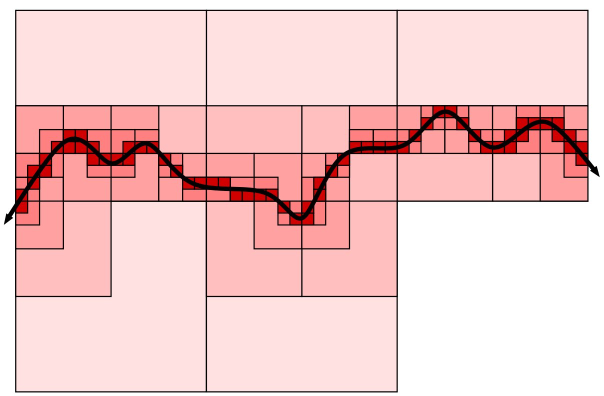
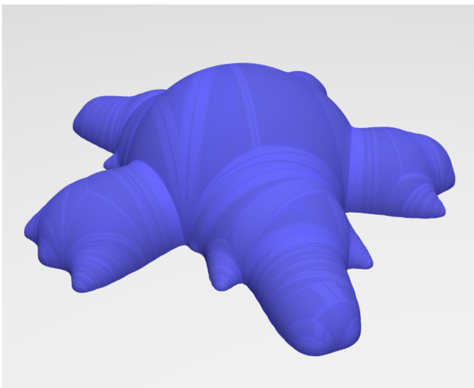
WEIL-PETERSSON CURVES, β-NUMBERS AND MINIMAL SURFACES

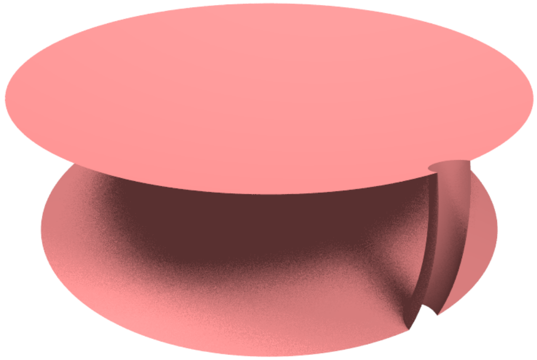
Christopher Bishop, Stony Brook

Hyperbolic Geometry Seminar
CUNY Graduate Center Seminar, Tuesday, 2/16/21

`www.math.stonybrook.edu/~bishop/lectures`







Goals for today:

(1) Define Weil-Peterson class of curves.

(2) Give motivation and connections to various areas.

(3) State almost half a theorem, sketch parts of the proof.

(4) Mention lots of open problems.

(1) String theory studies spaces of loops.

(2) Physicists like Hilbert spaces.

$\Rightarrow$ Physicists want the space of loops to look like a Hilbert space.

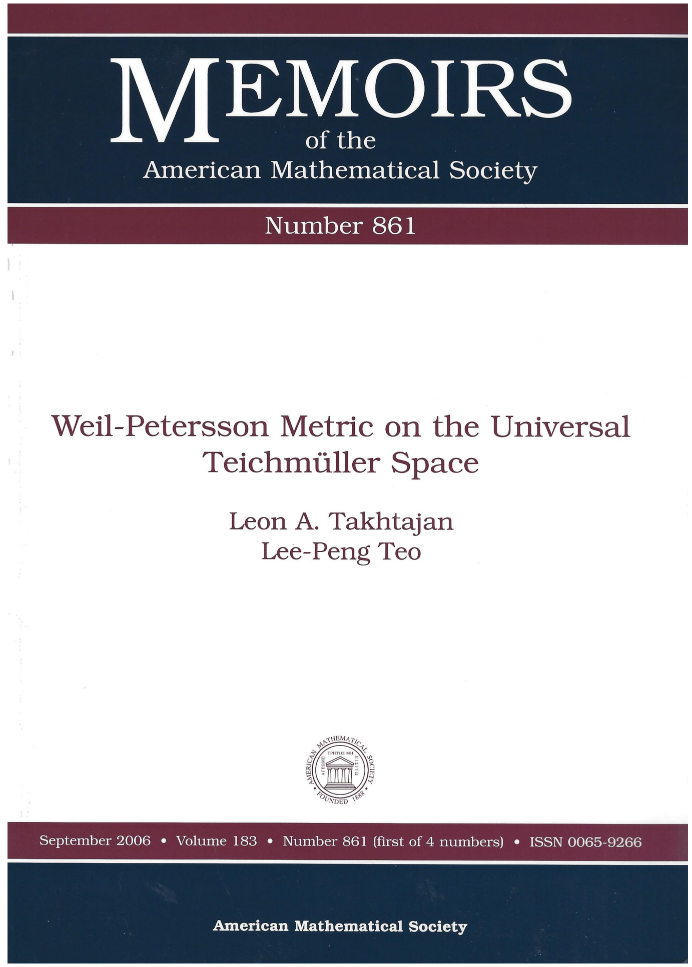
(1) Mathematicians study $T(1)$= universal Teichmüller space

(2) $T(1)$ = space of quasicircles

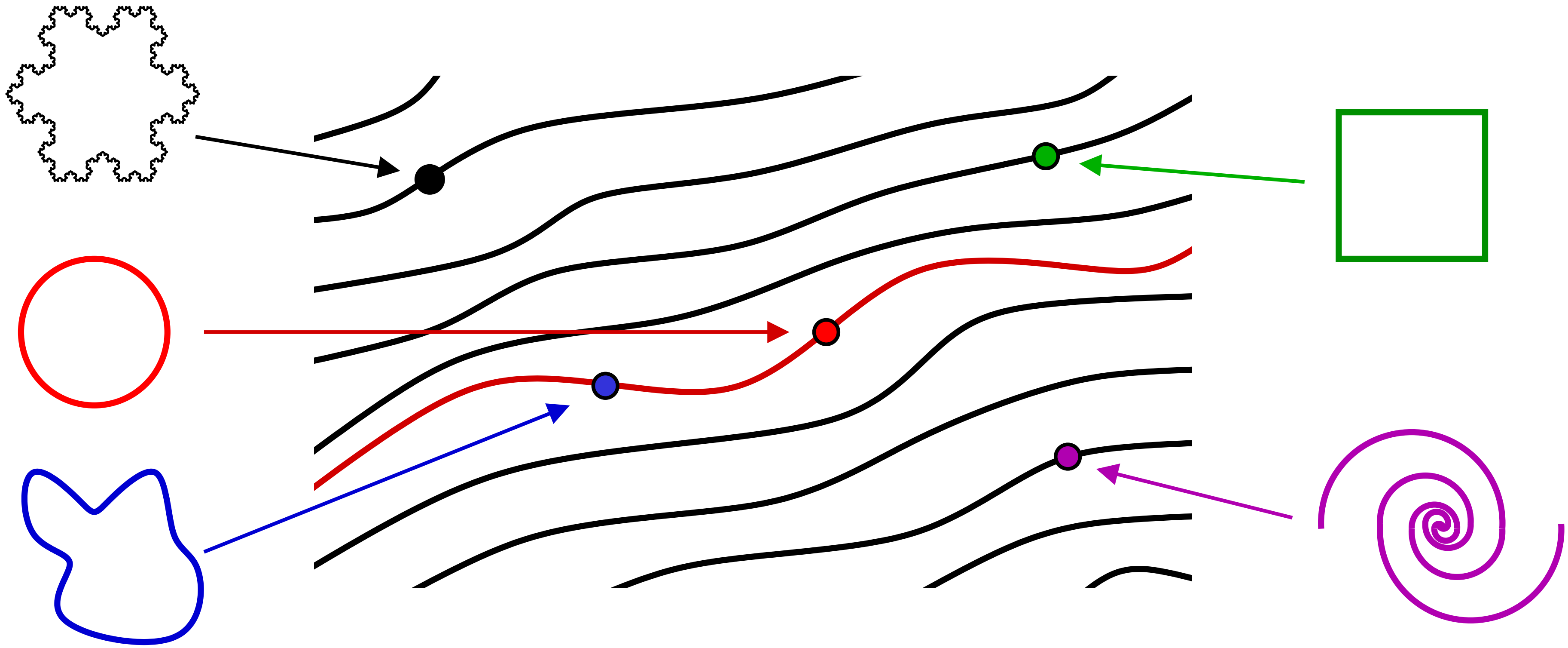
(3) Teichmüller metric based on L^∞.

$\Rightarrow$ $T(1)$ is Banach manifold, not Hilbert manifold.

$\Rightarrow$ Teichmüller metric not good for string theory.

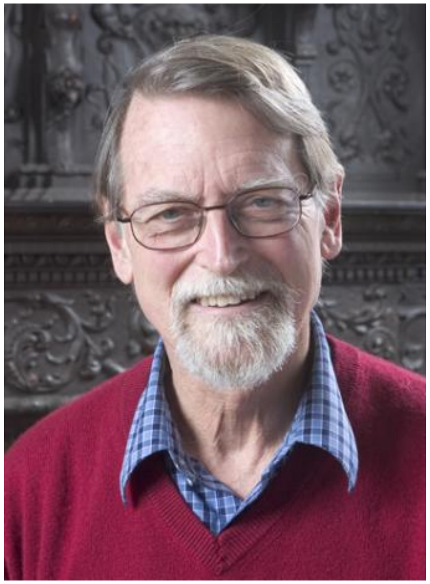


"In this memoir, we prove that the universal Teichmüller space $T(1)$ carries a new structure of a complex Hilbert manifold and show that the connected component of the identity of $T(1)$ — the Hilbert submanifold $T_0(1)$ — is a topological group. ..."



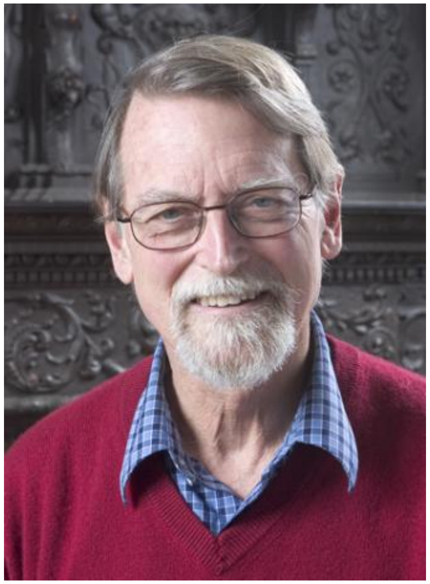
Takhtajan and Teo make $T(1)$ a (disconnected) Hilbert manifold.

$$
\begin{aligned}
T_0(1) &= \text{Weil-Petersson class} \\
&= \text{component containing the circle} \\
&= \text{closure of smooth curves} \\
&= \infty\text{-dim Kähler-Einstein manifold.}
\end{aligned}
$$



In Dec 2017 email David Mumford asked me which non-smooth curves are in WP? Motivated by computer vision and pattern recognition.

"Riemannian geometries on spaces of plane curves, Michor and Mumford, *J. Eur. Math. Soc. (JEMS)*, 2006.



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Jan 2019 IPAM workshop: Analysis and Geometry of Random Sets.

Lecture by Yilin Wang:

"Loewner energy via Brownian loop measure and action functional analogs of SLE/GFF couplings"

So the Weil-Petersson class is linked to:

- String theory
- Kähler-Einstein manifolds
- Teichmüller theory
- Pattern recognition
- Brownian motion, SLE, Gaussian free fields, ...

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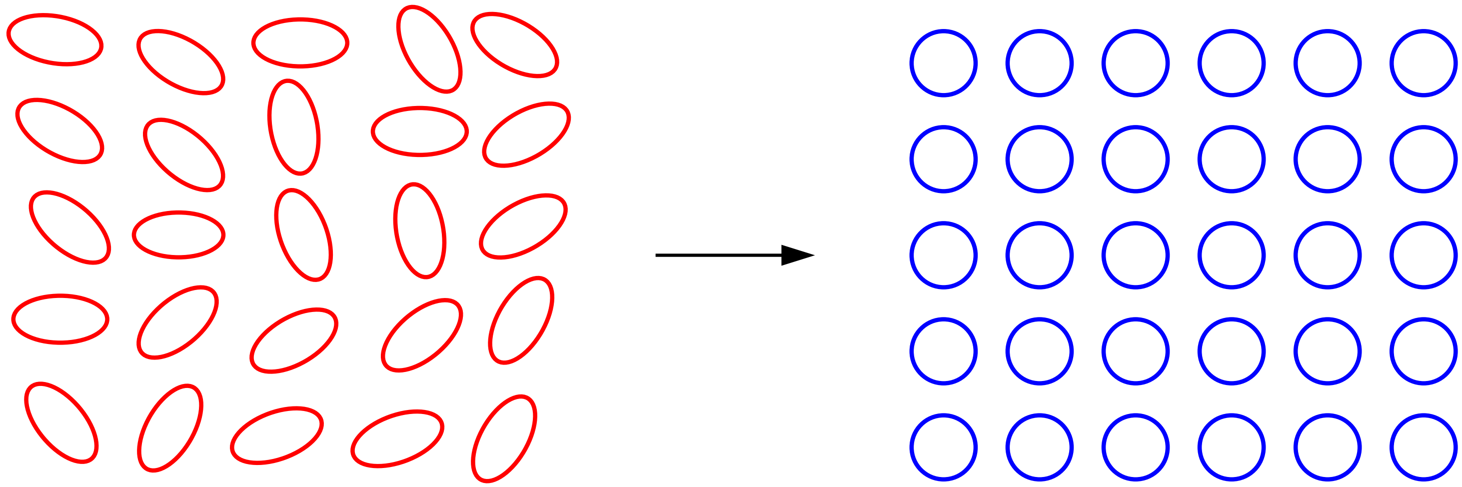
In today's talk I will discuss further connections to:

- Geometric function theory
- Sobolev spaces
- Geometric measure theorem (Peter Jones's TST)
- Convex hulls in hyperbolic space
- Minimal surfaces
- Isoperimetric inequalities
- Renormalized area

Quasiconformal (QC) map f sends infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

For K-QC maps, ellipses have eccentricity $\leq K$

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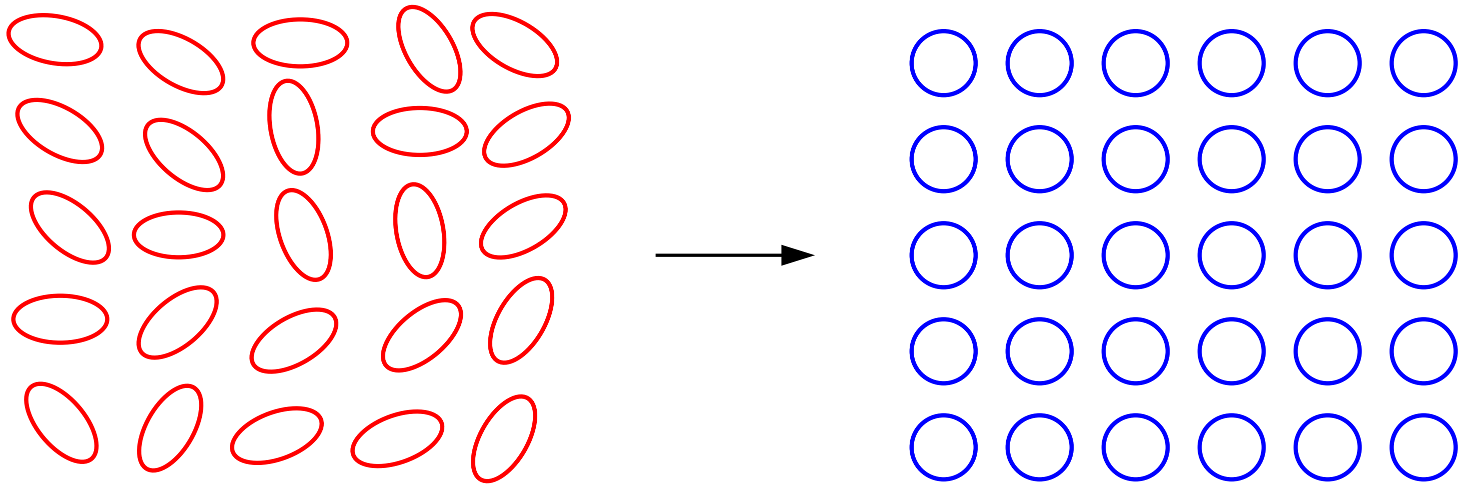

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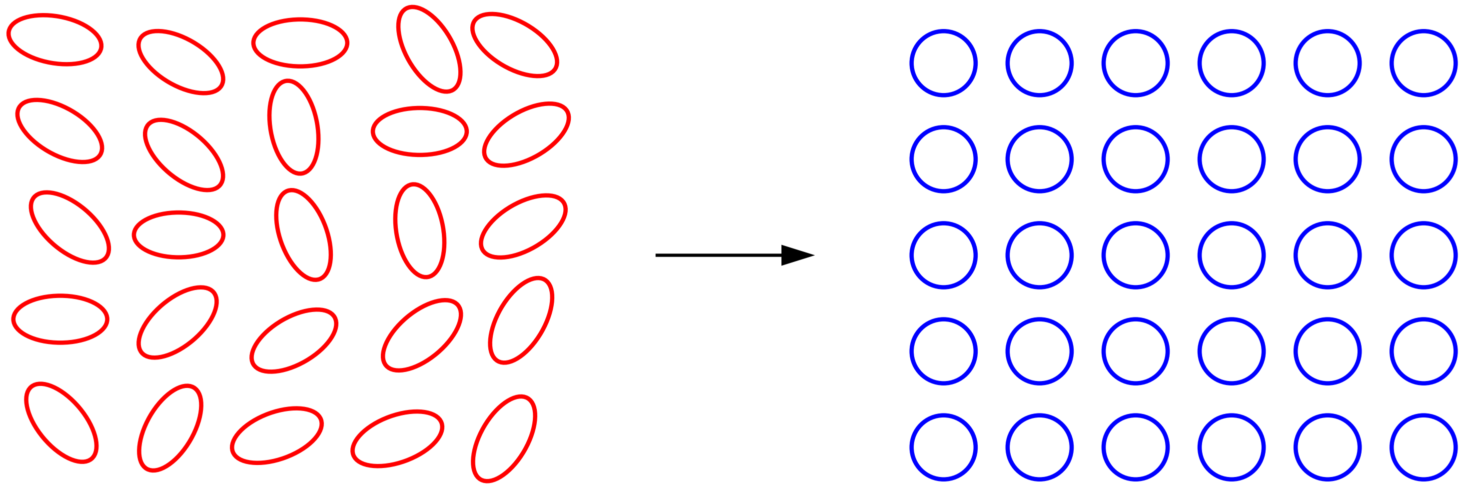
Ellipses determined a.e. by measurable dilatation $\mu = f_{\overline{z}}/f_z$ with

$$|\mu| \leq \frac{K-1}{K+1} < 1.$$

f is QC $\Leftrightarrow \|\mu\|_\infty < 1$. $\qquad$ f is conformal $\Leftrightarrow \mu \equiv 0$.

