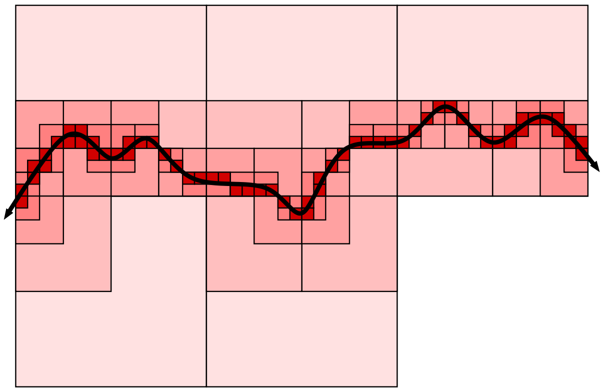
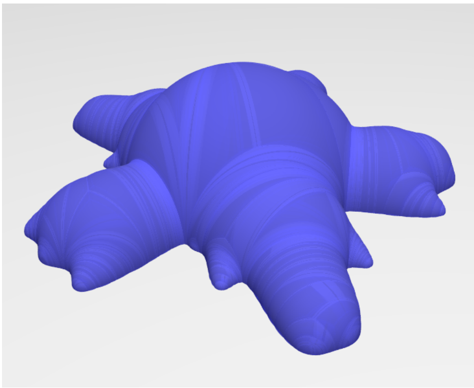
WEIL-PETERSSON CURVES, TRAVELING SALESMAN THEOREMS, AND MINIMAL SURFACES

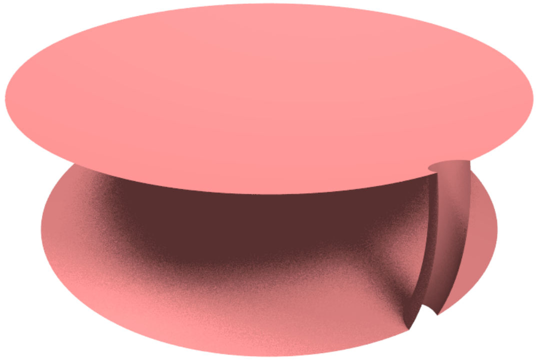
Christopher Bishop, Stony Brook

Differential Geometry and Geometric Analysis Seminar
Princeton University, Wednesday Nov 18, 2020

`www.math.sunysb.edu/~bishop/lectures`





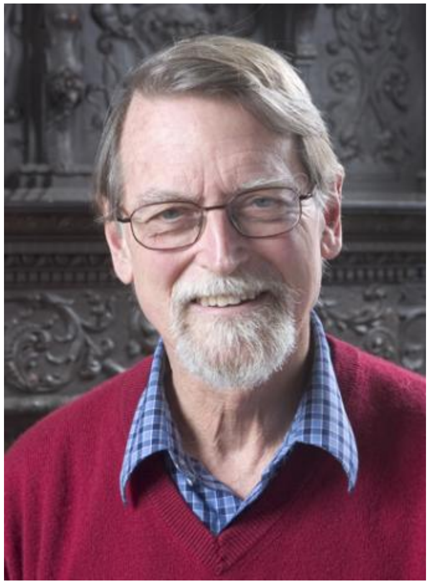


Dear Chris,

Perhaps I could get you more interested in the WP metric
if I put it in the following way. What T & T do is to take
the usual universal Teichmuller space (UTS) and foliate it
strata that are Hilbert manifolds. Via welding, we may view
UTS as plane curves mod rotations and translations. Now all
smooth plane curves belong to the leaf through the unit circle.
This leaf is really the essential object for the WP metric:
it is so marvelous, a homogeneous einstein-kahler manifold.
BUT WHAT ARE THE CURVES IN IT? How irregular can they be?
...

I'm not sure I am remembering things right but I think this set
of plane curves has not been characterized.

cheers, David



Dec 2017 email from
David Mumford

"Riemannian geometries on spaces of plane curves, Michor and Mumford, *J. Eur. Math. Soc. (JEMS)*, 2006.

"2D-Shape analysis using conformal mapping", Sharon and Mumford, *Int. J. Comput. Vis.*, 2006.

"Metric spaces of shapes and applications: compression, curve matching and low-dimensional representation", Feiszli, Kushnarev and Leonard, *Geom. Imaging Comput.*, 2014.

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Number 861

Weil-Petersson Metric on the Universal Teichmüller Space

Leon A. Takhtajan
Lee-Peng Teo

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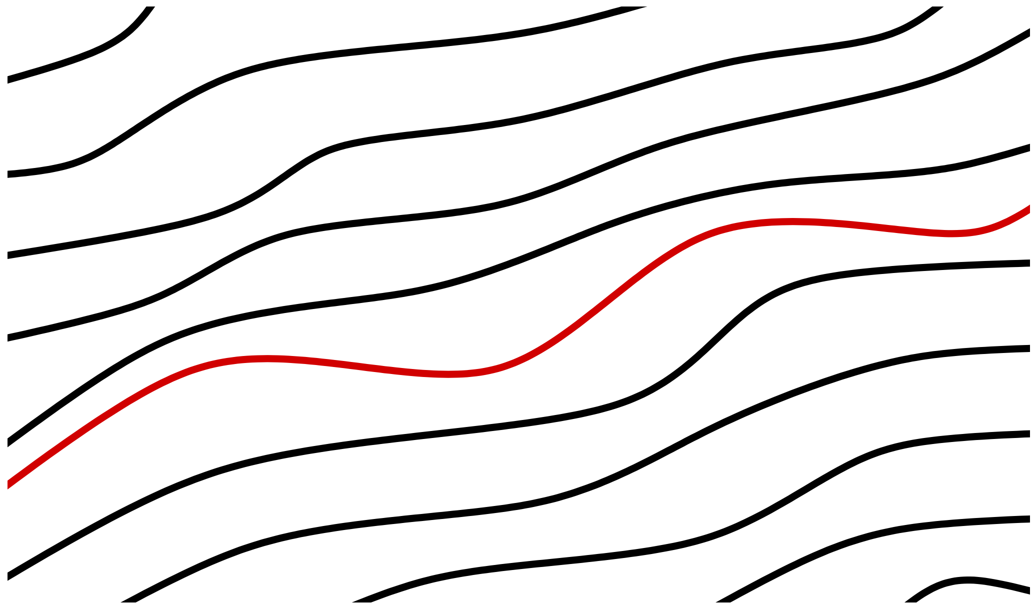
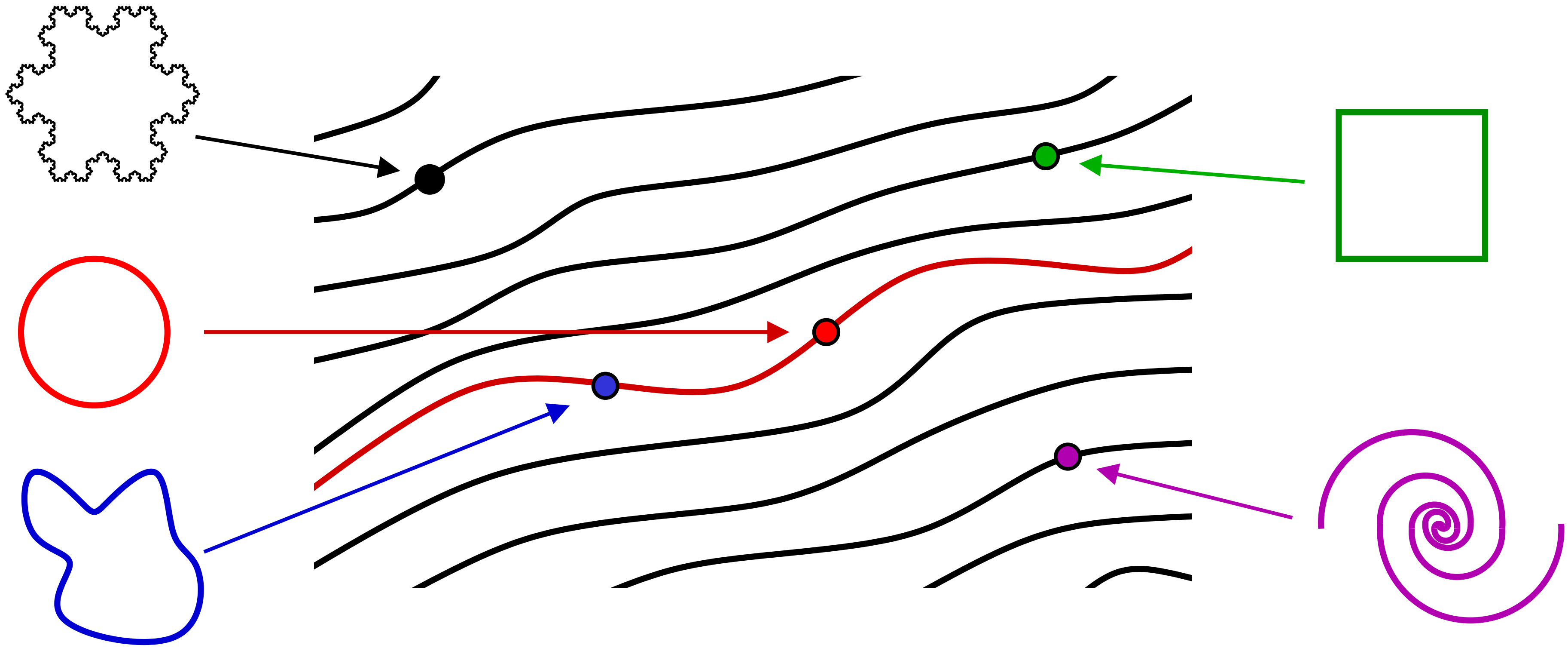


"In this memoir, we prove that the universal Teichmüller space $T(1)$ carries a new structure of a complex Hilbert manifold and show that the connected component of the identity of $T(1)$ — the Hilbert submanifold $T_0(1)$ — is a topological group. ..."



$T(1)$ = Universal Teichmüller Space = quasicircles modulo similarities.

Takhtajan and Teo metric makes this a (disconnected) Hilbert manifold.

$T_0(1)$ = Weil-Petersson class = component containing the circle.

$\quad$ = closure of smooth closed curves.

What non-smooth curves are in $T_0(1)$?

Chris,

In old physics papers [BR87a,BR87b] by Bowick and Rajeev an attempt was made to define a non-perturbative bosonic string theory. Configuration space for closed strings is loop space -- the space of all smooth maps from S^1 to R^d, d-dimensional Minkowski space. The space of loops passing starting at the origin is a complex manifold with a complex structure given by Fourier series. However, Diff(S^1) acts on the loops and M=Diff(S^1)/S^1 is the space of all complex structures on the loop space.... These papers, as well as the work of A.A. Kirillov in 1980s, serve as our motivation.

...

I hope this puts the things in perspective.

Best wishes, Leon

Part 1: Loewner energy via Brownian loop measure and action functional analogs of SLE/GFF couplings

Yilin Wang
ETH Zurich

The study of Loewner energy lies at the interface of random conformal geometry, geometric function theory, and Teichmüller theory. Loewner energy of a loop is defined as Dirichlet energy of its Loewner driving function, motivated by the action functional of SLE.

In the first part of my talk, I will give an overview of the connections of Loewner energy to determinants of Laplacians and Weil-Petersson class of universal Teichmüller space.
In the second part, I will present some consequences of these connections. Including an identity of Loewner energy with a renormalized Brownian loop measure attached to the curve, and (ongoing project with F. Viklund) a deterministic proof of the fact that analogs of both SLE/GFF couplings, namely the quantum zipper and flow-line coupling, hold for finite energy curves.

Jan 2019 IPAM workshop
"Analysis and Geometry of Random Sets".

So the Weil-Petersson class is linked to:

- Pattern recognition
- Infinite dimensional Kähler-Einstein manifolds
- Teichmüller theory
- String theory
- Brownian motion, SLE, Gaussian free fields, ...

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In today's talk I will discuss further connections to:

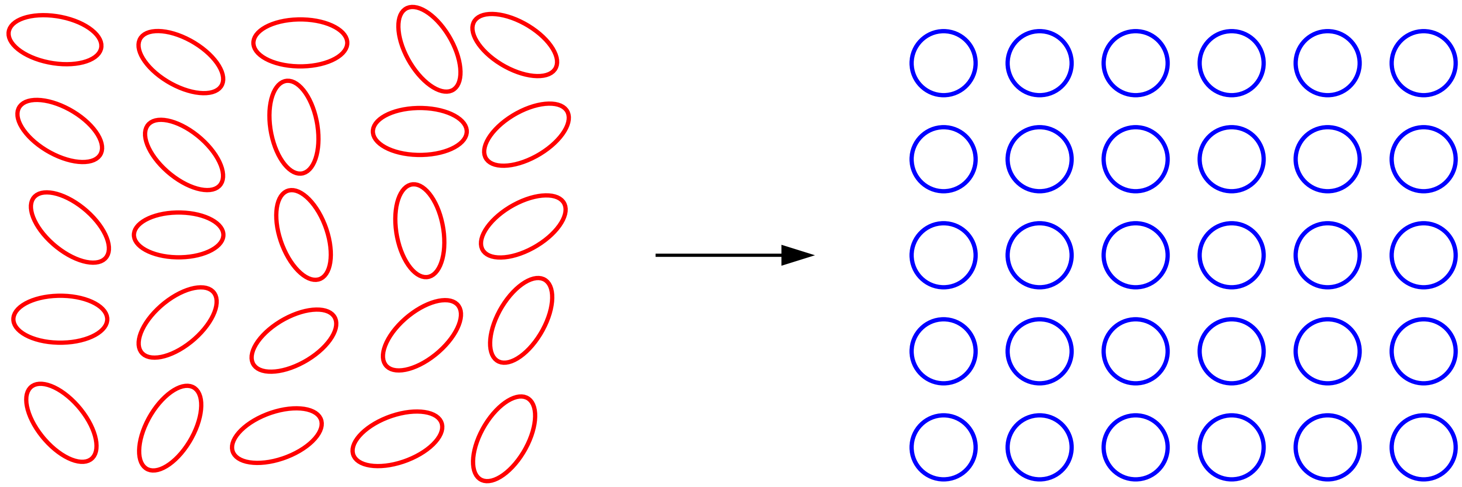
- Geometric function theory
- Sobolev spaces
- Knot theory
- Geometric measure theorem (Peter Jones's TST)
- Convex hulls in hyperbolic space
- Minimal surfaces
- Isoperimetric inequalities
- Renormalized area

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions
14	between disjoint disks

Definition	Description
15	thickness of convex hull
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder
21	closure of smooth curves in $T_0(1)$
22	P_φ^- is Hilbert-Schmidt
23	double hits by random lines
24	finite Loewner energy
25	large deviations of SLE(0^+)
26	Brownian loop measure

26 characterizations of Weil-Petersson curves.

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

For K-QC maps, ellipses have eccentricity $\leq K$

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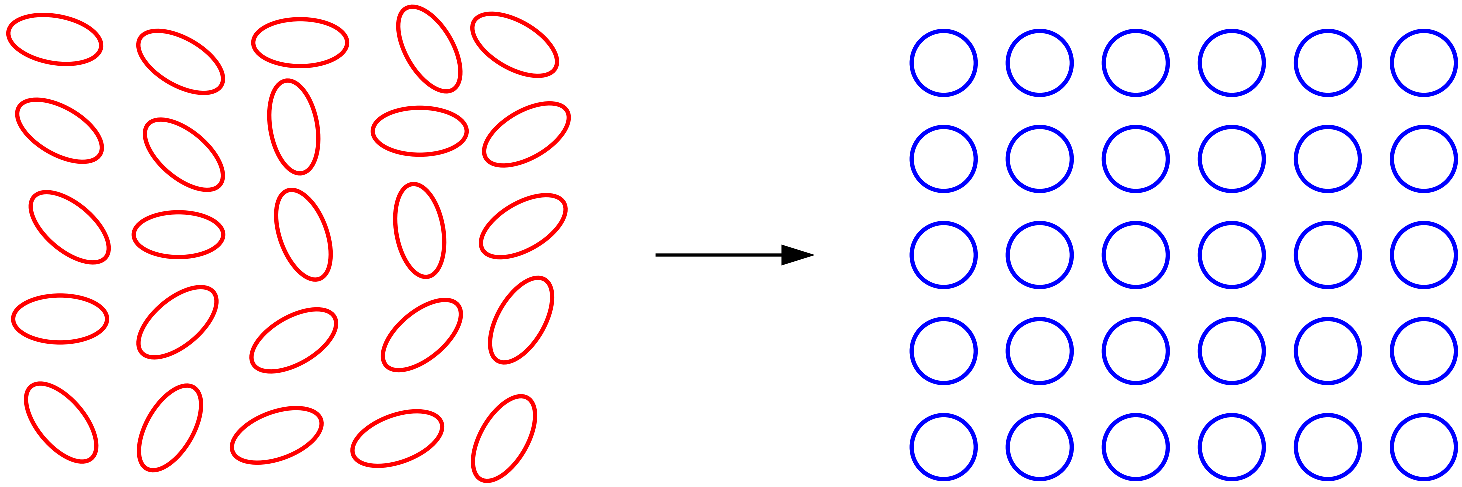

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For K-QC maps, ellipses have eccentricity $\le K$

Ellipses determined a.e. by measurable dilatation $\mu = f_{\overline{z}}/f_z$ with

$$|\mu| \le \frac{K-1}{K+1} < 1.$$

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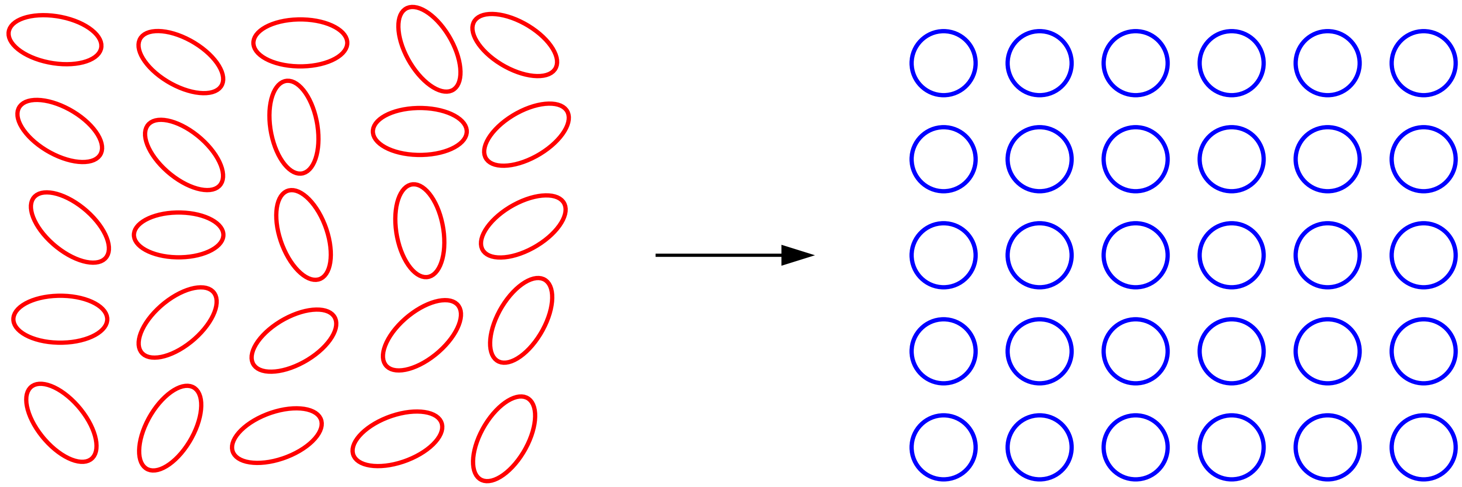

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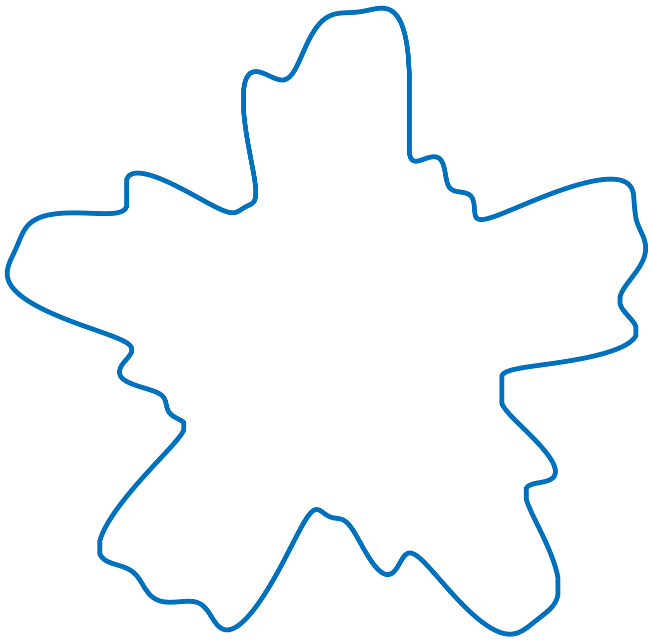
For K-QC maps, ellipses have eccentricity $\leq K$

A special case of QC maps are biLipschitz maps (all we need):

$$\frac{1}{C} \leq \frac{|f(x) - f(y)|}{|x - y|} \leq C.$$

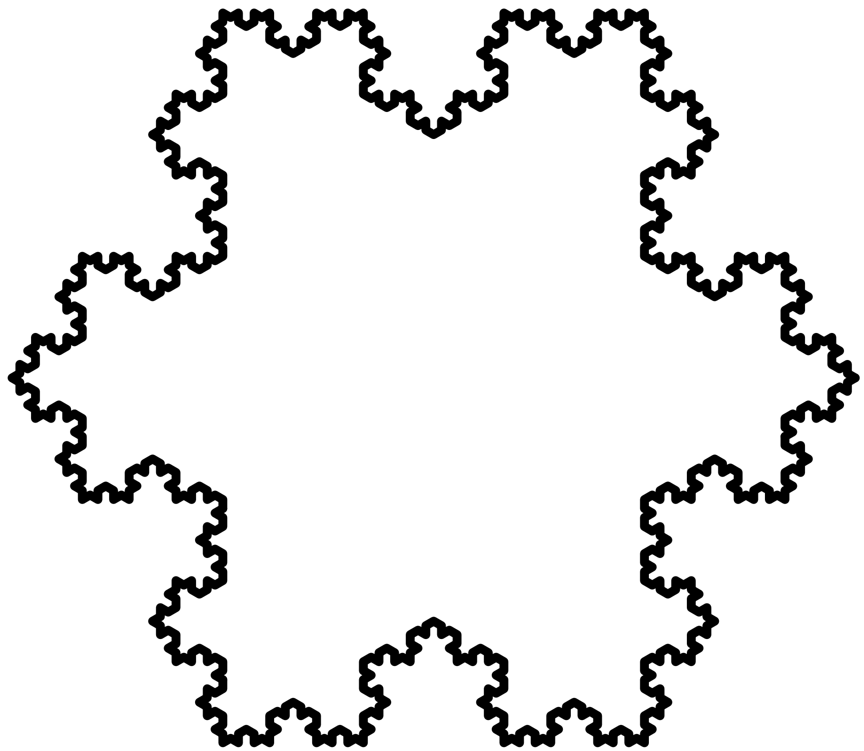
A **quasicircle** is the image of circle under a quasiconformal map of $\mathbb{R}^2$.

