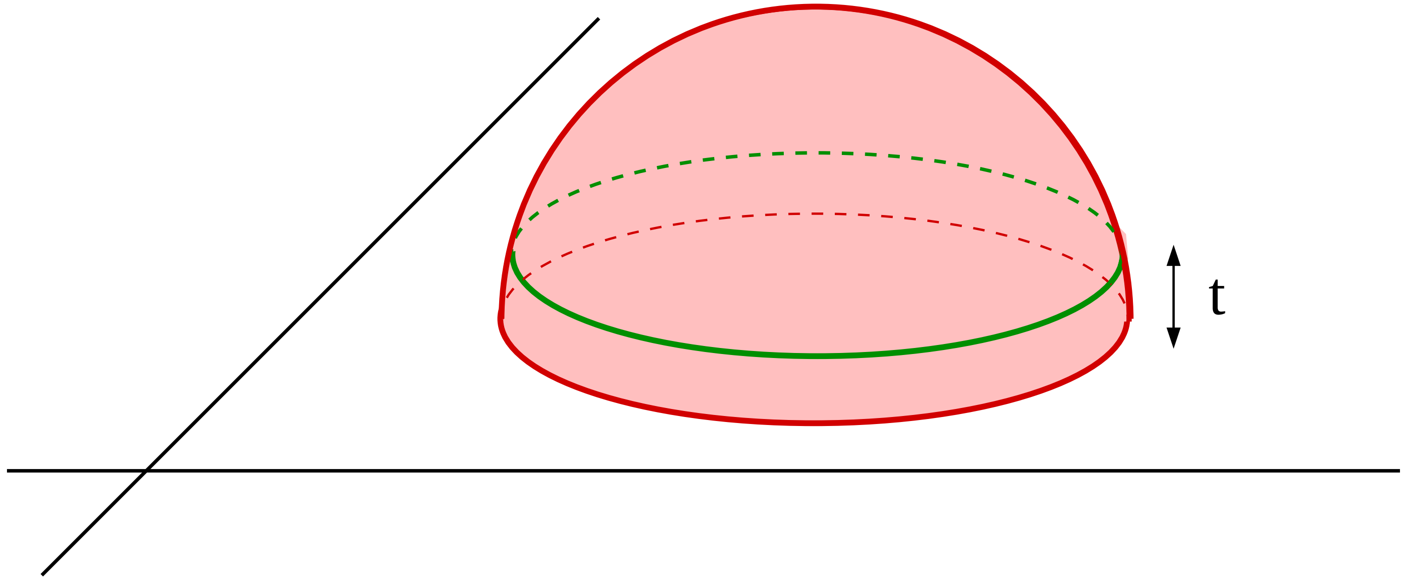
Gauss map = follow normal geodesics from surface to boundary.

Theorem (Charles Epstein, 1986): If S is a surface with $|K| < 1$, then the Gauss maps G_j, $j = 1, 2$ define a quasiconformal reflection across Γ with dilatation $|\mu(G(z))| = O(K(z))$.



If S is has finite total curvature, then $\int_{\mathbb{C}\setminus\Gamma} |\mu|^2 dA_\rho < \infty$. Thus:

Thm: Γ is WP iff it bounds a minimal surface of finite total curvature.



S has infinite hyperbolic area, but a "renormalized area" was introduced by Robin Graham and Ed Witten:

$$Y_t = Y \cap \{(x, y, z) : z > t\}, \quad \mathcal{A}_R(Y) = \lim_{t \searrow 0} \left[\mathrm{A}_\rho(Y_t) - \ell_\rho(\partial Y_t) \right]$$

Related to the AdS/CFT correspondence in string theory and quantum entanglement (e.g., analogous to surface area of event horizon measuring entropy of a black hole).

Finite renormalized area is closely related to finite total curvature.

Alexakis and Mazzeo proved that if $\Gamma \in C^{3,\alpha}$, then $\mathcal{A}_R(Y) < \infty$ and

$$\mathcal{A}_R(Y) = -2\pi\chi(Y) - \frac{1}{2}\int_Y |K_0|^2 d\mathrm{A}_\rho,$$

where K_0 is trace free 2nd fundamental form ($= K$ for minimal surfaces).

