Constructing entire functions by quasiconformal folding

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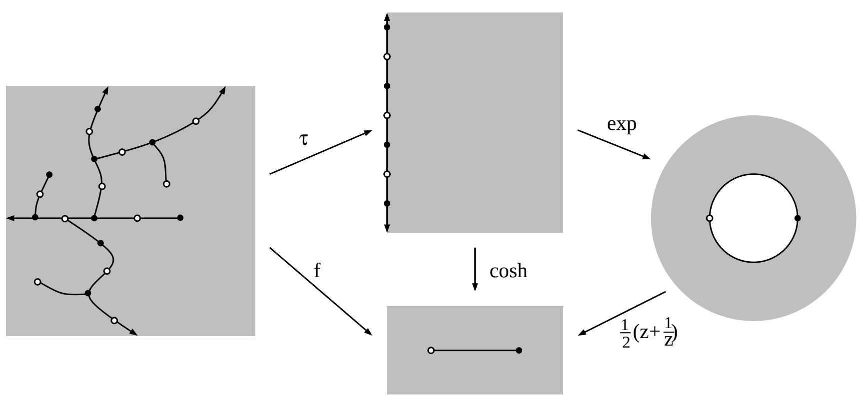

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$S(f)$ denotes the singular values of f, i.e.,

- critical values $= \{f(z) : f'(z) = 0\}$
- asymptotic values $=$ limits of f on curves to ∞

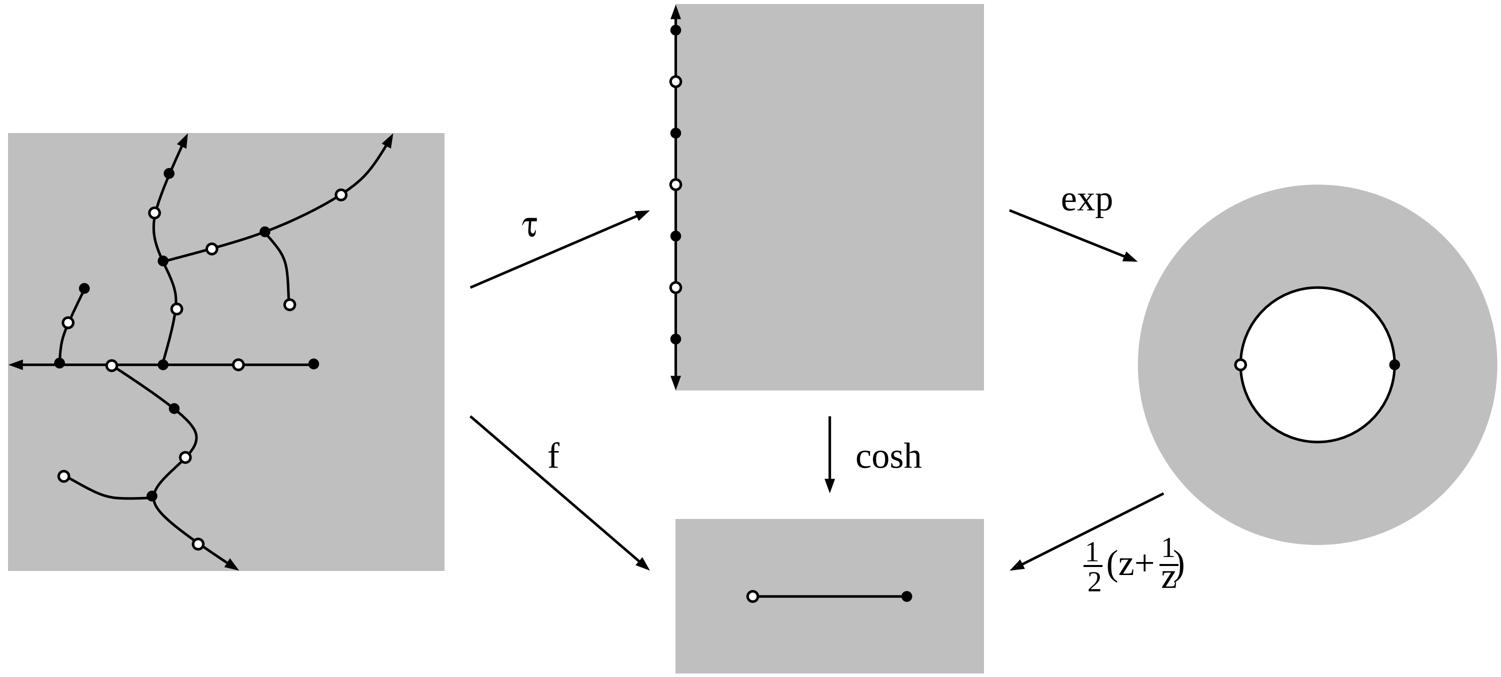
$\mathcal{B}$ denotes the **Eremenko-Lyubich** class of transcendental entire functions for which $S(f)$ is a bounded set.

$\mathcal{S} \subset \mathcal{B}$ is the **Speiser** class; $S(f)$ is a finite set.

How to construct such functions?

The basic idea is to construct a quasiregular function g with the desired property and singular set and then use the measurable Riemann mapping theorem to find a quasiconformal map ϕ so that $f = g \circ \phi$ is entire.

Since ϕ is a homeomorphism, the singular values of f are the same as for g and the tracts of f are quasiconformal images of the tracts for g; often we can get good estimates for ϕ and deduce f and g have very similar geometry.



We have a tree T with marked vertices (-1, +1).

τ is QC from components to half-plane.

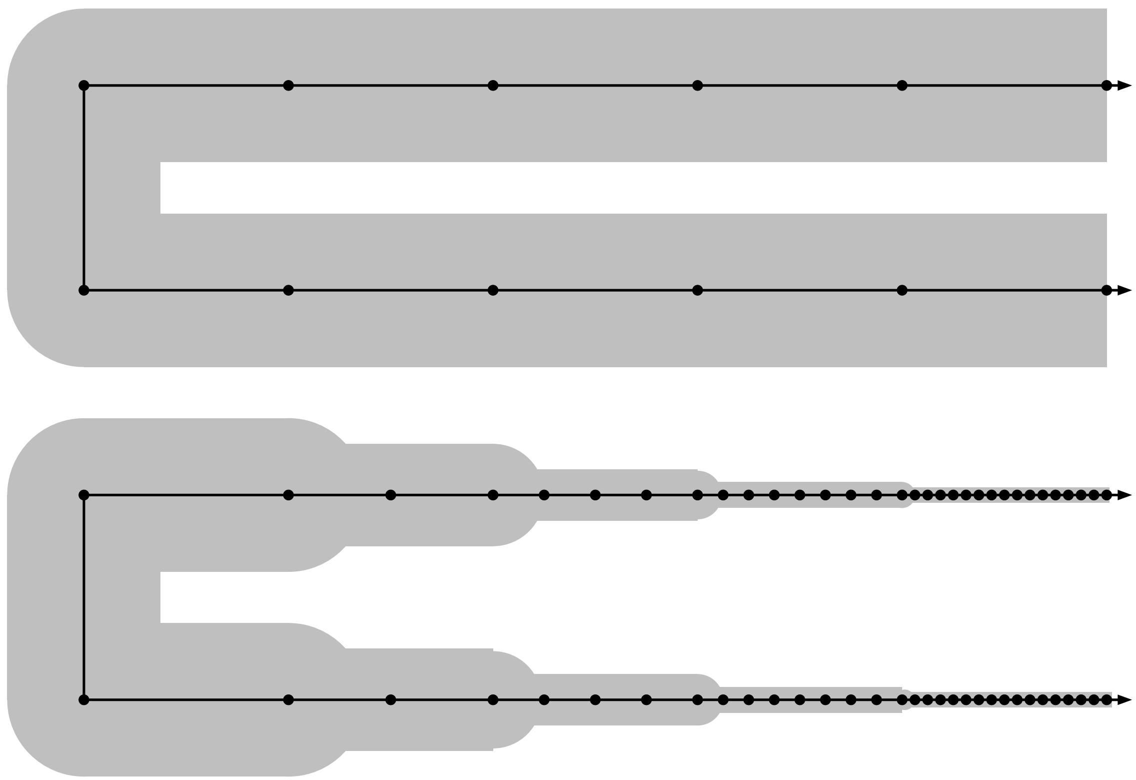
Vertices of tree map to $\pi i \mathbb{Z}$.

τ is not continuous across tree edges, but $\cosh \circ \tau$ is.

Given an edge e of T and $r > 0$ we define

$$e(r) = \{z : \mathrm{dist}(z, e) < r\mathrm{diam}(e)\},$$

Take union over all edges to get $T(r)$.

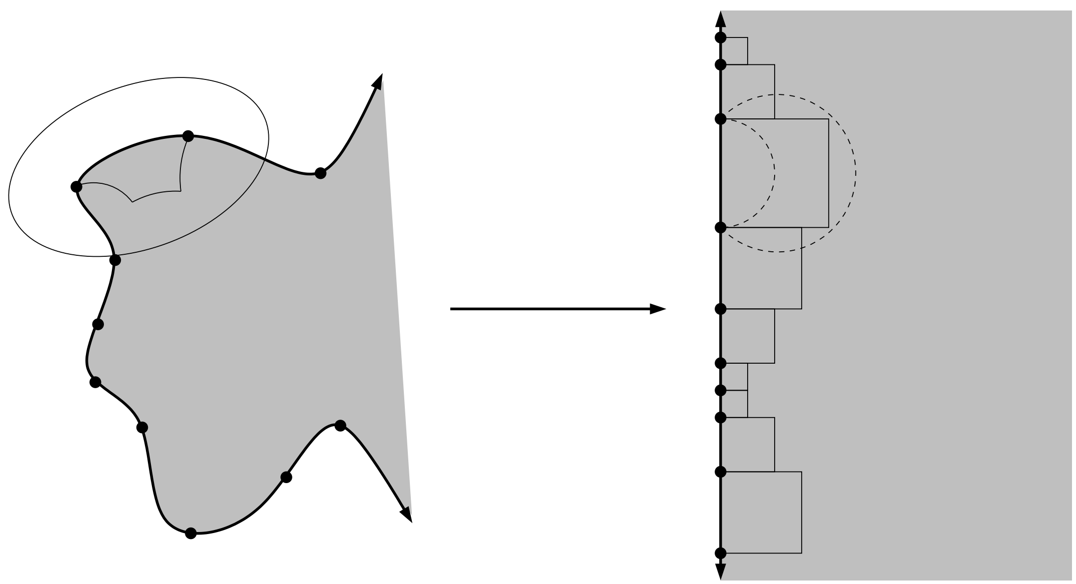


A conformal map $\tau : \Omega \to \mathbb{H}_r$ maps each edge e of $T = \partial\Omega$ to an interval of $\partial\mathbb{H}_r$.

These intervals partition $\partial\mathbb{H}_r$. Denote partition $\mathcal{I}$.

For each $I \in \mathcal{I}$, let $Q_I \subset \mathbb{H}_r$ be the open square with I as one side. Let $V_{\mathcal{I}} \subset \mathbb{H}_r$ be the union of these squares.

Lemma $\tau^{-1}(V_{\mathcal{I}}) \subset T(r)$ *for some fixed* $r > 0$.