Any smooth curve is a quasicircle.



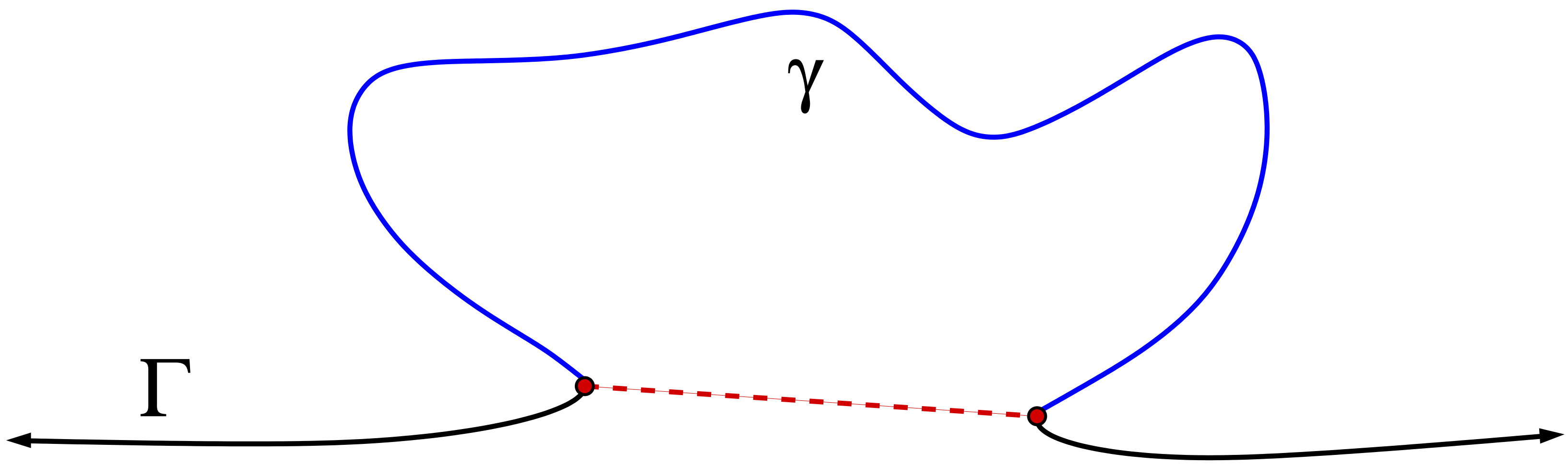
A **quasicircle** is the image of circle under a quasiconformal map of $\mathbb{R}^2$.

Some fractals are quasicircles.



A **quasicircle** is the image of circle under a quasiconformal map of $\mathbb{R}^2$.

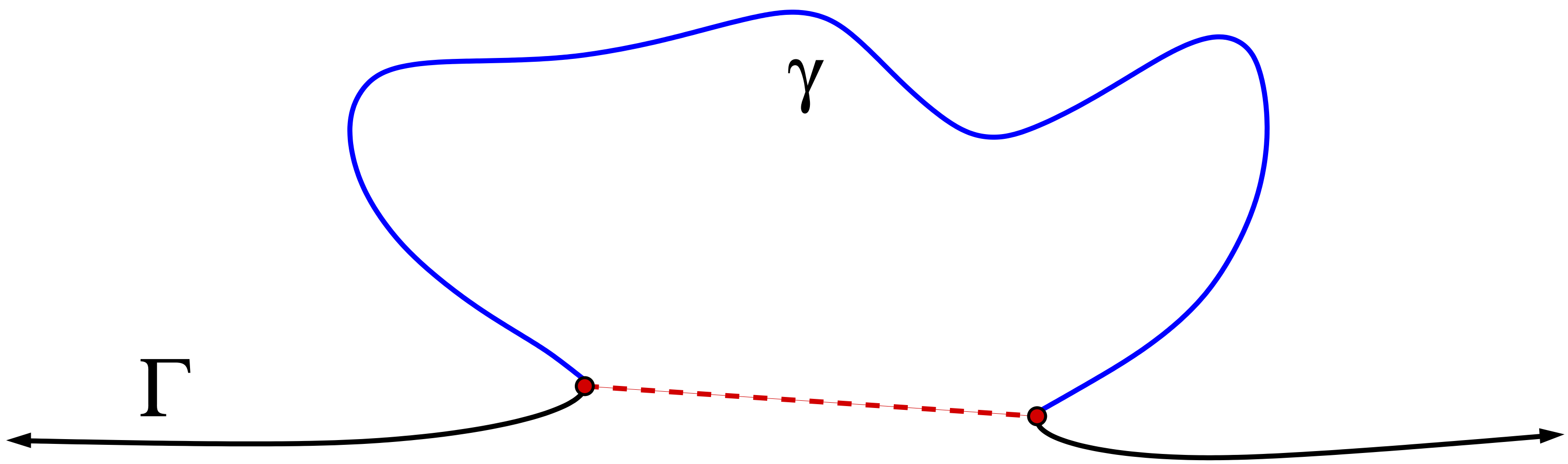
Γ is a quasicircle iff $\mathrm{diam}(\gamma) = O(\mathrm{crd}(\gamma))$ for all $\gamma \subset \Gamma$.



$\mathrm{crd}(\gamma) = |z - w|$, z, w, endpoints of γ.

A **quasicircle** is the image of circle under a quasiconformal map of $\mathbb{R}^2$.

Γ is a **chord-arc** iff $\ell(\gamma) = O(\mathrm{crd}(\gamma))$ for all $\gamma \subset \Gamma$.



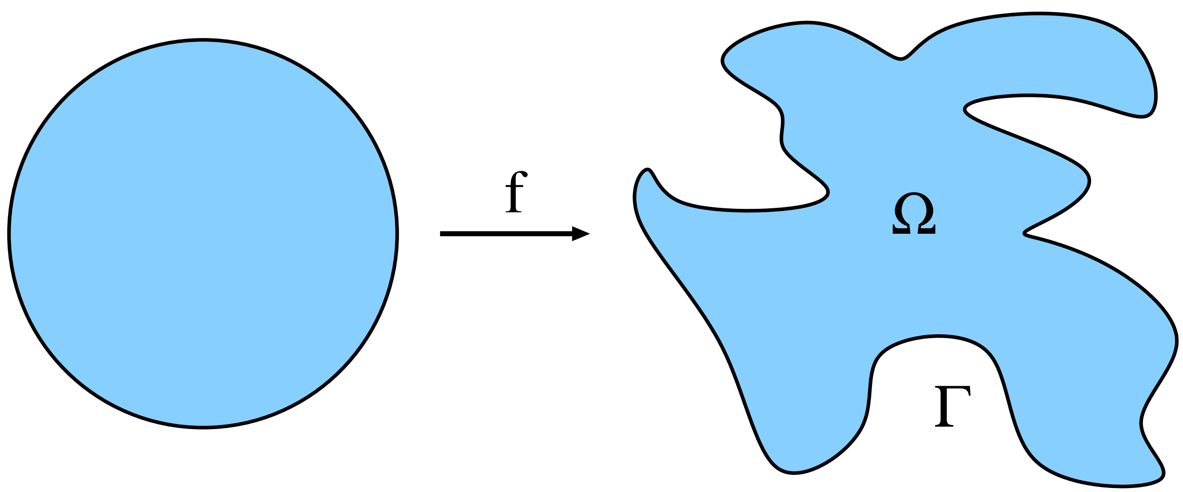
Chord-arc curves = biLipschitz images of circle.

A **quasicircle** is the image of circle under a quasiconformal map of $\mathbb{R}^2$.

$$\begin{aligned} T(1) &= \textbf{Universal Teichmüller space} \\ &= \text{quasicircles (modulo similarities).} \end{aligned}$$

A **Weil-Petersson curve** is $\Gamma = f(\mathbb{T})$ where f is quasiconformal on the plane, conformal outside $\mathbb{D}$, and $\mu \in L^2(dA_\rho)$.

$dA_\rho = \frac{dxdy}{(1-|z|^2)^2}$ = hyperbolic area.

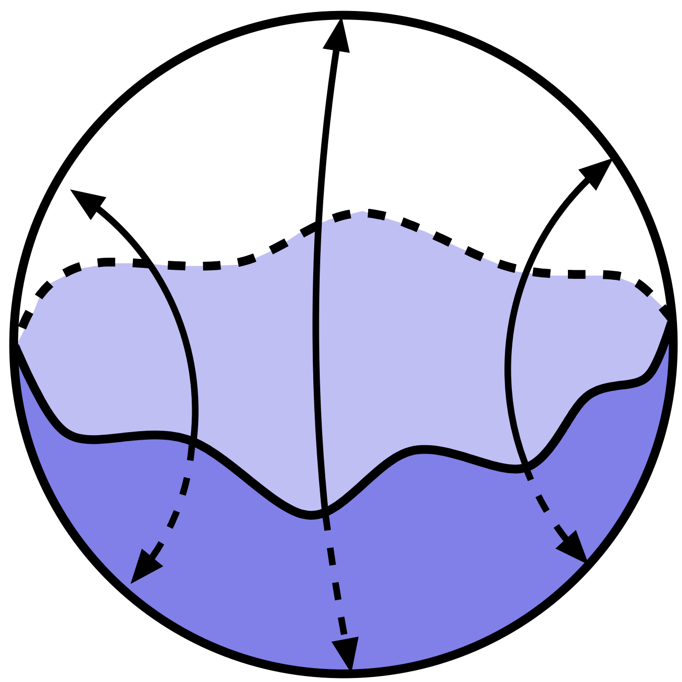


Quasicircles correspond to μ in open unit ball of L^∞.

WP corresponds to L^2 intersected with open unit ball of L^∞

A **Weil-Petersson curve** is $\Gamma = f(\mathbb{T})$ where f is quasiconformal on the plane, conformal outside $\mathbb{D}$, and $|\mu| \in L^2(dA_\rho)$.

$\Leftrightarrow$ Γ is fixed by QC involution with $\mu \in L^2$ for hyperbolic area on $\mathbb{S}^2 \setminus \Gamma$.



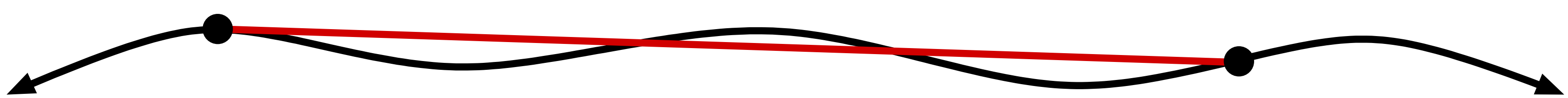
This extends to higher dimensions using biLipschitz involutions.

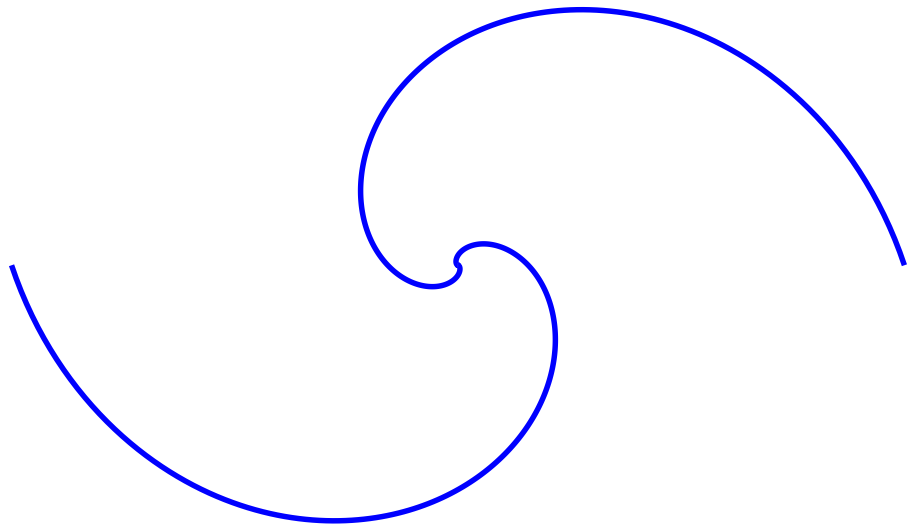
A **Weil-Petersson curve** is $\Gamma = f(\mathbb{T})$ where f is quasiconformal on the plane, conformal outside $\mathbb{D}$, and $|\mu| \in L^2(dA_\rho)$.

Every smooth curve is Weil-Petersson. Every WP curve is chord-arc.

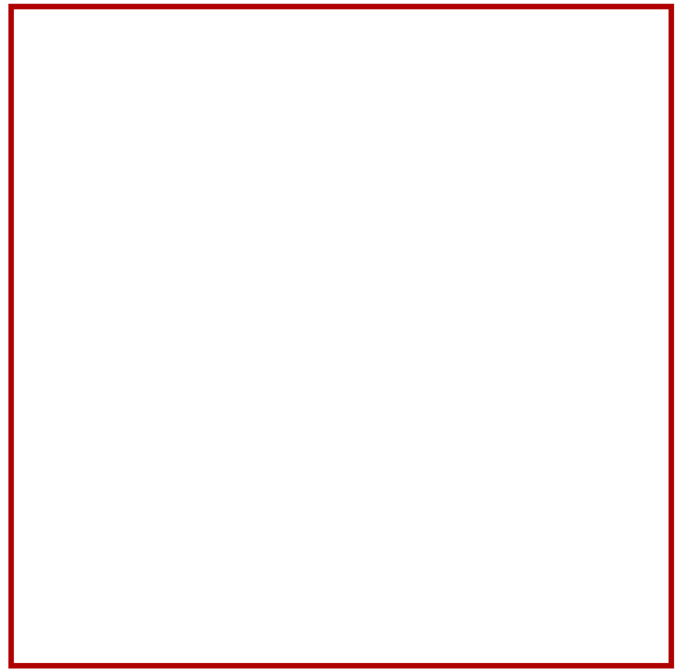
Weil-Petersson curves are almost C^1 (but not quite).

WP $\Rightarrow$ Asymptotically smooth $= \gamma \subset \Gamma$, $\ell(\gamma) \to 0$ implies $\frac{\ell(\gamma)}{\mathrm{crd}(\gamma)} \to 1$.





Weil-Petersson curves need not be C^1.
$z(t) = \exp(-t + i \log t)$, infinite spiral.

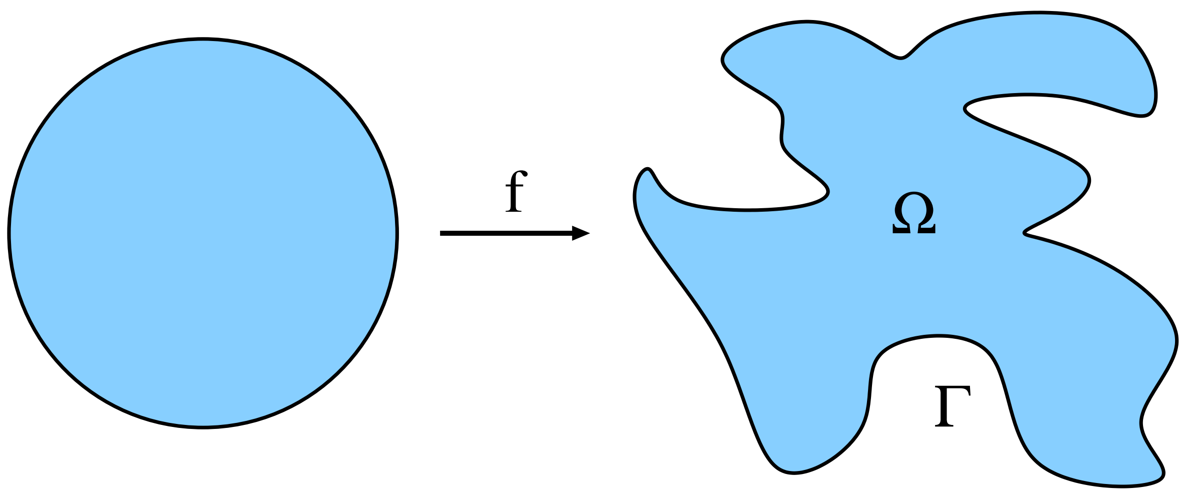


Not Weil-Petersson

In their memoir, Takhtajan and Teo prove:

Theorem: Γ is Weil-Petersson iff $\Gamma = f(\mathbb{T})$ where f is conformal map on $\mathbb{D}$ so that $u = \log f'$ is in the Dirichlet class.

Dirichlet class $= \{u : |\nabla u| \in L^2(dxdy)\} = \{u \in W^{1,2}(\mathbb{D})\}$.



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Yilin Wang proved the Dirichlet norm is finite iff **Loewner energy** of Γ is finite (defined by her and Steffen Rohde). This gives connections to large deviation theory of SLE (Schramm-Loewner evolutions) and the Brownian Loop Soup of Lawler and Werner.






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In 1990's Astala, Zinsmeister invented "BMO-Teichmüller theory."

P. Jones and I characterized curves where $\log f' \in$ BMO.

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Dirichlet class $= \{u : |\nabla u| \in L^2(dxdy)\} = \{u \in W^{1,2}(\mathbb{D})\}$.

Corollary: Γ is WP iff $u = \log f' \in H^{1/2}(\mathbb{T})$ (Sobolev trace thm).

$H^{1/2}$ = Sobolev space = half a derivative in L^2.

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|u(z) - u(w)|^2}{|z - w|^2} |dz||dw| < \infty.$$

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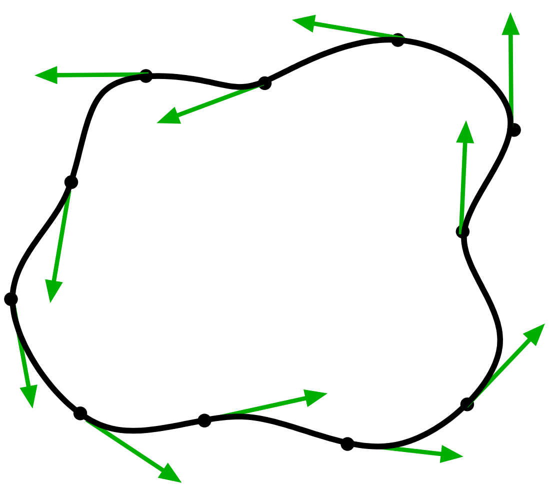
Corollary: Γ is WP iff $u = \log f' \in H^{1/2}(\mathbb{T})$ (Sobolev trace thm).

Easy to show:

$$\begin{aligned} \log f' \in H^{1/2} &\Rightarrow \arg f' \in H^{1/2} \\ &\Rightarrow \exp(i \arg f') \in H^{1/2} \\ &\Rightarrow f'/|f'| \in H^{1/2} \end{aligned}$$

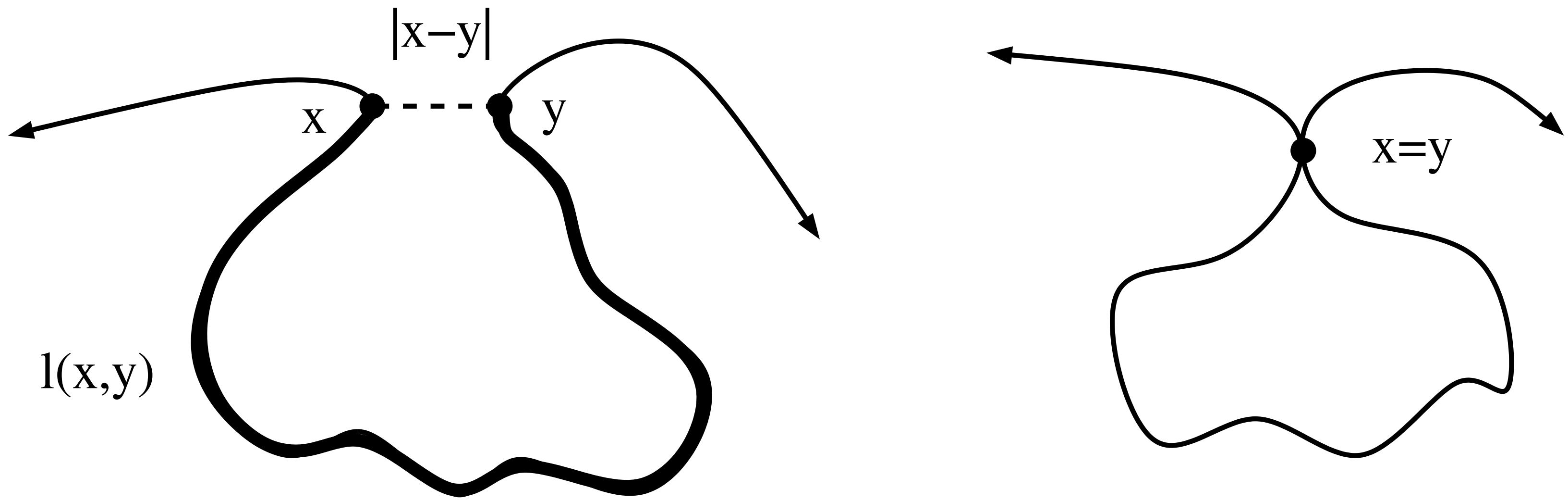


Theorem: Γ is WP iff the arc-length parameterization is in $H^{3/2}(\mathbb{T})$.

Theorem: Γ is WP iff the arc-length parameterization is in $H^{3/2}$.