Quasiconformal (QC) map f sends infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

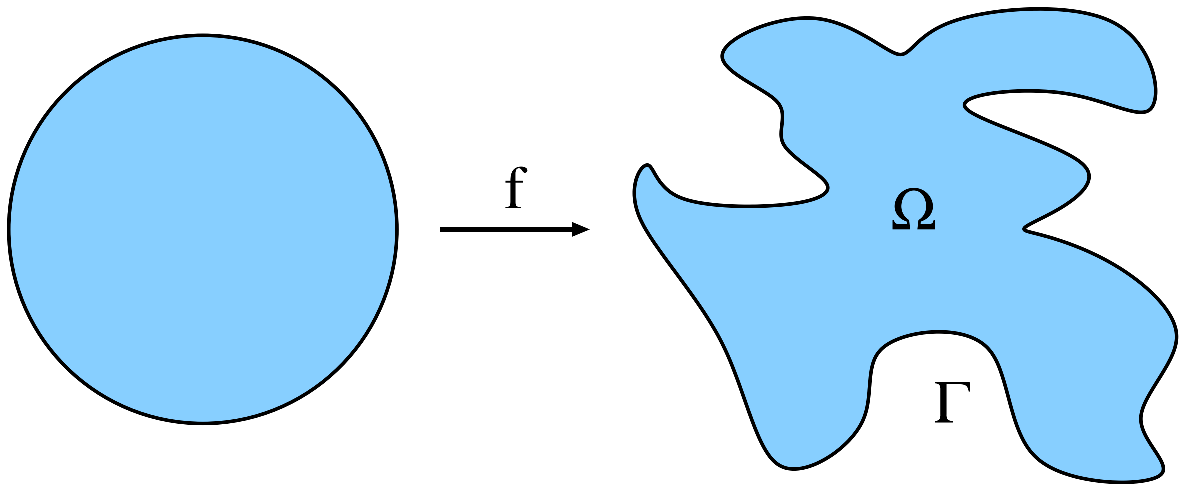
For K-QC maps, ellipses have eccentricity $\leq K$

A special case of QC maps are biLipschitz maps (all we need):

$$\frac{1}{C} \leq \frac{|f(x) - f(y)|}{|x - y|} \leq C.$$

It is unknown which dilatations correspond to biLipschitz maps.

A **quasicircle** is the image of circle under a quasiconformal map of $\mathbb{R}^2$.

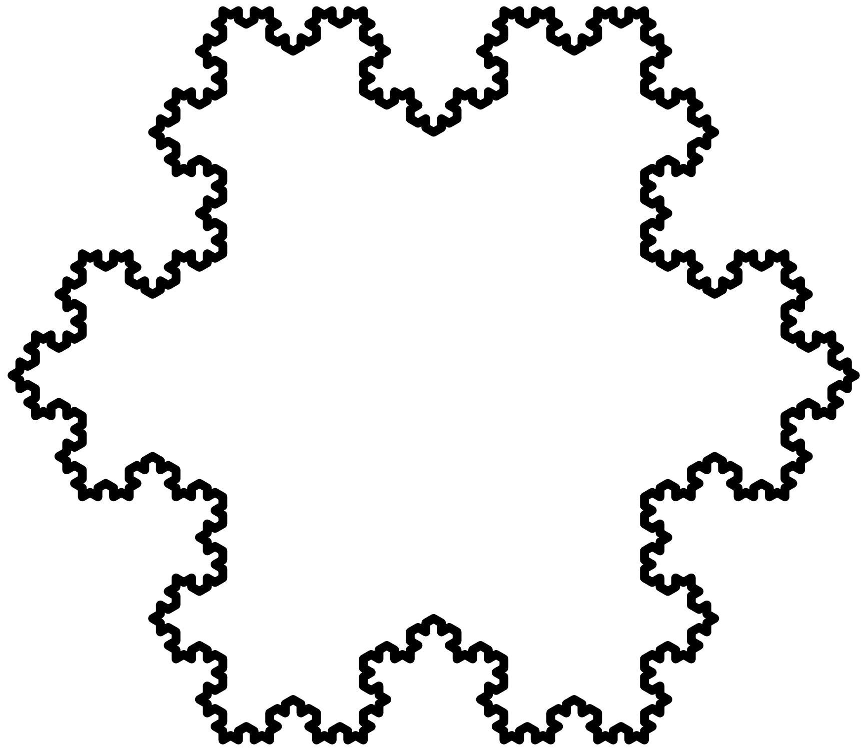


We can assume f is conformal outside disk and μ supported inside.

(Or vice versa.)

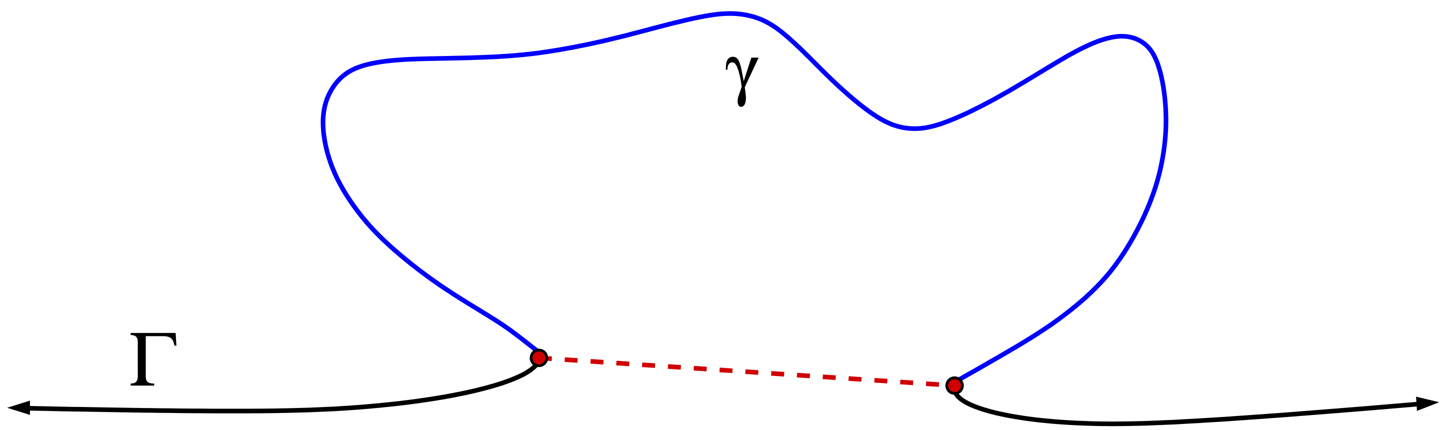
All smooth closed curves are quasicircles.

Some fractals are quasicircles.



Geometric characterization:

Γ is a quasicircle iff $\operatorname{diam}(\gamma) = O(\operatorname{crd}(\gamma))$ for all $\gamma \subset \Gamma$.

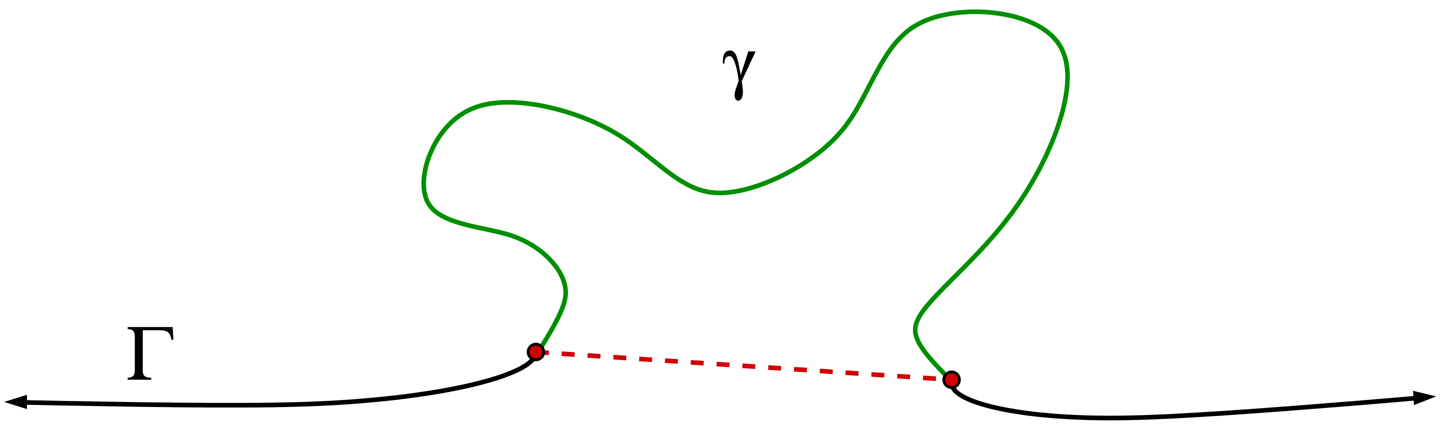


$\operatorname{crd}(\gamma) = |z - w|$, z, w, endpoints of γ.

Defn: Chord-arc curves = biLipschitz images of circle.

Geometric characterization:

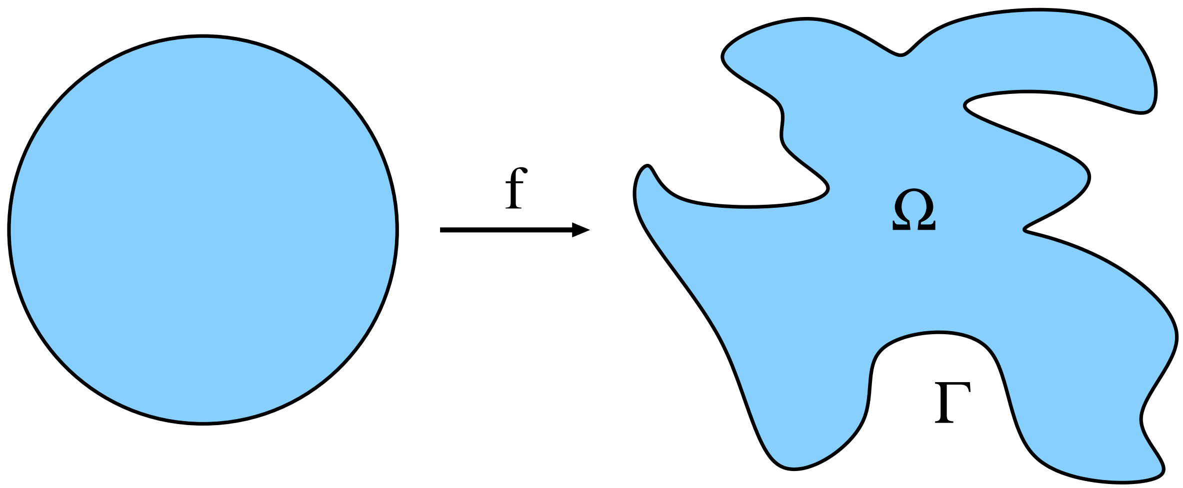
Γ is a **chord-arc** iff $\ell(\gamma) = O(\text{crd}(\gamma))$ for all $\gamma \subset \Gamma$.



Suppose f is QC with μ supported on $\mathbb{D}$, f conformal outside $\mathbb{D}$,

Defn: $\Gamma = f(\mathbb{T})$ is **Weil-Petersson** if $\mu \in L^2(dA_\rho)$.

Here $dA_\rho = \frac{dxdy}{(1-|z|^2)^2}$ = hyperbolic area.



Quasicircles = $\{\|\mu\|_\infty < 1\}$. WP = $\{\|\mu\|_2 < \infty\} \cap \{\|\mu\|_\infty < 1\}$.

Informally: WP is to L^2, as QC is to L^∞.

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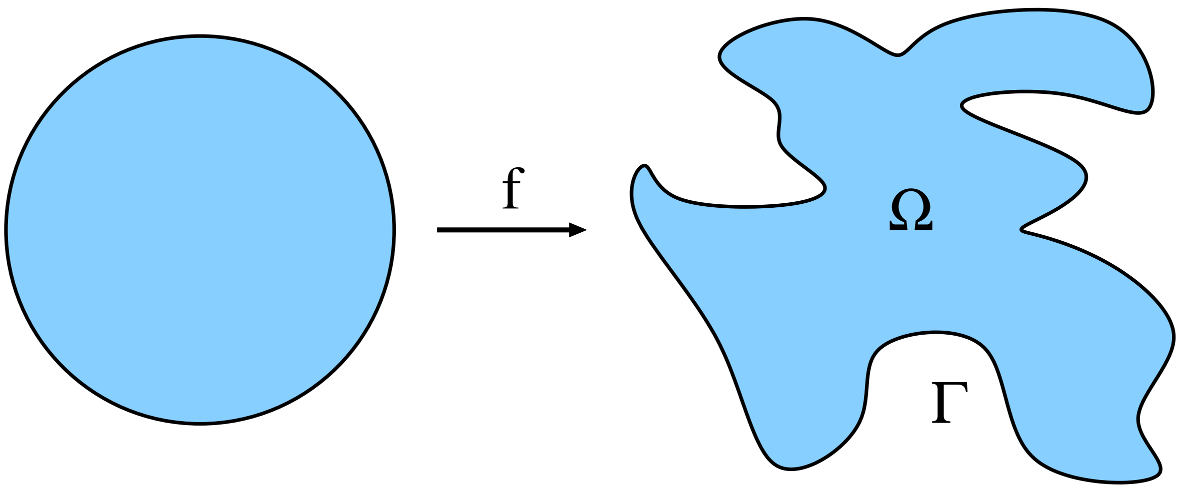

Quasicircles denoted by $T(1)$. Weil-Petersson class = $T_0(1)$.

1 = trivial group. $T(G)$ = Teichmüller space for a Fuchsian group G.