A homeomorphism τ between rectifiable curves γ_1, γ_2 **respects length** if it is absolutely continuous with respect to arclength and $|\tau'|$ is a.e. constant, i.e.,

$$\ell(\tau(E)) = \ell(E)\ell(\gamma_2)/\ell(\gamma_1),$$

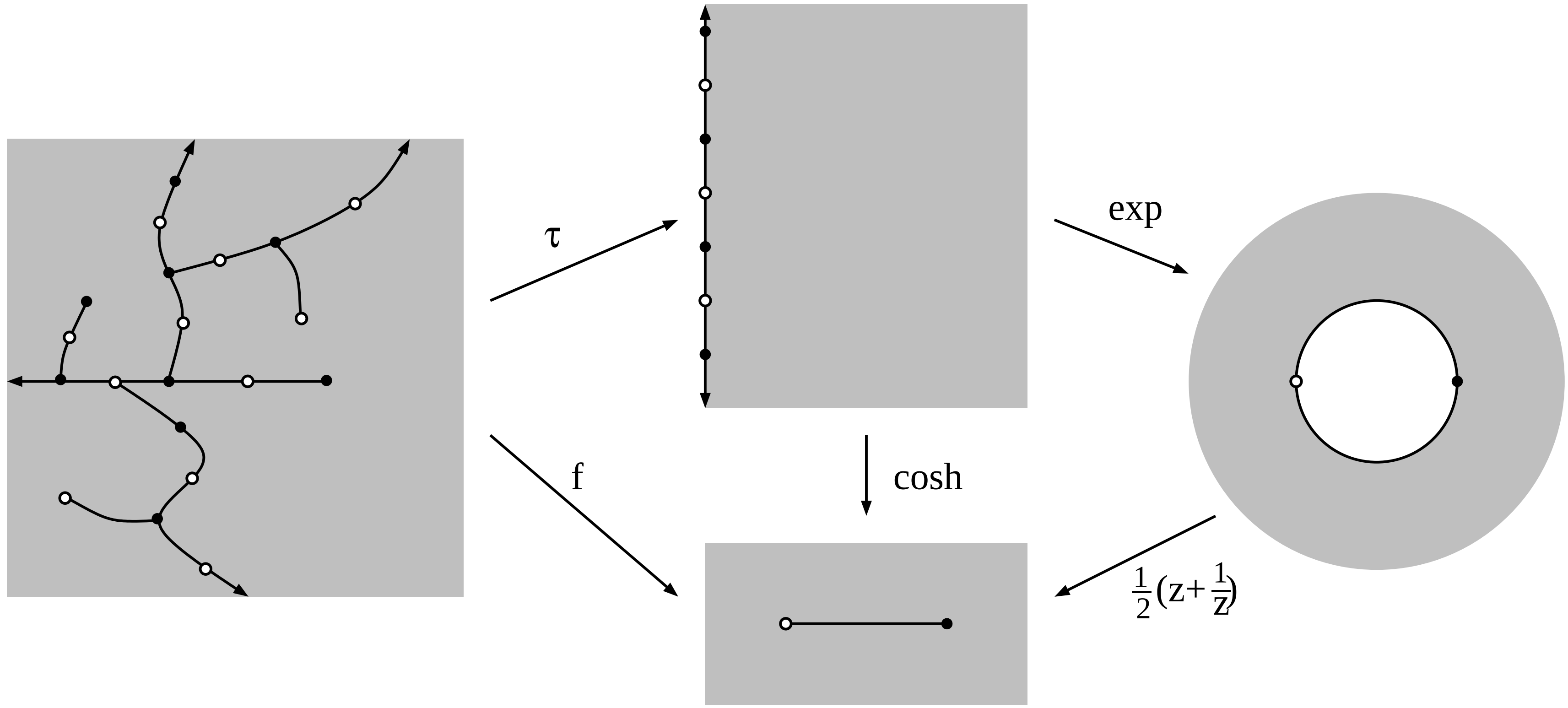
for every $E \subset \gamma_1$.

Generalizes linear map between line segments.

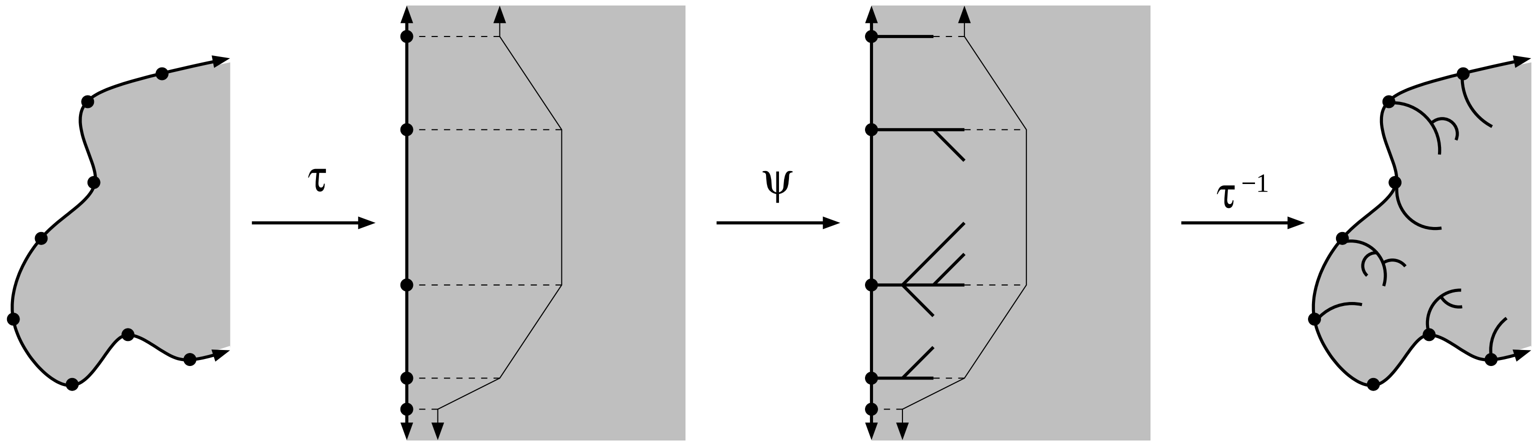
Theorem: *Suppose T is an unbounded, locally finite tree in $\mathbb{C}$ with rectifiable edges, and suppose each of its complementary components $\{\Omega_j\}$ has a K-quasiconformal map $\tau_j : \Omega_j \to \mathbb{H}_r$ that respects length on each edge and such that ("the integer images condition"):*

$$\ell(\tau_j(e)) = 2\pi n \tag{1}$$

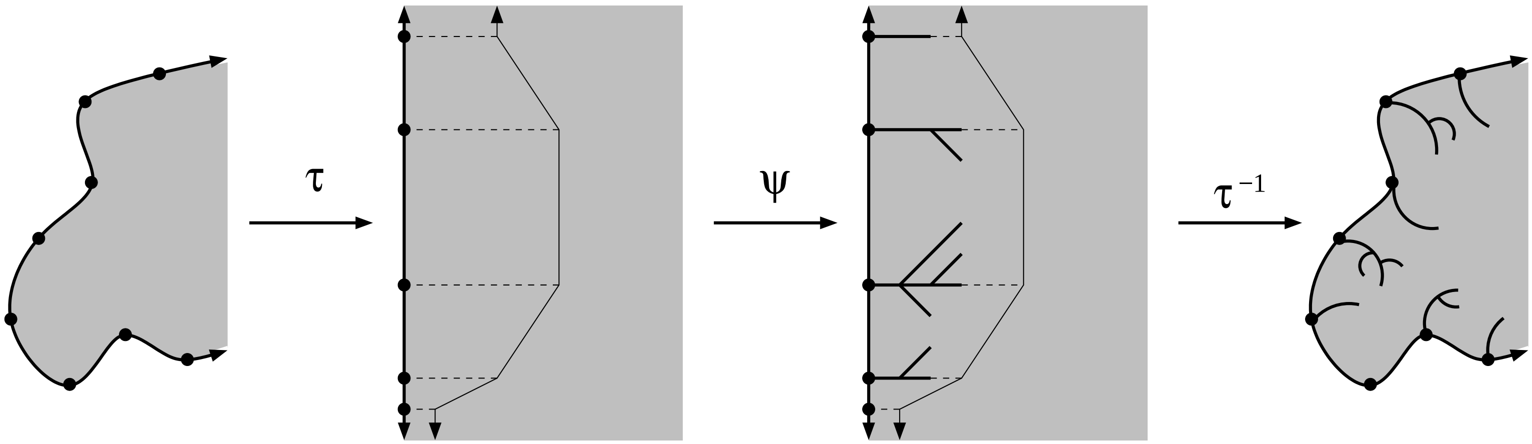
for some odd integer n; assume adjacent intervals have lengths within a fixed factor M of each other. Then there is a $f \in \mathcal{S}$ and a quasiconformal ϕ so that $f \circ \phi = \cosh \circ \tau$ off $T(r)$. The quasiconstant of ϕ depends only on K and M.



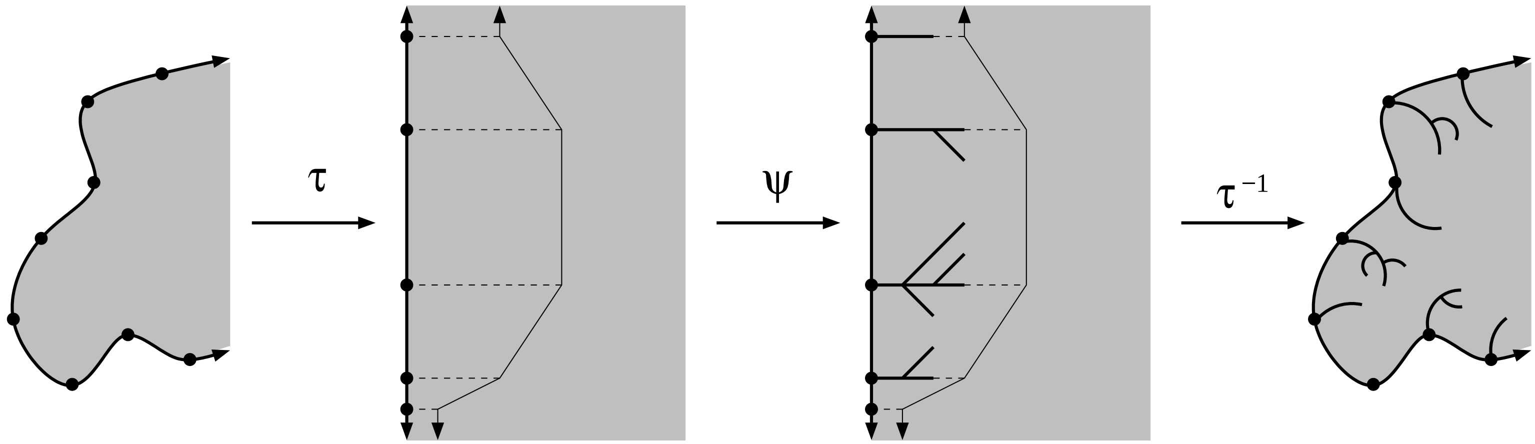
τ
exp
f
cosh
$\frac{1}{2}(z+\frac{1}{z})$



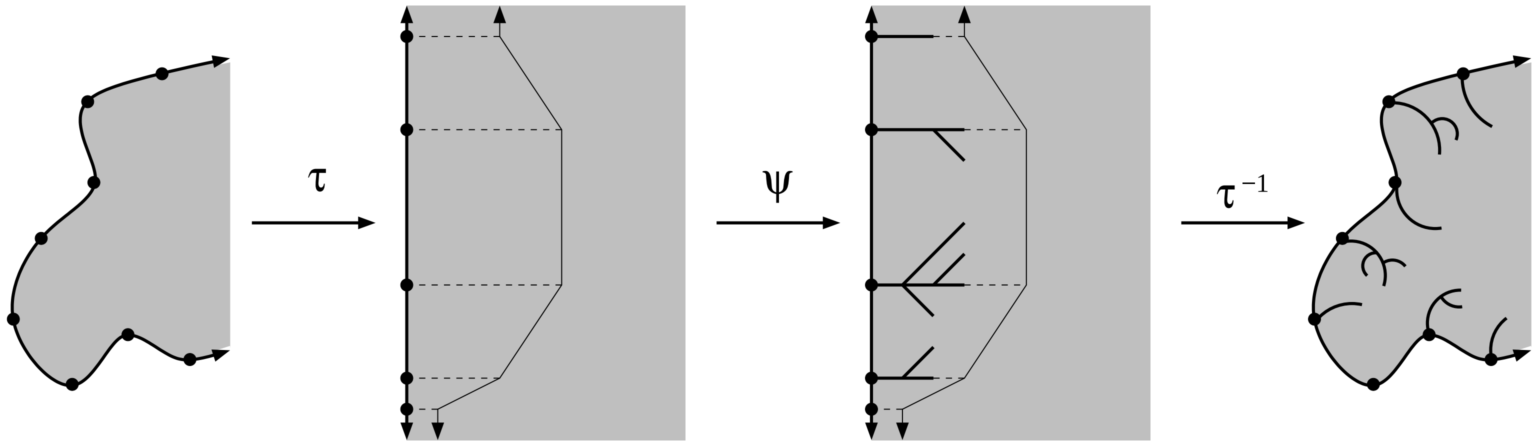
For each complementary component of T, we build a QC map ψ of $\mathbb{H}_r$ into a subdomain $W \subset \mathbb{H}_r$.



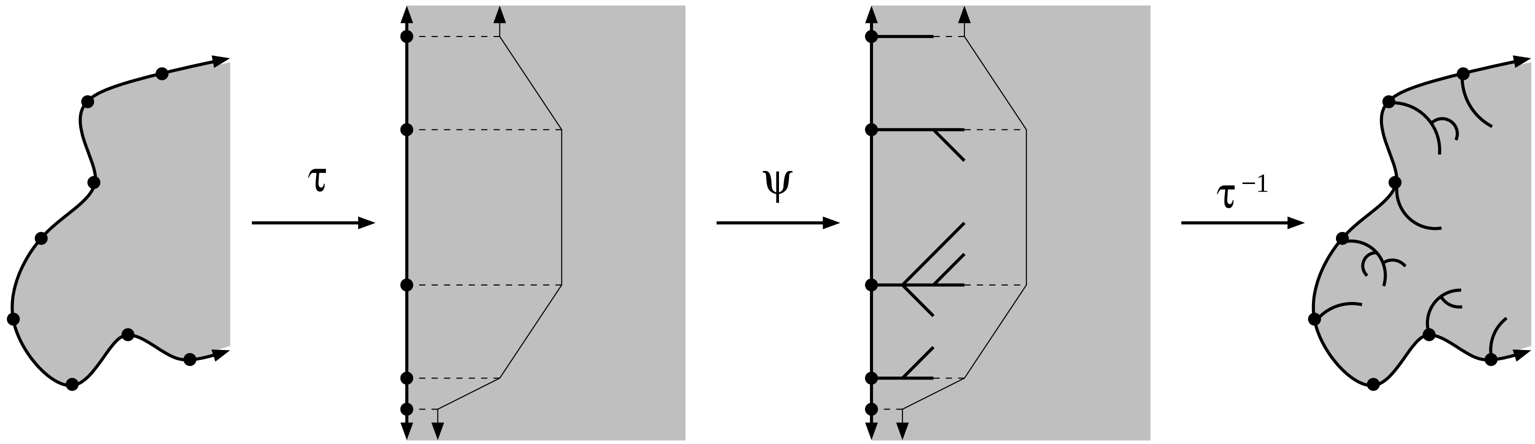
Let $\mathcal{I}$ be the partition consisting of τ-images of edges in T. We may assume each element of $\mathcal{I}$ consists of a odd number of elements of $\mathcal{Z}$.