This was proven implicitly in early version of my paper, but I didn't notice. David Mumford pointed out it follows from other characterizations.

$H^{3/2}$ curves arise in other areas, e.g., knot theory.



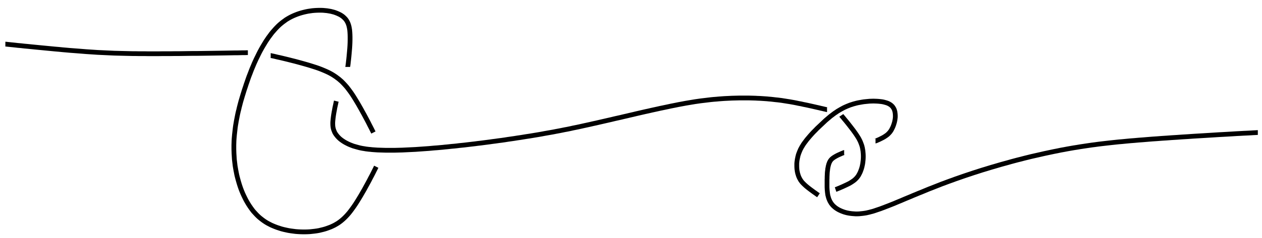
The Möbius energy of a curve $\Gamma \in \mathbb{R}^n$ is

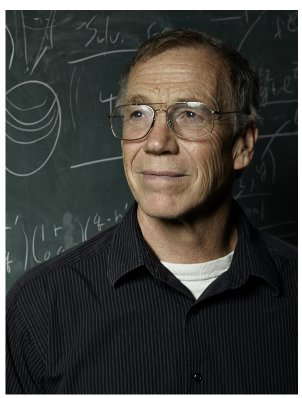
$$\text{Möb}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|x-y|^2} - \frac{1}{\ell(x,y)^2} \right) dx dy.$$

Blows up if curve self-intersects. Represents renormalized “self-repelling” force.

Möbius energy is one of several "knot energies" due to Jun O'Hara.

Studied by Freedman, He and Wang. They showed:
- Möb(Γ) is Möbius invariant (hence the name),
- that finite energy curves are chord-arc,
- and in $\mathbb{R}^3$ they are topologically tame.





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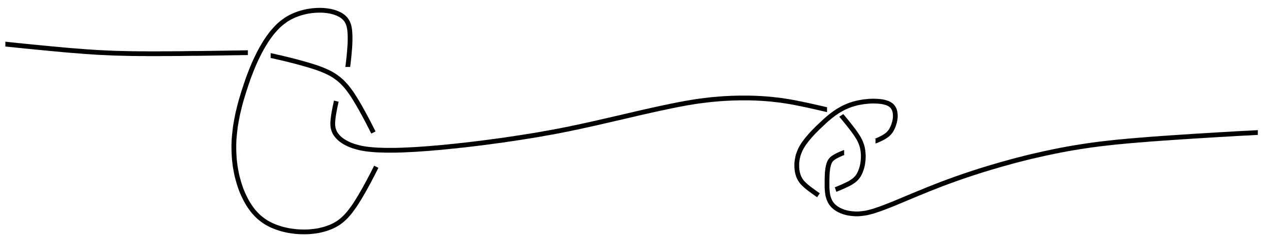
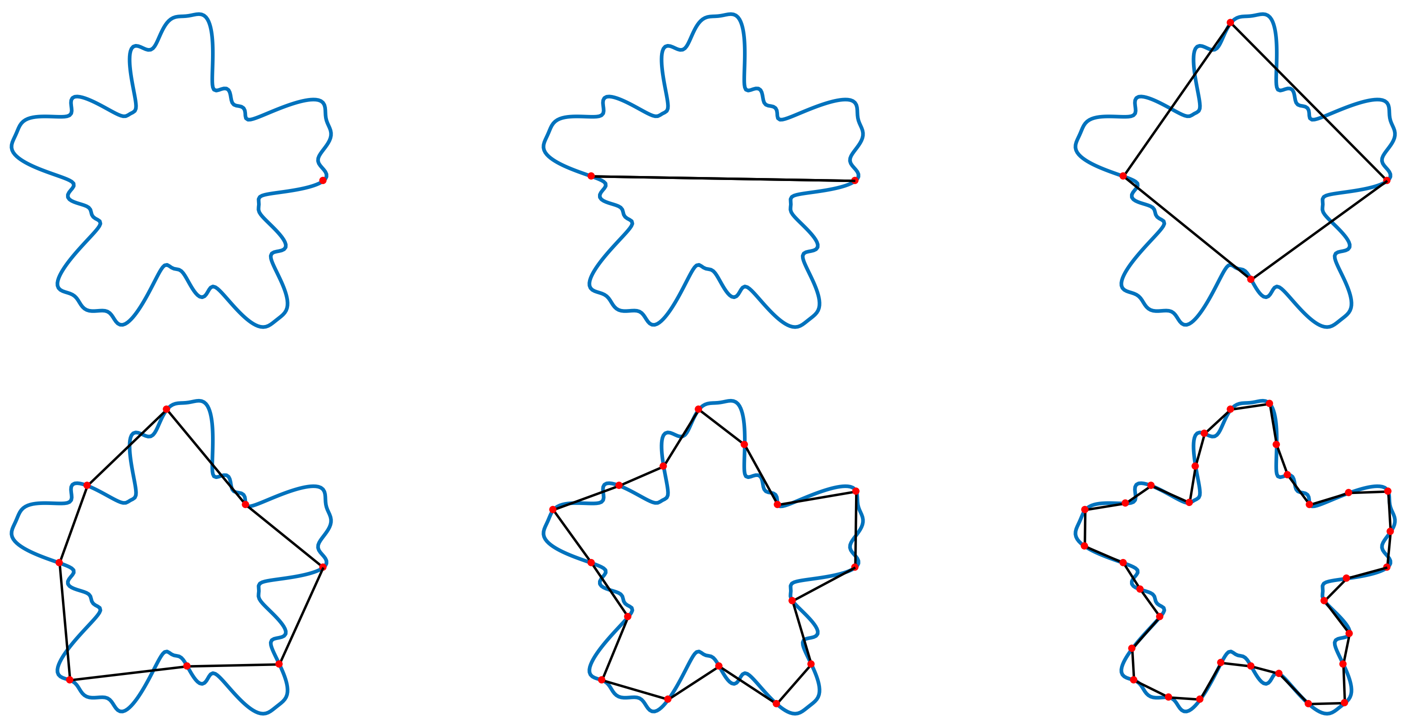


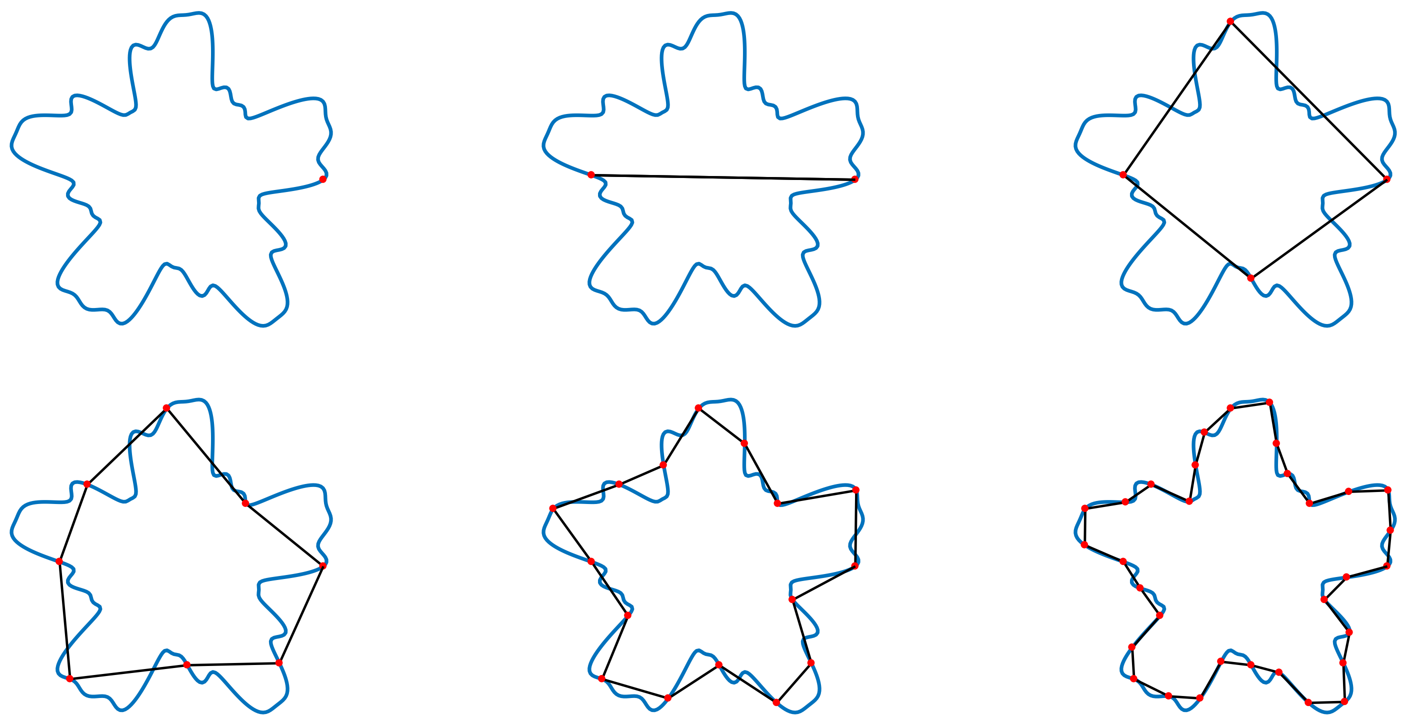
Theorem (Blatt): Möb(Γ) $< \infty$ iff arclength parameterization is $H^{3/2}$.

Thus WP curve = finite Möbius energy.

Dyadic decomposition. Choose a base point $z_1^0 \in \Gamma$ and forenrch $n \geq 1$, let $\{z_j^n\}$, $j = 1, \ldots, 2^n$ be the unique set of ordered points with $z_1^n = z_1^0$ that divides Γ into 2^n equal length intervals (called the nth generation dyadic subintervals of Γ).



Let Γ_n be the inscribed 2^n-gon with these vertices. Clearly $\ell(\Gamma_n) \nearrow \ell(\Gamma)$.



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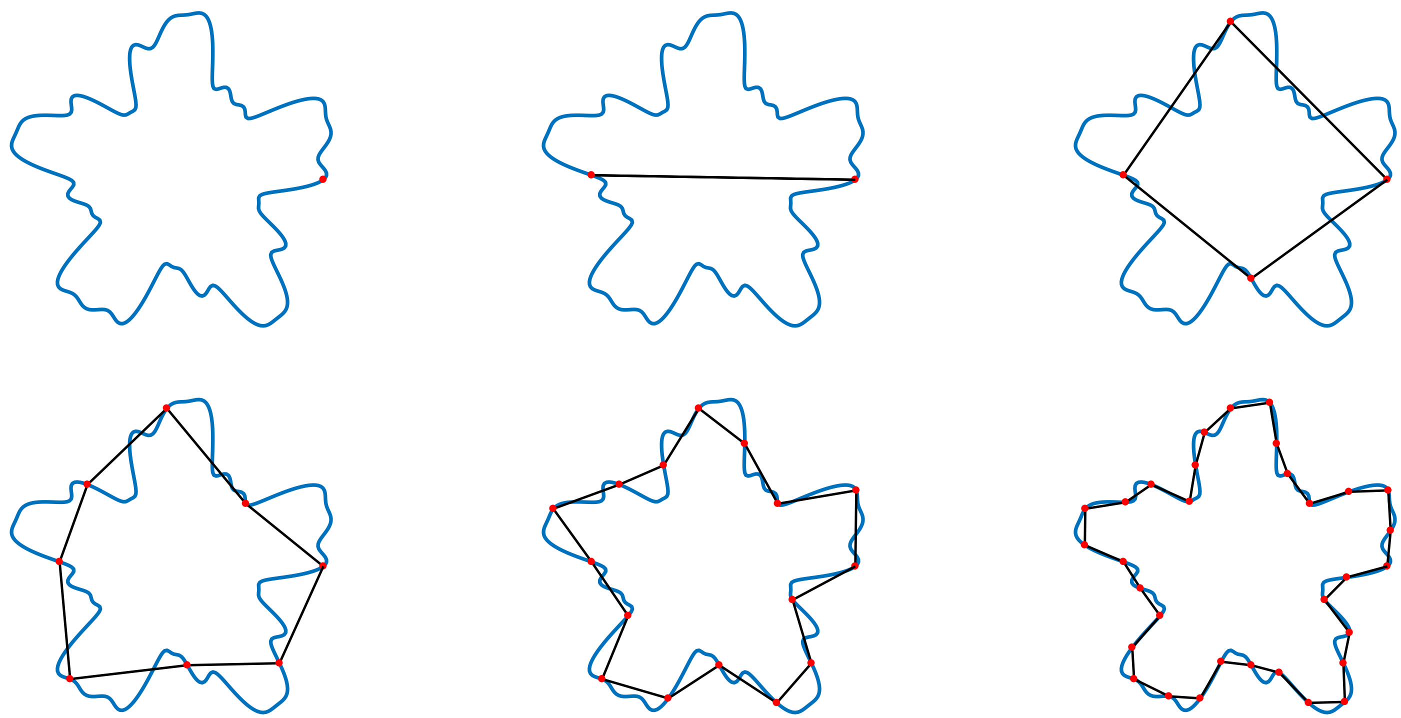


Theorem: Γ is Weil-Petersson if and only if

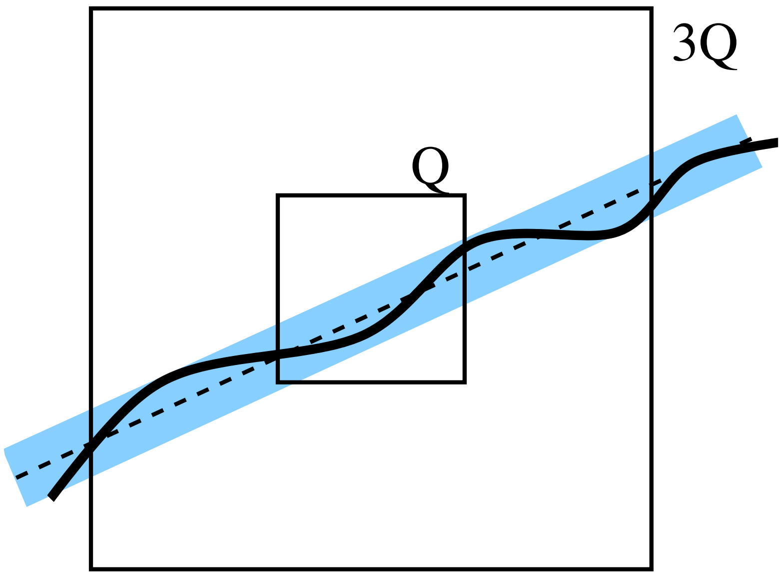
$$\sum_{n=1}^{\infty} 2^n \left[\ell(\Gamma) - \ell(\Gamma_n)\right] < \infty$$

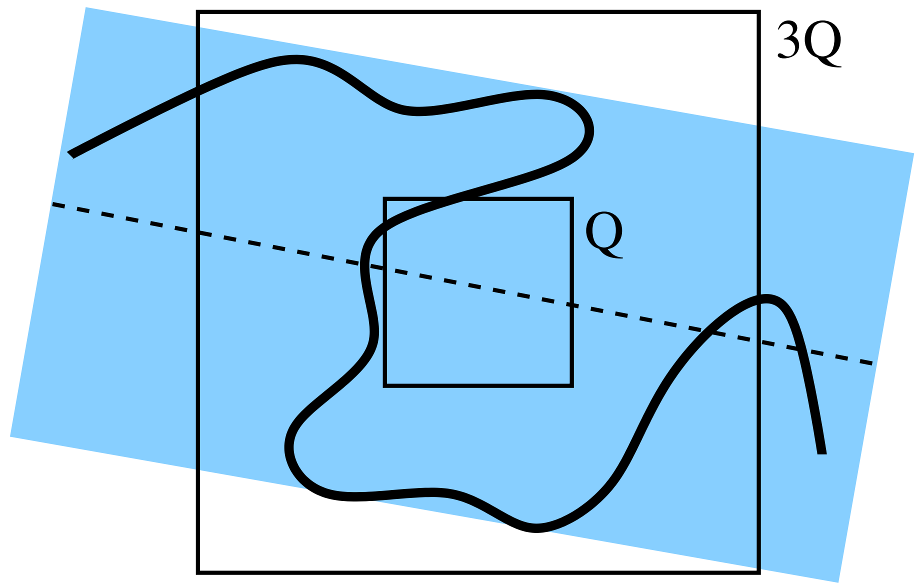
with a bound that is independent of the choice of the base point.

Peter Jones's β-numbers:

$$\beta_\Gamma(Q) = \inf_L \sup\{\frac{\text{dist}(z, L)}{\text{diam}(Q)} : z \in 3Q \cap \Gamma\},$$

where the infimum is over all lines L that hit $3Q$.

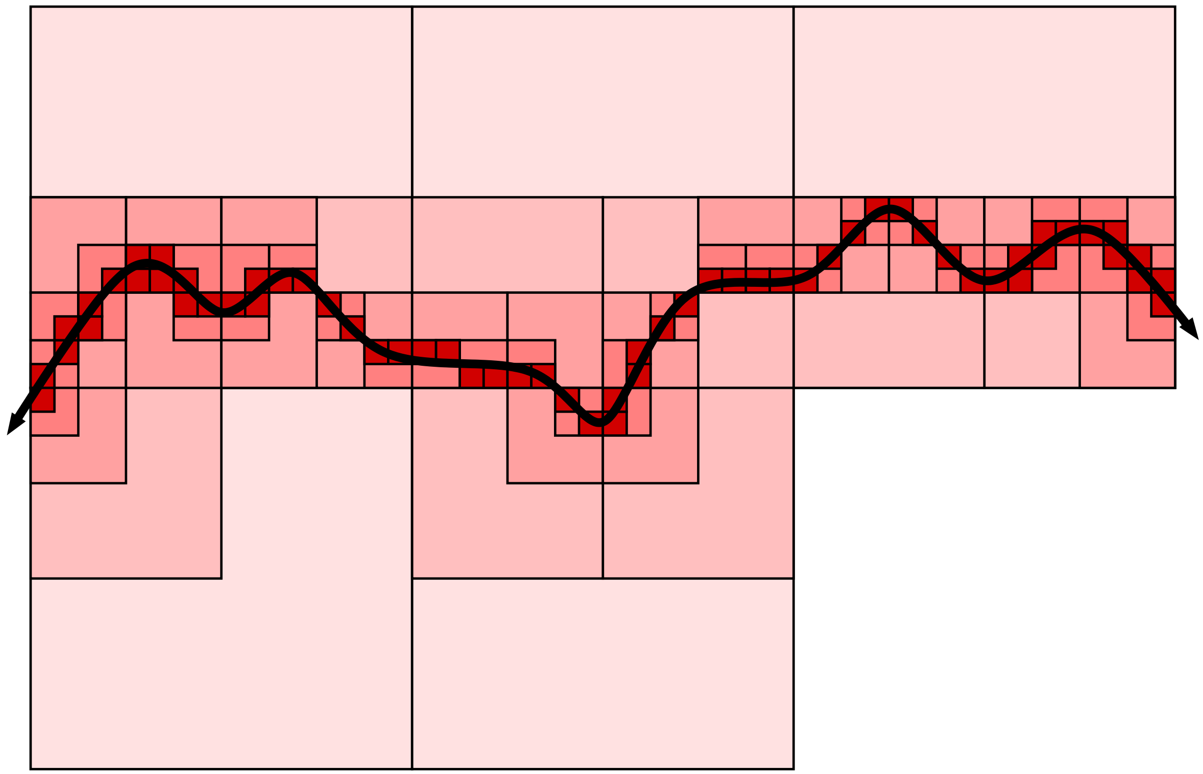




Jones invented the β-numbers as part of his traveling salesman theorem:

$$\ell(\Gamma) \simeq \operatorname{diam}(\Gamma) + \sum_{Q} \beta_\Gamma(Q)^2 \operatorname{diam}(Q),$$

where the sum is over all dyadic cubes Q in $\mathbb{R}^n$.



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Theorem: Γ is Weil-Petersson iff

$$\sum_Q \beta_\Gamma(Q)^2 < \infty,$$

where the sum is over all dyadic cubes.

WP curves have "curvature in L^2, integrated over all positions and scales".

Jones invented the β-numbers as part of his traveling salesman theorem:

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Theorem: Γ is Weil-Petersson iff

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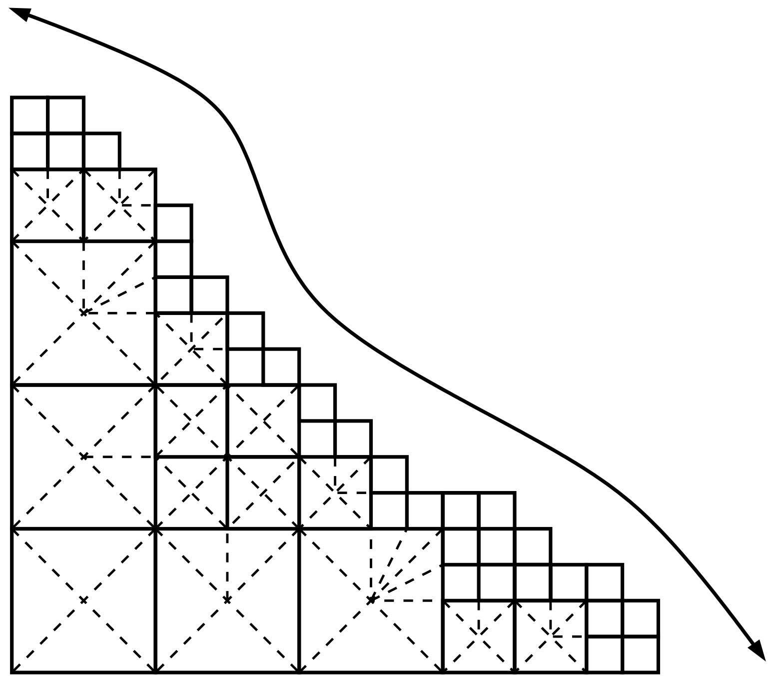
where the sum is over all dyadic cubes.