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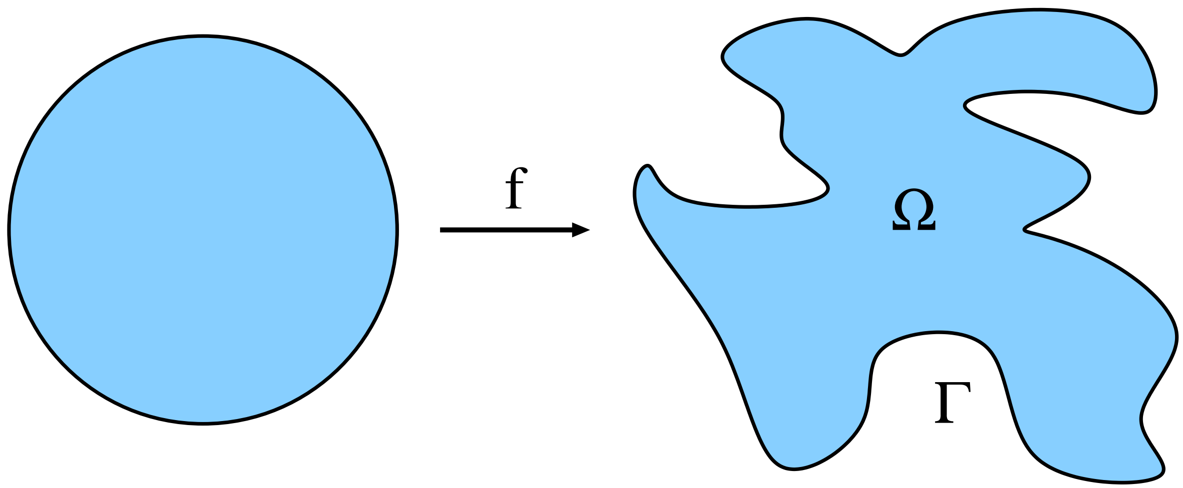

The Weil-Petersson class is Möbius invariant.

Suppose f is QC with μ supported on $\mathbb{D}$, f conformal outside $\mathbb{D}$,

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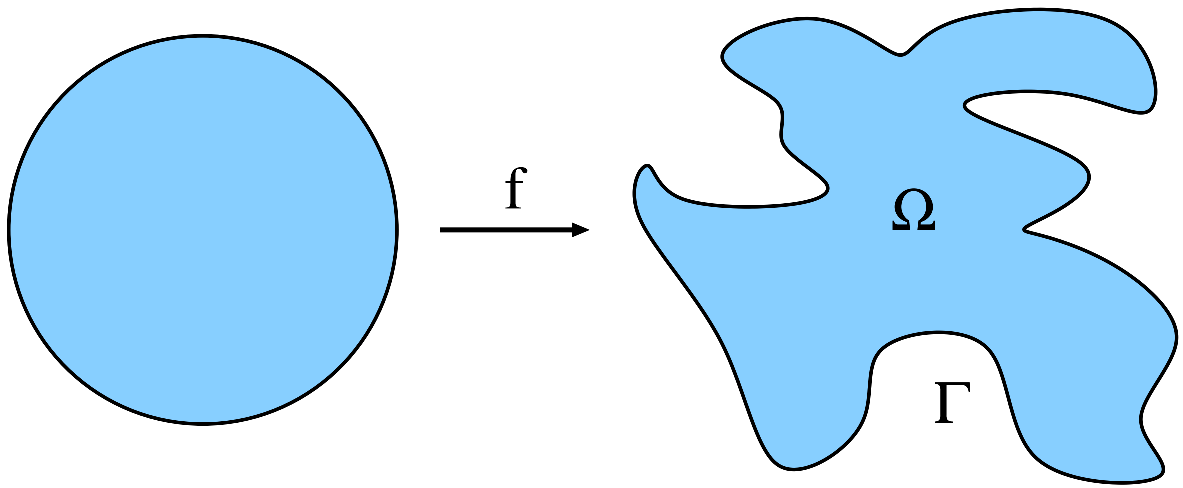
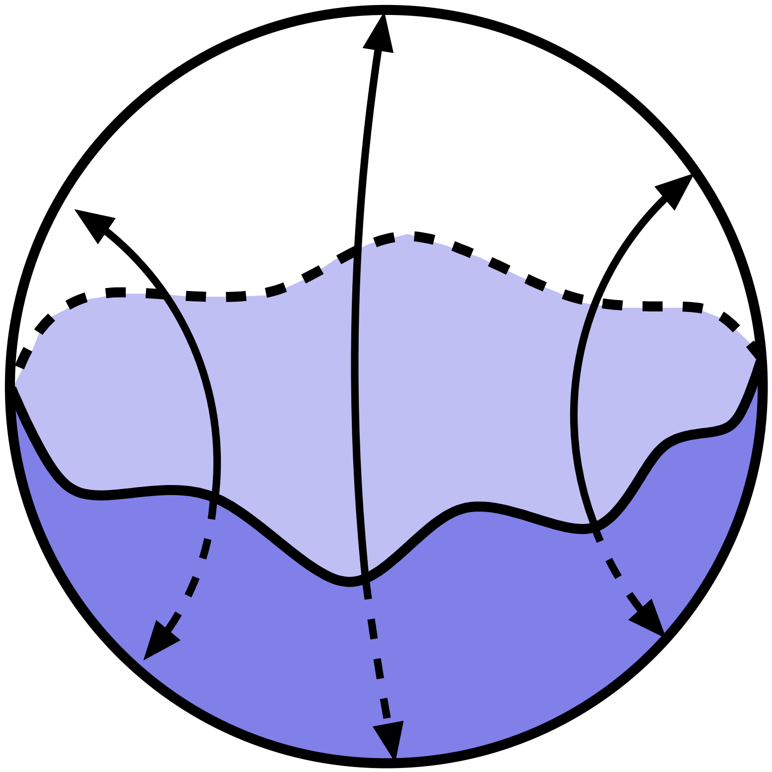


Define $F(z) = f(1/\overline{f^{-1}(z)})$.

F is a biLipschitz involution fixing Γ, and swaps the two sides of Γ.

Alternate Defn: $\Gamma = f(\mathbb{T})$ is **Weil-Petersson** if Γ is pointwise fixed for biLipschitz involution with $\mu \in L^2$ for hyperbolic area on $\mathbb{S}^2 \setminus \Gamma$.

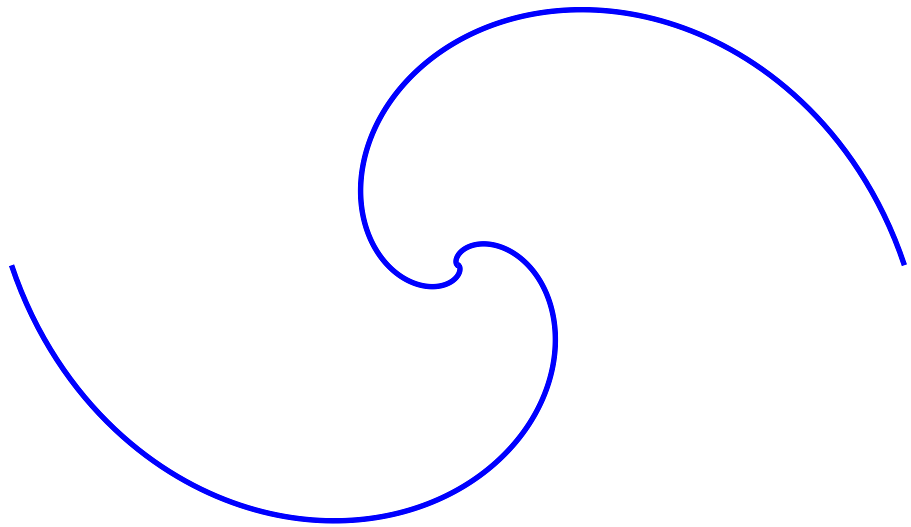


This extends to higher dimensions.

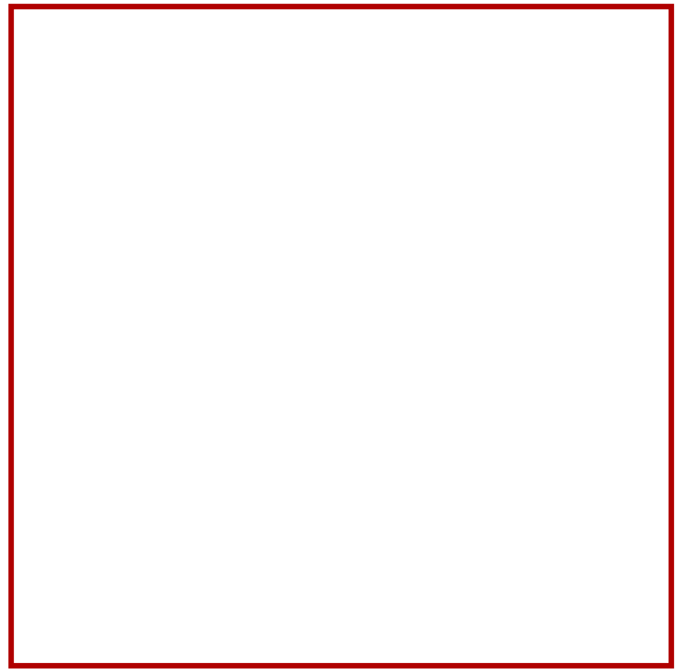
Smooth $\Rightarrow$ Weil-Petersson $\Rightarrow$ Asymptotically smooth $\Rightarrow$ chord-arc.

Asymptotically smooth $=$ $\gamma \subset \Gamma$, $\ell(\gamma) \to 0$ implies $\frac{\ell(\gamma)}{\mathrm{crd}(\gamma)} \to 1$.

Weil-Petersson curves are almost C^1 (but not quite).



Weil-Petersson curves need not be C^1.
$z(t) = \exp(-t + i \log t)$, infinite spiral.

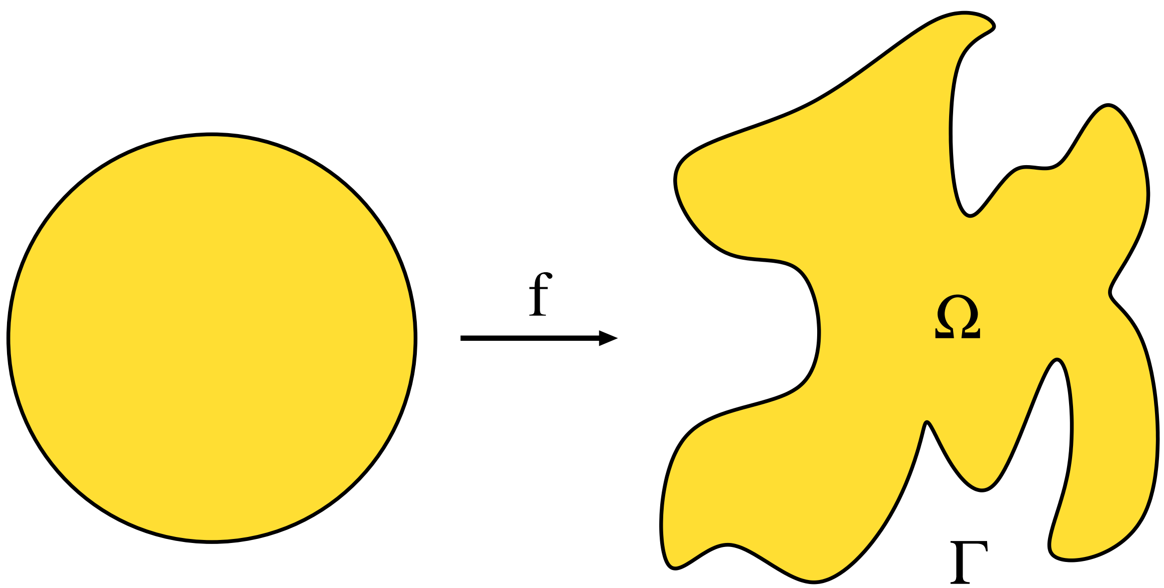


Not Weil-Petersson

In their memoir, Takhtajan and Teo prove:

Theorem: Γ is Weil-Petersson iff $\Gamma = f(\mathbb{T})$ where f is conformal map on $\mathbb{D}$ so that $u = \log f'$ is in the Dirichlet class.

Dirichlet class $= \{u : |\nabla u| \in L^2(dxdy)\} = \{u \in W^{1,2}(\mathbb{D})\}$.



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Yilin Wang proved $u \in W^{1,2}$ iff **Loewner energy** of Γ is finite (as defined by her and Steffen Rohde). Her IPAM talk connected WP to **large deviations of Schramm-Loewner evolutions** and the **Brownian loop soup** of Lawler and Werner.






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In 1990's Astala, Zinsmeister invented "BMO-Teichmüller theory" where $\log f' \in \text{BMO}$.

Jones and I characterized such Γ.

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Dirichlet class $= \{u : |\nabla u| \in L^2(dxdy)\} = \{u \in W^{1,2}(\mathbb{D})\}$.

Corollary: Γ is WP iff $u = \log f' \in H^{1/2}(\mathbb{T})$ (Sobolev trace thm).

$H^{1/2} =$ Sobolev space $=$ half a derivative in L^2.

$$u(e^{i\theta}) = \sum a_n e^{in\theta}, \quad \text{where } \sum n|a_n|^2 < \infty.$$

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|u(z)-u(w)|^2}{|z-w|^2}|dz||dw| < \infty,$$

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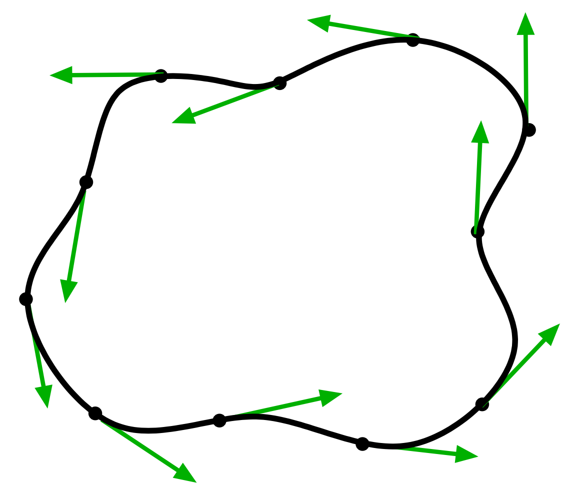
Since $\arg z = \mathrm{Im}(\log \mathrm{z})$, we have:

$$\begin{aligned} \log f' \in H^{1/2} &\Rightarrow \arg f' \in H^{1/2} \\ &\Rightarrow \exp(i \arg f') \in H^{1/2} \\ &\Rightarrow f'/|f'| \in H^{1/2} \end{aligned}$$



Theorem: Γ is WP iff the arc-length parameterization is in $H^{3/2}(\mathbb{T})$.

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This was proven implicitly in early version of my paper, but I didn't notice. David Mumford pointed out it followed from other characterizations.

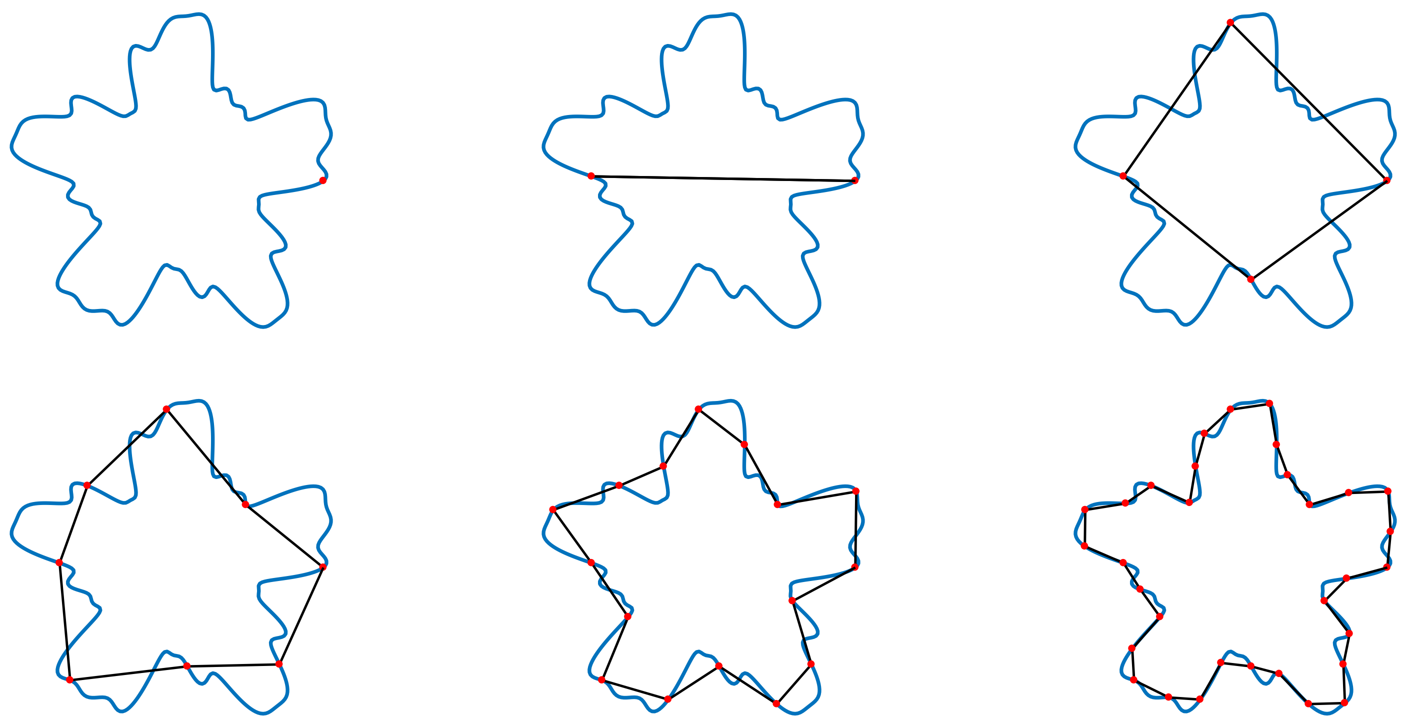
Takhtajan and Teo proved tangent space of $T_0(1)$ is $H^{3/2}$.