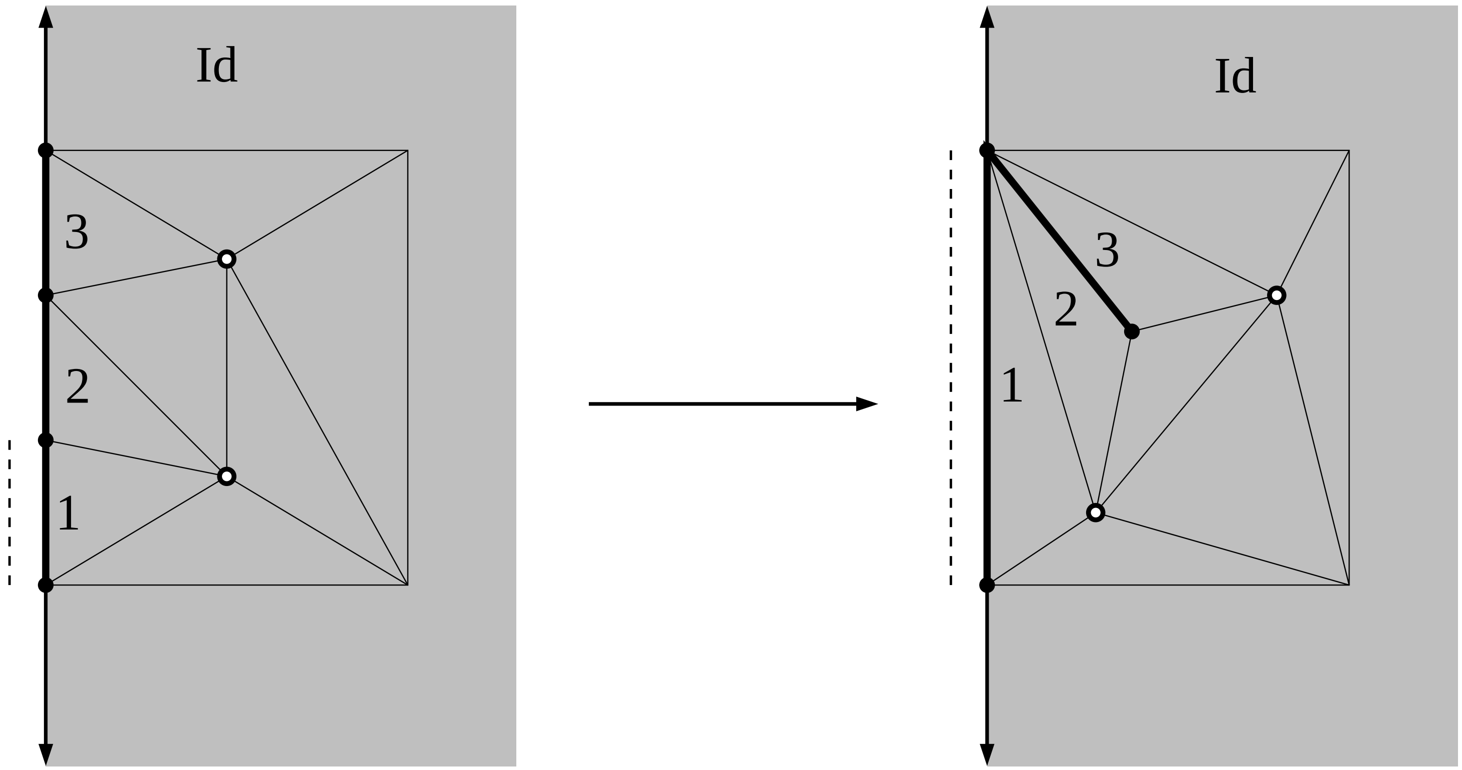
Our map ψ will be linear on each element of $\mathcal{Z}$ and hence length respecting on these intervals. It will also be the identity off $V_{\mathcal{I}}$. For each $I \in \mathcal{I}$, it will map exactly one subinterval of I in $\mathcal{Z}$ to an element of $\mathcal{Z}$. The remaining subintervals of $\mathcal{Z}$ in I are mapped to segments with interiors inside $\mathbb{H}_r$.



We also require that if the ψ images of two open $\mathcal{Z}$ intervals intersect, then they have the same image and are traversed in opposite directions by $\psi(iy)$ as y increases. Moreover, every interval of $\mathcal{Z}$ that is mapped into $\mathbb{H}_r$ is paired with another interval of $\mathcal{Z}$ in this way.



Thus $\partial W \cap \mathbb{H}_r$ consists of finite trees rooted at the points $Z = i\pi\mathbb{Z}$. If two points are identified by ψ, then the values of cosh at these points must be the same, i.e., $\cosh \circ \psi^{-1}$ is continuous on $\overline{\mathbb{H}_r}$.



Here is a simple fold for $n = 3$. One interval is expanded and the othe two are folded to form opposite sides of a one edge tree rooted on the boundary.

In general we do this for large n, but uniform QC bounds on the folding map (this is hard part).

Lemma *If* τ *is QC from* Ω *to* $\mathbb{H}_r$ *so that*

$$\ell(\tau(e)) \geq 2\pi \tag{2}$$

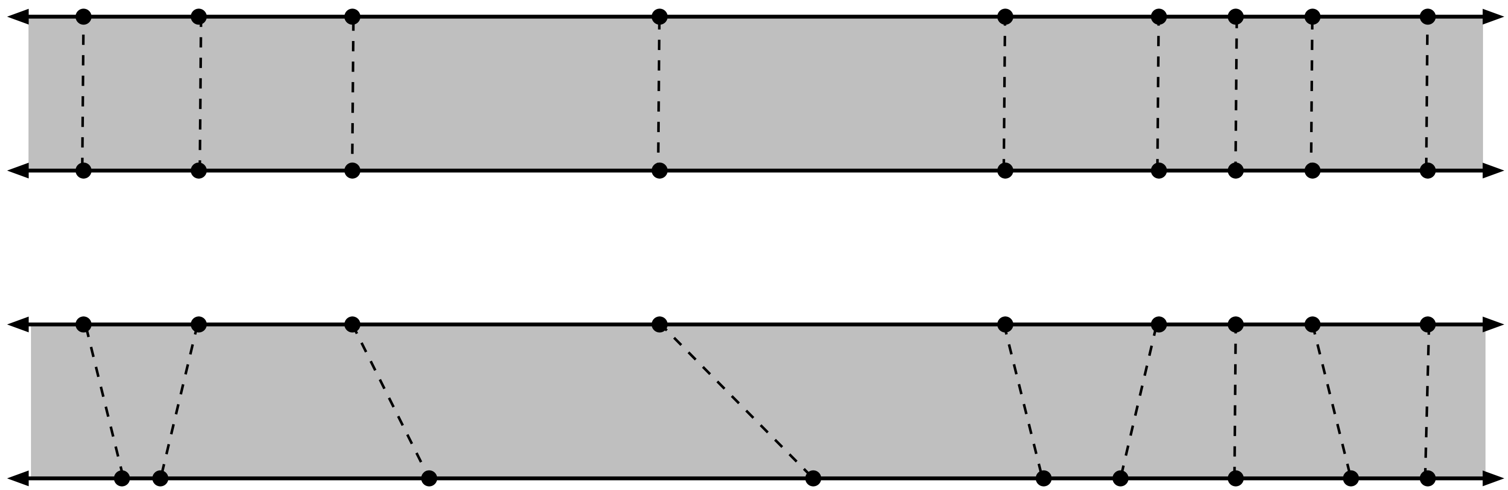
and adjacent intervals have comparable lengths, then there is a QC $\psi : \mathbb{H}_r \to \mathbb{H}_r$ *so that* $\psi \circ \tau$ *satisfies (1) (integer images).* $\psi = \mathrm{Id}$ *on* $\{z : \Re(\tau(z)) > 1\}$ *and has uniformly bounded QC-constant.*

Easy to prove.

Lemma *Suppose* $\mathcal{I} = \{I_j\}$ *is a bounded geometry partition of the real numbers (i.e., adjacent intervals have comparable lengths) so that every interval has length* ≥ 1*. Then there is second partition* $\mathcal{J} = \{J_j\}$ *so that*

1. *Every endpoint of* $\mathcal{J}$ *is an integer.*
2. *The length of* J_j *is an odd integer.*
3. I_j *and* J_j *have lengths differing by* ≤ 2.
4. *The left endpoints of* I_j *and* J_j *are within* 2.5.

Let J_0 be the maximal interval in I_0 with (1) and (2). For $j > 0$, let the left endpoint of J_j be the right endpoint of J_{j-1}. Choose its right endpoint so J_j is maximal satisfying (1), (2). Then (3) and (4) follow.



We can interpolate between the two partitions in a biLipschitz way on a unit width strip and take ψ the identity elsewhere.

A locally finite graph T has "bounded geometry" if:

1. every edge is twice differentiable with uniform bounds.
2. edges meet at angles bounded away from zero.
3. non-adjacent edges e, f satisfy $\frac{\text{dist}(e,f)}{\text{diam}(e)} > c > 0$.

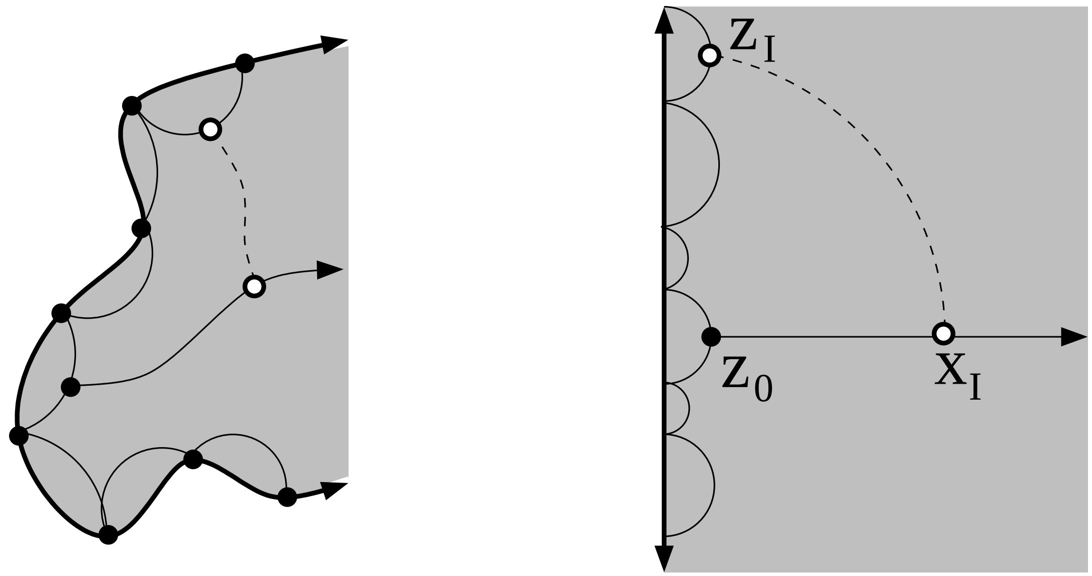
The only consequence of bounded geometry we really need is that if τ maps two edges on $\partial\Omega$ to adjacent intervals of $\partial\mathbb{H}_r$, then the images have comparable length (with a constant independent of the edges and τ.) In this case we say the induced partition of $\partial\mathbb{H}_r$ has "bounded geometry".