Proof requires improvement of TST for curves:

$$\ell(\Gamma) - \operatorname{crd}(\Gamma) \simeq \sum_Q \beta_\Gamma(Q)^2 \operatorname{diam}(Q),$$

Easy to see $\sum \beta^2 < \infty$ implies Weil-Petersson.



- Triangulate one side of Γ (e.g., triangulate Whitney squares).

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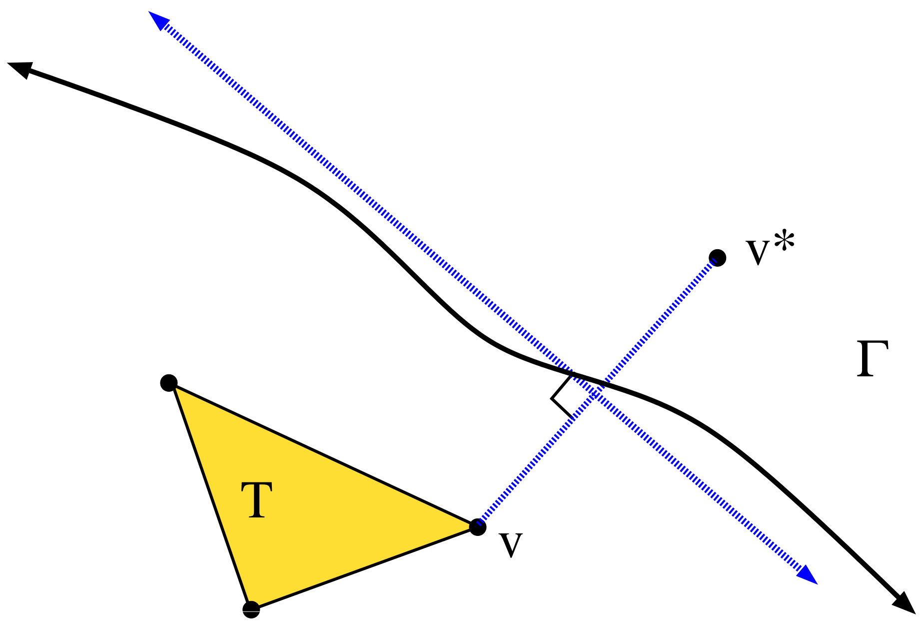

- Triangulate one side of Γ (e.g., triangulate Whitney squares).
- Use approximating lines to reflect vertices.

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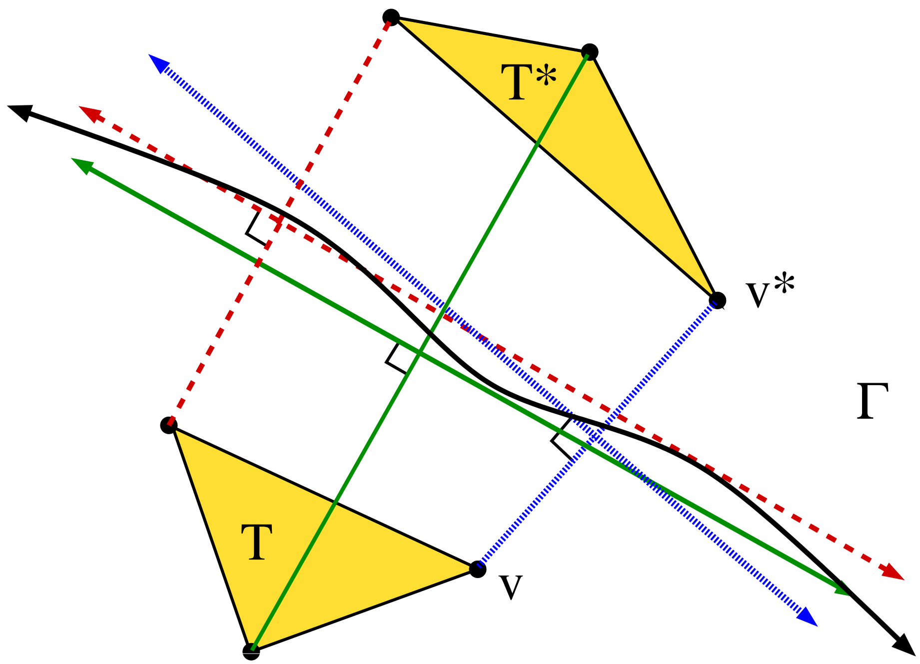


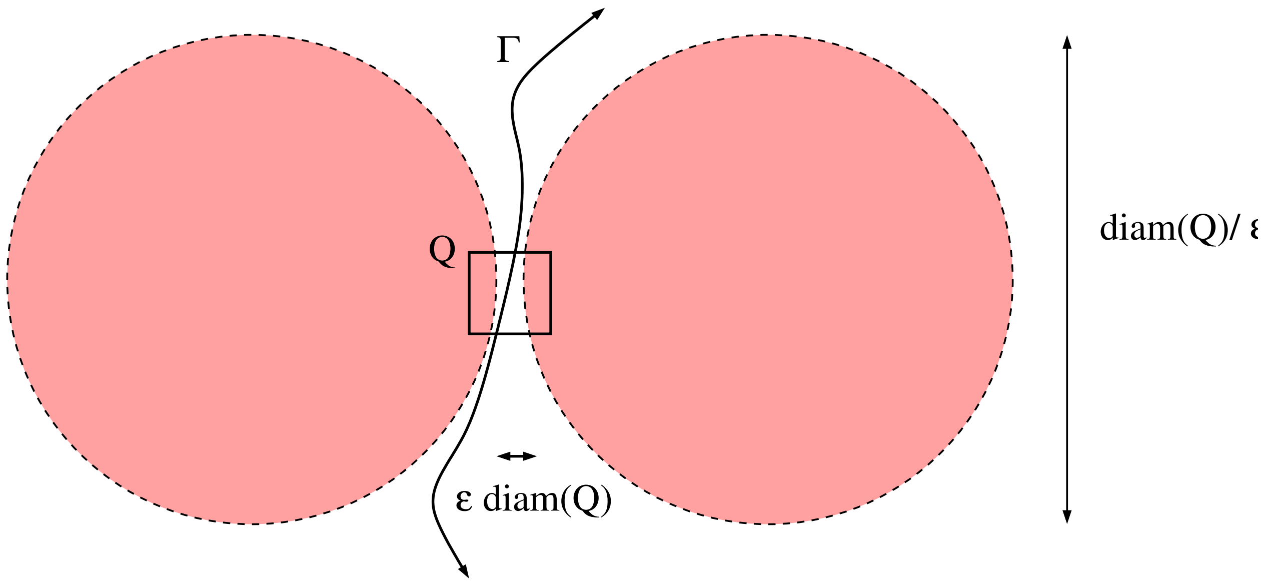
- Triangulate one side of Γ (e.g., triangulate Whitney squares).
- Use approximating lines to reflect vertices.
- Define piecewise linear map.
- $|\mu| = O(\beta)$.
- Get involution fixing Γ with $|\mu| \in L^2(dA_\rho) \Rightarrow$ Weil-Petersson.

The Weil-Petersson class is Möbius invariant.

The β-numbers are not.

What is a Möbius invariant version of the β-numbers?

For a dyadic square Q let $\varepsilon_\Gamma(Q)$ be the infimum of the $\epsilon \in (0,1]$ so that there are disks D, D' of radius $\ell(Q)/\epsilon$ on opposite sides of Γ so that $\mathrm{dist}(Q \cap D, Q \cap D') \leq \epsilon \ell(Q)$.

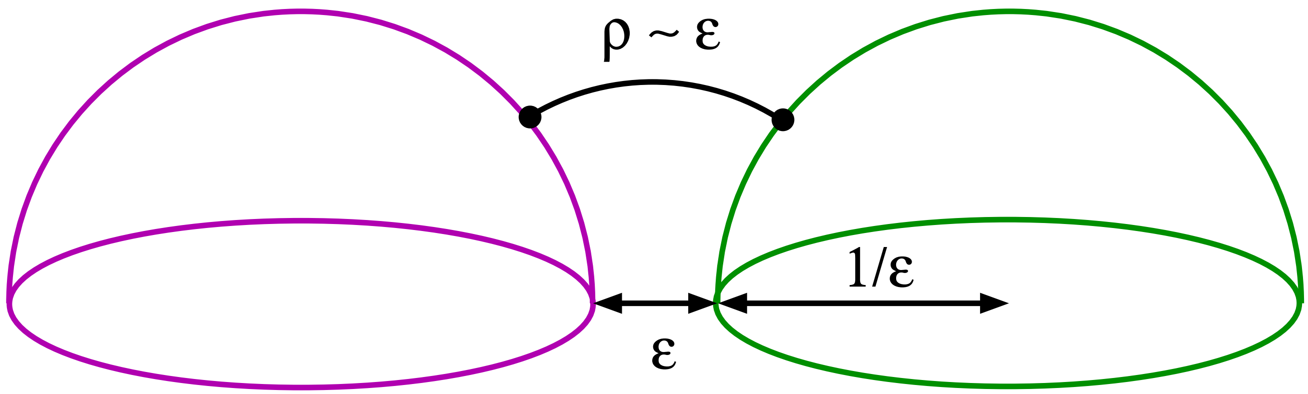


Easy to check $\beta(Q) \lesssim \varepsilon(Q)$. Converse can fail, but

Theorem: $\sum_Q \varepsilon^2(Q) < \infty$ iff $\sum_Q \beta^2(Q) < \infty$.

Each disk is the base of a hemisphere in the upper half-space $\mathbb{H}^3 = \mathbb{R}^3_+$.

The hyperbolic distance between these hemispheres is $\lesssim \varepsilon(Q)$.

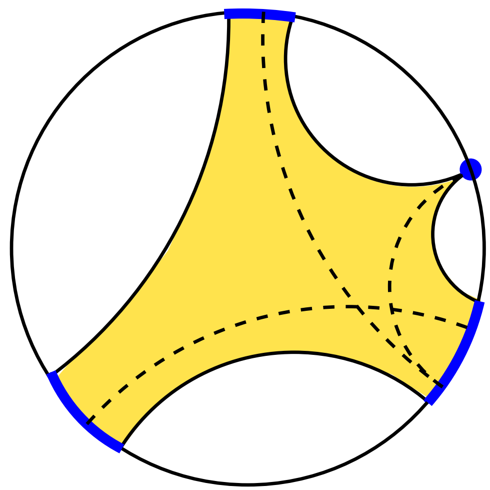


The hyperbolic length of a (Euclidean) rectifiable curve in the unit disk $\mathbb{D}$ or in the n-dimensional ball $\mathbb{B}^n$ is given by integrating

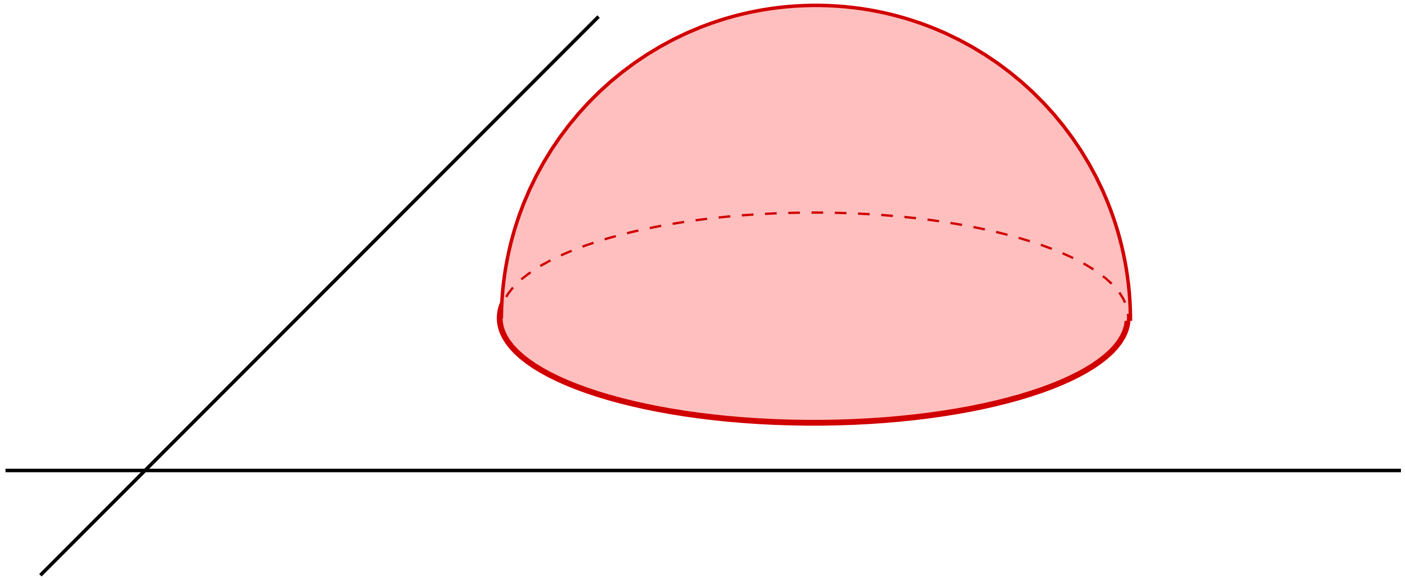
$$d\rho = \frac{ds}{1 - |z|^2},$$

along the curve. In the upper half-space $\mathbb{H}^n$ we integrate $d\rho = ds/2t$.

Geodesics are circles (or lines) perpendicular to boundary.



Convex hull of boundary set



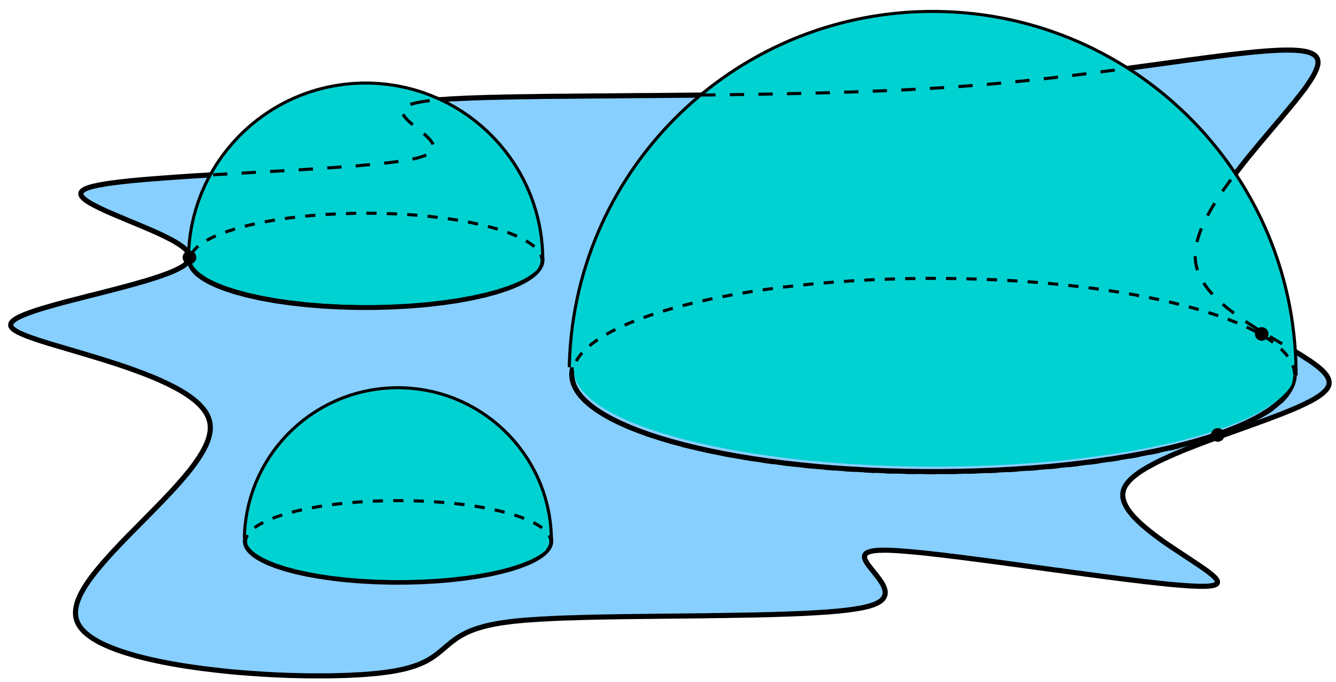
The hyperbolic convex hull of $\Gamma \subset \mathbb{R}^2$, denoted $\mathrm{CH}(\Gamma)$, is the smallest convex set in $\mathbb{R}^3_+$ that contains all (infinite) hyperbolic geodesics with both endpoints in Γ.

For a circle in plane, hyperbolic convex hull is a hemisphere.

In general, $\mathrm{CH}(\Gamma)$ has non-empty interior.

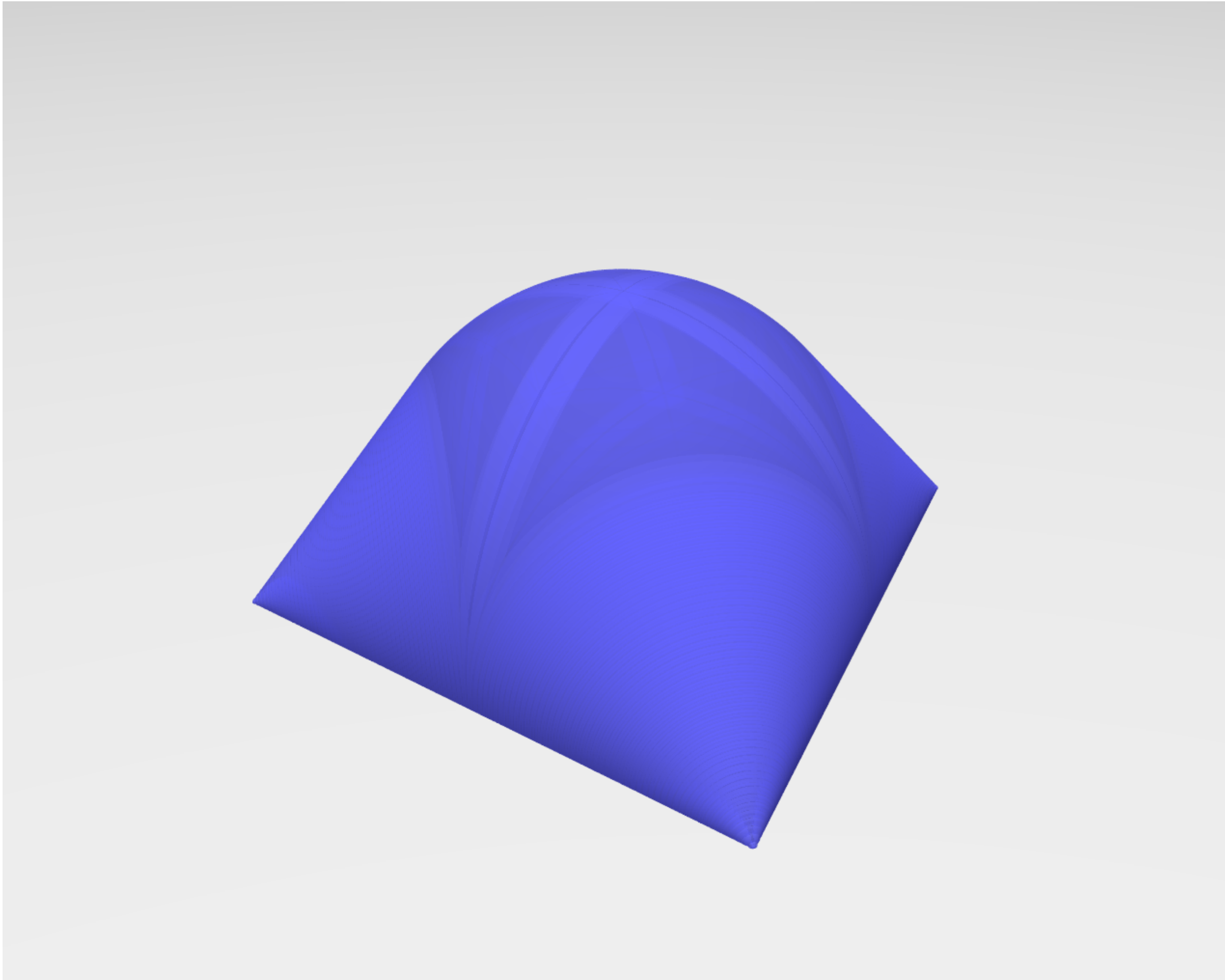
Suppose Ω is Jordan domain with boundary Γ.