$H^{3/2}$ curves arise naturally in other areas, e.g., knot theory (ask me later).

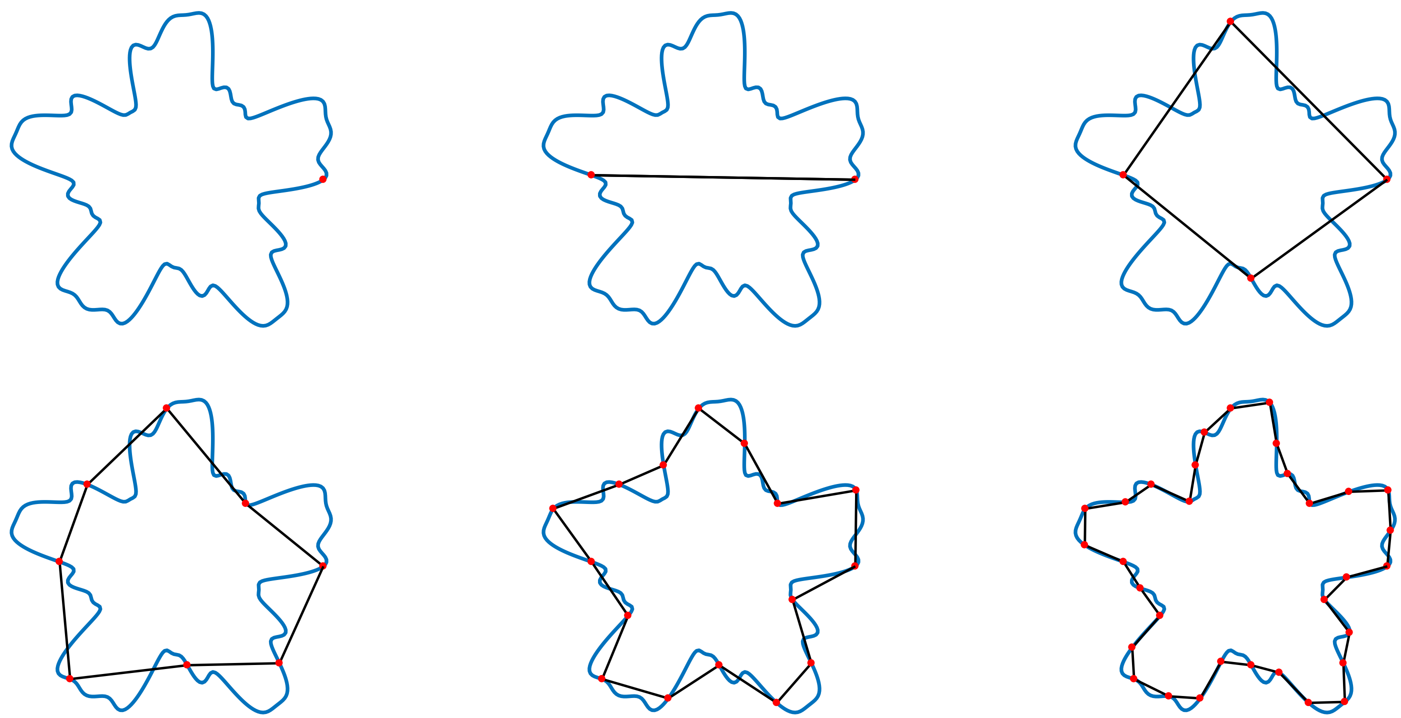
Teichmüller spaces and $H^{3/2}$, are pretty sophisticated.

How can you describe WP curves to a calculus student?

Dyadic decomposition. Choose a base point $z_1^0 \in \Gamma$ and for each $n \geq 1$, let $\{z_j^n\}$, $j = 1, \ldots, 2^n$ be the unique set of ordered points with $z_1^n = z_1^0$ that divides Γ into 2^n equal length intervals (called the nth generation dyadic subintervals of Γ).



Let Γ_n be the inscribed 2^n-gon with these vertices. Clearly $\ell(\Gamma_n) \nearrow \ell(\Gamma)$.



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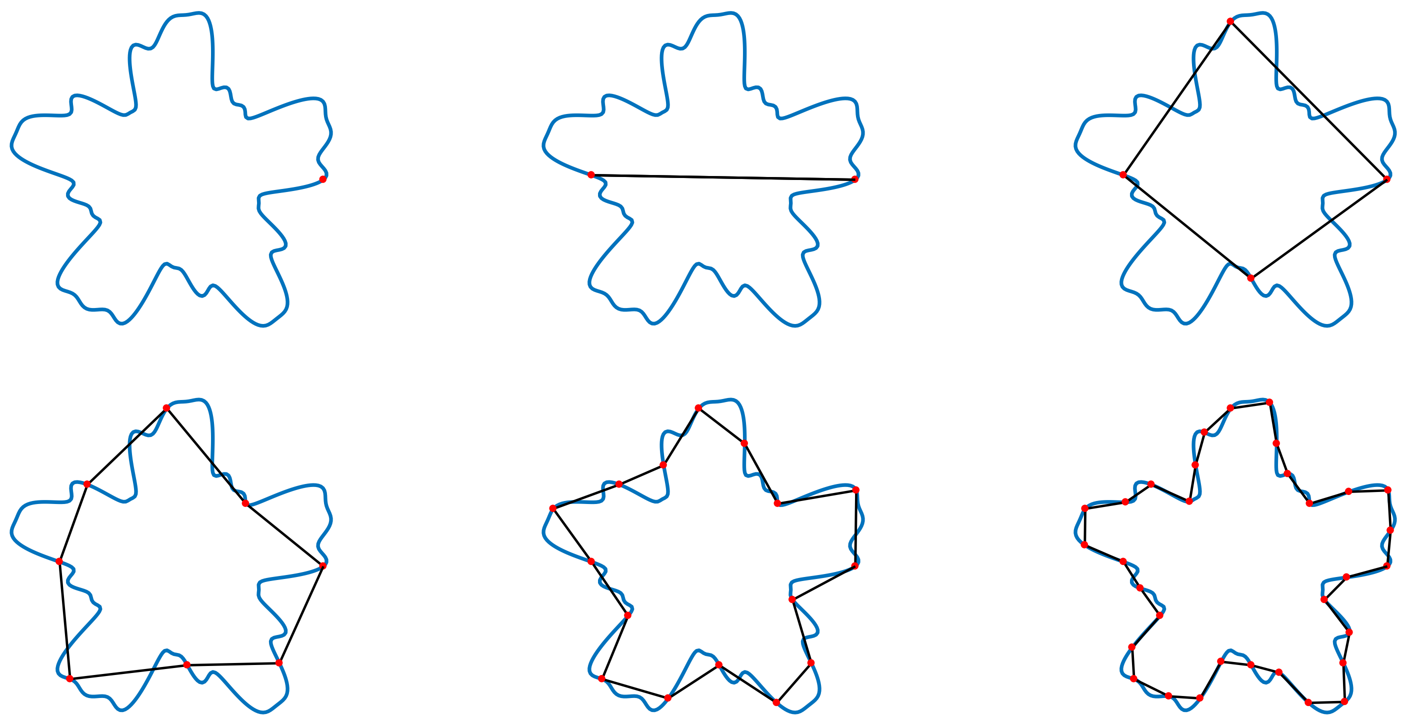


Theorem: Γ is Weil-Petersson if and only if

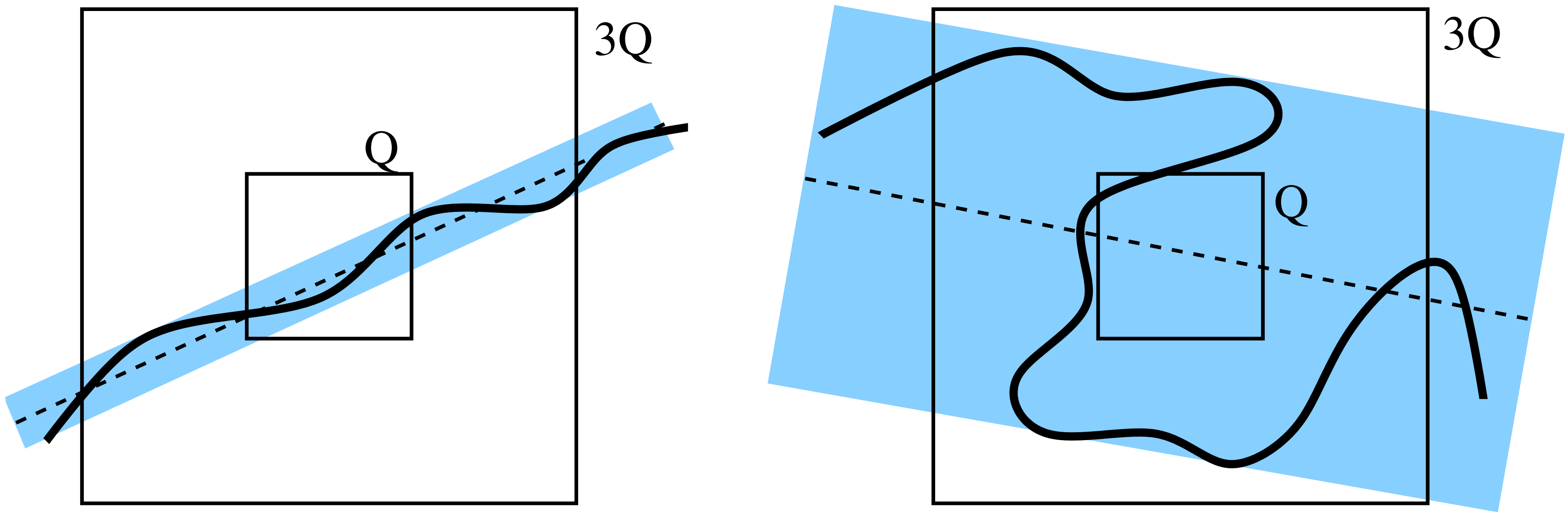
$$\sum_{n=1}^{\infty} 2^n \left[\ell(\Gamma) - \ell(\Gamma_n)\right] < \infty$$

with a bound that is independent of the choice of the base point.

Peter Jones's β-numbers:

$$\beta_\Gamma(Q) = \inf_L \sup\{\frac{\mathrm{dist}(z,L)}{\mathrm{diam}(Q)} : z \in 3Q \cap \Gamma\},$$

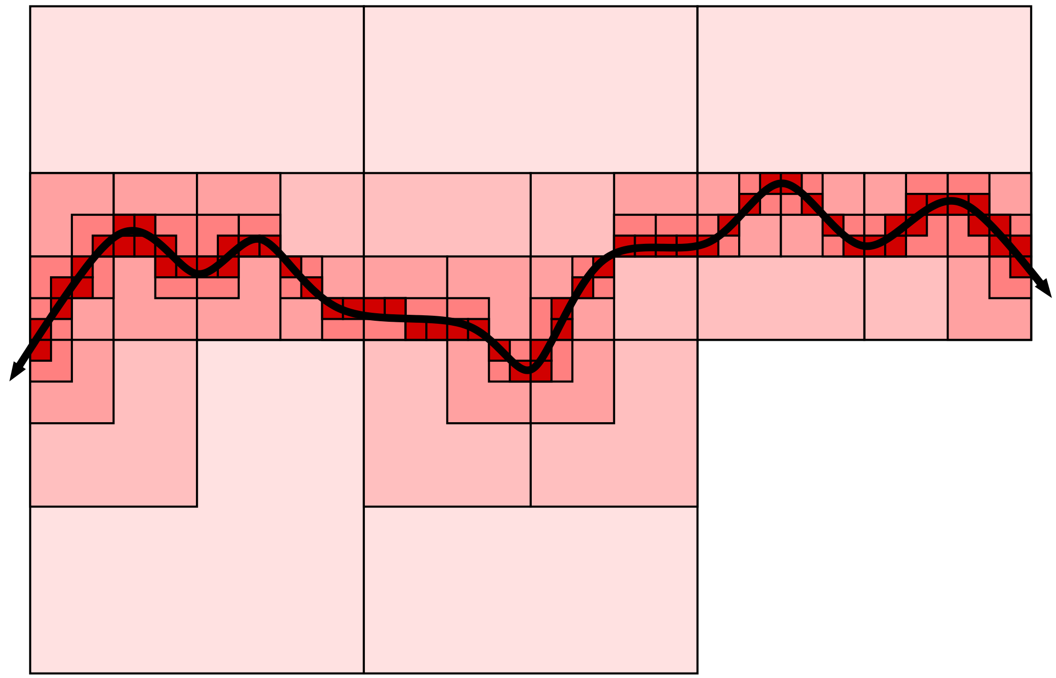
where the infimum is over all lines L that hit $3Q$.



Jones invented the β-numbers as part of his traveling salesman theorem:

$$\ell(\Gamma) \simeq \mathrm{diam}(\Gamma) + \sum_Q \beta_\Gamma(Q)^2 \mathrm{diam}(Q),$$

where the sum is over all dyadic cubes Q in $\mathbb{R}^n$.



Previous theorem on length should have translation into β-numbers.

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Theorem: Γ is Weil-Petersson iff

$$\sum_Q \beta_\Gamma(Q)^2 < \infty,$$

where the sum is over all dyadic cubes.

WP curves have "curvature in L^2, integrated over all positions and scales".

Jones invented the β-numbers as part of his traveling salesman theorem:

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Theorem: Γ is Weil-Petersson iff

$$\sum_Q \beta_\Gamma(Q)^2 < \infty,$$

where the sum is over all dyadic cubes.

Proof requires improvement of TST for curves:

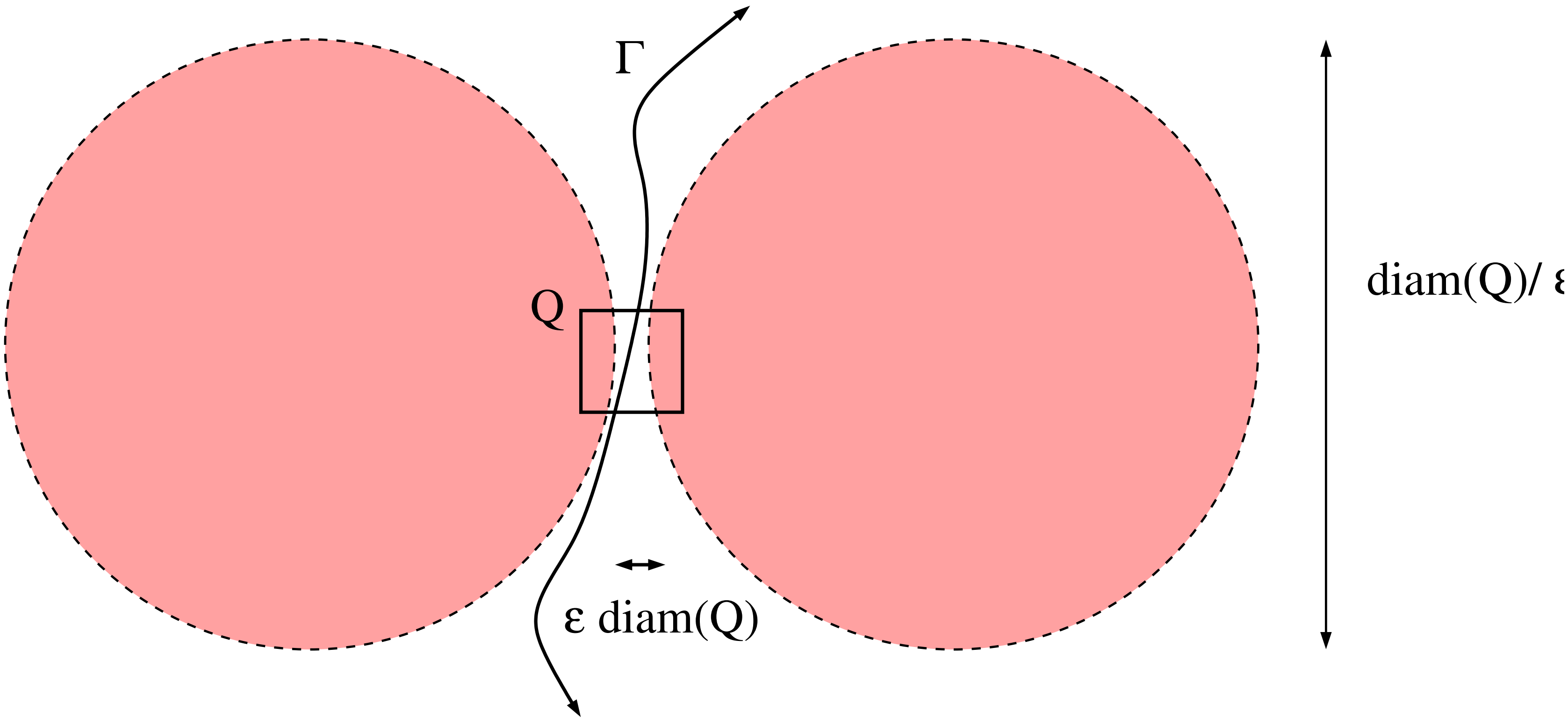
$$\ell(\Gamma) - \mathrm{crd}(\Gamma) \simeq \sum_Q \beta_\Gamma(Q)^2 \mathrm{diam}(Q),$$

The Weil-Petersson class is Möbius invariant.