Theorem: *Suppose* T *has bounded geometry with conformal maps* $\tau : \Omega \to \mathbb{H}_r$ *associated to each complementary component and suppose*

$$\inf \ell(\tau(e)) > 0,$$

where the infimum is over all edges of T *and both possible* τ *images of* e*. Then there exists* $f \in \mathcal{S}$ *and a* K*-quasiconformal* ϕ *so that* $f \circ \phi = e^{\tau}$ *off* $T(r)$*.* K *only depends on the bounded geometry constants.*

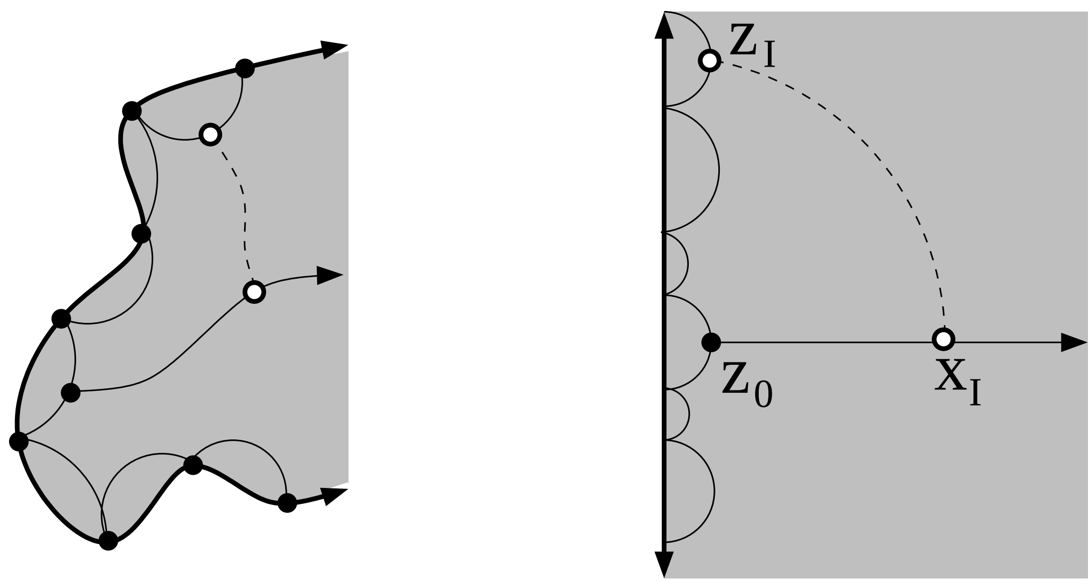
The "big images" condition is usually easy to check.



For each integer j, let $x_j \in \gamma_\infty$ be the point that is closest (in the hyperbolic geometry of $\mathbb{H}_r$) to $\gamma_j = \gamma_{I_j}$.

Then $\ell(I_j) \geq \ell(I_0)$ if

$$\rho(\gamma_j, x_j) \leq \rho(x_j, x_0).$$



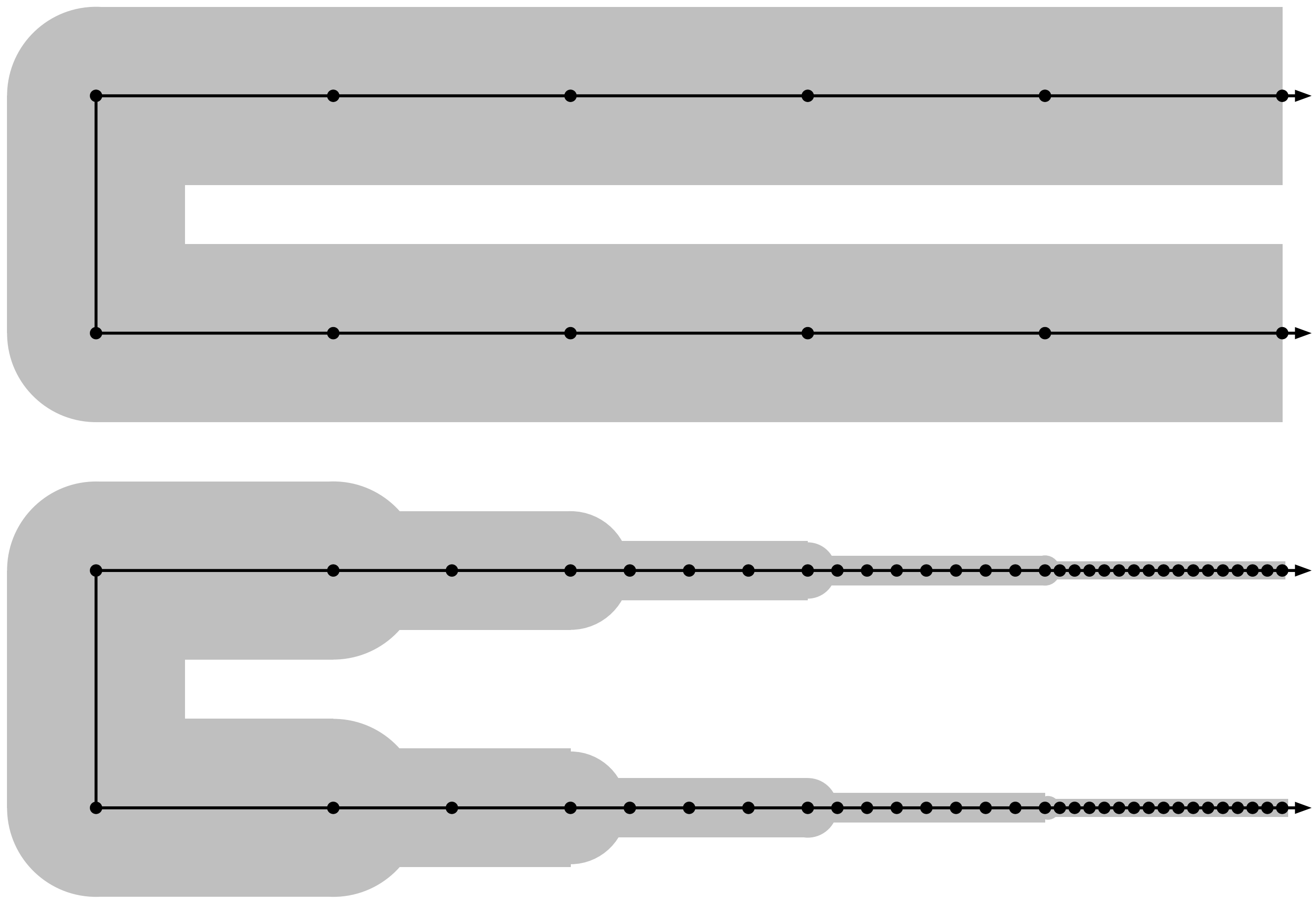
In many examples, we can verify stronger estimates

$$\rho(\gamma_j, x_j) \leq \lambda \rho(x_j, x_0).$$

for some $0 \leq \lambda < 1$ or

$$\rho(\gamma_j, x_j) \leq C,$$

These imply exponential growth of partition intervals. This means we can insert more vertices and decrease size of $T(r)$ exponentially.

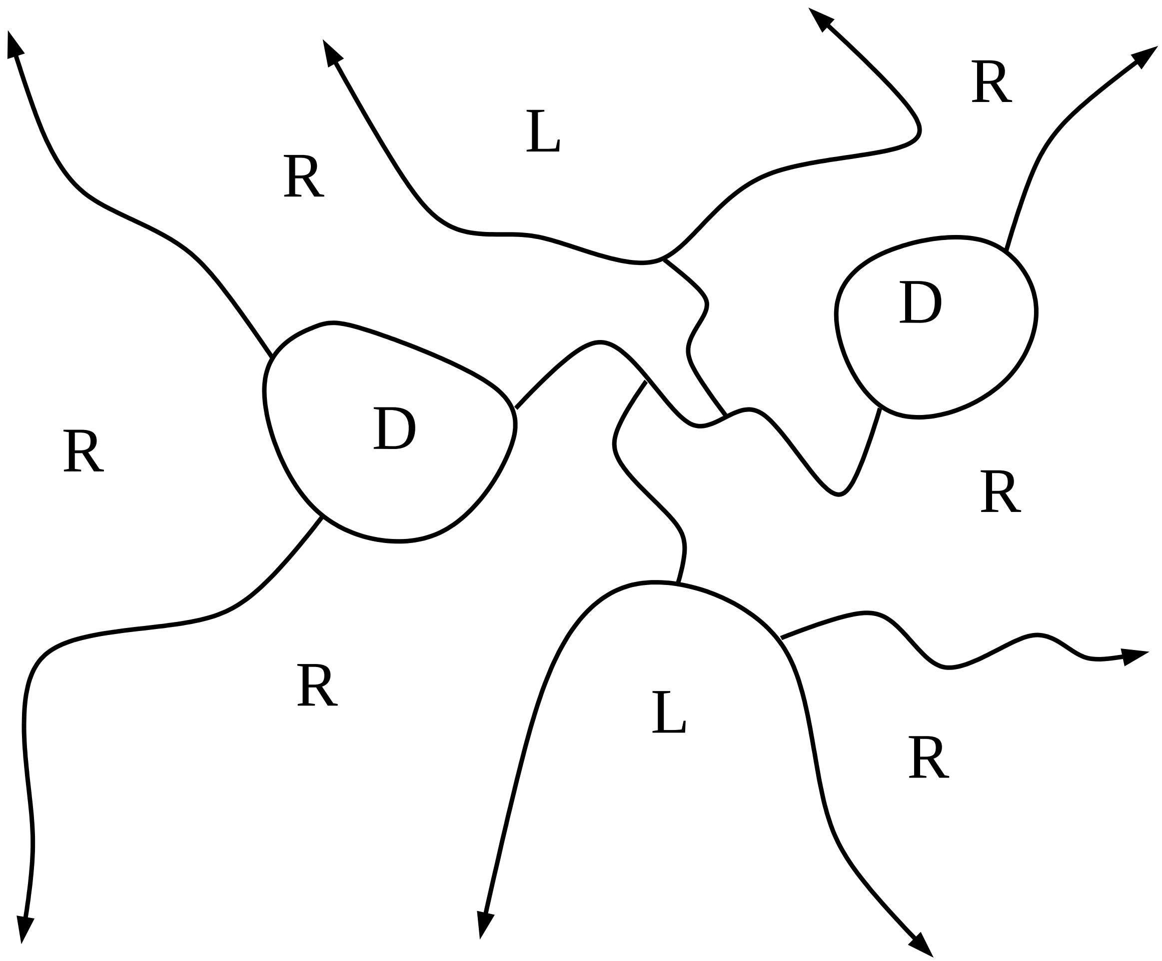


To allow finite asymptotic values or critical points with arbitrarily high degree, the tree T is replaced by a connected graph whose complementary components are each mapped to one of three possible standard domains:

1. a disk
2. a left half-plane
3. a right half-plane

So far have seen only right half-planes.

A disk or left half-plane can only share an edge with a right half-plane. A right half-plane can share edges with any of the three types.



Disk components: Ω is bounded and $\partial\Omega$ is a closed Jordan curve that is the union of a finite number of edges of T, say d.

We are given a length respecting quasiconformal map $\tau : \Omega \to \mathbb{D}$. The map $\sigma : \mathbb{D} \to \mathbb{D}$ is $z \to z^d$ followed by a quasiconformal map $\rho : \mathbb{D} \to \mathbb{D}$, to place critical value where we want.

If a critical value a is desired, then ρ is chosen so $\rho(0) = a$. If $|a| < 1/2$, then ρ can be chosen to be conformal on $\{|z| < 3/4\}$, so, in this case, the dilatation of ρ is supported on $\{z : \frac{3}{4} < |z| < 1\}$. The dilatation of σ is bounded by $O(|a|)$ and is supported on

$$\{z : 1 - \frac{1}{d}\log 4 < |z| < 1\}.$$

Left half-plane components: Here Ω is an unbounded Jordan domain and we are given a length respecting, quasiconformal $\tau : \Omega \to \mathbb{H}_l$.