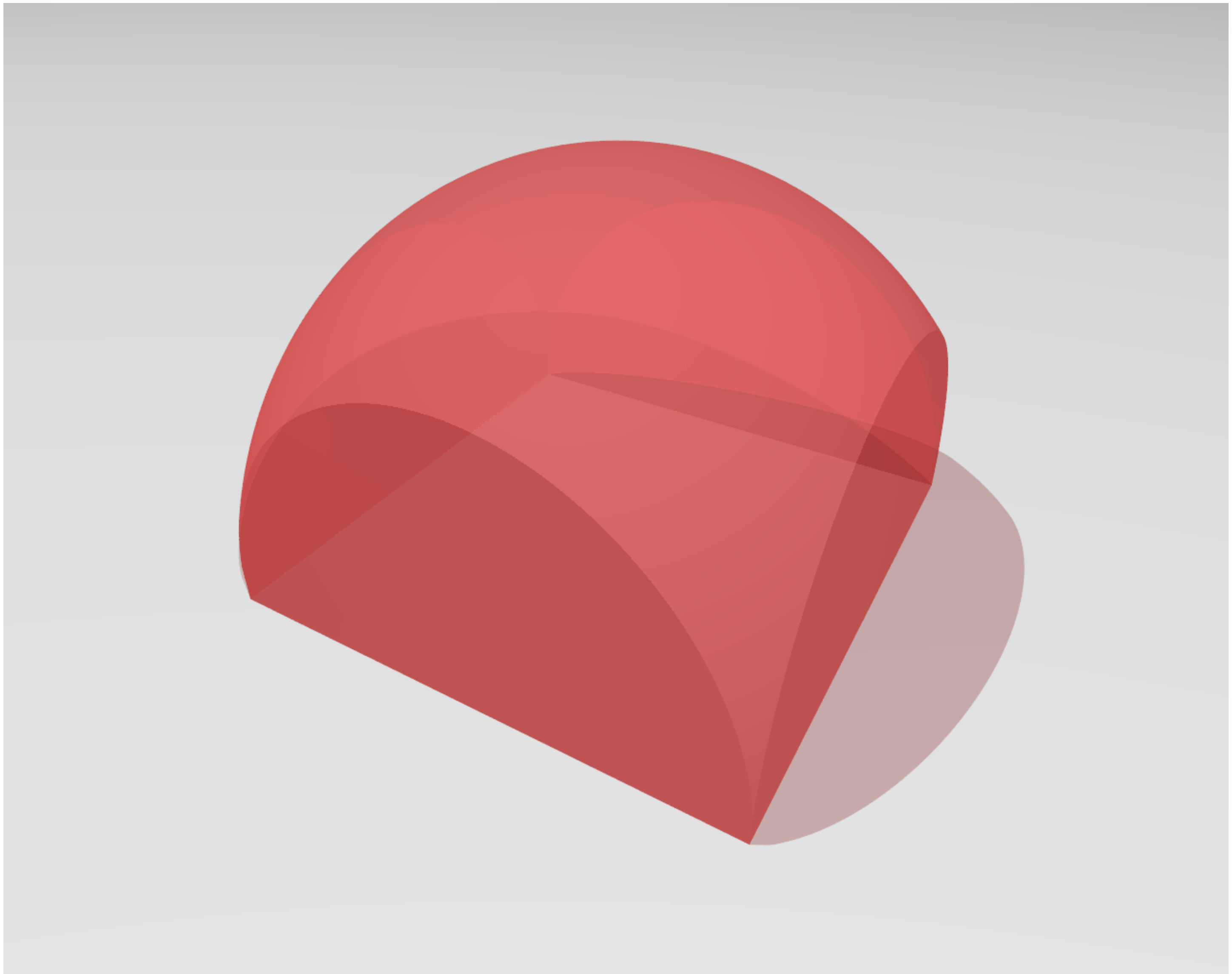
The dome of Ω is upper envelope of all hemispheres with base disk in Ω.

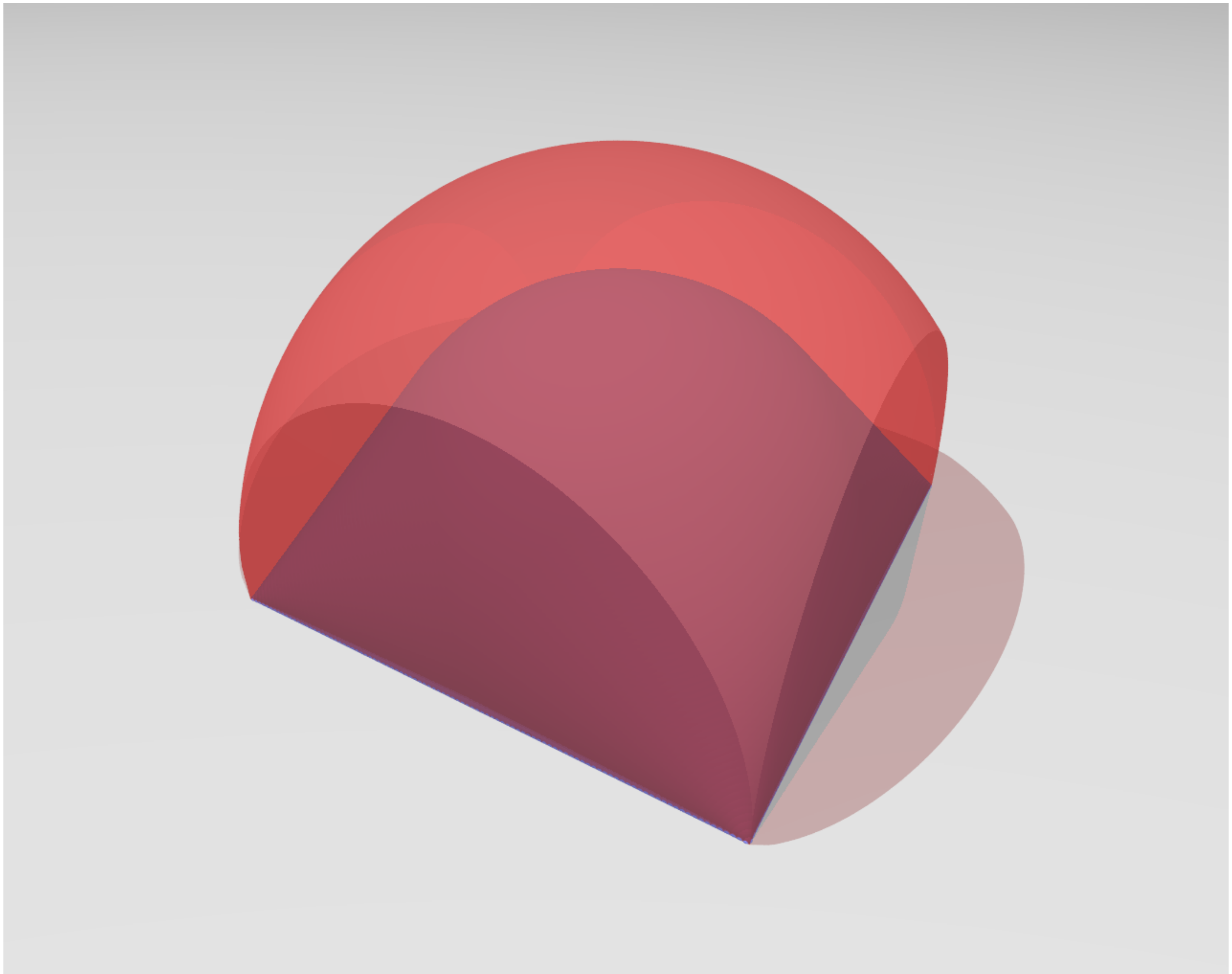


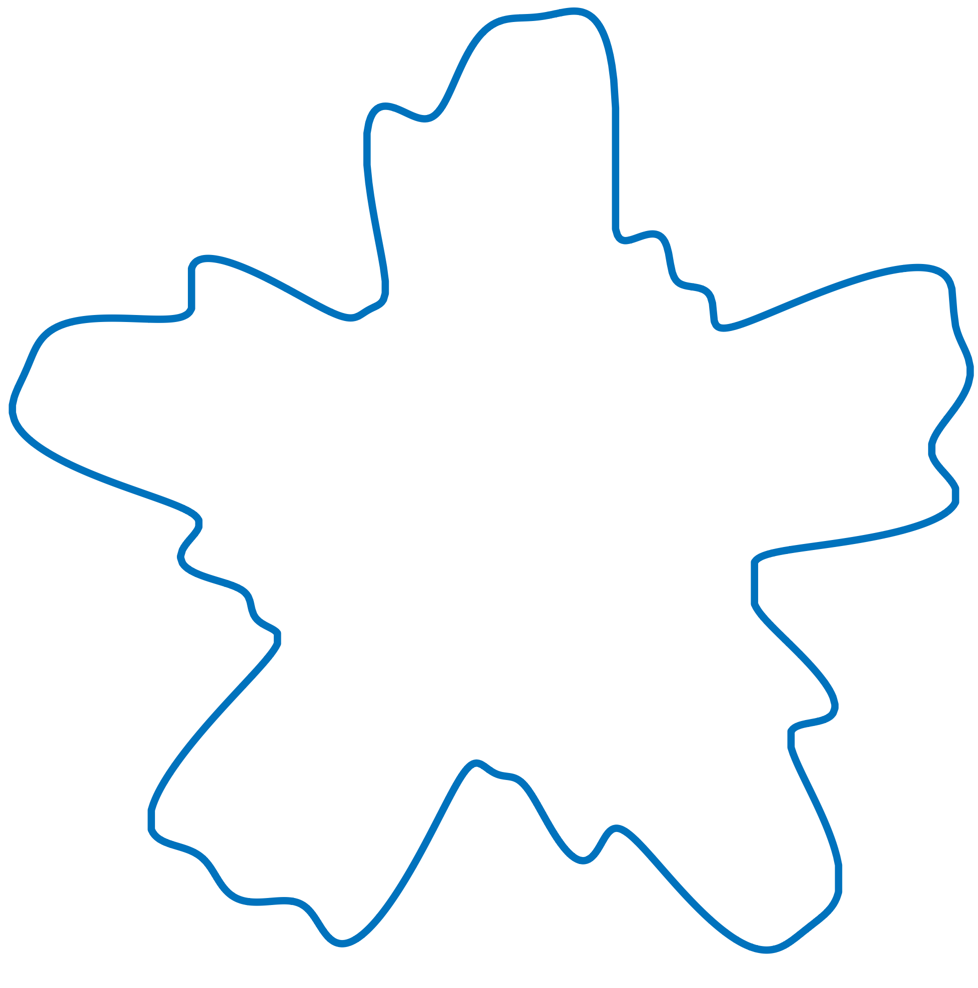
Region above dome is intersection of half-spaces, hence convex.

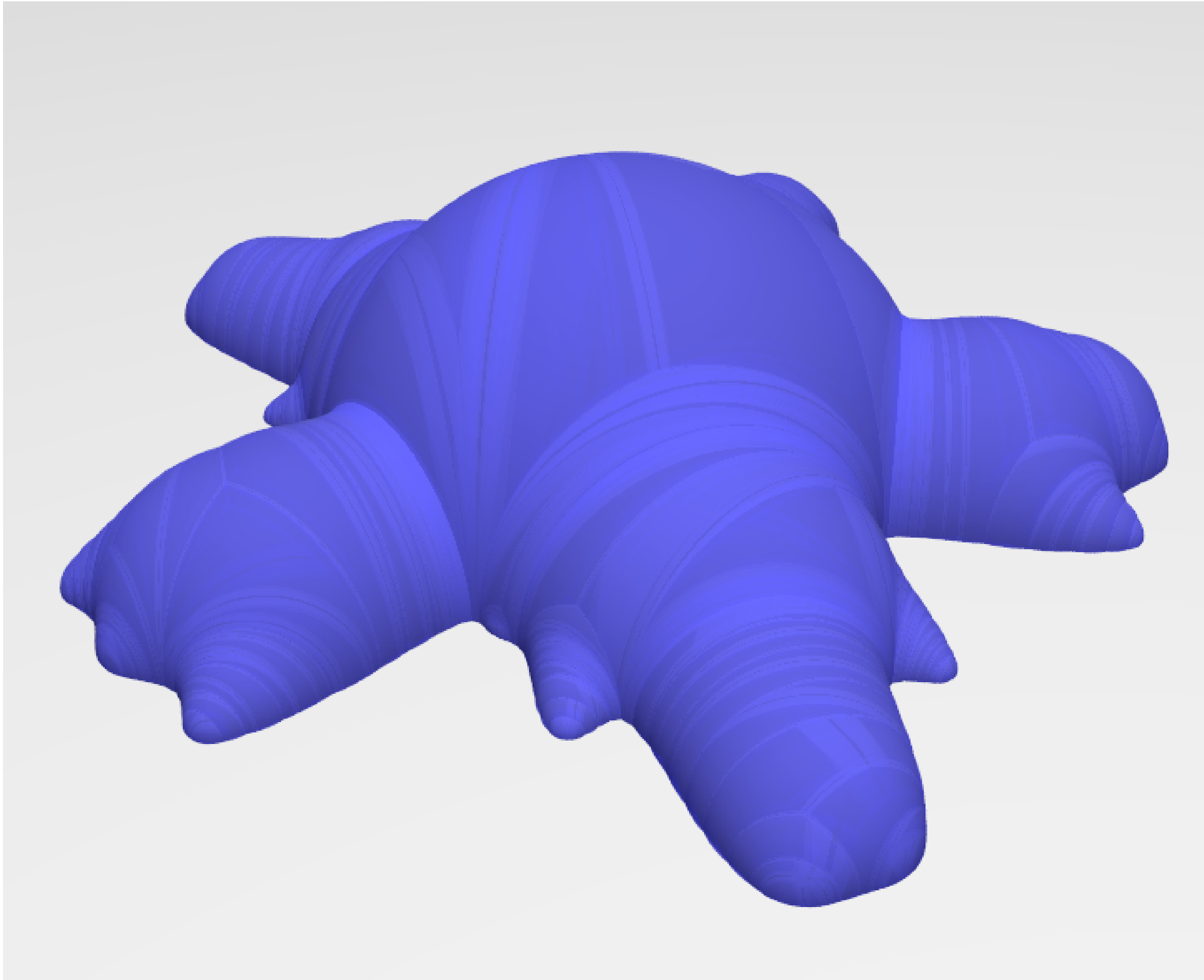
CH(Γ) is region between domes of "inside" and "outside" of Γ.