The β-numbers are not: lines ($\beta = 0$) can map to circles ($\beta > 0$).

What is a Möbius invariant version of the β-numbers?

First idea: let $\varepsilon_\Gamma(Q)$ be smallest ϵ so that there are disks D, D' of radius $\ell(Q)/\epsilon$, separated by Γ, and with $\mathrm{dist}(Q \cap D, Q \cap D') \le \epsilon \ell(Q)$.

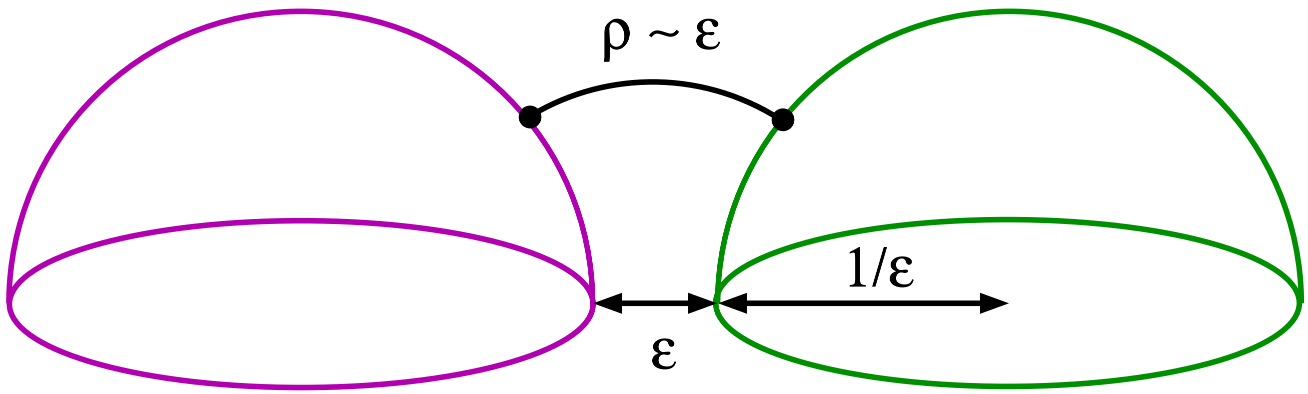


Easy to check $\beta(Q) \lesssim \varepsilon(Q)$. Converse can fail, but

Theorem: $\sum_Q \varepsilon^2(Q) < \infty$ iff $\sum_Q \beta^2(Q) < \infty$ iff Γ is WP.

Each disk is the base of a hemisphere in the upper half-space $\mathbb{H}^3 = \mathbb{R}^3_+$.

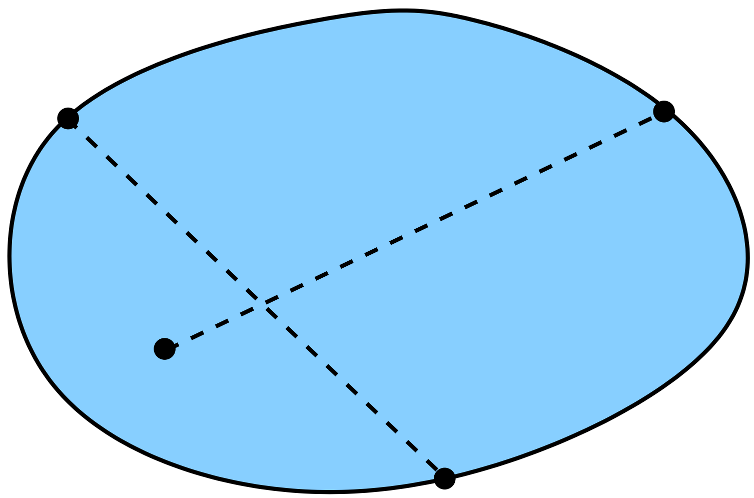
The hyperbolic distance between these hemispheres is $\lesssim \varepsilon(Q)$.



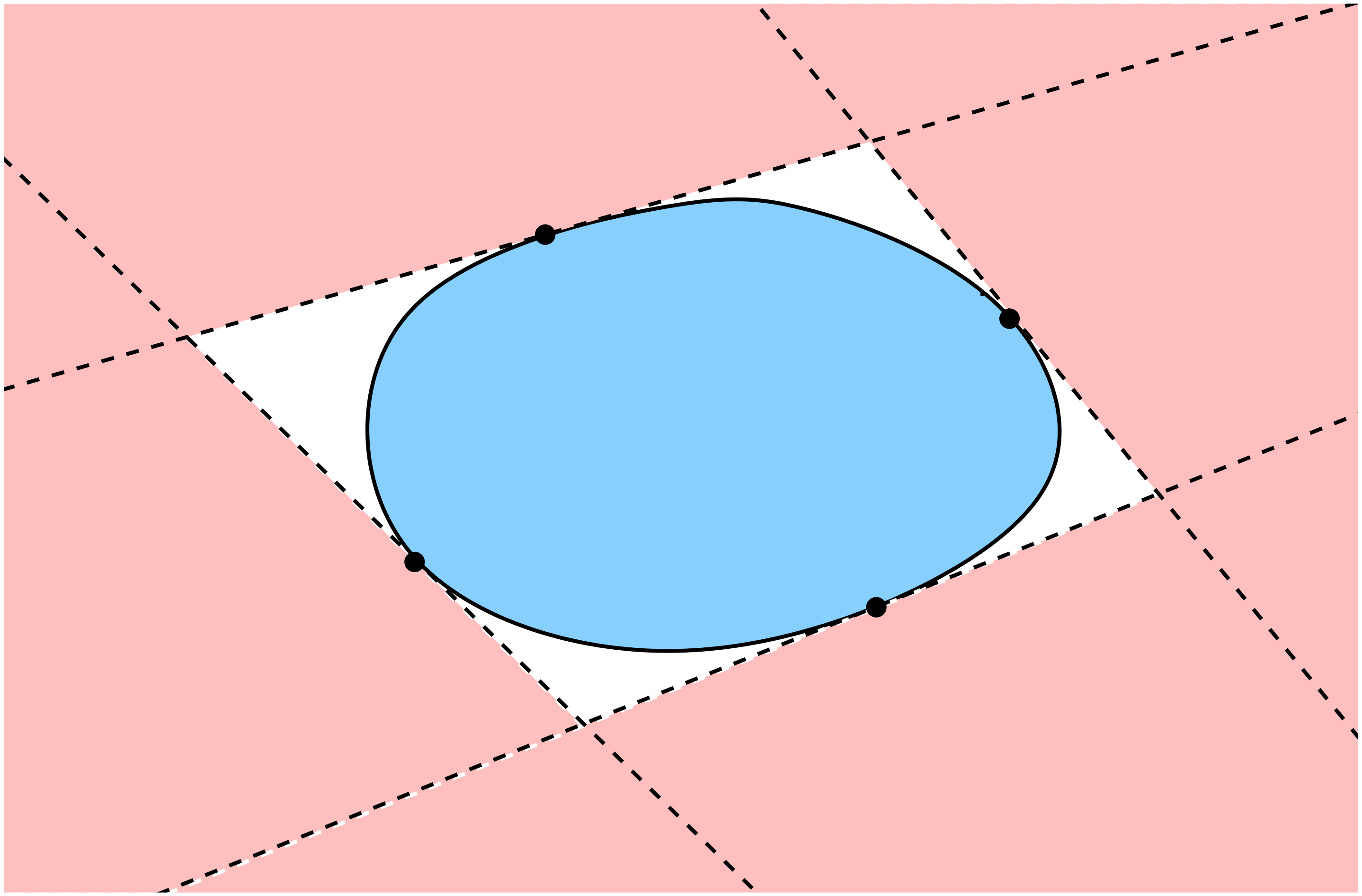
These Euclidean hemispheres are hyperbolic half-spaces.

This connects WP class to convex sets in hyperbolic upper half-space.

Usual definition of convex: contains geodesic between any two points.



More useful for us: complement is a union of half-spaces.

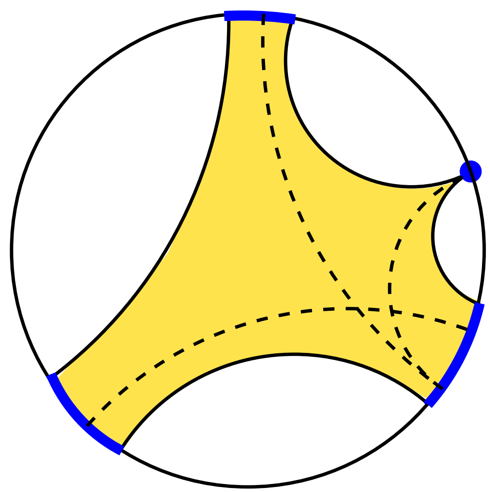


The hyperbolic length of a (Euclidean) rectifiable curve in the unit disk $\mathbb{D}$ or in the n-dimensional ball $\mathbb{B}^n$ is given by integrating

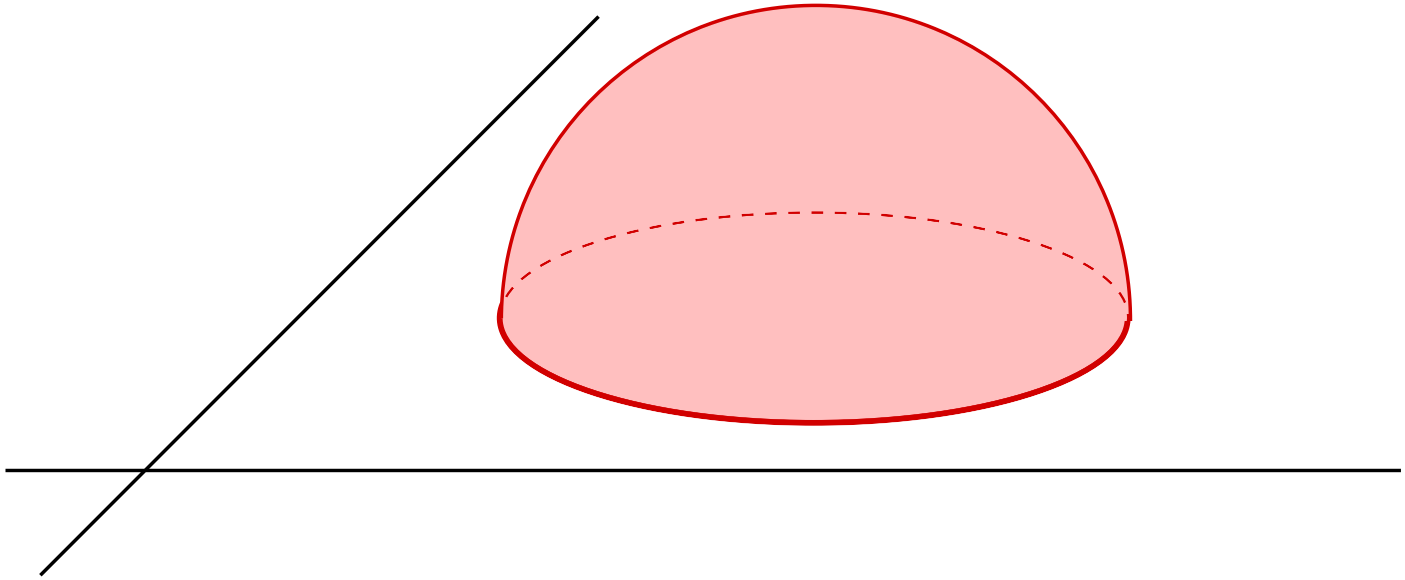
$$d\rho = \frac{ds}{1-|z|^2},$$

along the curve. In the upper half-space $\mathbb{H}^n$ we integrate $d\rho = ds/2t$.

Geodesics are circles (or lines) perpendicular to boundary.



Hyperbolic convex hull of a boundary set



In $\mathbb{R}^3_+$, a hyperbolic half-space = hemisphere.

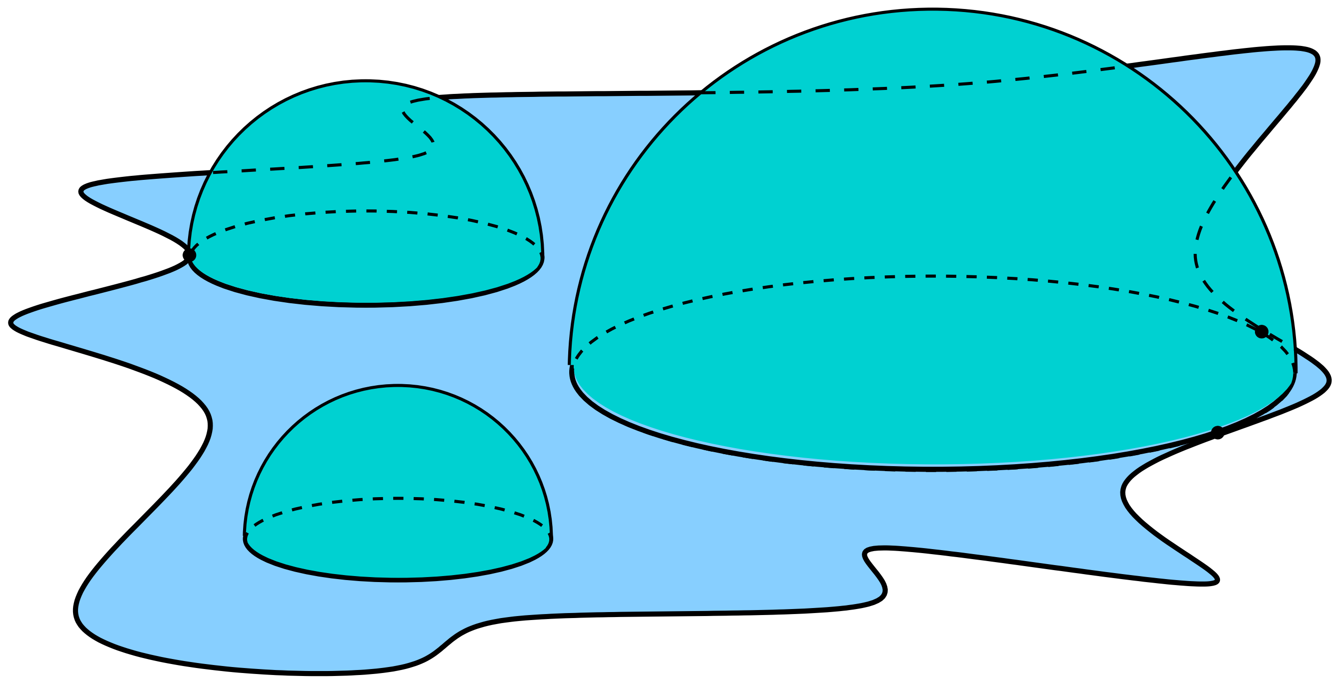
The hyperbolic convex hull of $\Gamma \subset \mathbb{R}^2$, is the smallest convex set containing all geodesics with endpoints in Γ.