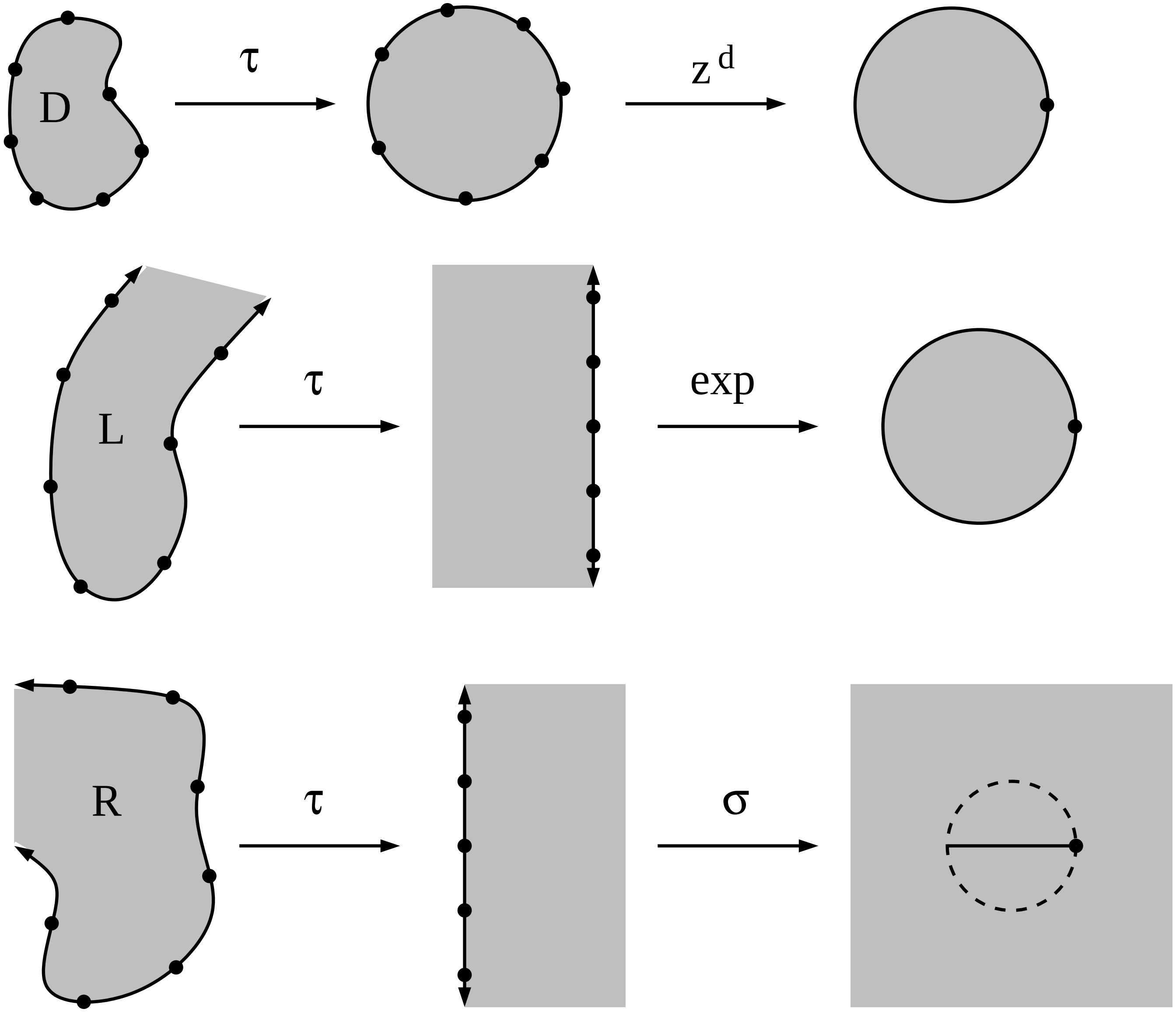
The map $\sigma : \Omega \to \mathbb{D} \setminus \{0\}$ is just $z \to \exp(z)$. This gives a component with finite asymptotic value 0.

If a different asymptotic value a with $|a| < 1/2$ is desired, we post-compose this map with quasiconformal map $\rho : \mathbb{D} \to \mathbb{D}$ such that $\rho(0) = a$ and ρ is the identity on $\partial\mathbb{D}$.

Right half-plane components: Here Ω is simply connected and unbounded and we are given a length respecting, quasiconformal map $\tau : \Omega \to \mathbb{H}_r$. The boundary may be a tree instead of a Jordan curve.

After folding, intervals $I \in \mathcal{Z}$ will be identified either with another arc on the boundary of a right half-plane component (type I) or a disk or left half-plane (type II).

On the type 1 intervals we let $\sigma(iy) = \cosh(iy)$ and on the type 2 intervals we let $\sigma(iy) = \exp(iy)$. We then extend σ to be quasiregular on all of $\mathbb{H}_r$ and equal to cosh on $\{z : \Re(z) > 2\pi\}$.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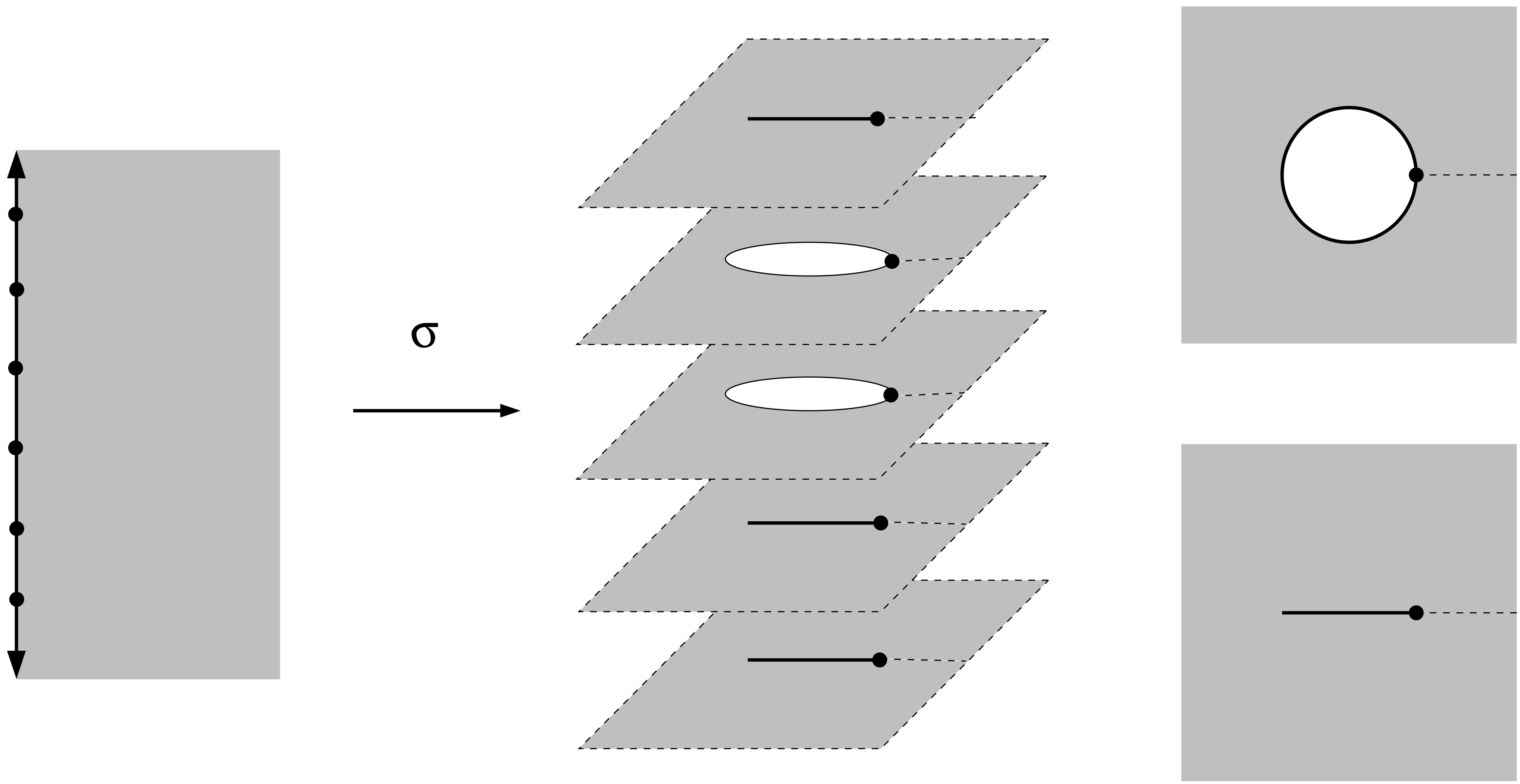
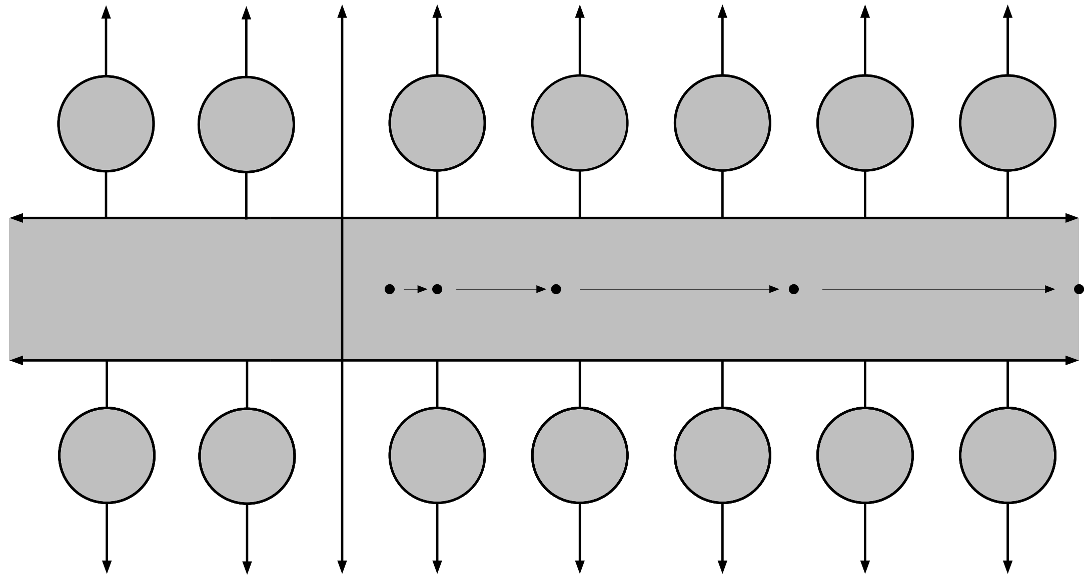

σ

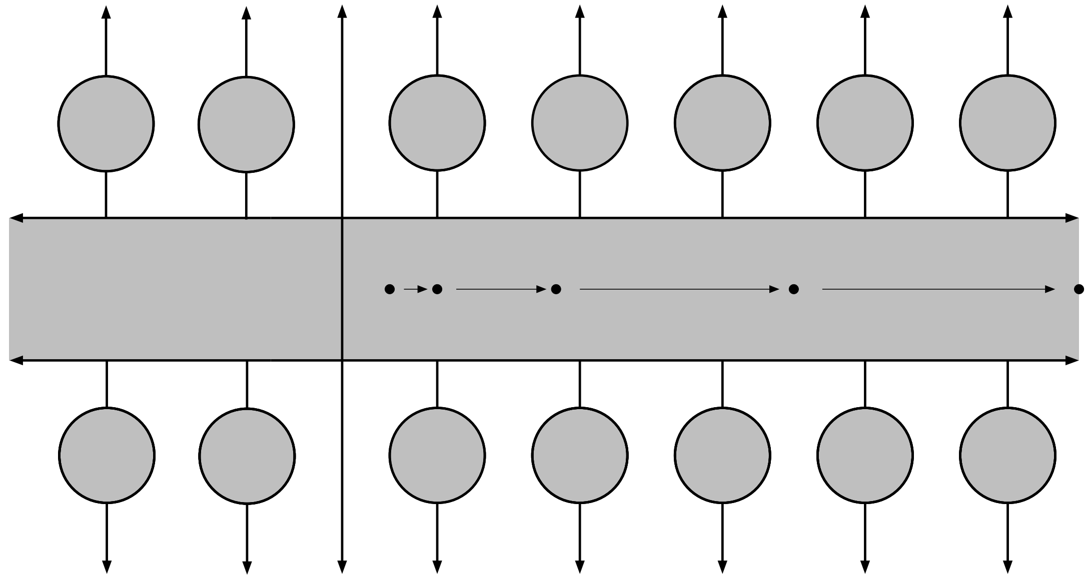
Theorem: *Suppose T is a bounded geometry graph and τ is conformal from each complementary component to its standard version. Assume τ maps*

- *the vertices of disk comps. to roots of unity,*
- *the vertices of LHP comps. to $\{2\pi i\mathbb{Z}\}$*
- *the edges of RHP comps. to length $> \epsilon > 0$.*

Then there is an entire function f and a quasiconformal map ϕ of the plane so that $f \circ \phi = e^\tau$ off $T(r)$. The only singular values of f are ± 1 (critical values coming from the vertices of T) and the critical values and singular values assigned by the disk and left half-plane components.