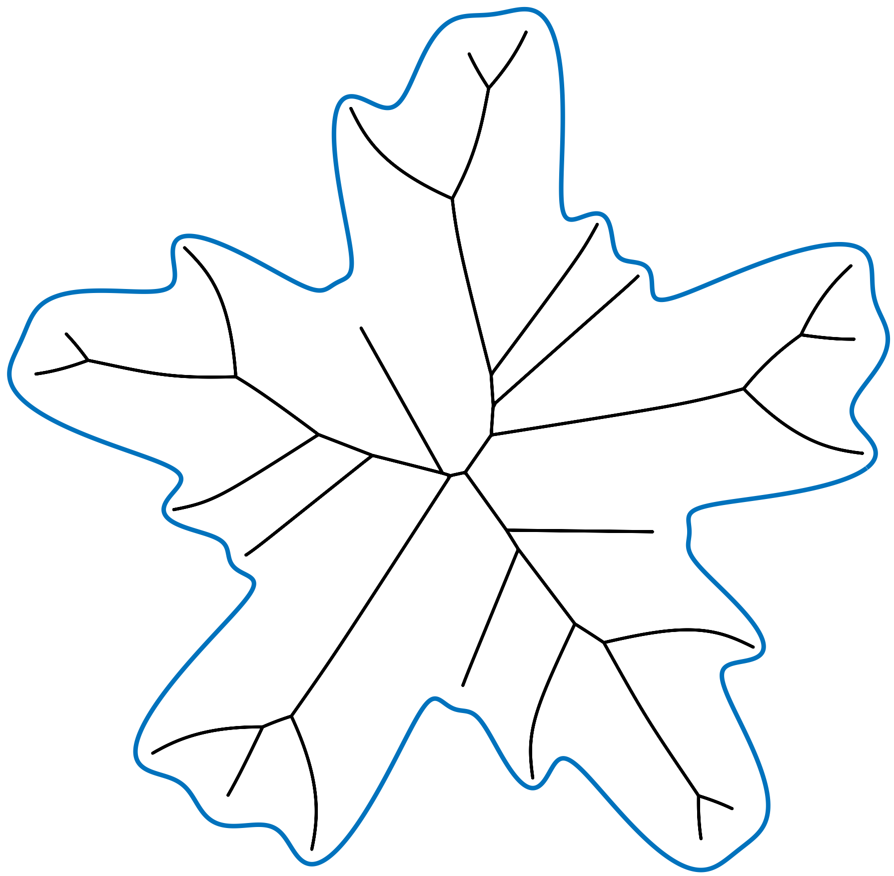




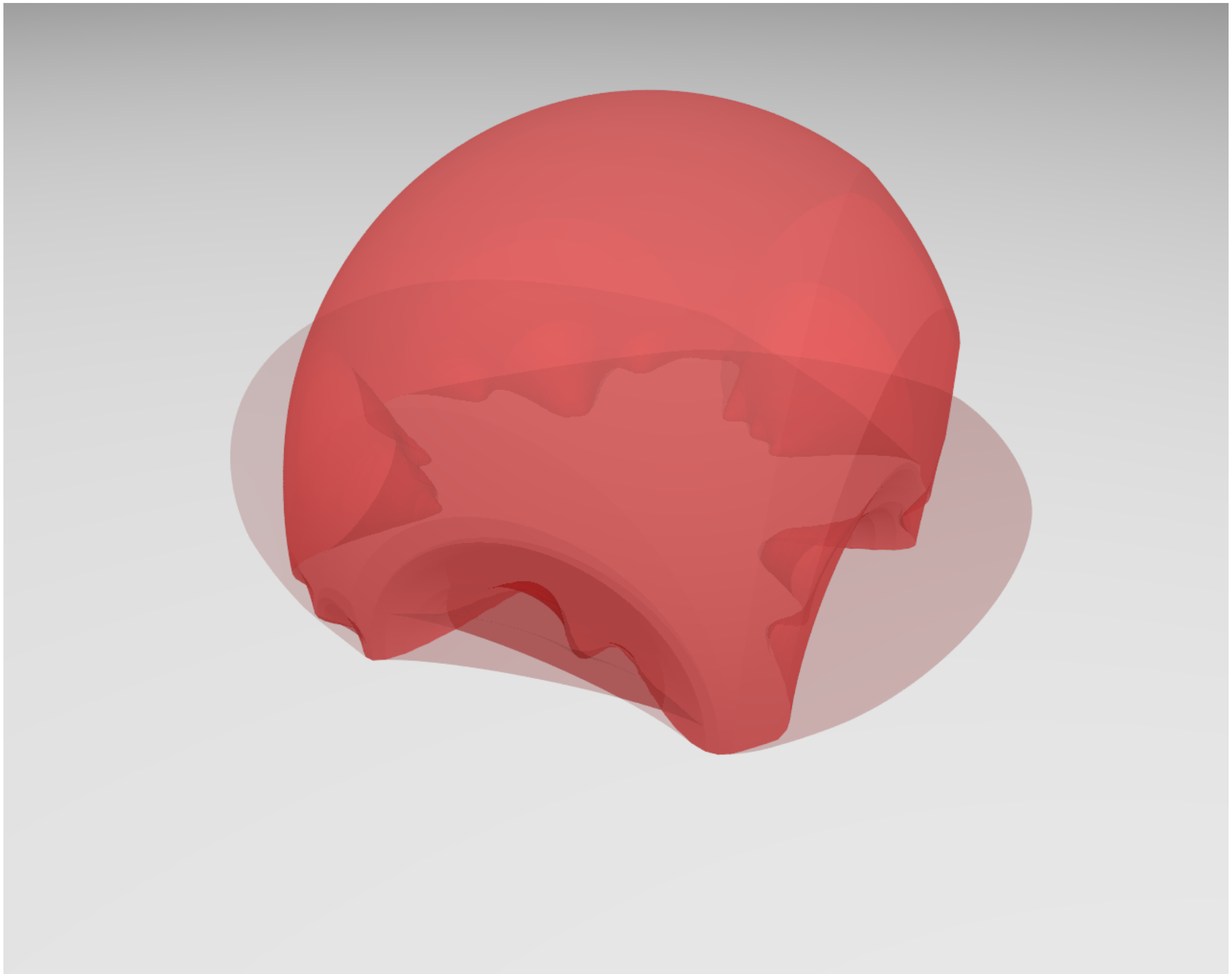


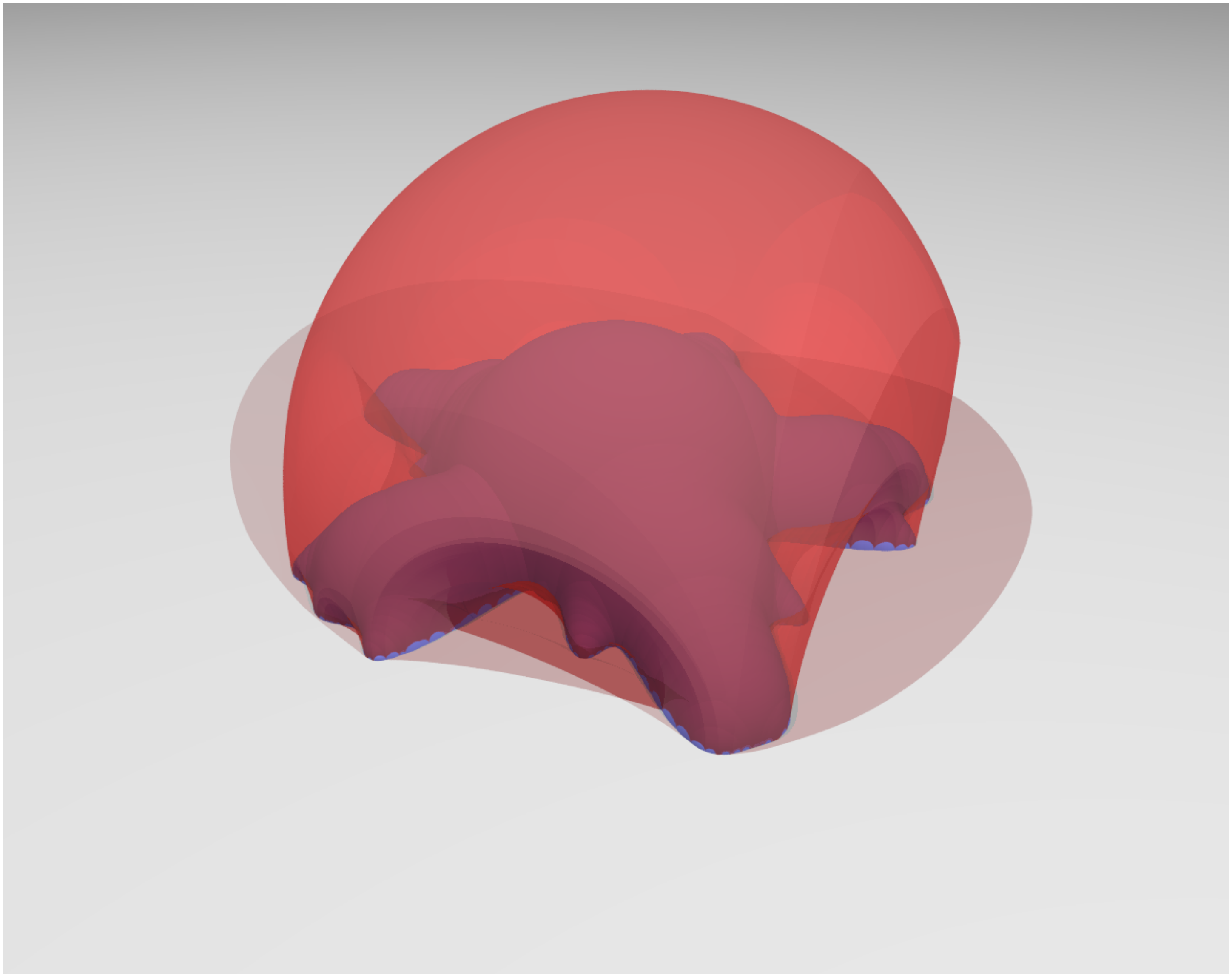


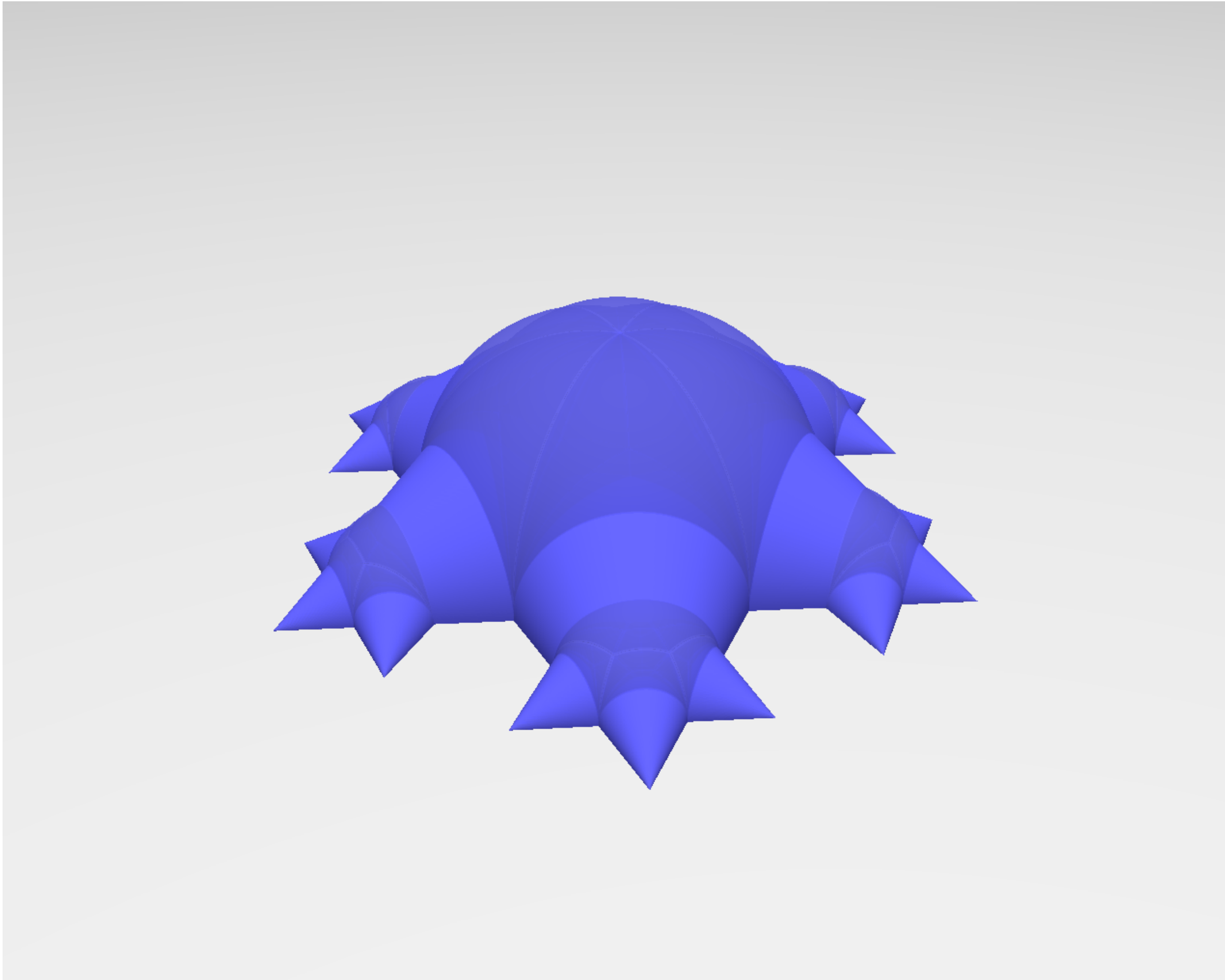


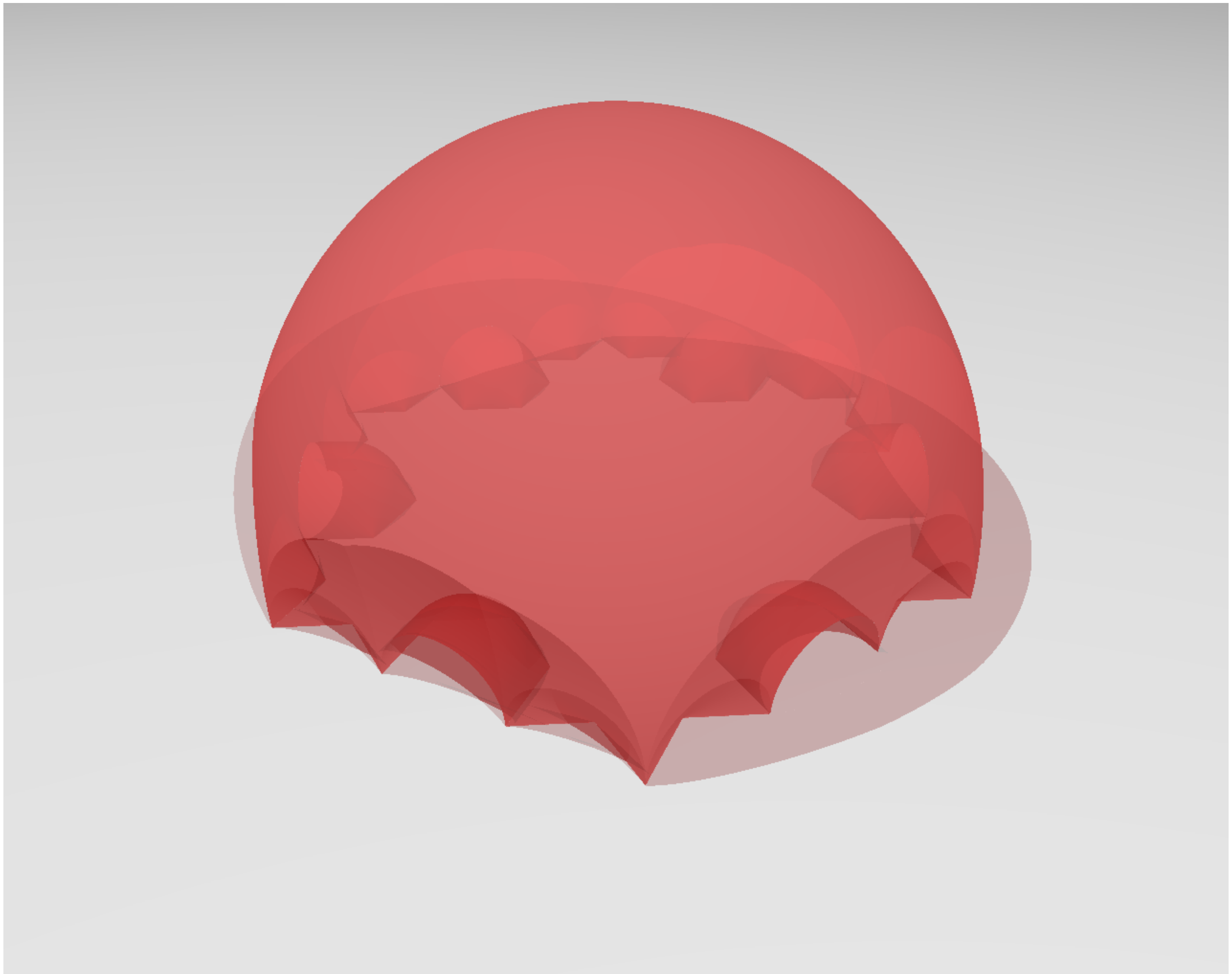


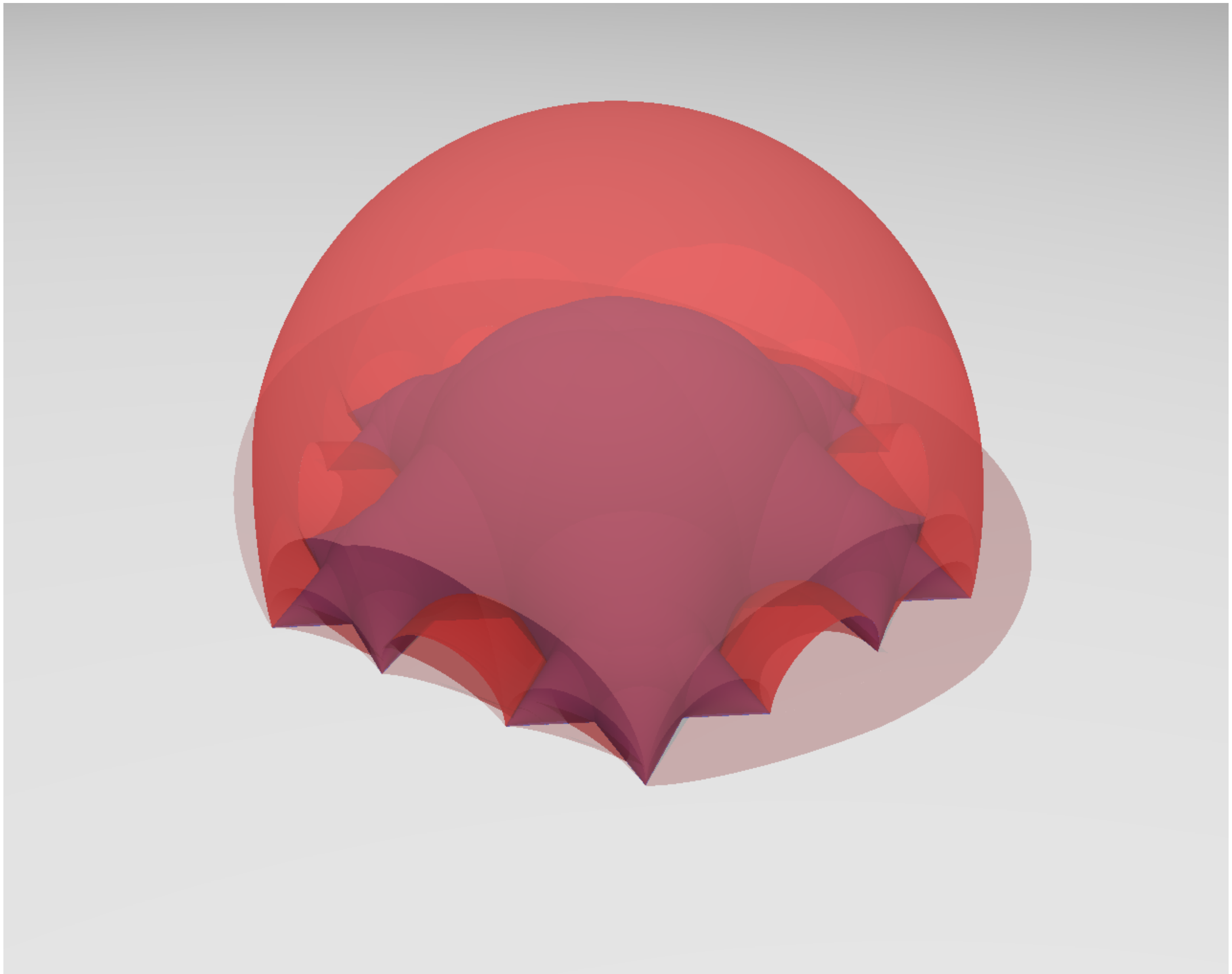
The medial axis. Equidistant from at least two boundary points.
Corresponding hemispheres give the dome.



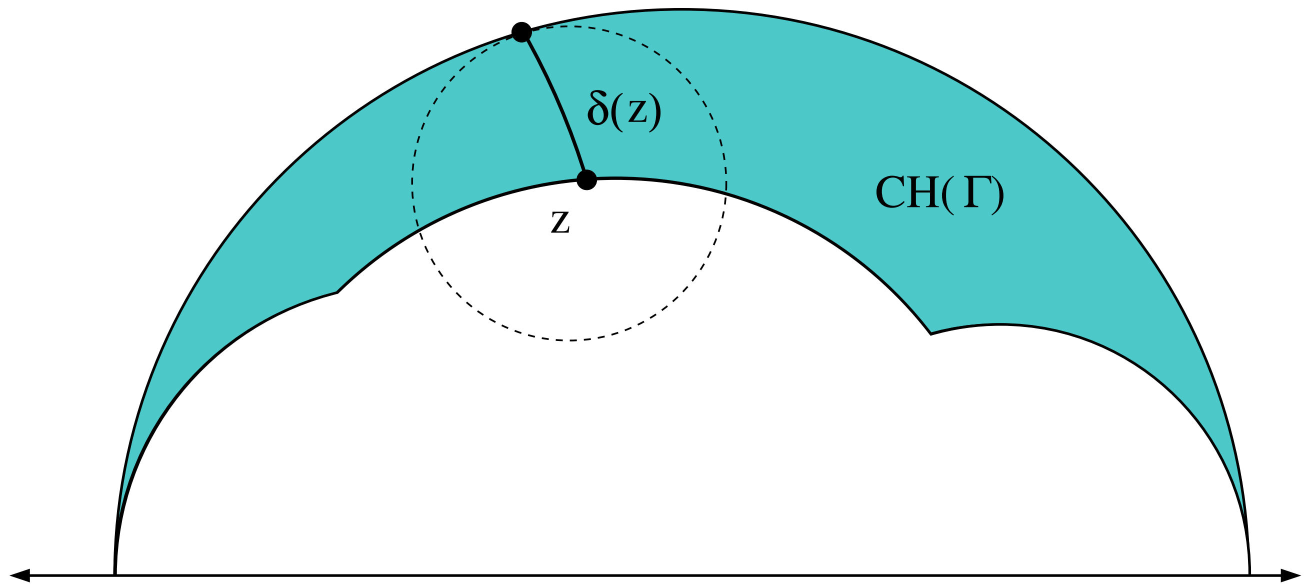








We define $\delta(z)$ to be the maximum of the hyperbolic distances from z to the two boundary components of $\mathrm{CH}(\Gamma)$.



Theorem: Γ is Weil-Petersson iff $\int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z) dA_\rho < \infty$.

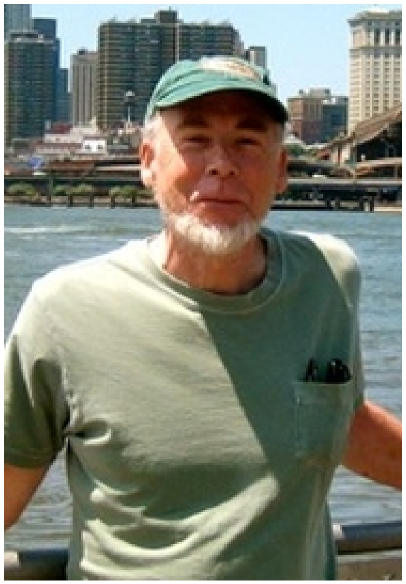
Let S be a surface in $\mathbb{H}^3$ that has asymptotic boundary Γ.

We let $K(z)$ denote the Gauss curvature of S at z.

Gauss equation says $K(z) = -1 + \kappa_1(z)\kappa_2(z)$ (principle curvatures).

S is a **minimal surface** if $\kappa_1 = -\kappa_2$ (the mean curvature is zero).

Theorem (Anderson, 1983): Every closed Jordan curve $\Gamma \subset \mathbb{R}^2$ bounds a minimal disk $S \subset \mathrm{CH}(\Gamma) \subset \mathbb{H}^3$.



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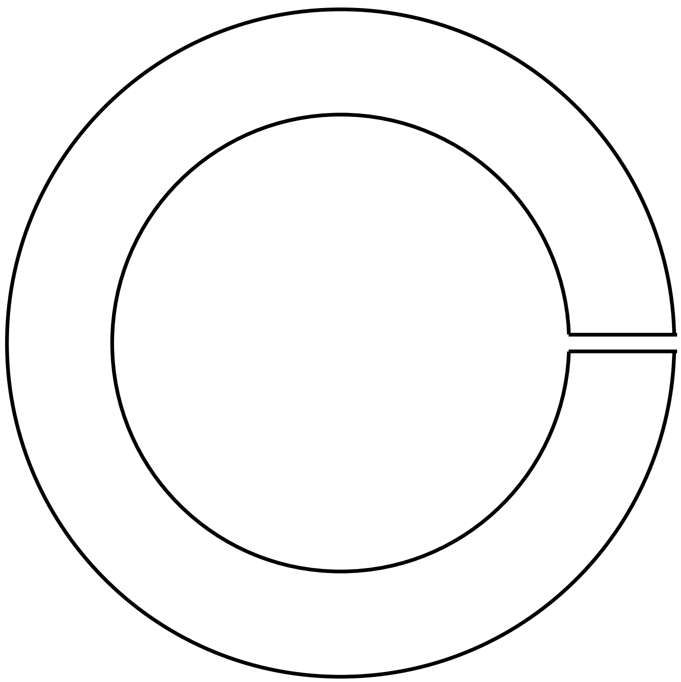

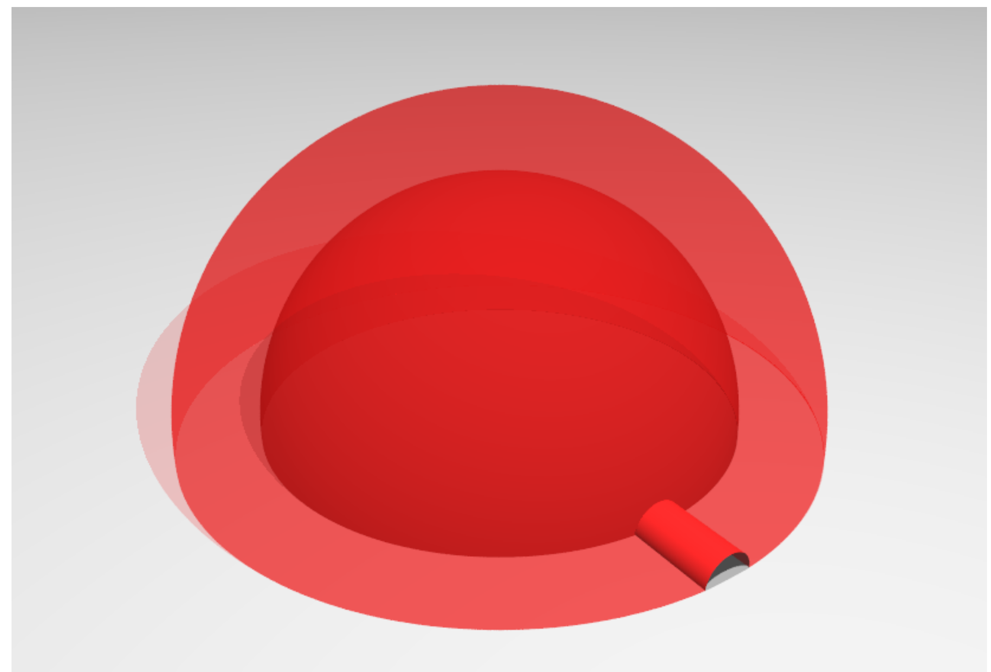

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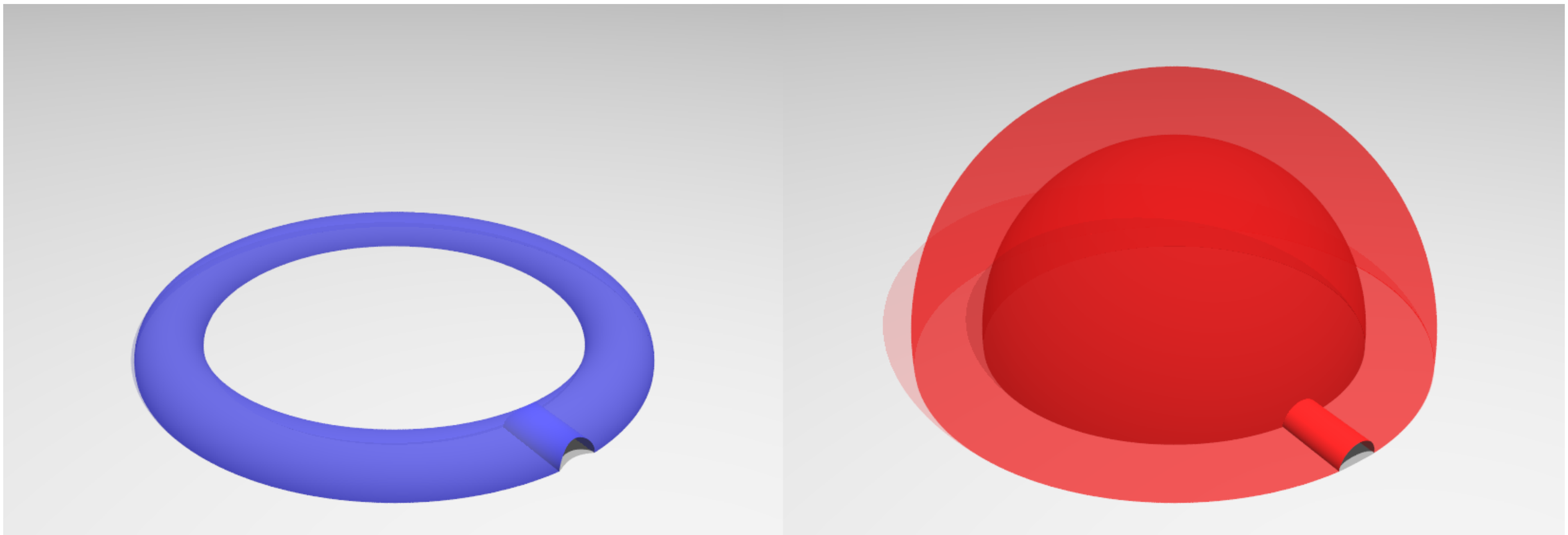