Thus we expect:

$$\partial S \in \mathrm{WP} \quad \Rightarrow \quad \int_S K^2 dA_\rho < \infty \quad \Rightarrow \quad \mathcal{A}_R(S) < \infty,$$

but WP curves are too rough to apply their result (not even C^1).

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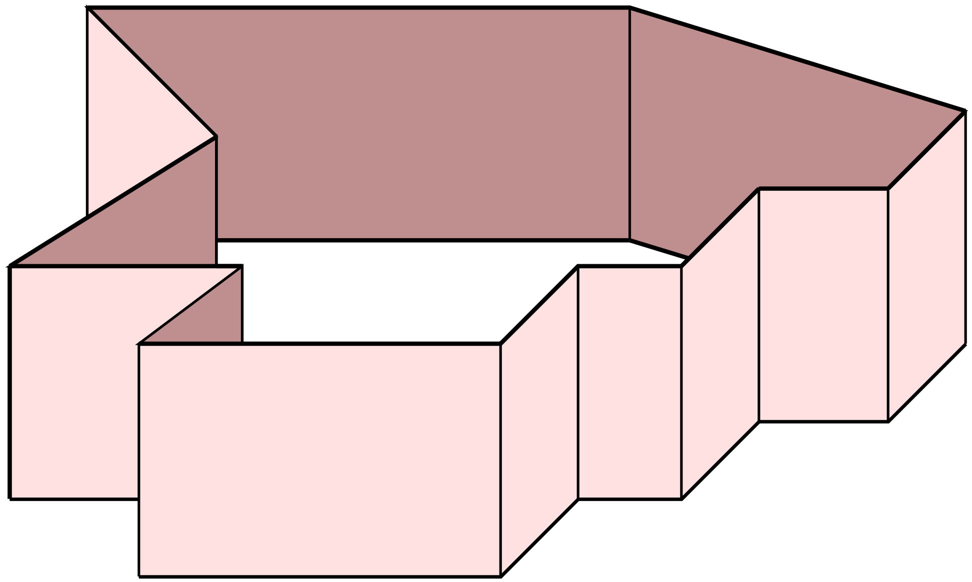
but WP curves are too rough to apply their result (not even C^1).

However, the result it is still true anyway:

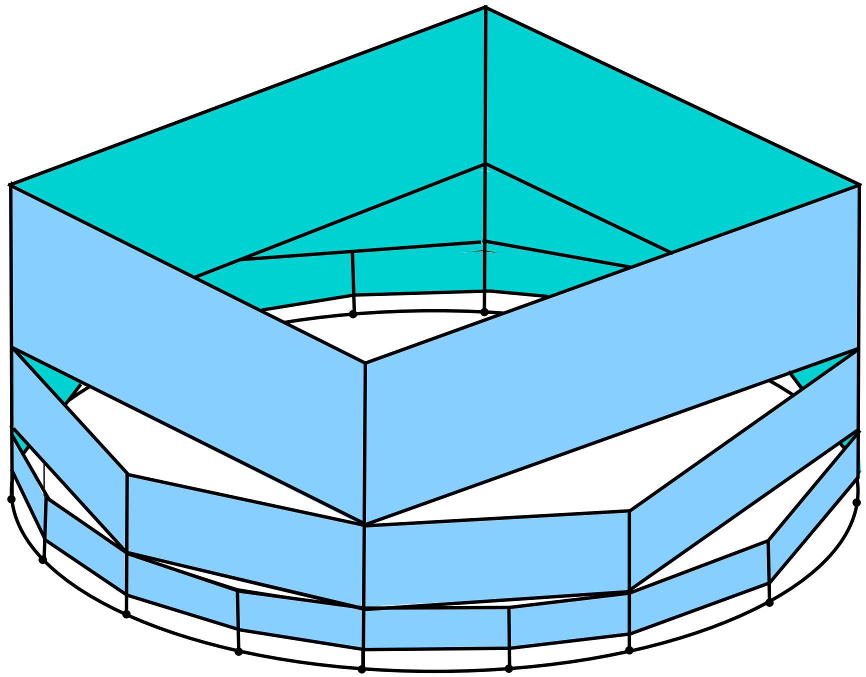
Thm: S minimal + $\partial S \in \mathrm{WP} \Rightarrow S$ has finite renormalized area.