Theorem: *There is an* $f \in \mathcal{B}$ *whose Fatou set contains a wandering domain.*



Lemma: *There is an* $f \in \mathcal{B}$*, a disk* D_0 *and an increasing sequence of integers* $\{n_k\} \nearrow \infty$ *so that if we set* $D_n = f(D_{n-1})$ *for* $n \geq 1$*, then*

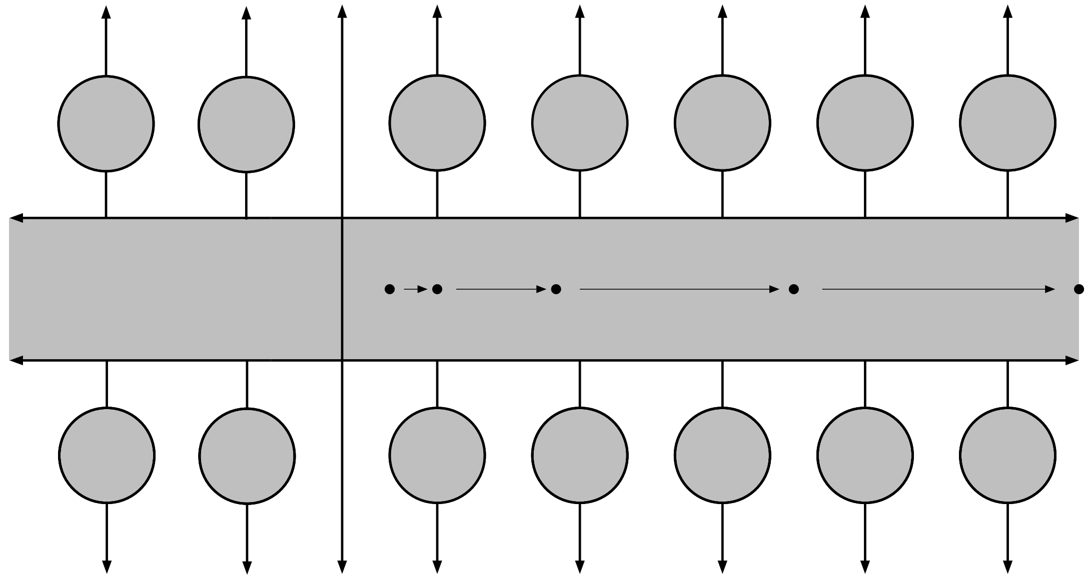
1. *The diameter of* D_n *tends to zero.*
2. $\text{dist}(0, D_{n_k}) \nearrow \infty$*, but* $\text{dist}(0, D_{n_k+1}) \leq 1$ *for all* $k = 1, 2, \ldots$.

(1) implies every subsequence has a subsequence that either approaches ∞ or converges to a finite constant. Thus D_0 and all its images are in the Fatou set.

Suppose $n < m$ and D_n and D_m were in the same component Ω of $\mathcal{F}(f)$. The hyperbolic distance between D_m and D_n cannot increase under iteration.

Iterate n_k+1-m times; then D_m maps to D_{n_k+1} near the origin, but D_n maps to $D_{n_k+1-m+n}$ whose distance from 0 grows to ∞ with k.

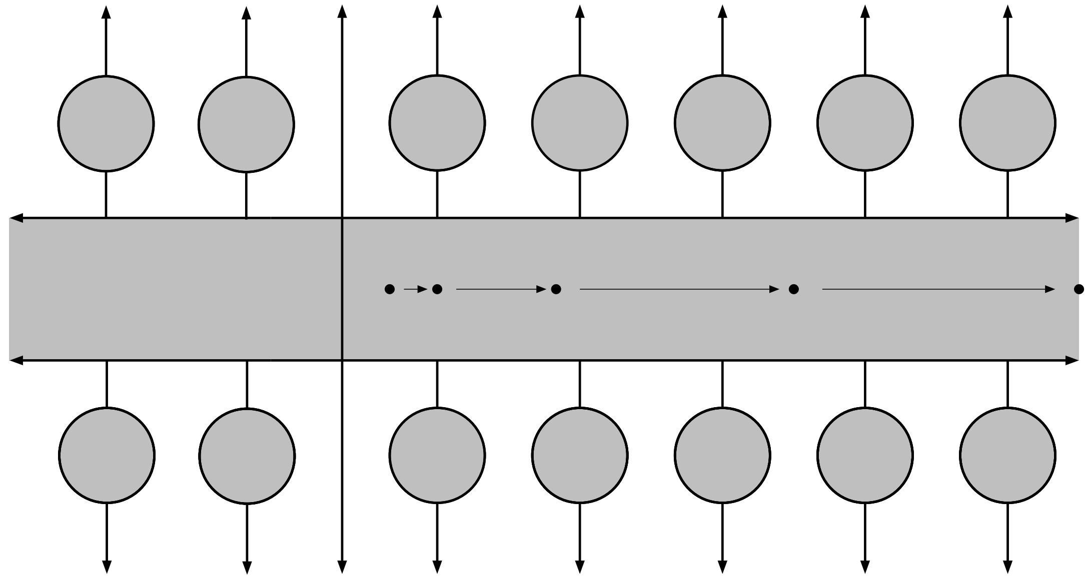
But then hyperbolic distance between the iterates of D_n and D_m increases to ∞ with k. By contradiction, all D_n's are in different components of the Fatou set.



Proof of Lemma:

Use $g = \cosh \circ \sinh$ in horizontal strip S_+ and $g = \sigma_k(z - z_k)^{d_k}$ in disks D_k. Use folding construction in vertical strips to get $f = g \circ \phi$, where ϕ is QC and close to identity.

Verify 1/2 iterates to ∞ along real axis.



Proof of Lemma:

Let D_k be disk nearest kth iterate of $1/2$.

Let U_k be component of $f^{-k}(D_k)$ near $1/2$.

Choose d_k so large that $f(\frac{1}{2}D_k)$ is small compared to U_{k+1}. Check this is not circular.

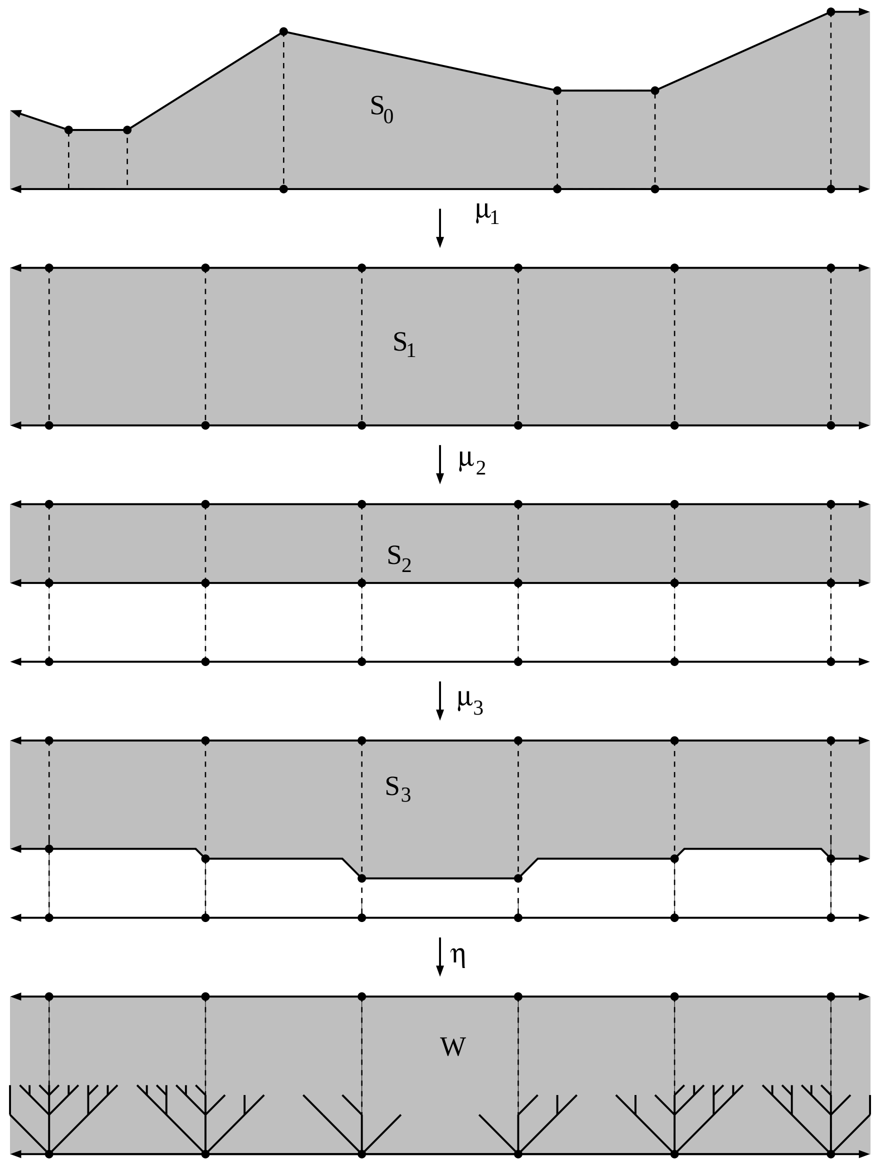
Adjust σ_k so that $f(\frac{1}{2}D_k) \subset U_{k+1}$. Check does not effect earlier containments.

Proof of folding theorem

The proof is an explicit construction of piecewise linear map of a half-plane into a subset of itself.

The data is a partition of the boundary into odd, integer length intervals.