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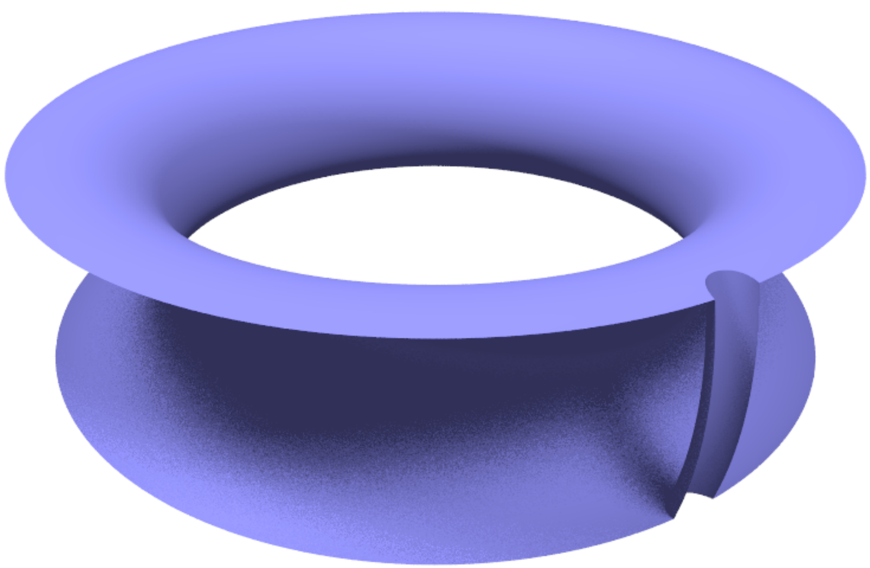

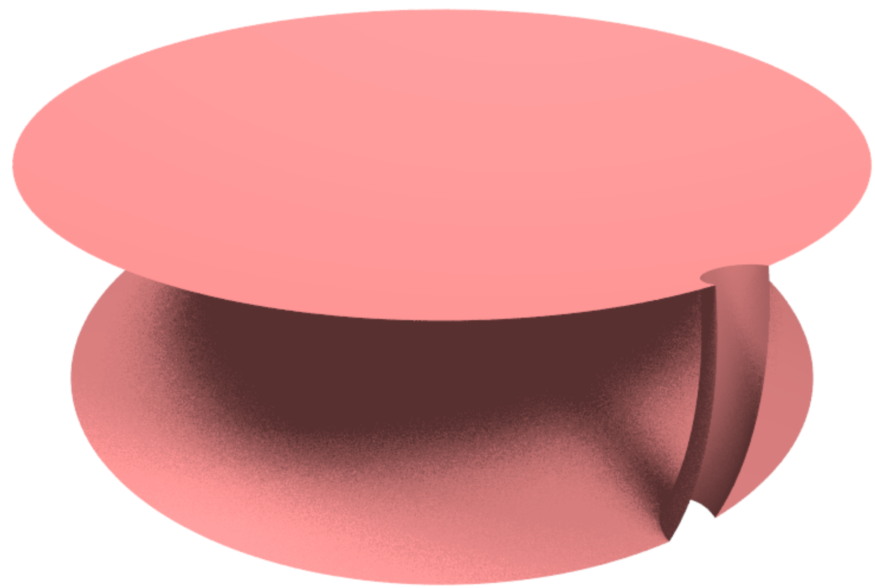

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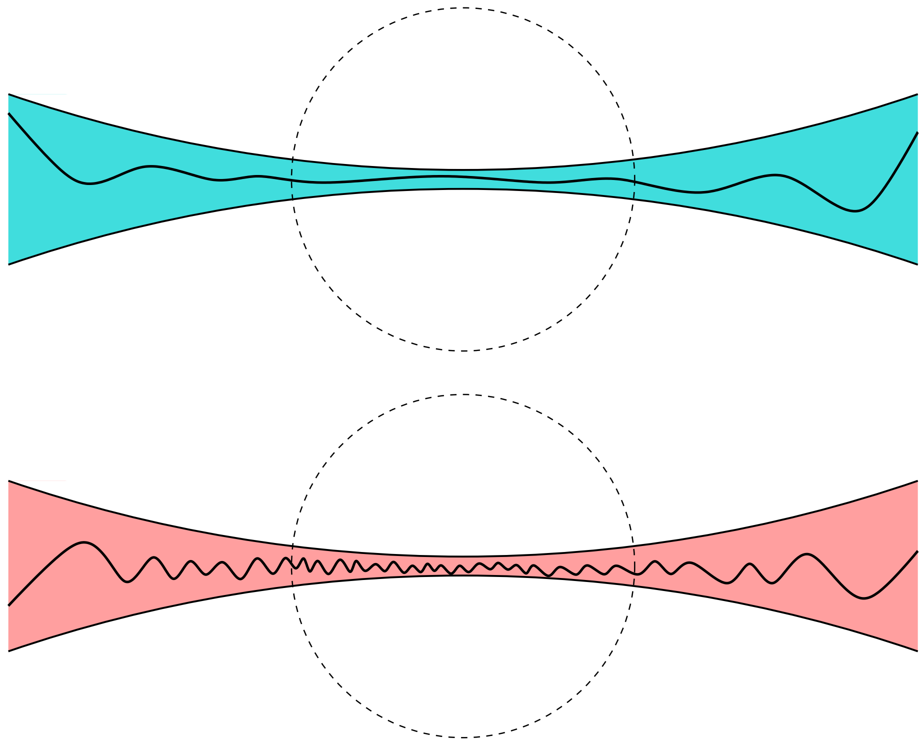
Theorem (Seppi, 2016): Principle curvatures satisfies $\kappa(z) = O(\delta(z))$.



$\sinh(\mathrm{dist}(z, P))$ satisfies $\Delta_S u - 2u = 0$. Use Schauder estimate $\|\nabla^2 u\|_\infty \leq C\|u\|_\infty$.

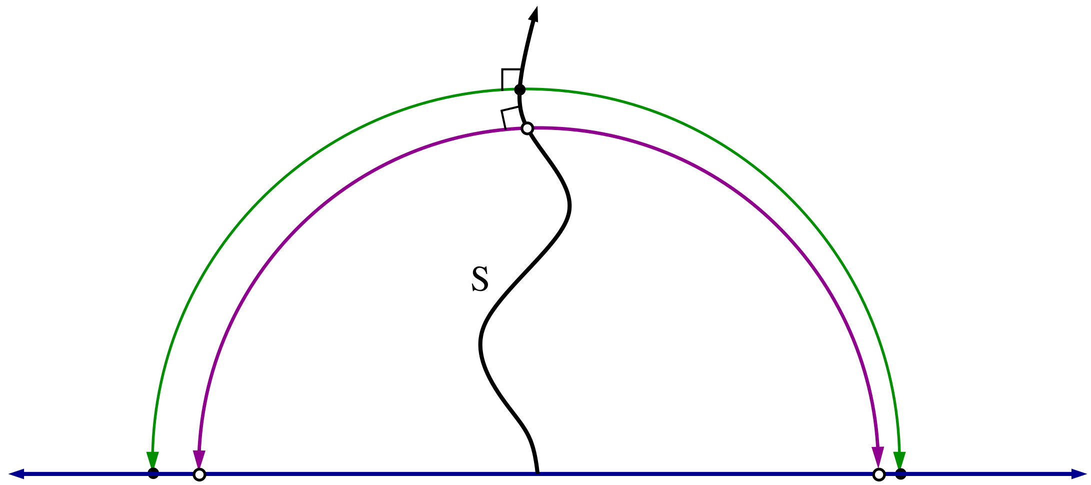
Theorem (Anderson, 1983): Every closed Jordan curve $\Gamma \subset \mathbb{R}^2$ bounds a minimal disk $S \subset \mathrm{CH}(\Gamma) \subset \mathbb{H}^3$.

Theorem (Seppi, 2016): Principle curvatures satisfies $\kappa(z) = O(\delta(z))$.

Theorem: Γ is WP iff it bounds a minimal disk with finite total curvature

$$\int_S (-K-1)\,dA_\rho = \int_S \kappa^2(z)\,dA_\rho < \infty.$$

Theorem (C. Epstein, 1986): If S is a surface with $|K| < 1$, then the Gauss maps G_j, $j = 1, 2$ define a quasiconformal reflection across Γ with dilatation $|\mu(G(z))| = O(|\kappa_1|(z) + |\kappa_2|(z))$.



If S is has finite total curvature, then $\int_{\mathbb{C}\setminus\Gamma} |\mu|^2 d\mathrm{A}_\rho < \infty$.

$\Rightarrow \Gamma$ is fixed by a QC involution with $\mu \in L^2(dA_\rho) \Rightarrow$ Weil-Petersson.

Isoperimetric inequality:

For a domain $\Omega \subset \mathbb{R}^2$ of area A and boundary length L:

$$L^2 \geq 4\pi A,$$

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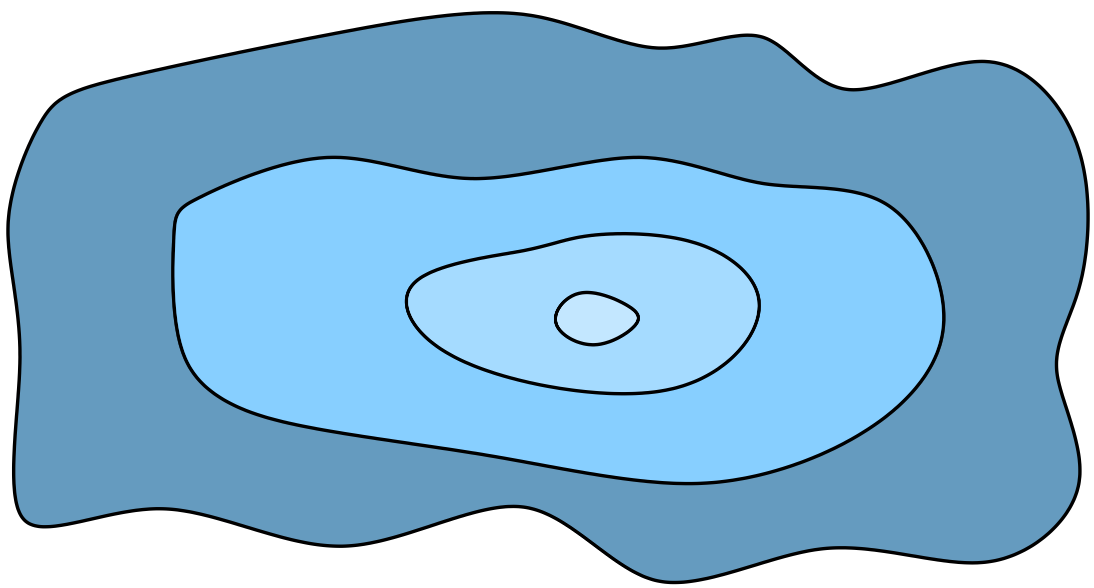
$$L^2 - A^2 \geq 4\pi\chi A$$

$$L - A \geq \frac{4\pi\chi A}{L + A} > 4\pi\chi$$

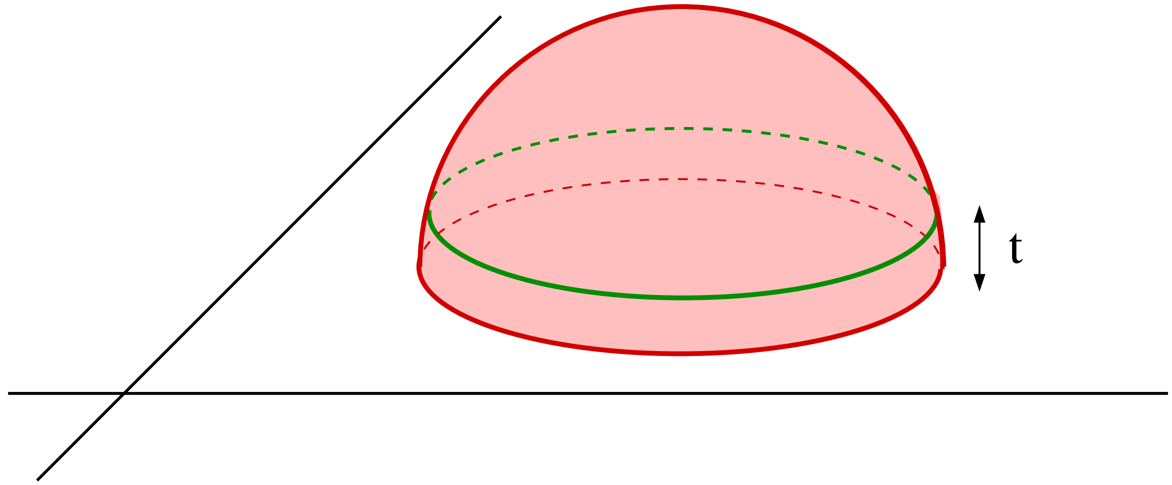
When do we have $L - A = O(1)$?

Theorem: Suppose $S \subset \mathbb{H}^3$ with finite Euler characteristic and asymptotic boundary a Jordan curve $\Gamma \subset \mathbb{R}^2$. Then Γ is Weil-Petersson iff S can be exhausted by compact Jordan domains $\Omega_1 \subset \Omega_2 \subset \cdots$ so that

$$\limsup_{n \nearrow \infty} [L(\Omega_n) - A(\Omega_n)] < \infty.$$



For surface in upper half-space with boundary on $\mathbb{R}^2$, we can form sub-domains by cutting at a certain height.



Truncate $S \subset \mathbb{R}^3_+$ at a fixed height above the boundary, i.e.,

$$S_t = S \cap \{(x, y, s) \in \mathbb{R}^3_+ : s > t\}, \quad \partial S_t = S \cap \{(x, y, s) \in \mathbb{R}^3_+ : s = t\}$$

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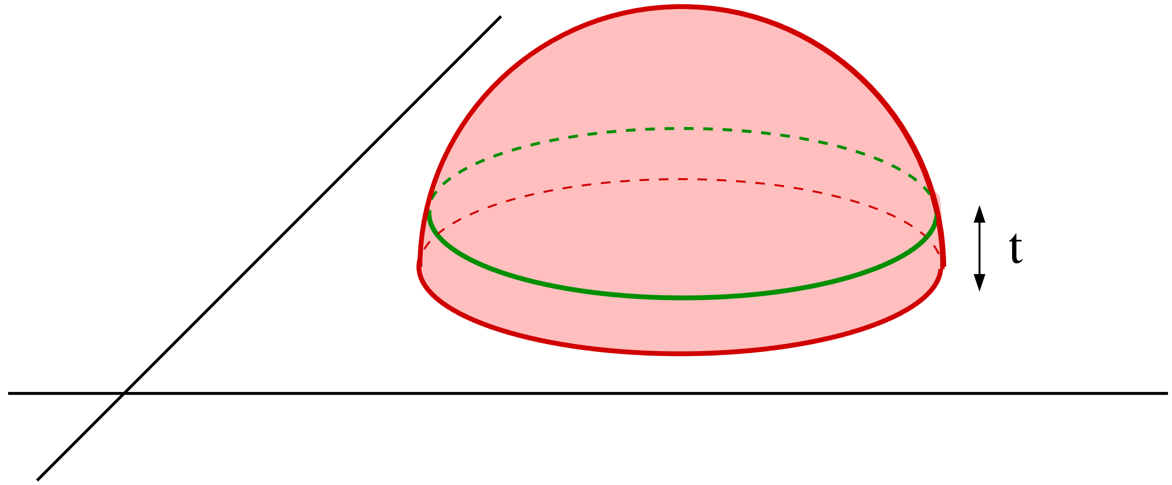
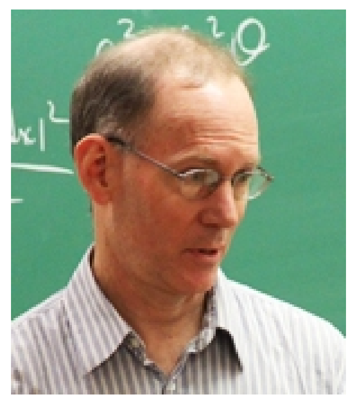


Define the **renormalized area**: $\mathcal{A}_R(S) = \lim_{t \searrow 0} \left[\mathrm{A}_\rho(S_t) - \ell_\rho(\partial S_t) \right]$.

Due to Graham and Witten.
Related to quantum entanglement, ...
we already know:

$$\mathcal{A}_R(S) < \infty \Rightarrow \text{Weil-Petersson.}$$



Theorem: S has finite renormalized area iff Γ is Weil-Petersson.

I know two proofs of Weil-Petersson $\Rightarrow \mathcal{A}_R(S) < \infty$.

- Use Gauss-Bonnet, Seppi's estimate and $\int \delta^2 < \infty$.
- Use "dyadic cylinder", a discrete version of minimal surface S.

Using the Gauss-Bonnet theorem

$$\begin{aligned}
\mathrm{A}_\rho(S_t) - \ell_\rho(\partial S_t) &= \int_{S_t} 1 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= \int_{S_t} (1+\kappa^2) d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= -\int_{S_t} K d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= -2\pi\chi(S_t) + \int_{\partial S_t} \kappa_g d\ell_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= -2\pi\chi(S_t) - \int_{S_t} \kappa^2 d\mathrm{A}_\rho + \int_{\partial S_t} (\kappa_g - 1) d\ell_\rho
\end{aligned}$$

Can prove $\kappa_g(z) = 1 + O(\delta^2(z))$, so WP implies last term $\to 0$.

Theorem: For any closed curve $\Gamma \subset \mathbb{R}^2$ and for any minimal surface $S \subset \mathbb{R}^3_+$ with finite Euler characteristic and asymptotic boundary Γ,

$$\mathcal{A}_R(S) = -2\pi\chi(S) - \int_S \kappa^2(z)\,d\mathrm{A}_\rho,$$

Both sides are $-\infty$ iff Γ is not Weil-Petersson.

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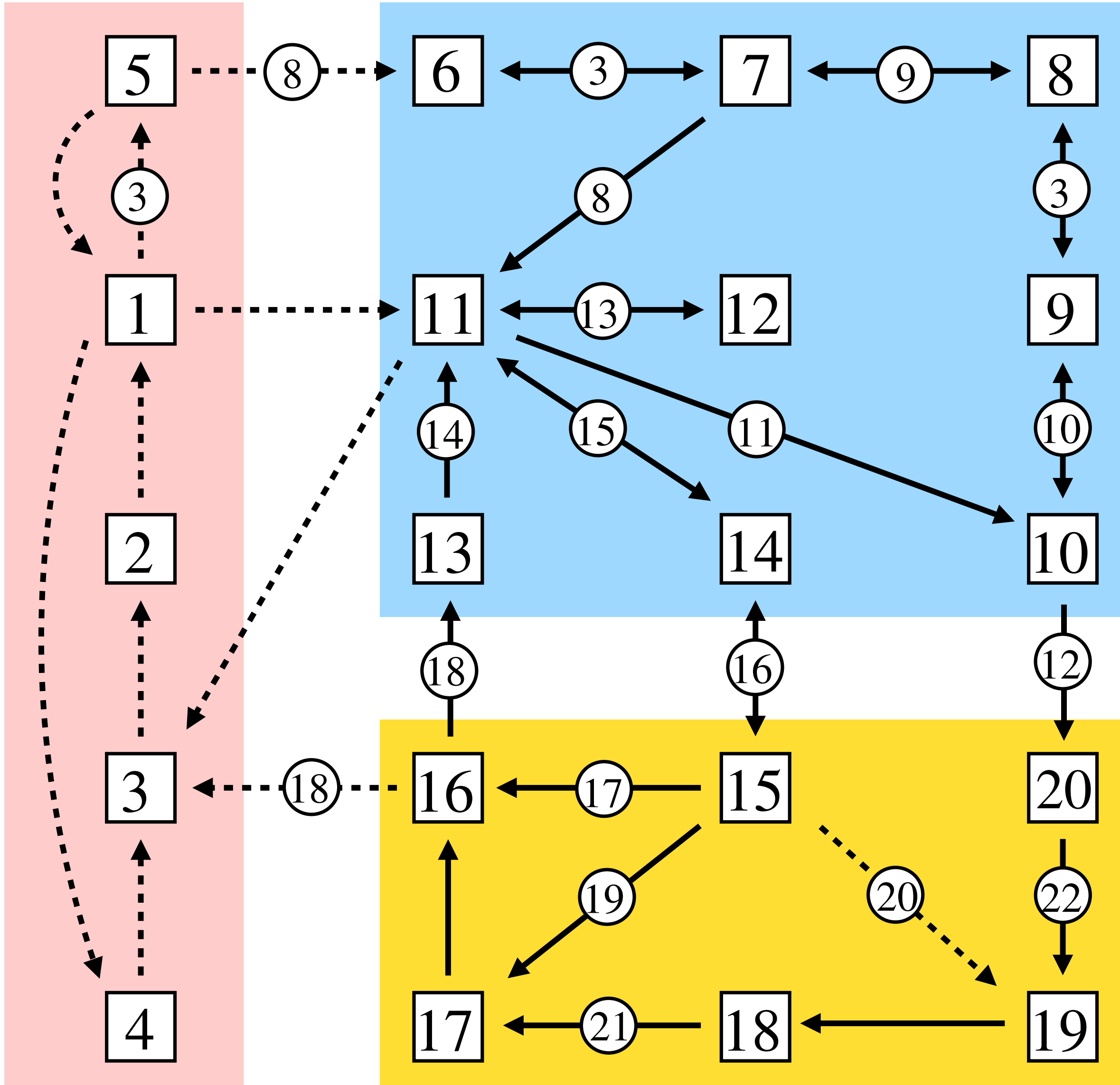
This formula is due to Alexakis and Mazzeo in the setting of n-dimensional Poincaré-Einstein manifolds (that formula also contains a term involving the Weyl curvature), assuming that Γ is $C^{3,\alpha}$. My result was inspired by trying to understand their theorem and remove smoothness.

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions
14	between disjoint disks
15	thickness of convex hull
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder

Weil-Petersson curves

André Weil

Hans Petersson



THANKS FOR LISTENING. QUESTIONS?