Idea is that cylinders $X = \Gamma \times (0, 1]$ have finite renormalized area:

$$\mathrm{A}_\rho(X_t) = \int_\Gamma \int_t^1 \frac{1}{r^2} dr ds = \ell(\Gamma)(\frac{1}{t} - 1), \qquad \ell_\rho(\partial X_t) = \frac{1}{t}\ell(\Gamma)$$

$$\Rightarrow \qquad \mathcal{A}_R(X) = \mathrm{A}_\rho(X_t) - \ell_\rho(X_t) = -\ell(\Gamma)$$

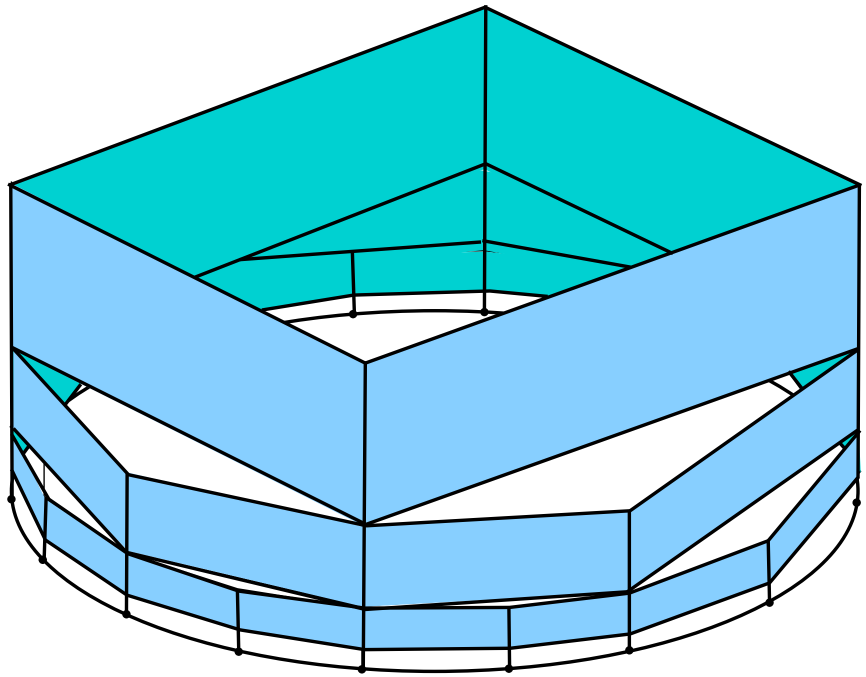


Define a dyadic cylinder in the upper half-space:

$$X = \bigcup_{n=1}^{\infty} \Gamma_n \times [2^{-n}, 2^{-n+1}),$$

where $\{\Gamma_n\}$ are inscribed dyadic polygons in Γ.