The result is a piecewise linear map of the half-plane into a subset of itself that expands one unit interval from each partition interval to the whole interval, and maps all others to edges of a finite tree in the half-plane, rooted on the boundary.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S2
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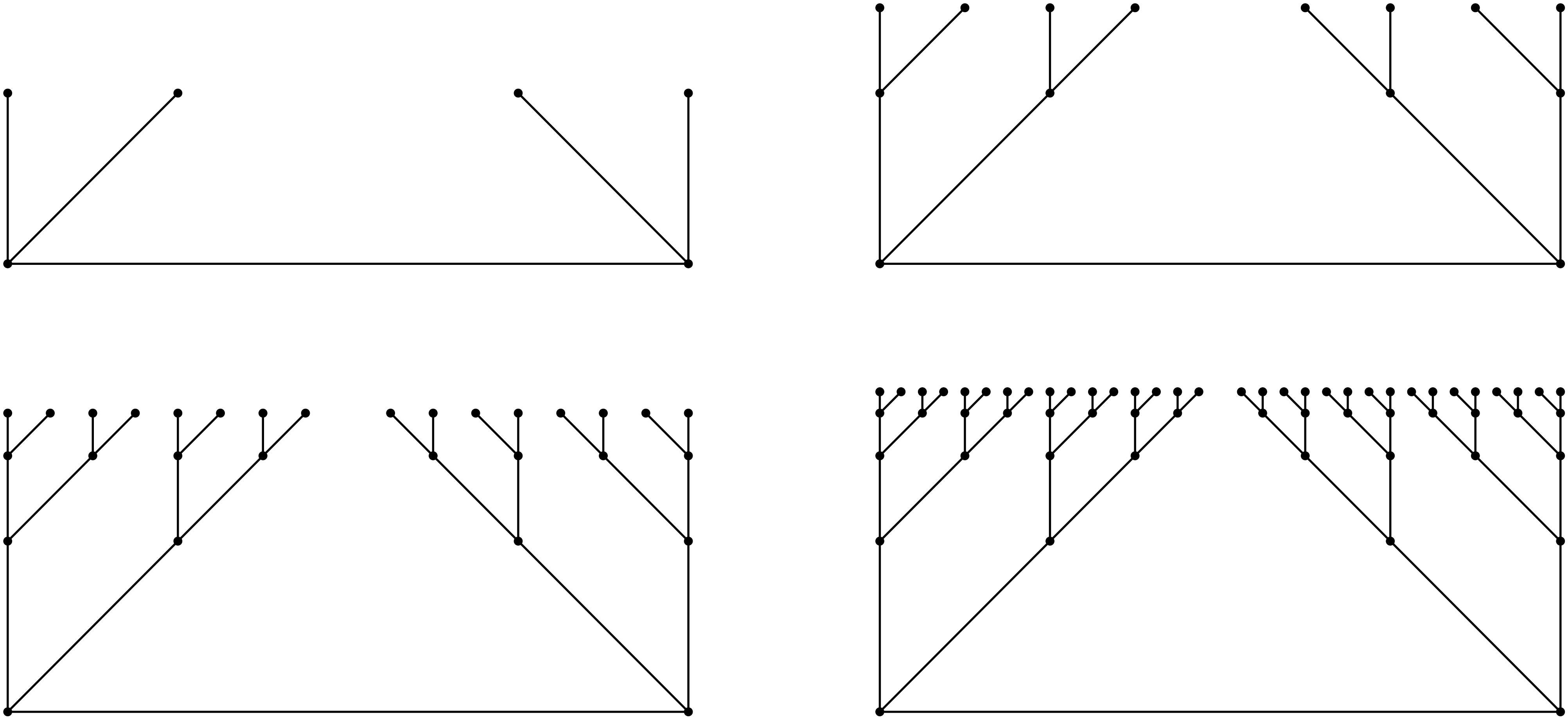
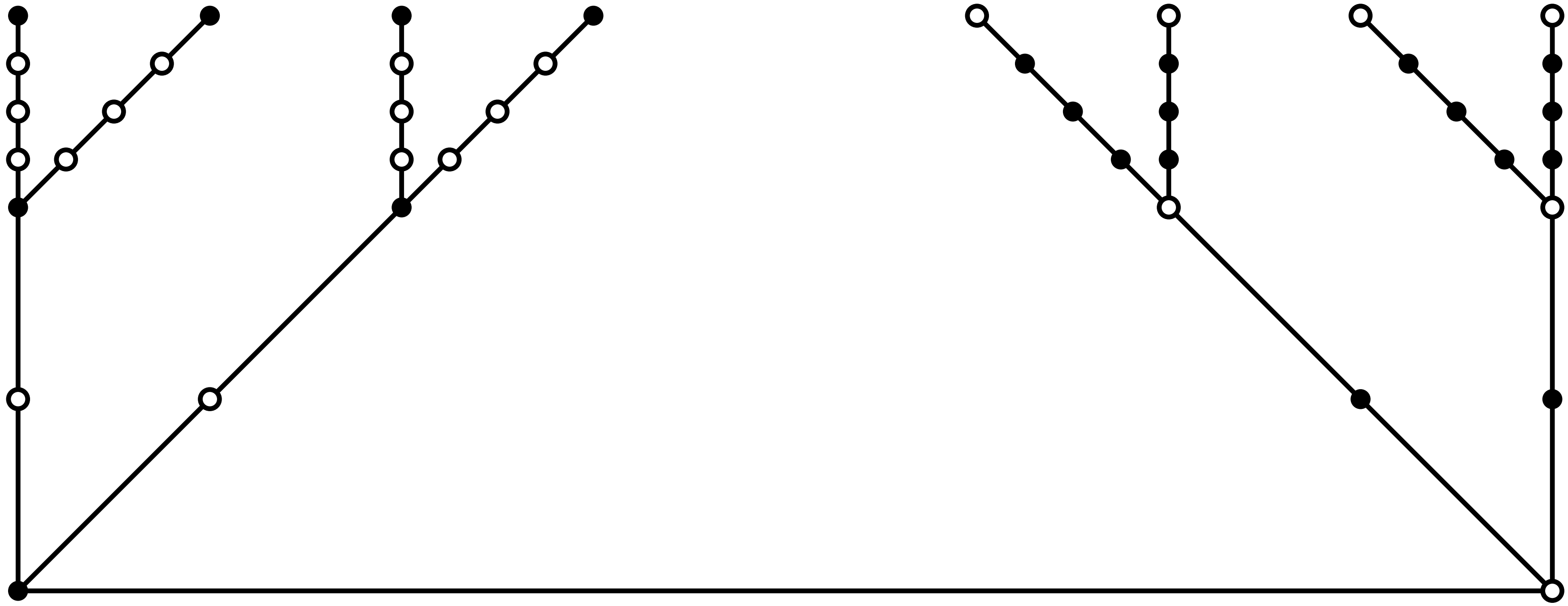
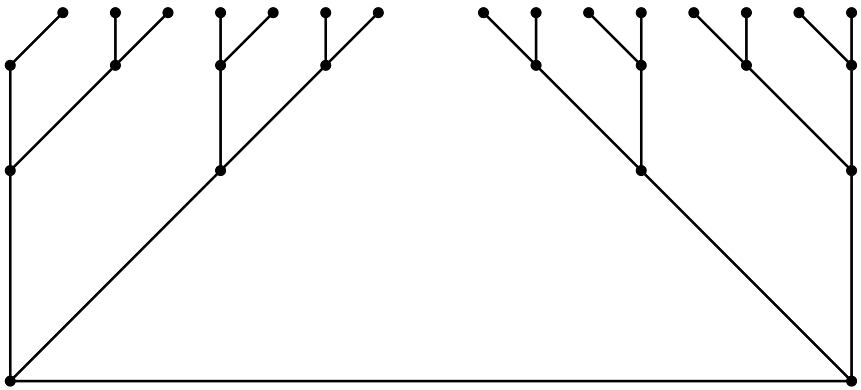
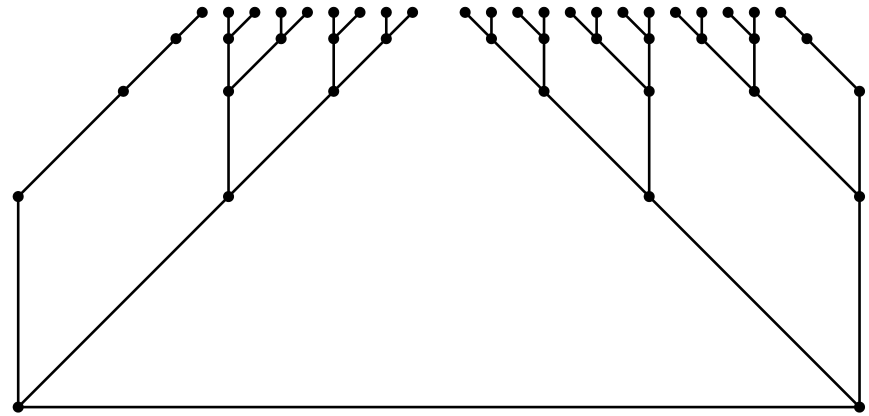
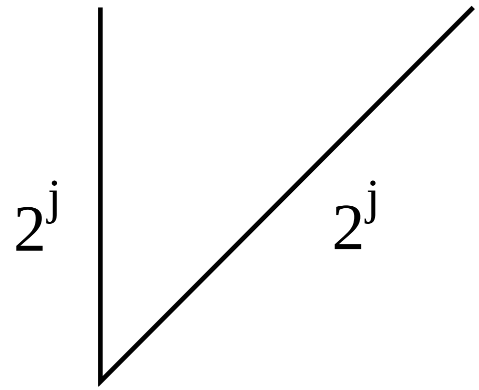


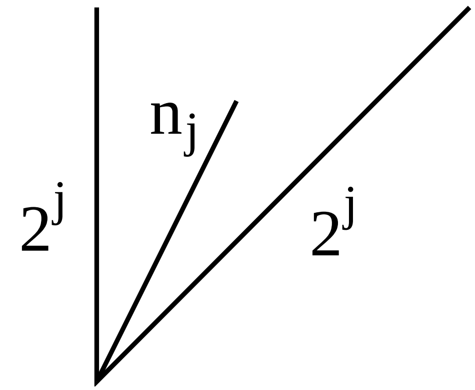








2^j
2^j



n_j
2^j
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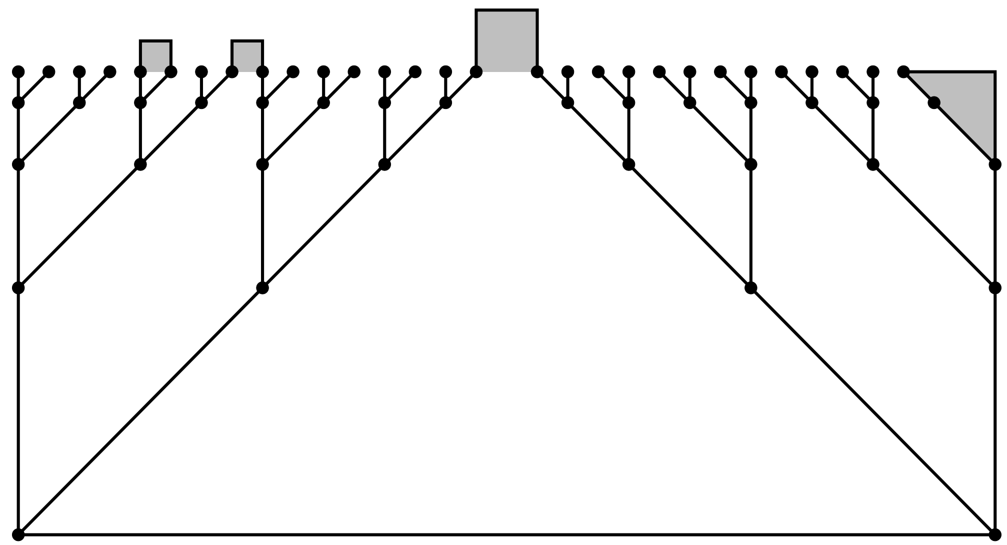