$=$ CH(Γ) = complement of all open half-spaces that miss Γ.

In general, CH(Γ) has non-empty interior and 2 boundary components.

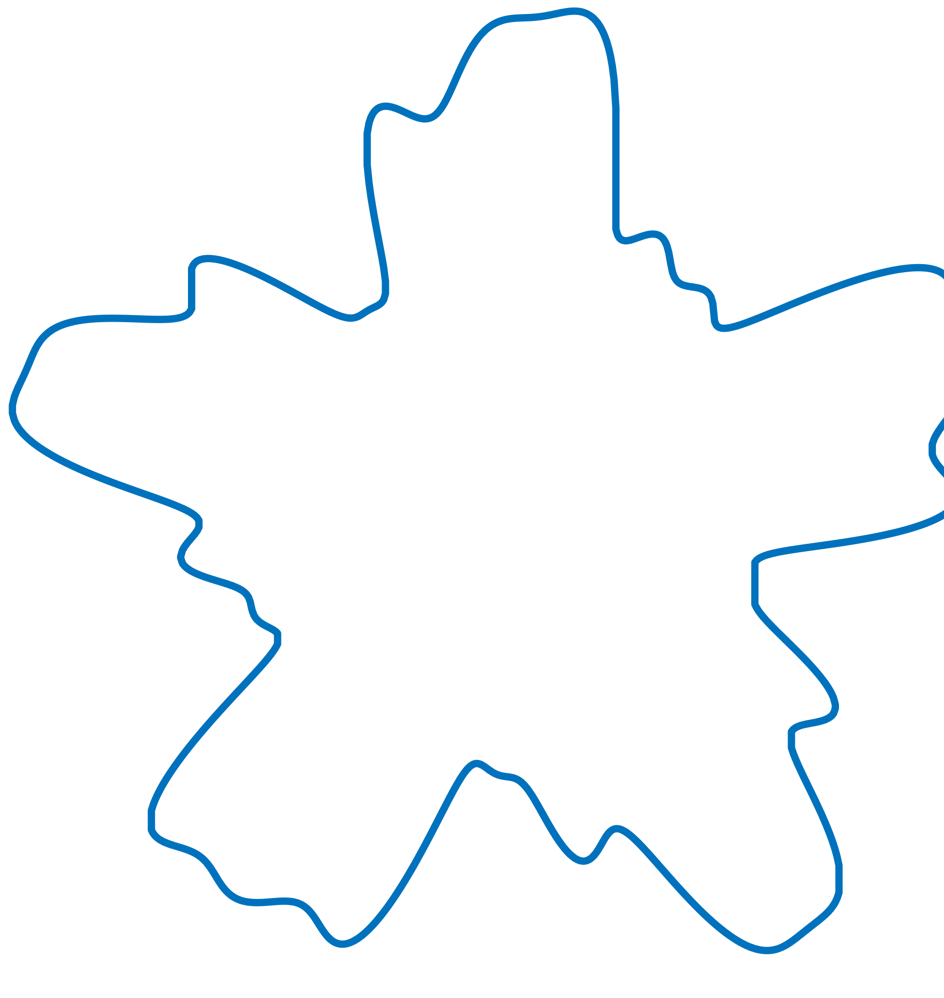
A hyperbolic half-space missing CH(Γ) has boundary disk missing Γ.

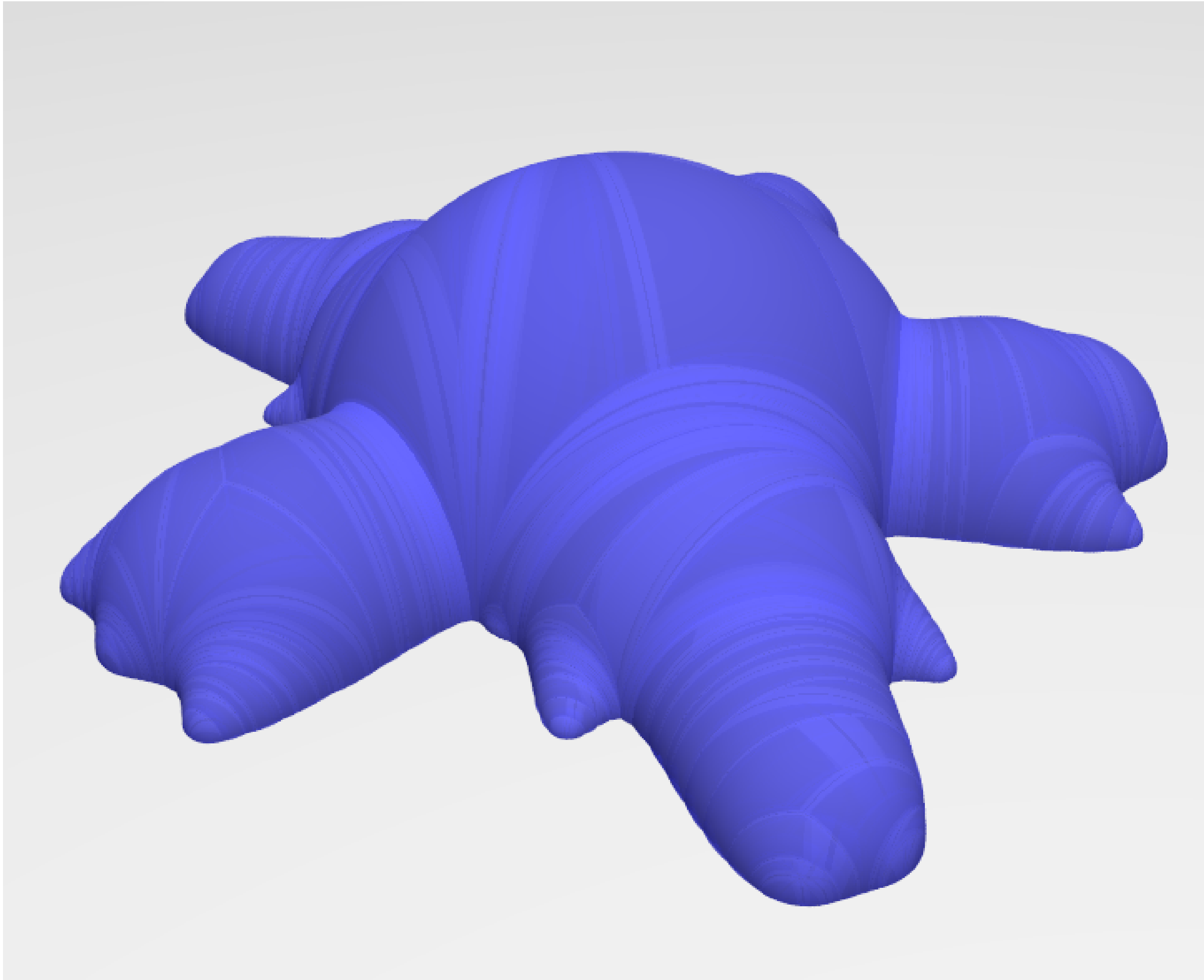
This disk is inside or outside Γ. Dome(Ω) is union over "inside" disks.

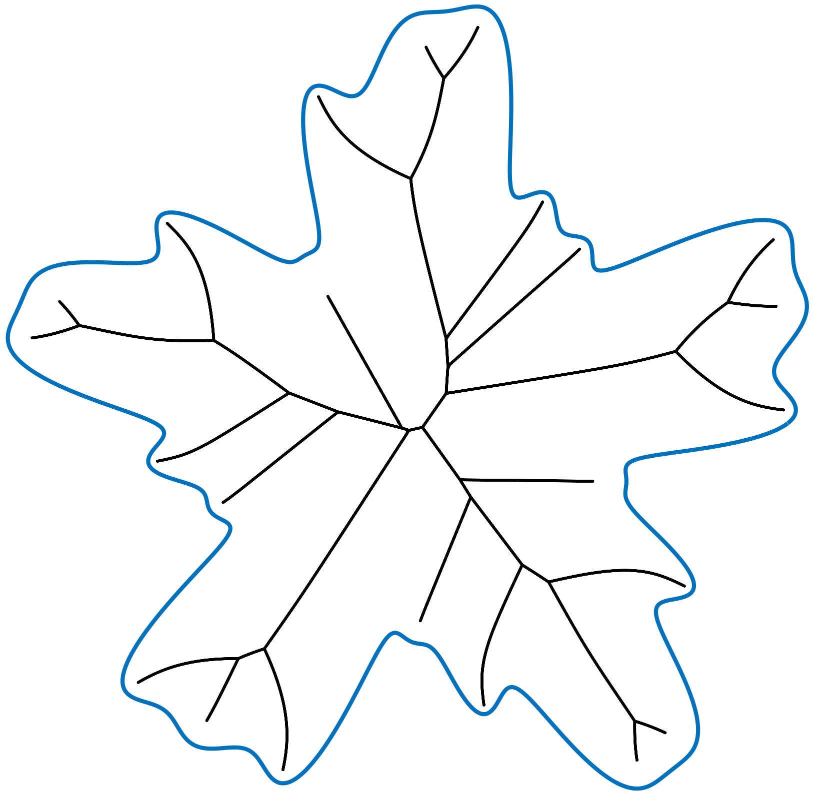


Region above dome is intersection of half-spaces, hence convex.

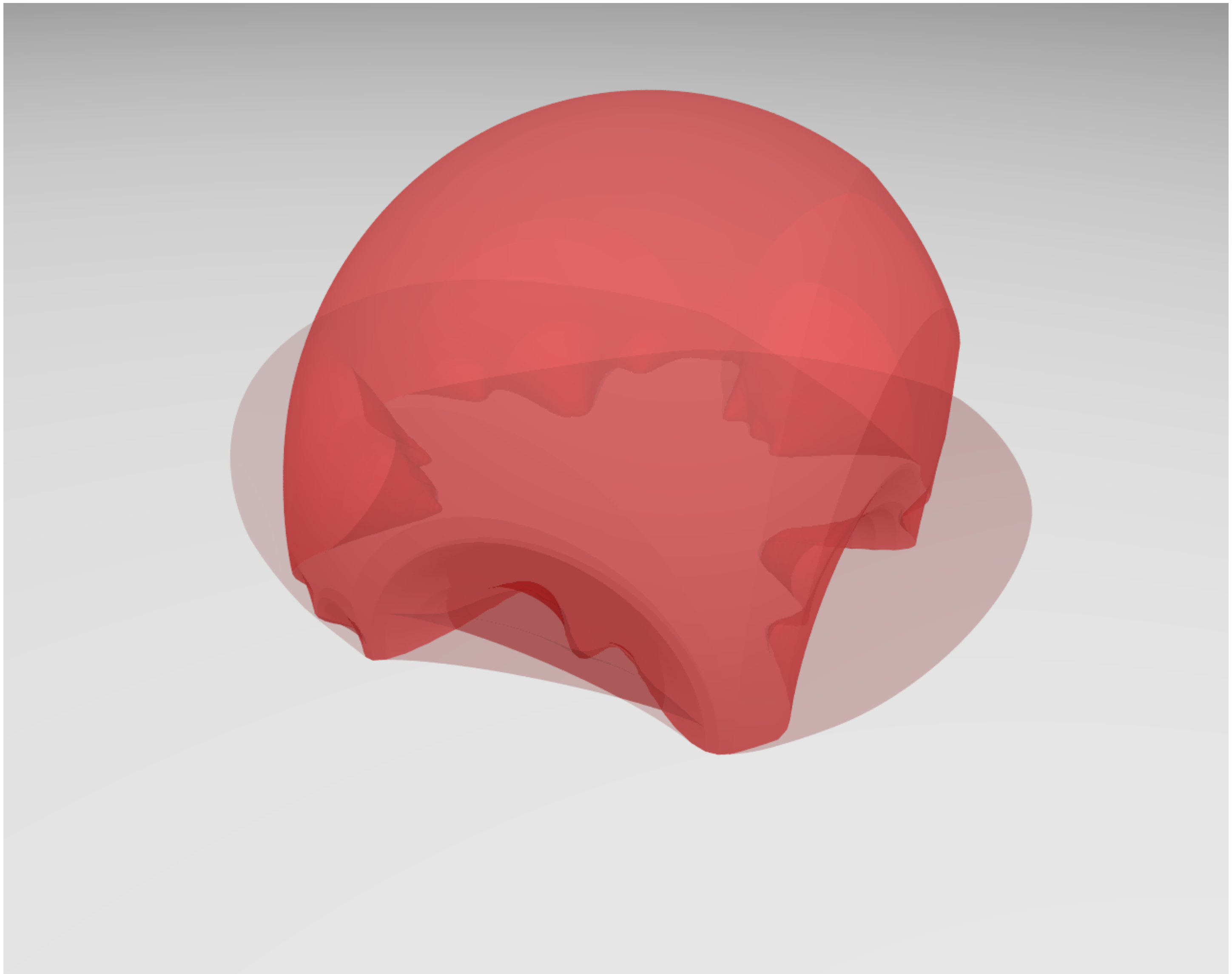
CH(Γ) is region between domes for "inside" and "outside" of Γ.