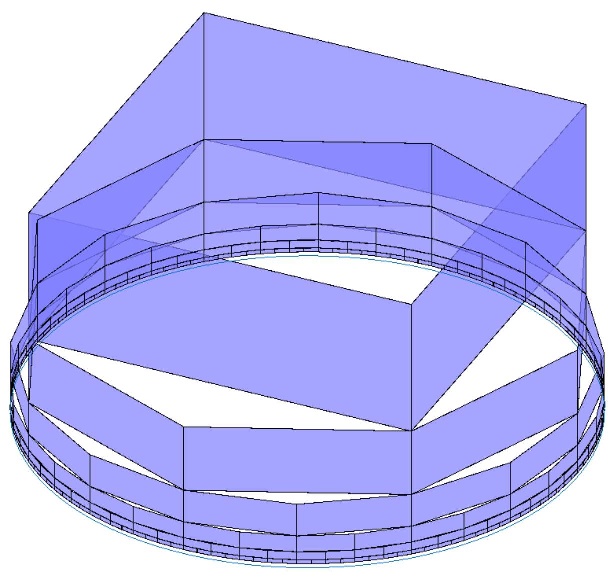
An idea connecting Euclidean and hyperbolic results.

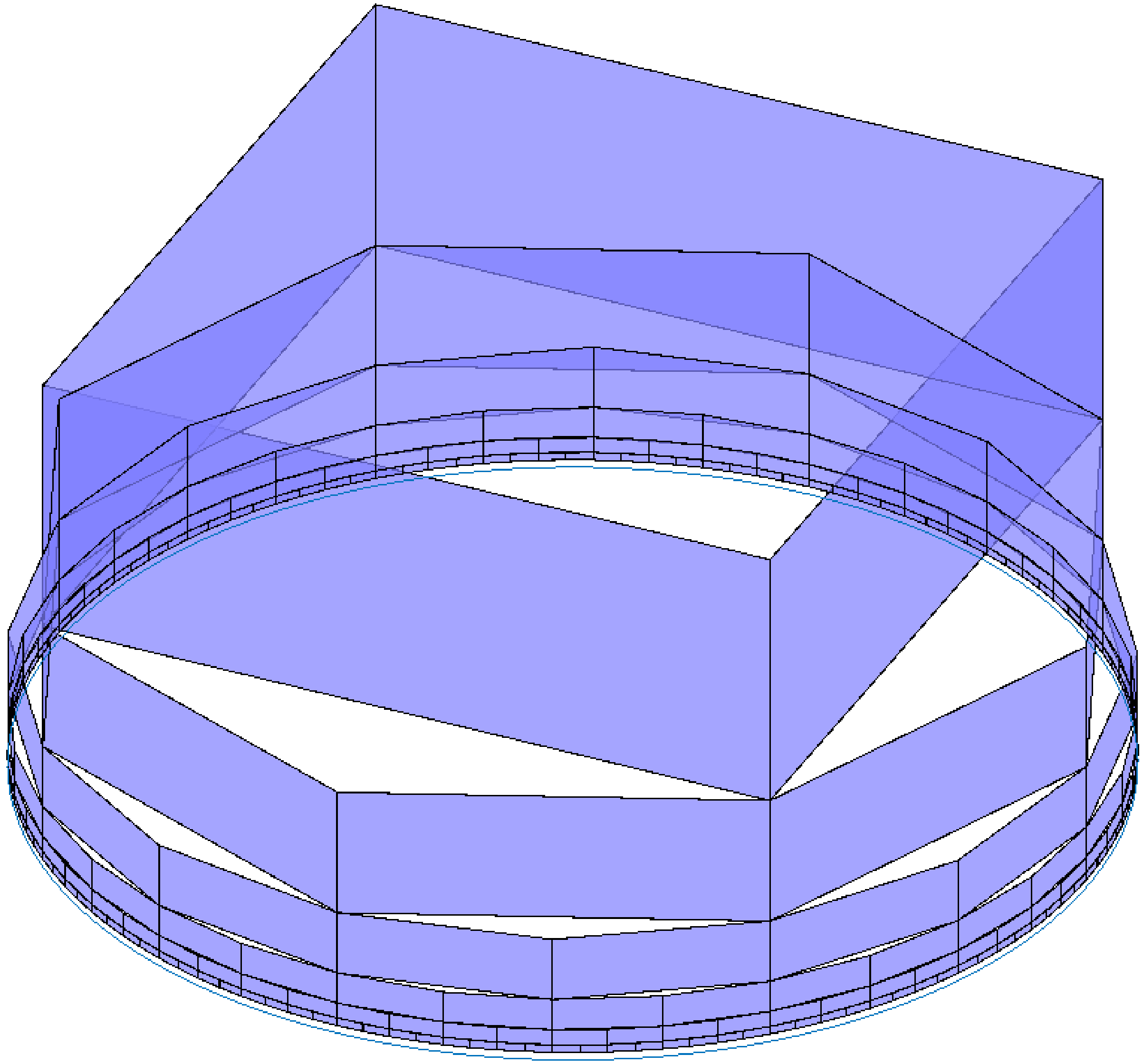
Define a dyadic cylinder in the upper half-space:

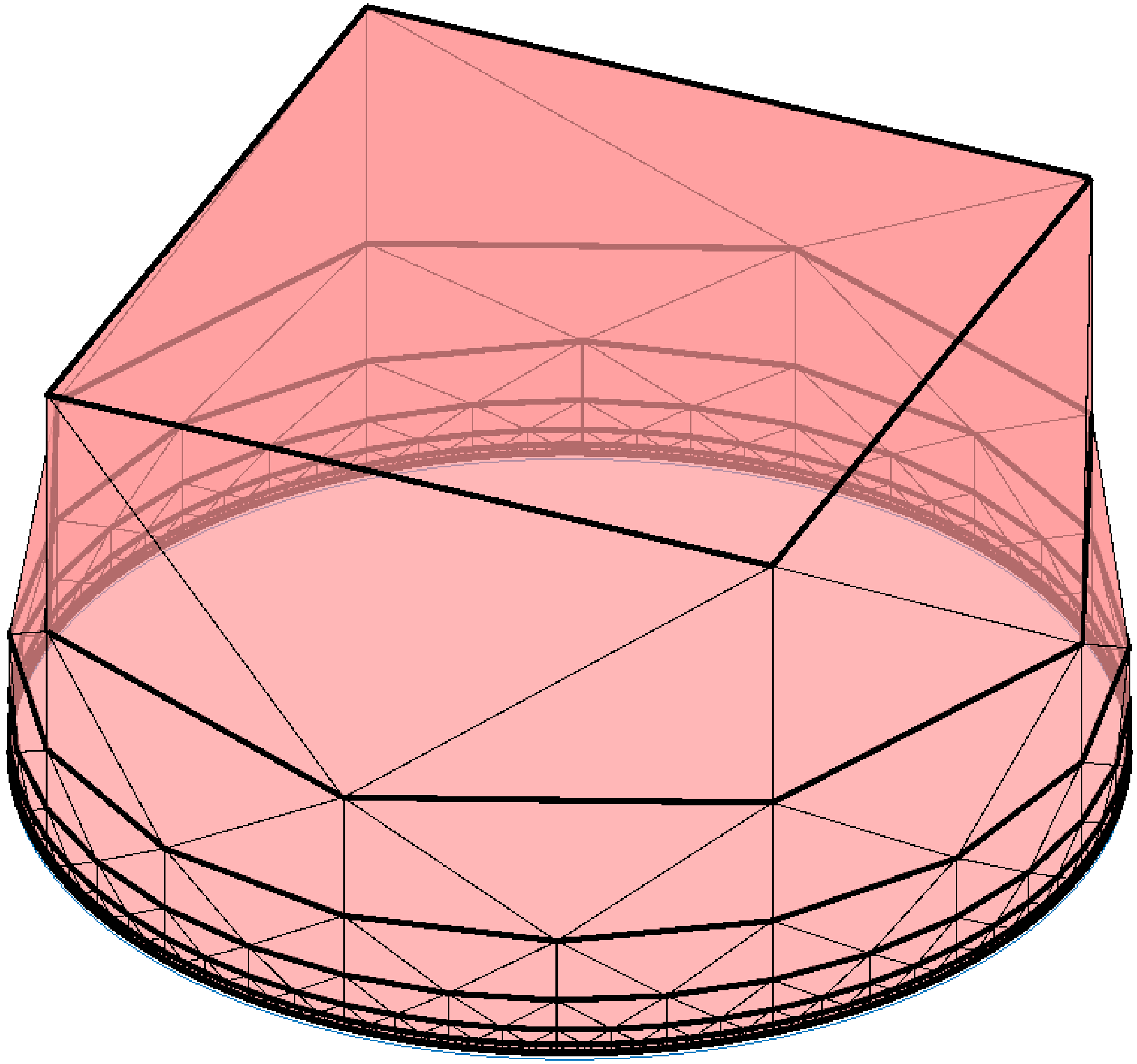
$$X = \bigcup_{n=1}^{\infty} \Gamma_n \times [2^{-n}, 2^{-n+1}),$$

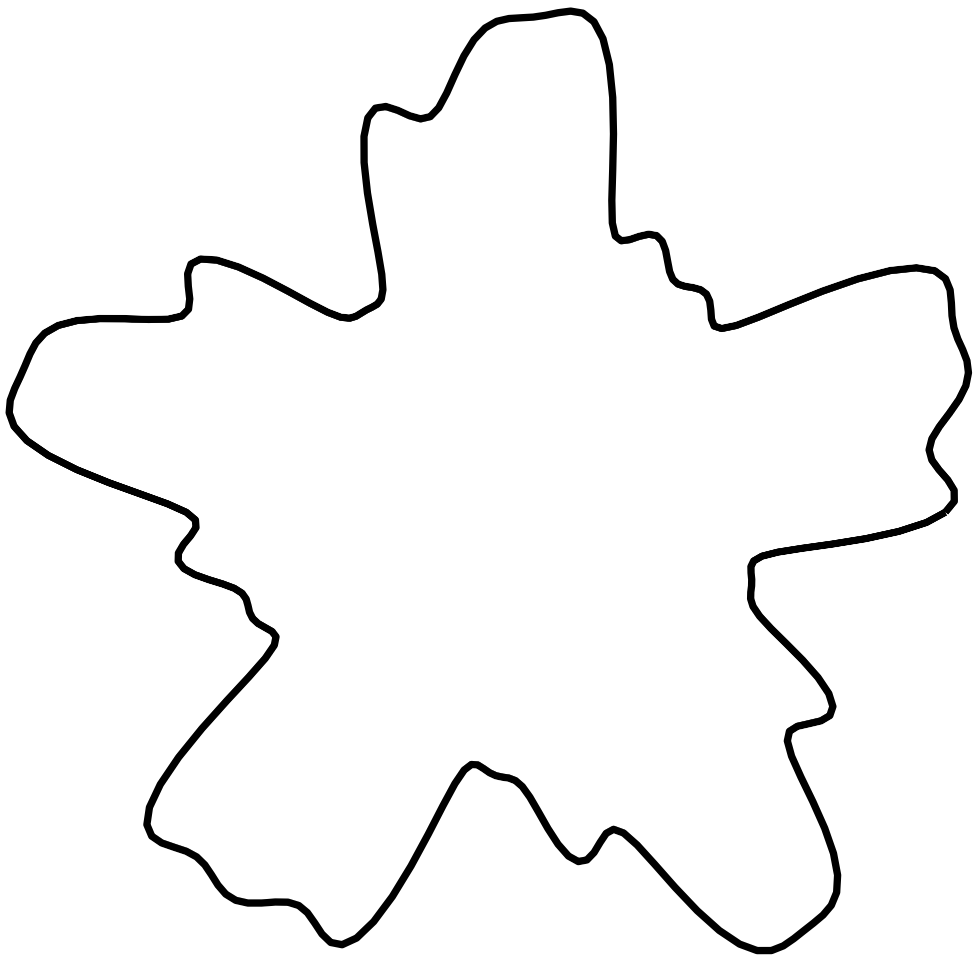
where $\{\Gamma_n\}$ are inscribed dyadic polygons in Γ.

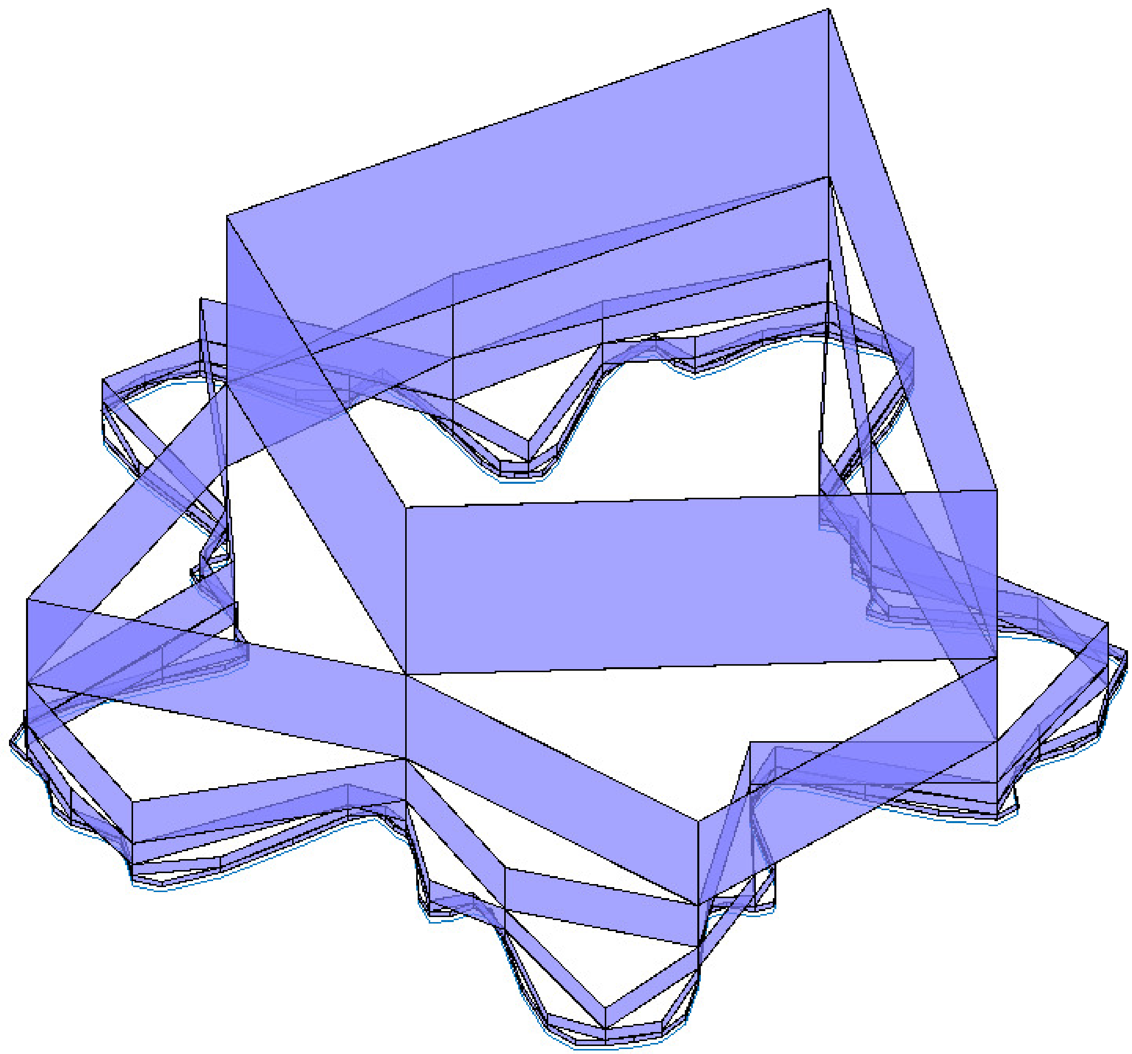
Discrete analog of minimal surface with boundary Γ.

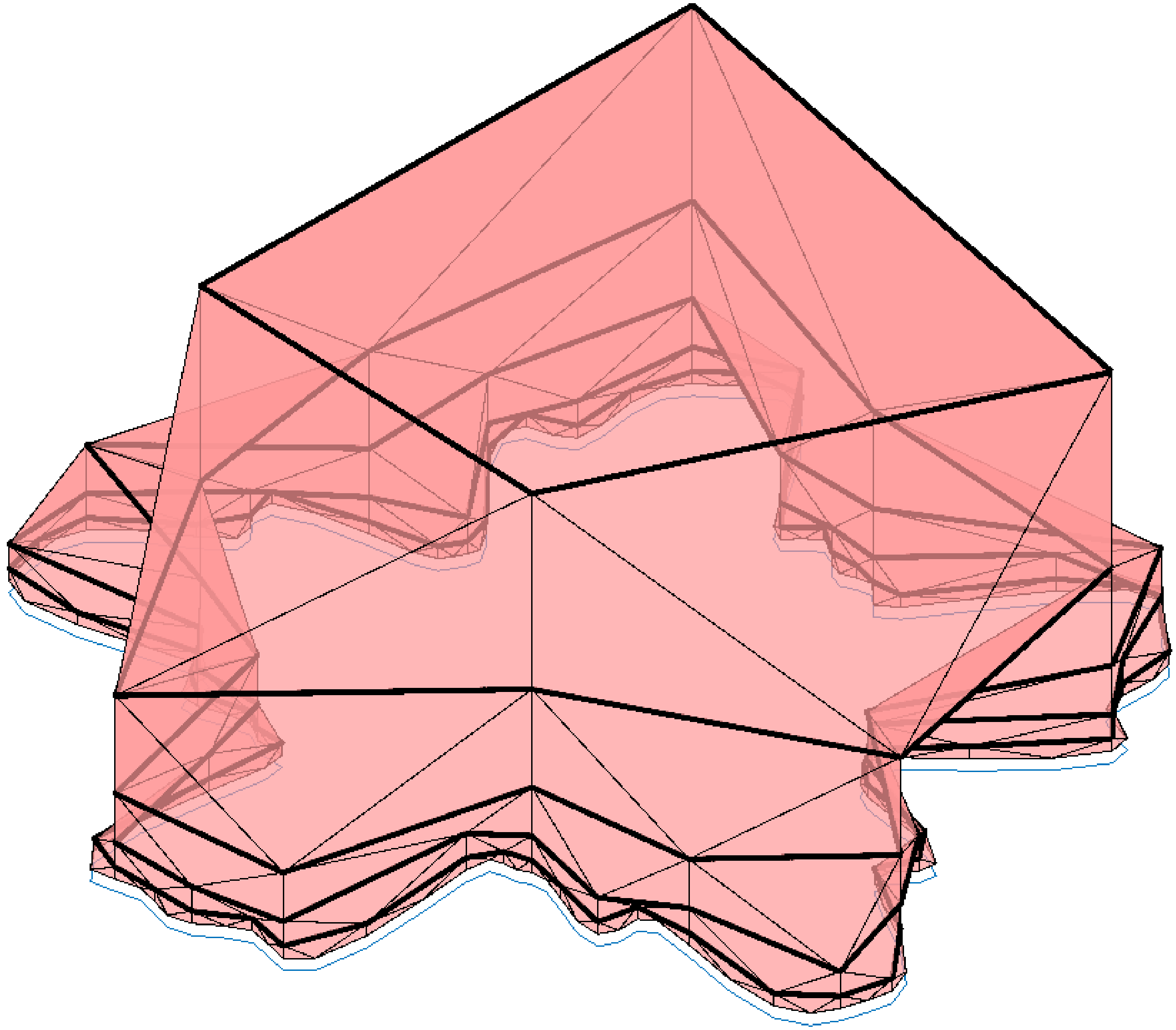












Our earlier estimate

$$\sum_n 2^n(\ell(\Gamma) - \ell(\Gamma_n)) < \infty$$

is equivalent to the dyadic cylinder having finite renormalized area.

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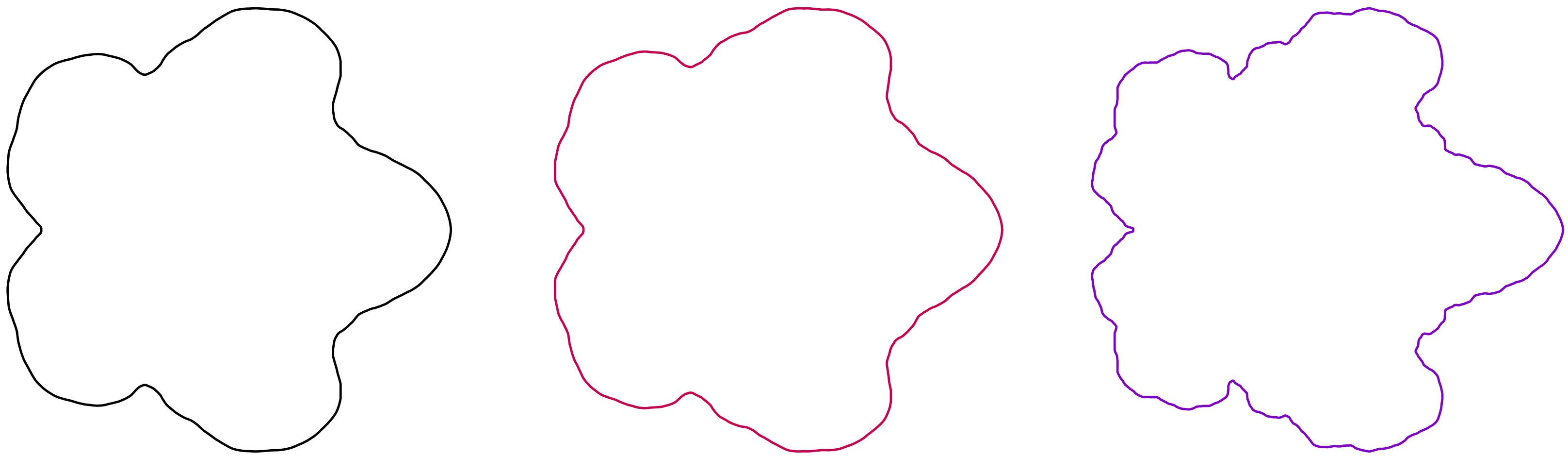
is equivalent to the dyadic cylinder having finite renormalized area.

Obvious "normal projection" from the dyadic cylinder to minimal surface, distorts length and area each by a bounded additive error.

We can deduce finite renormalized area for the minimal surface from the same result for the dyadic cylinder.

$F(z) = \sum_1^\infty a_n z^n$ is Dirichlet class iff $\sum n|a_n|^2 < \infty$.

If $\log f' = \sum \sqrt{\frac{b_k}{\lambda^k}}\, z^{\lambda^k}$ then $\Gamma = f(\mathbb{T})$ is WP iff $\sum b_k < \infty$.



$\sum \frac{1}{k \log^2 k} < \infty$ $\quad$ $\sum \frac{1}{k \log k} = \infty$ $\quad$ $\sum 1 = \infty$