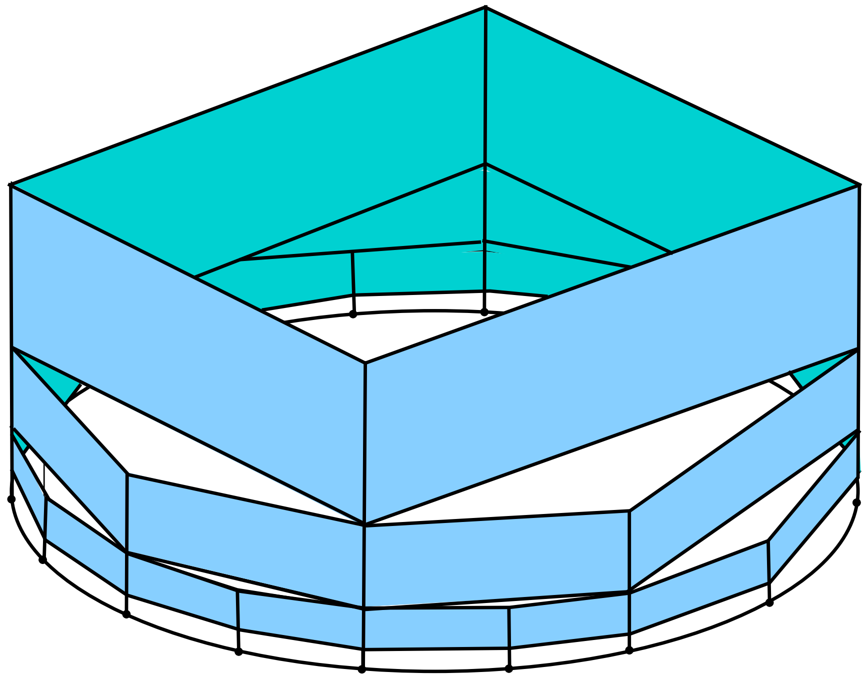
Discrete analog of minimal surface with boundary Γ.



Our earlier estimate

$$\sum_n 2^n(\ell(\Gamma) - \ell(\Gamma_n)) < \infty$$

implies Γ is WP iff the dyadic cylinder has finite renormalized area.