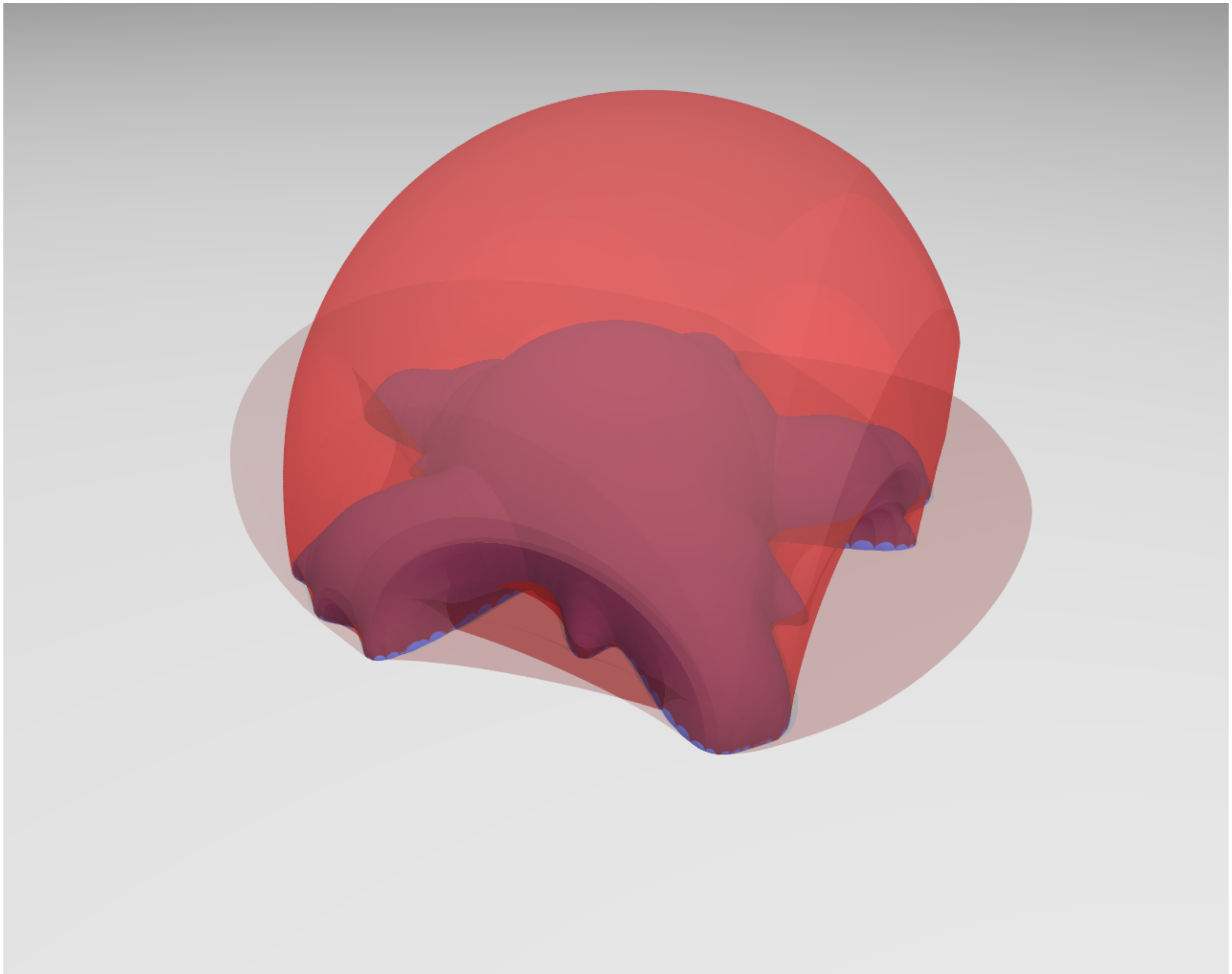


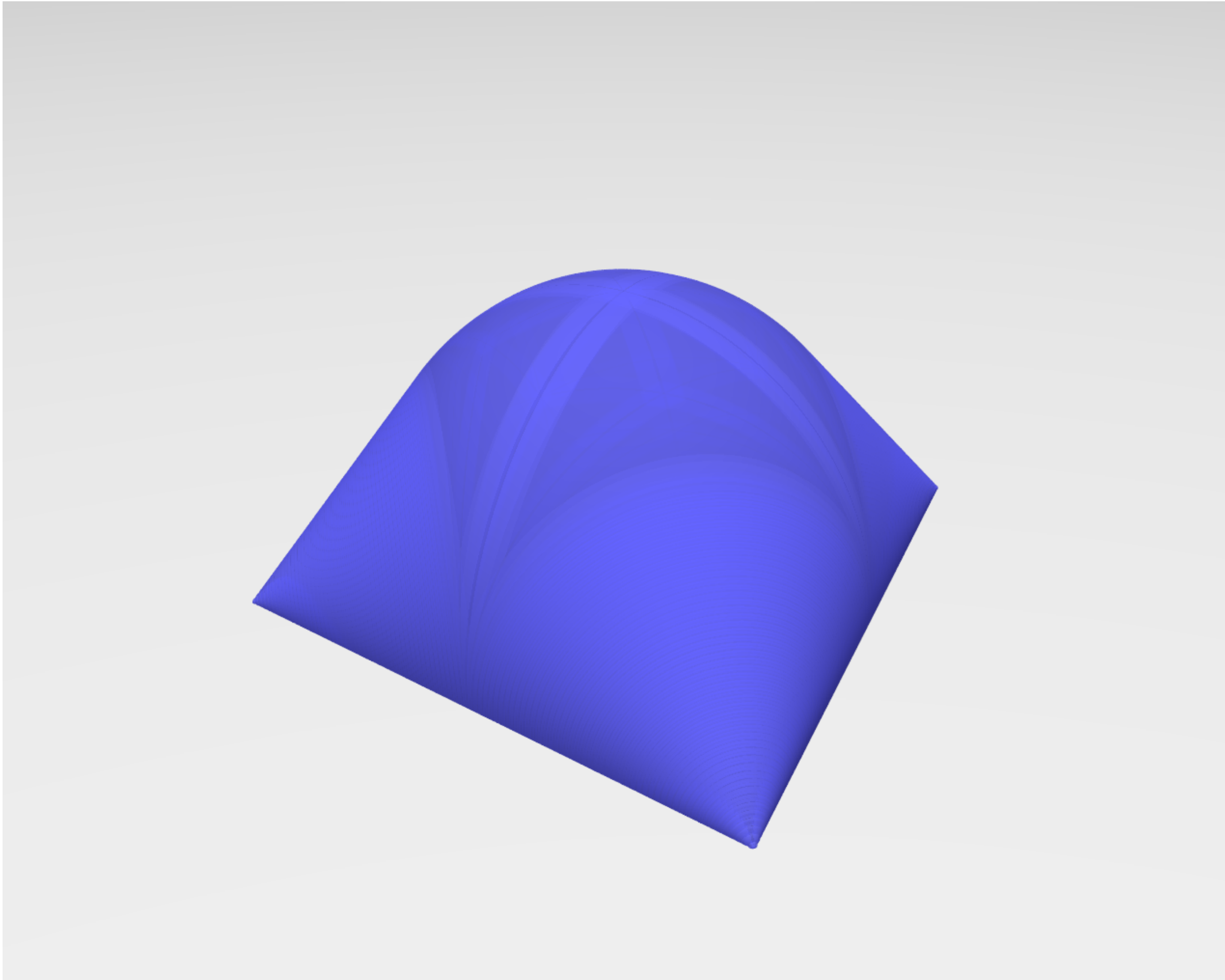


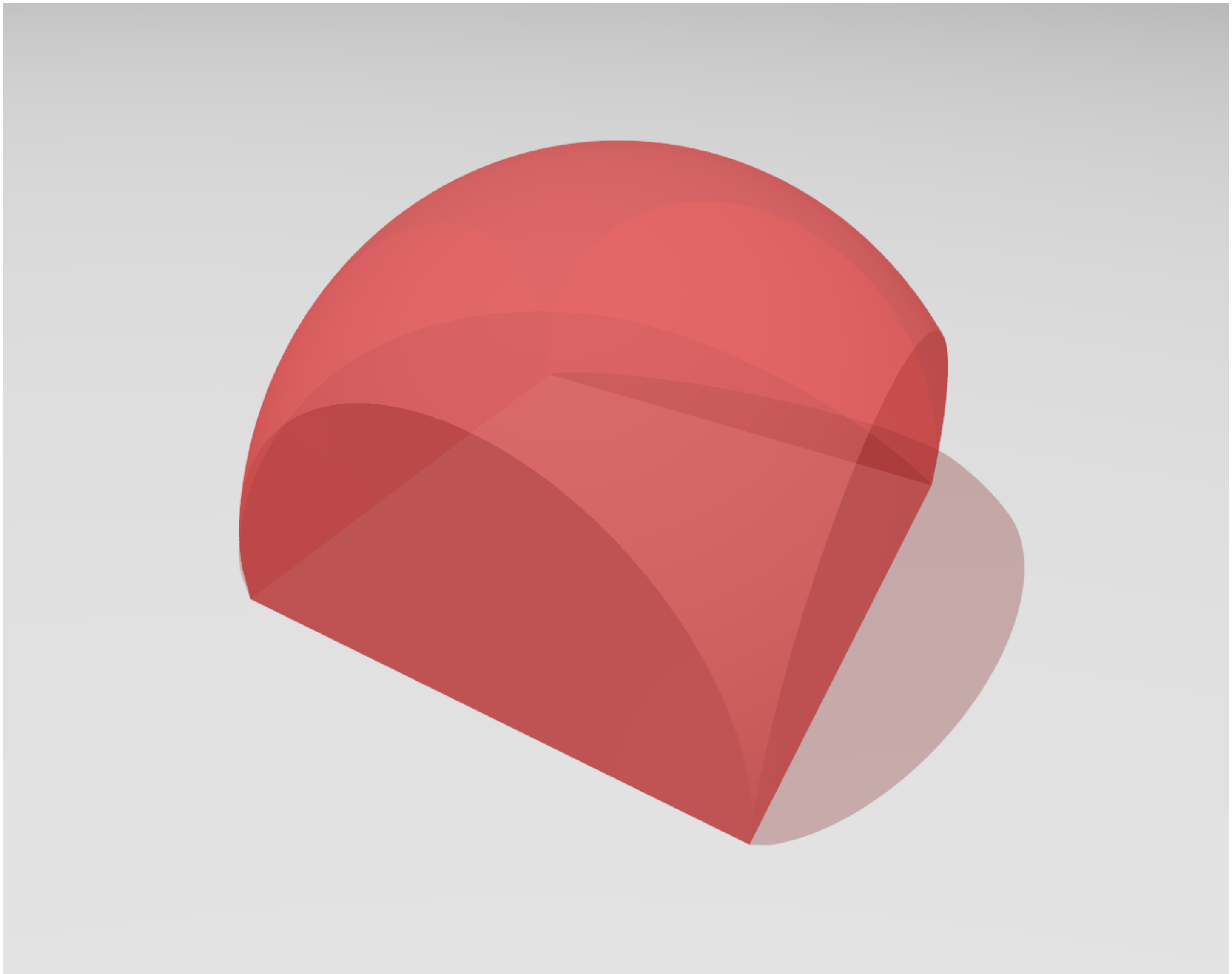


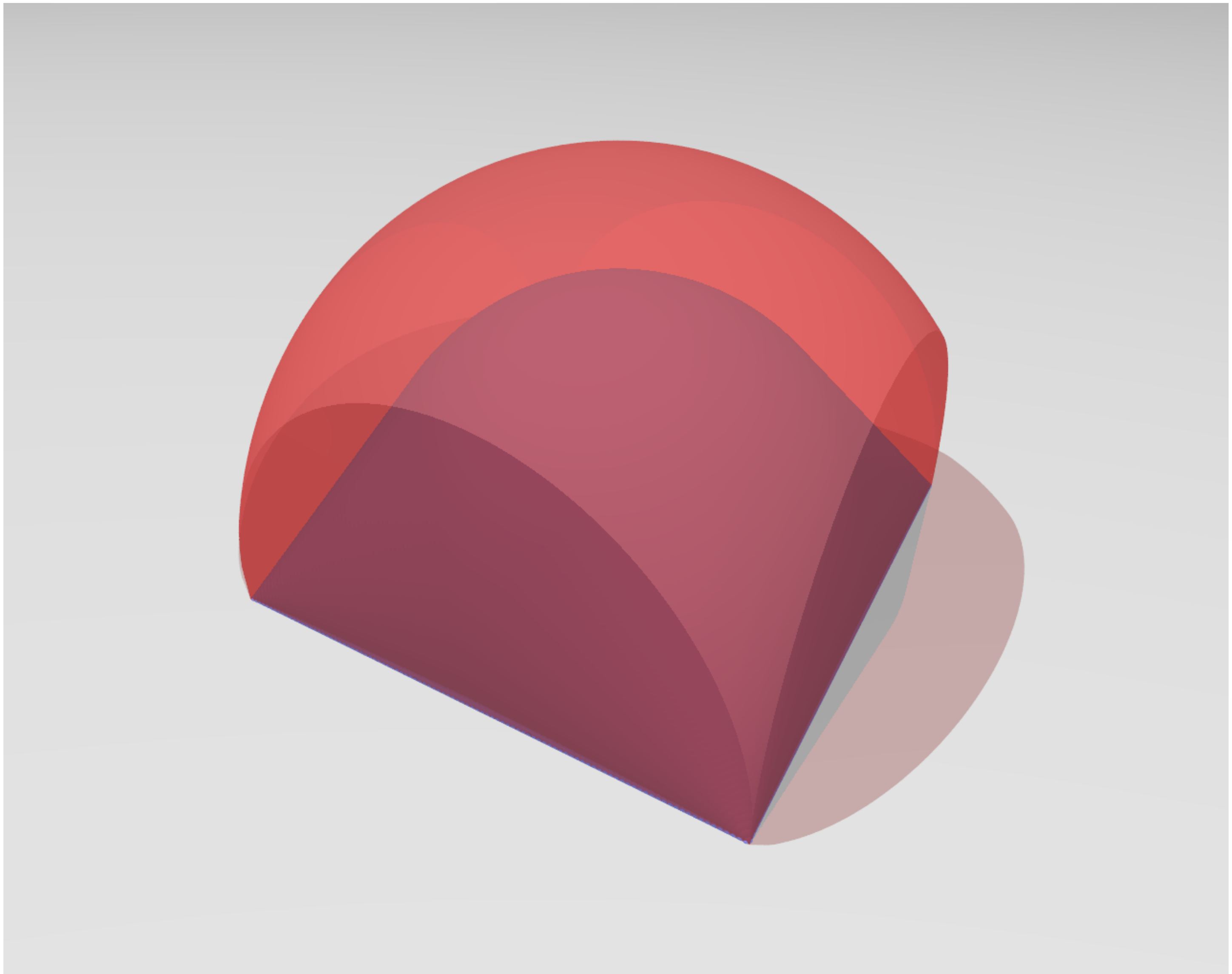
The medial axis. Equidistant from at least two boundary points.
Corresponding hemispheres give the dome.
Medial axis is widely studied in computational geometry.