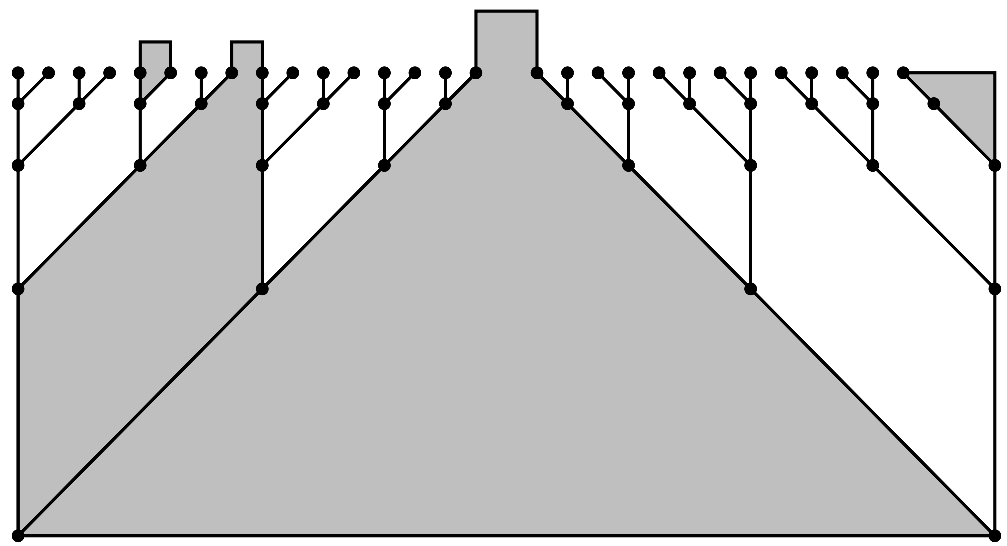

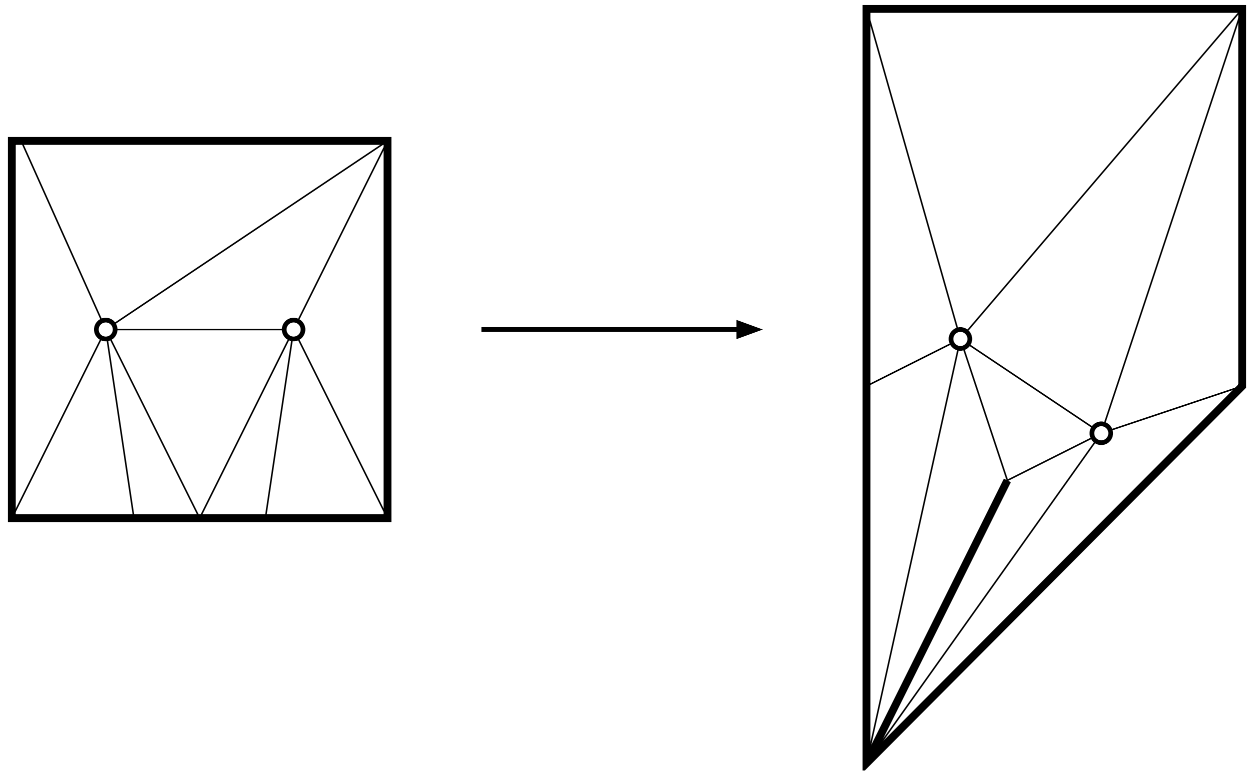

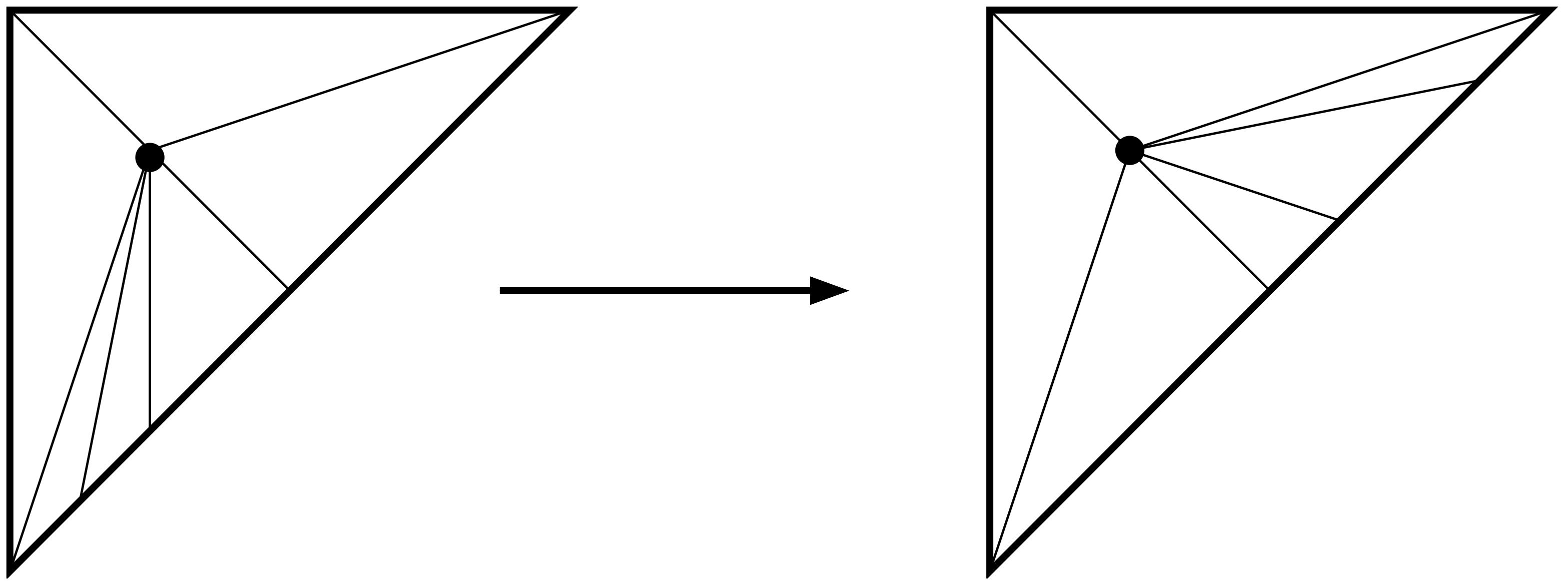

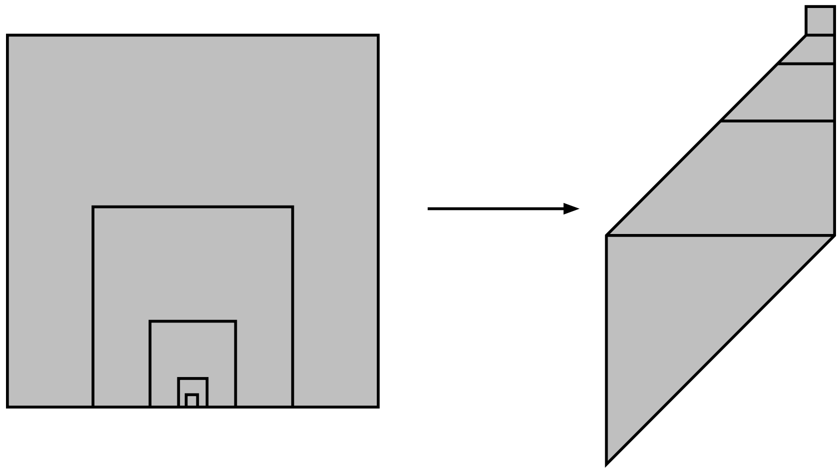

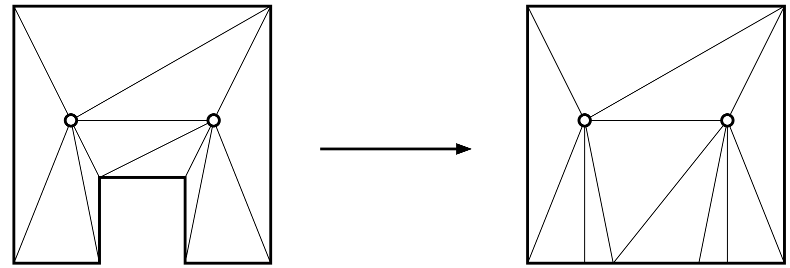

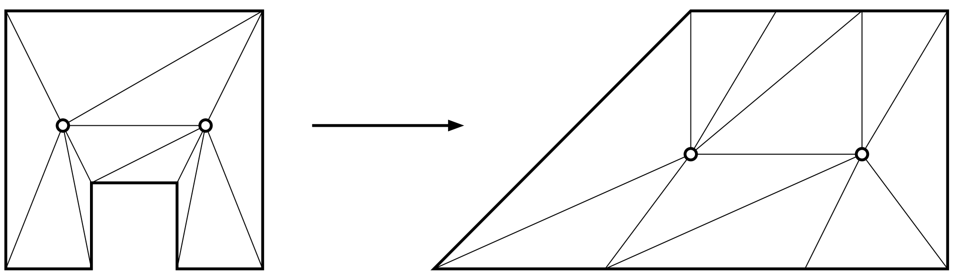

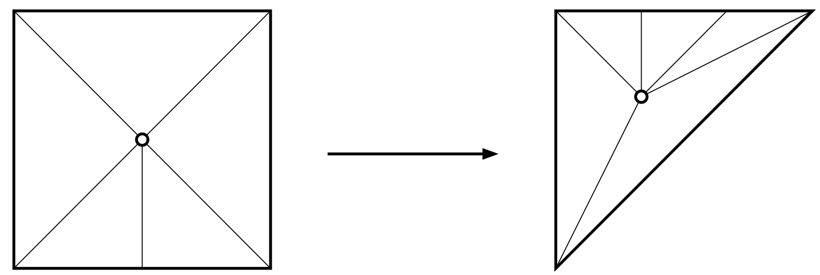

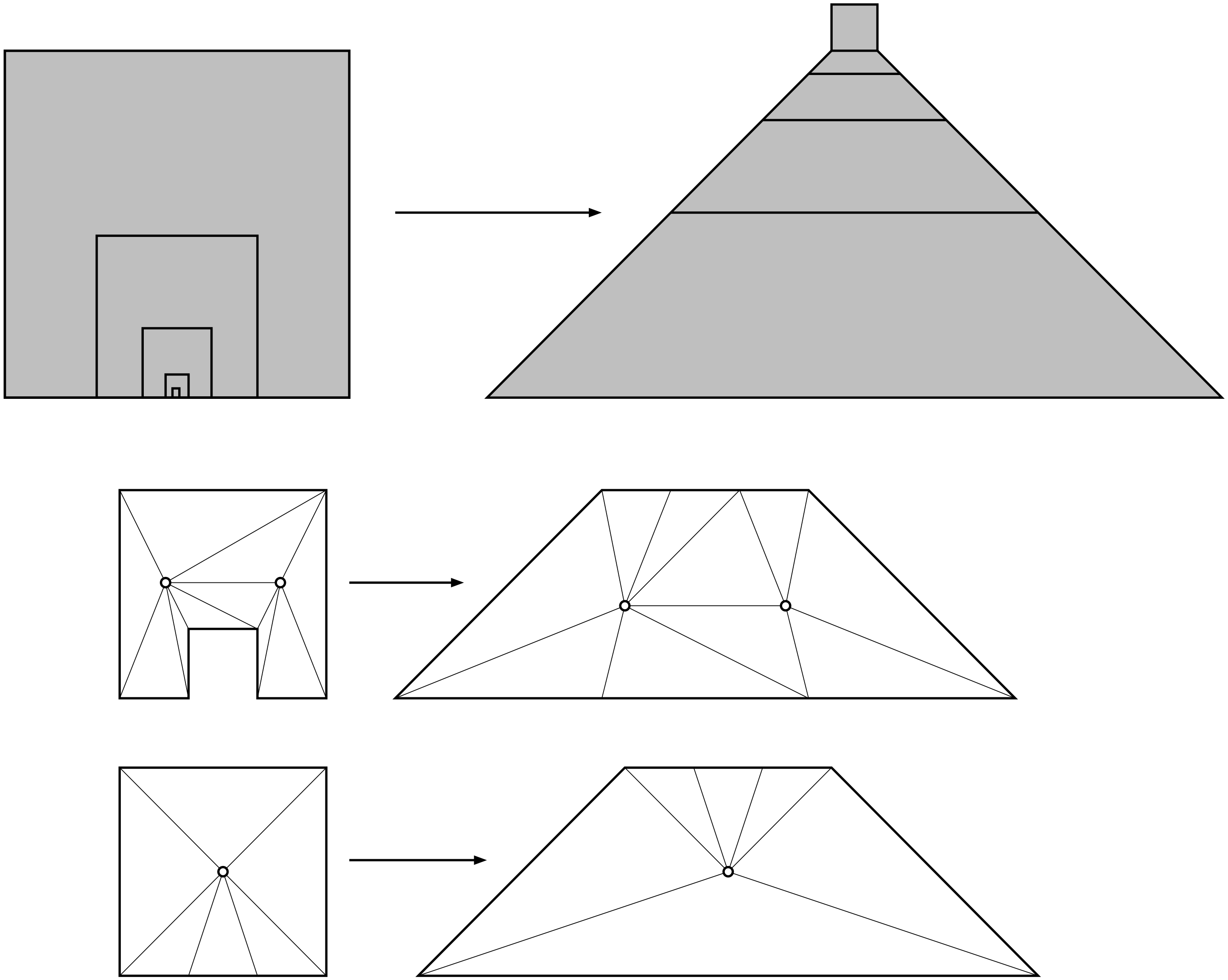

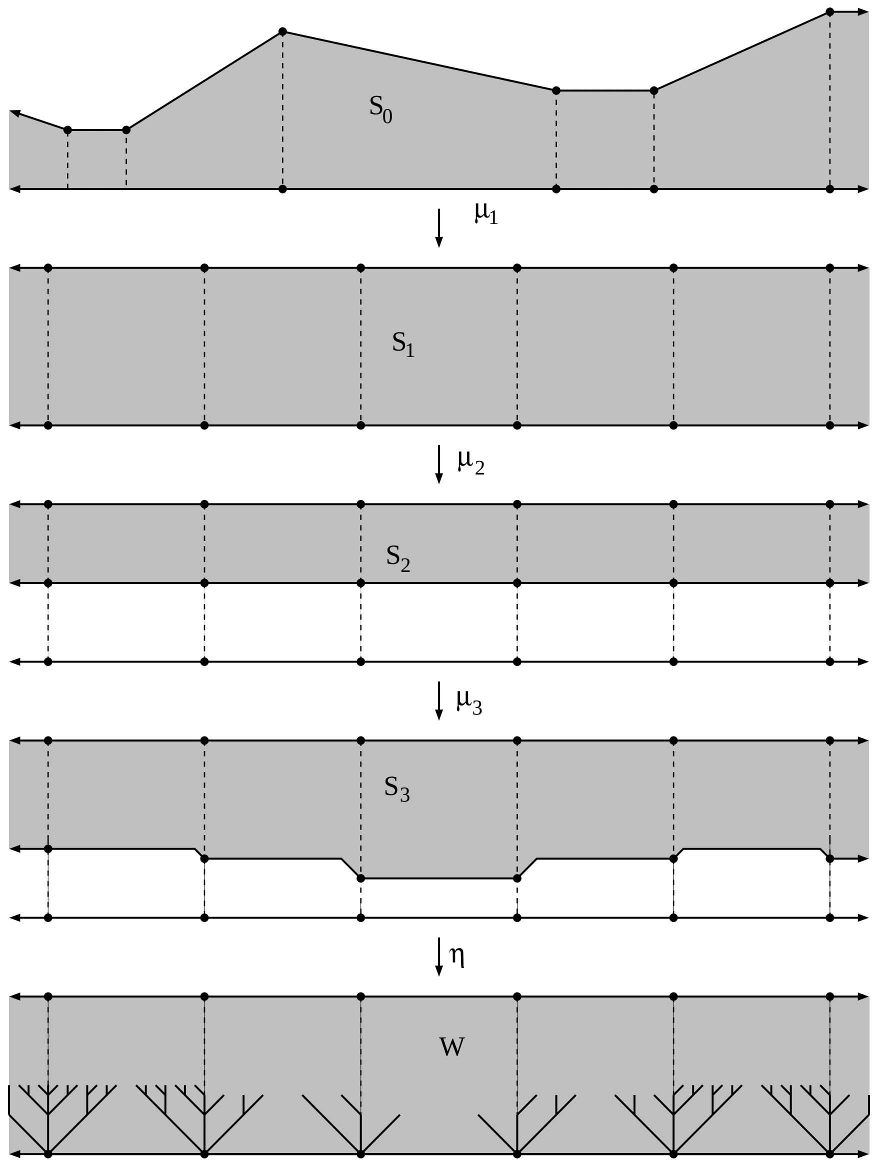

S_0
μ_1
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μ_3
S_3
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