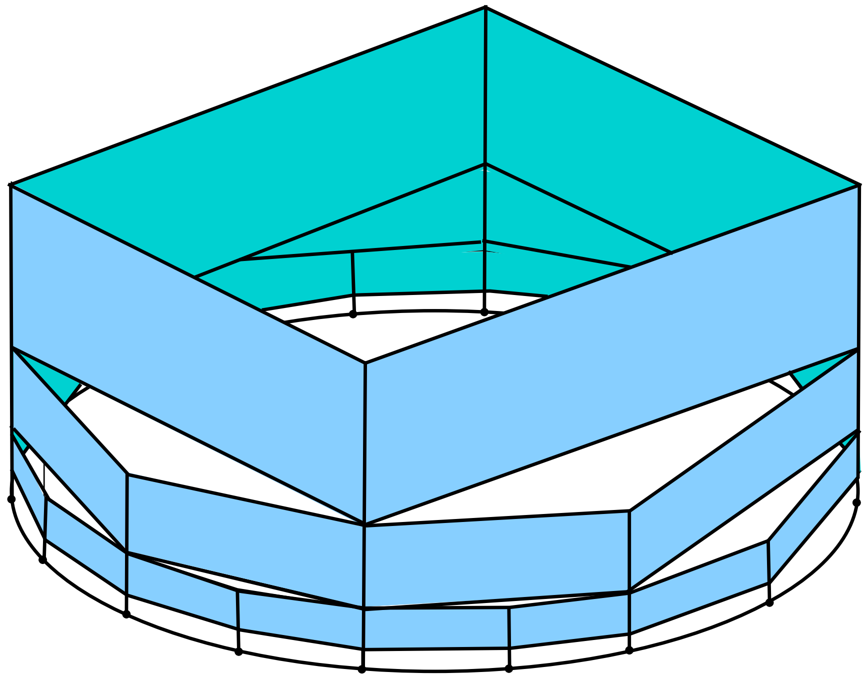


Lemma: If Γ is Weil-Petersson, then the limits

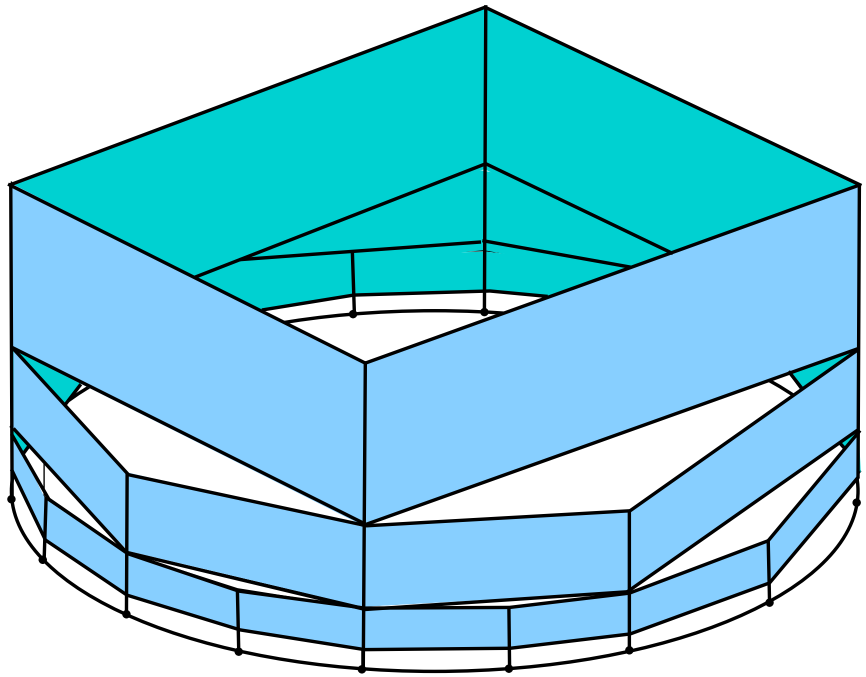
$$\lim_{t\to 0}[\mathrm{A}_\rho(S_t) - \mathrm{A}_\rho(X_t)], \qquad \lim_{t\to 0}[\ell_\rho(S_t) - \ell_\rho(X_t)],$$

both exist and are finite.

Proof: orthogonally project $X \to S$; use β, δ to estimate area/length.



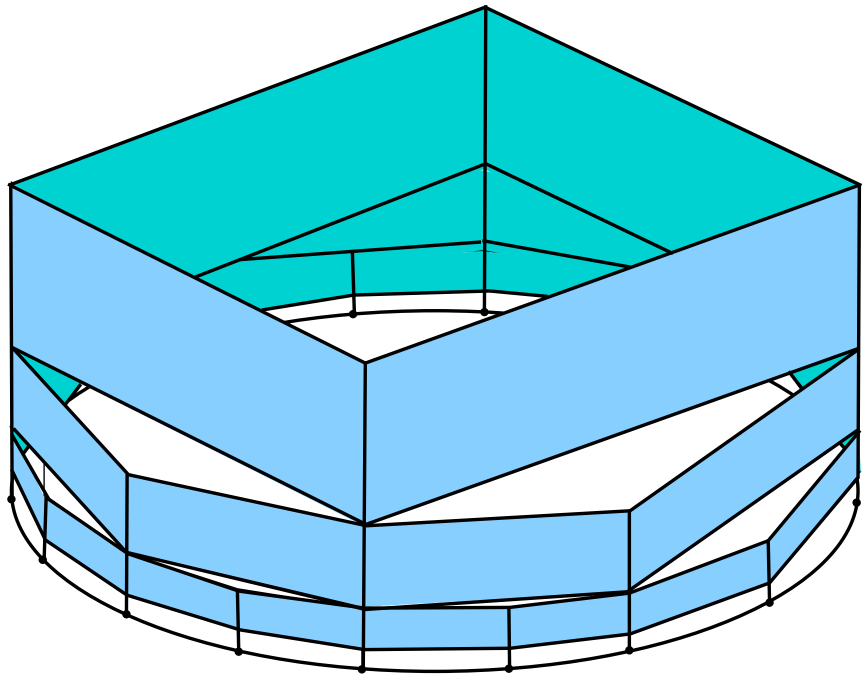
Thm: S minimal + $\partial S \in \mathrm{WP} \Rightarrow S$ has finite renormalized area.



Thm: S minimal + $\partial S \in \mathrm{WP} \Rightarrow S$ has finite renormalized area.

Does converse hold? A formula of Alexakis-Mazzeo may help:

$$\mathrm{A}_\rho(S_t) - \frac{1}{t}\ell(\partial S_t) = -2\pi\chi(S_t) - \frac{1}{2}\int_{S_t} |K|^2 d\mathrm{A}_\rho - \int_{\partial S_t}(\kappa_t - 1)d\ell_\rho$$



THANKS TO EVERYONE FOR LISTENING

BEST WISHES TO JOHN AND DON