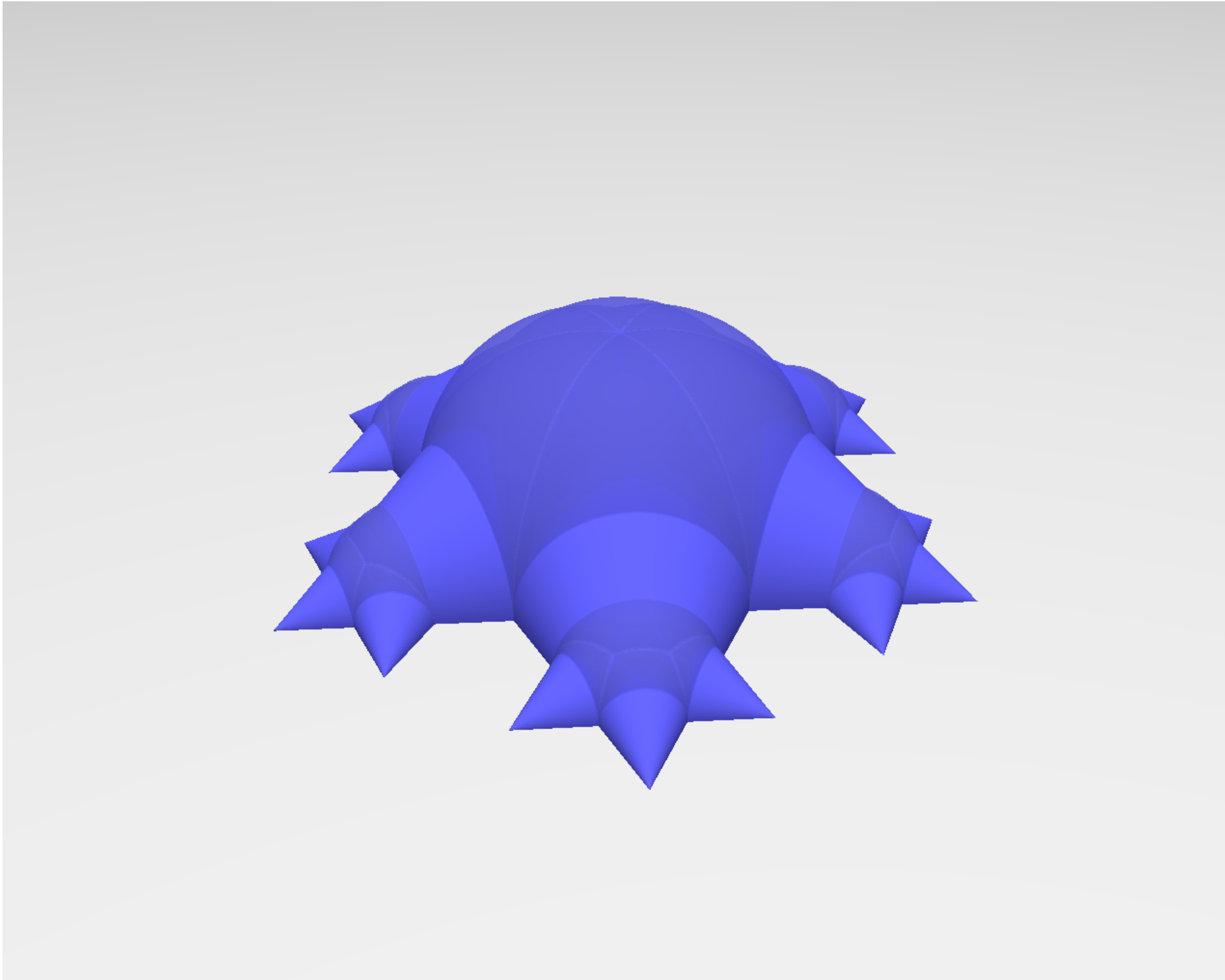


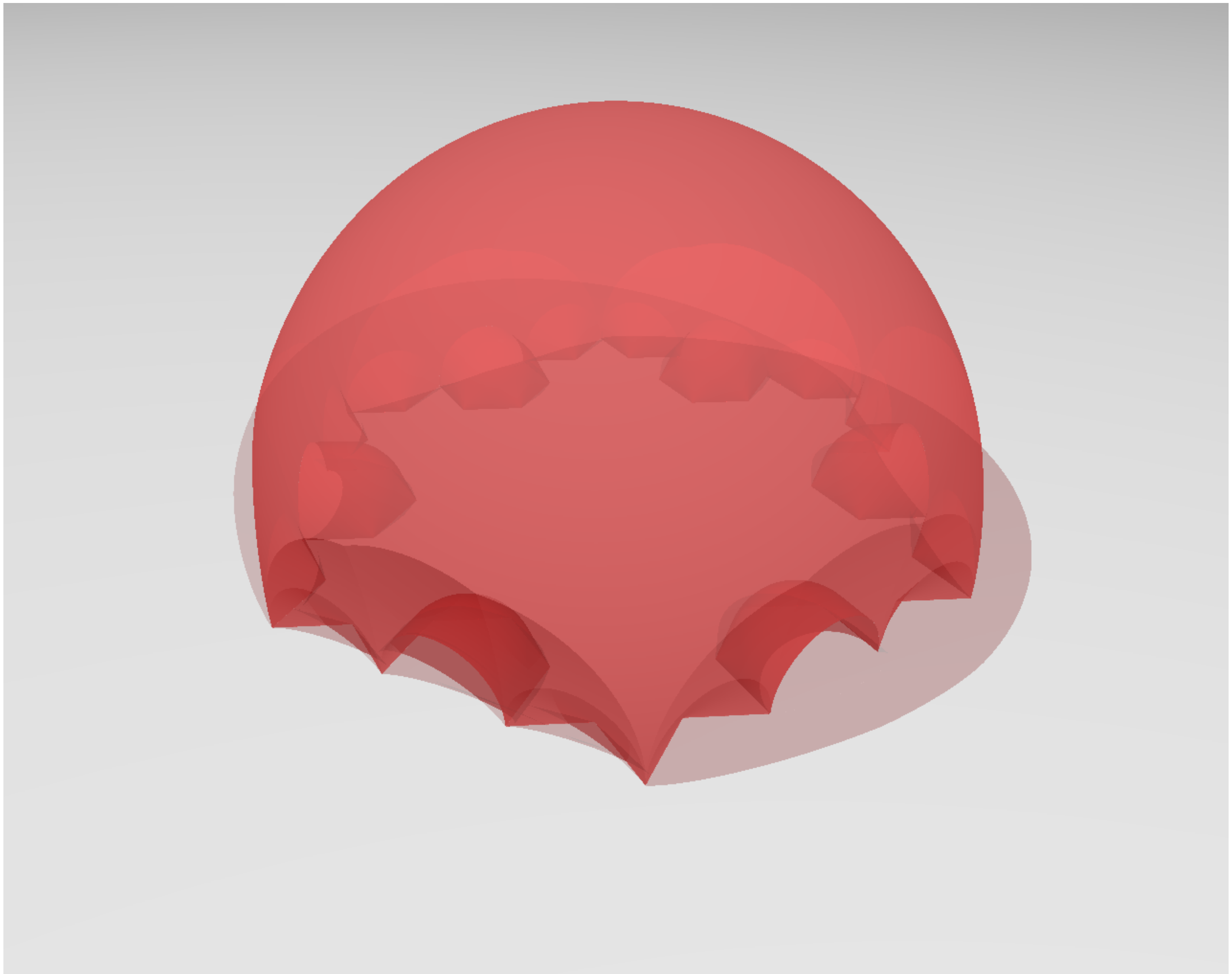


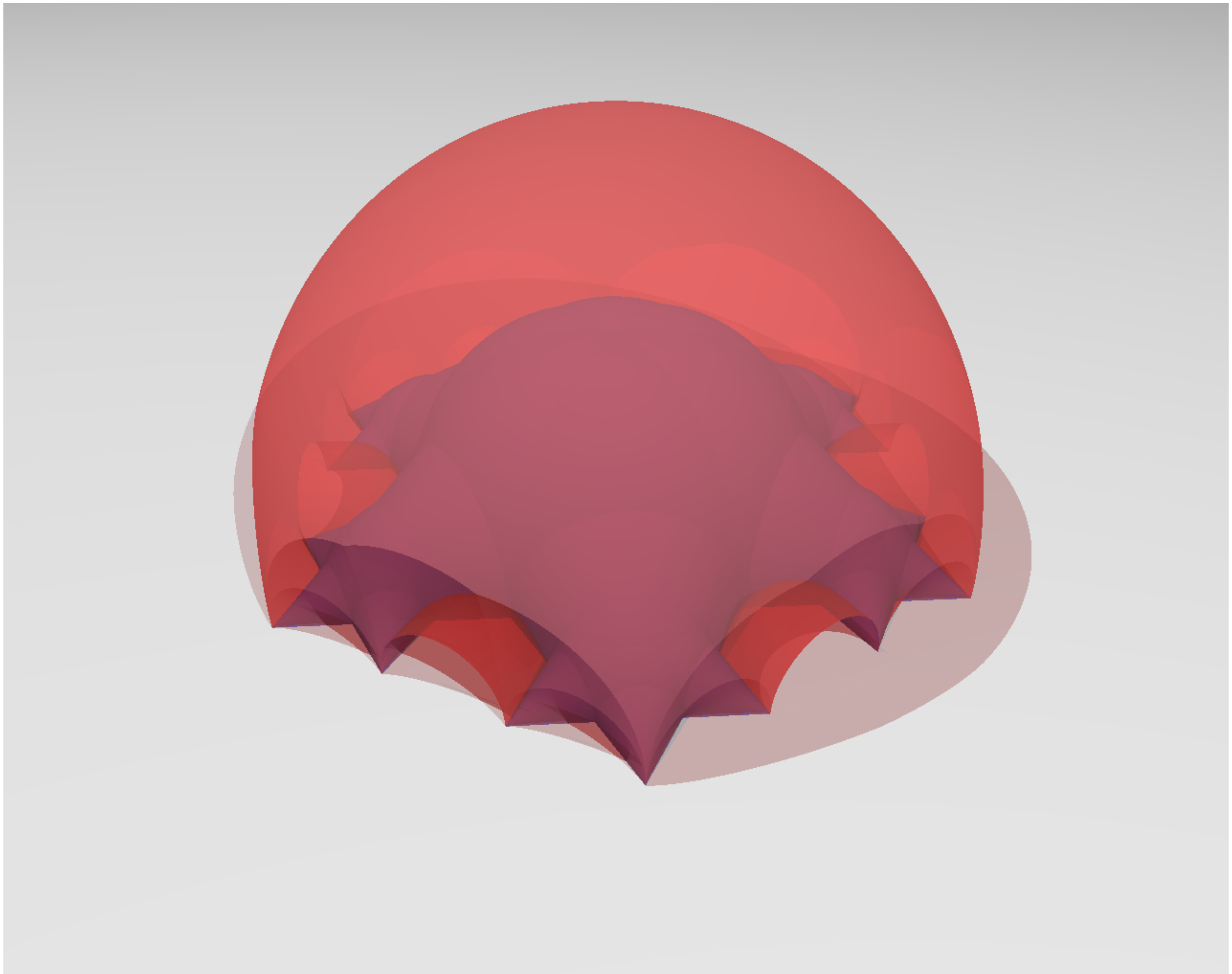




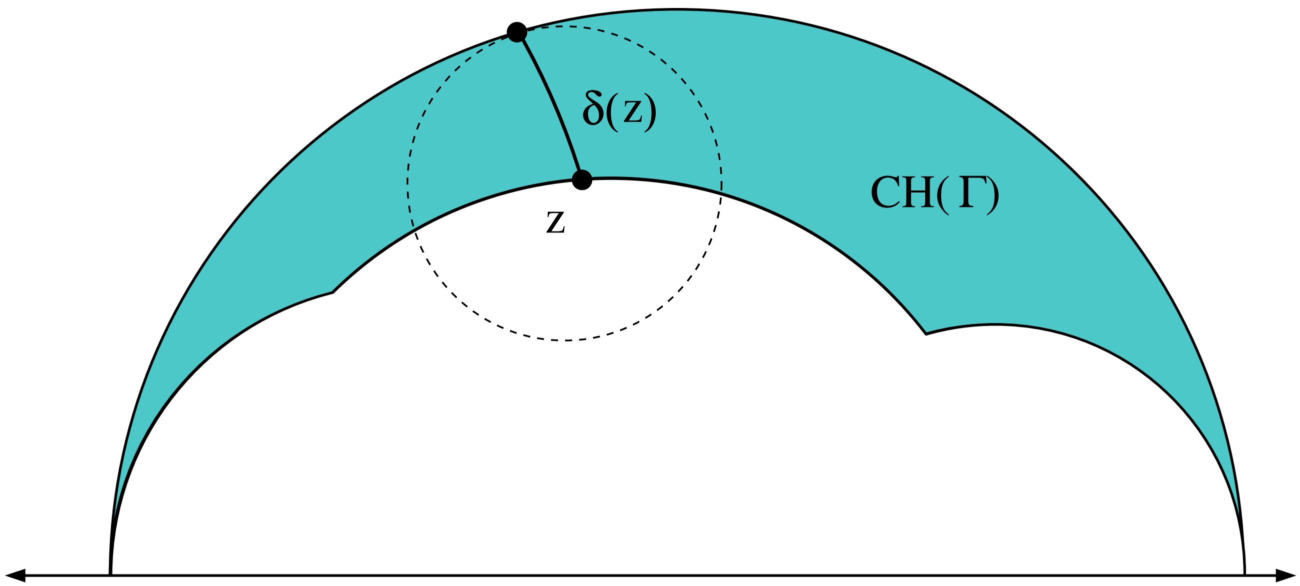








We define $\delta(z)$ to be the maximum distance from z to the components of $\partial \mathrm{CH}(\Gamma)$. Can show $\delta(z) \lesssim \varepsilon_\Gamma(Q)$, for some Q.



Theorem: Γ is Weil-Petersson iff $\int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z) dA_\rho < \infty$.

Easy to see: Quasicircle $\Rightarrow$ $\delta \in L^\infty$ $\not\Rightarrow$ Quasicircle.

Let S be a surface in $\mathbb{H}^3$ that has asymptotic boundary Γ.

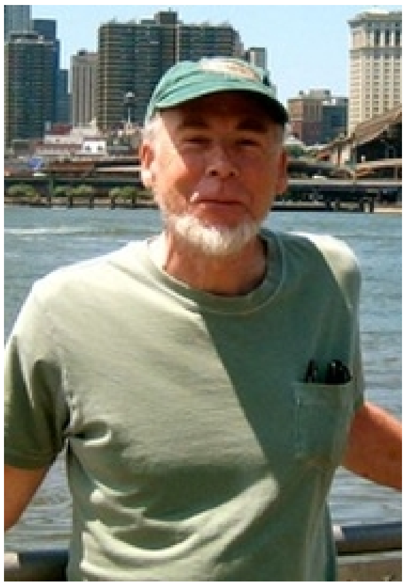
We let $K(z)$ denote the Gauss curvature of S at z.

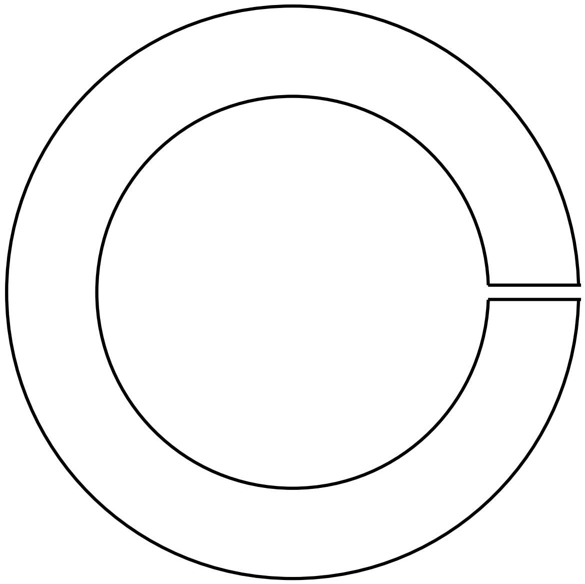
Gauss equation says $K(z) = -1 + \kappa_1(z)\kappa_2(z)$ (principle curvatures).

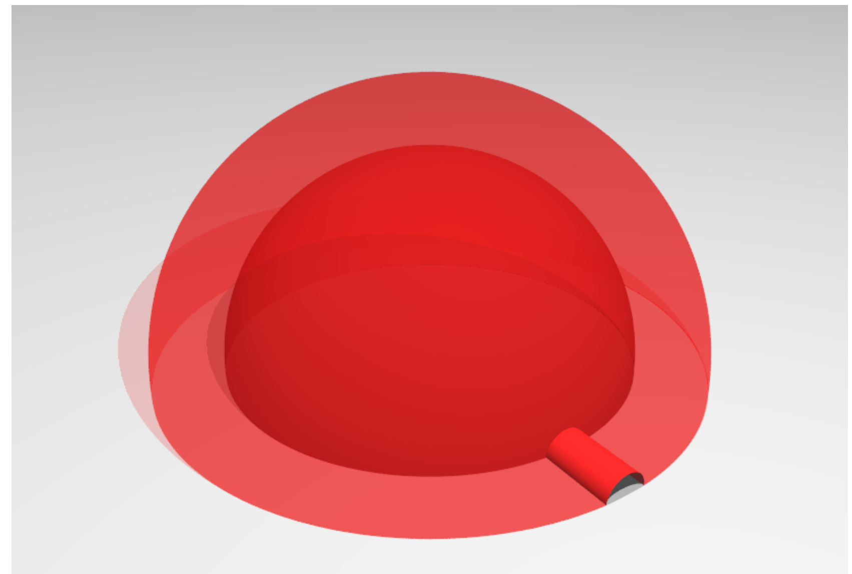
S is a **minimal surface** if $\kappa_1 = -\kappa_2$ (the mean curvature is zero).

In that case, $K(z) = -1 - \kappa^2(z) \leq -1$.

Theorem (Anderson, 1983): Every closed Jordan curve $\Gamma \subset \mathbb{R}^2$ bounds a minimal disk $S \subset \mathrm{CH}(\Gamma) \subset \mathbb{H}^3$.







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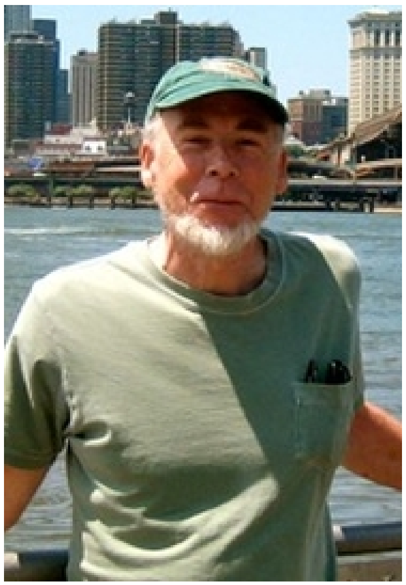

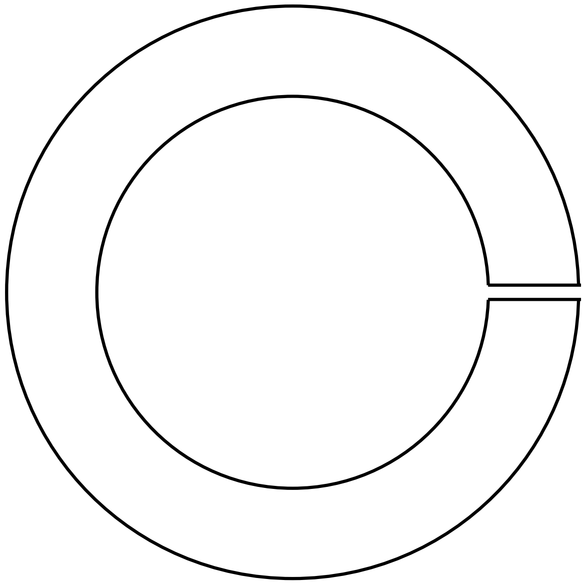

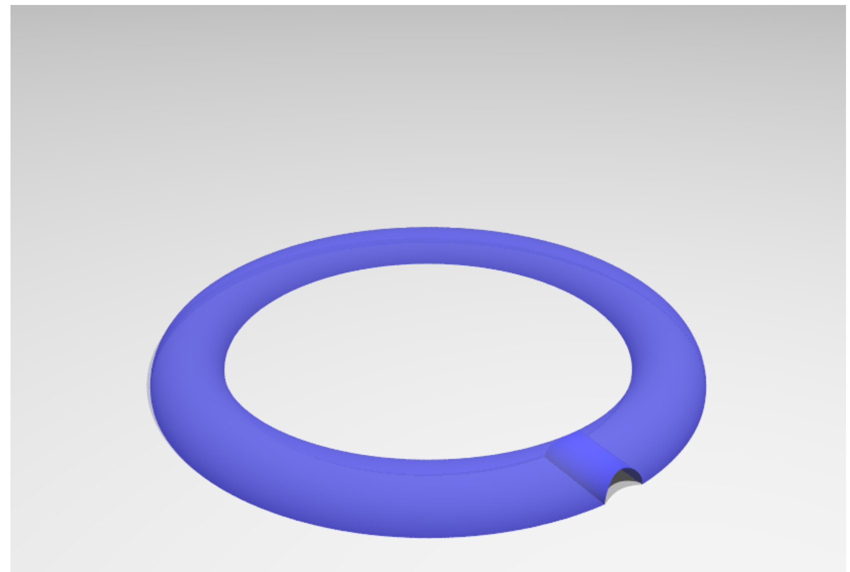

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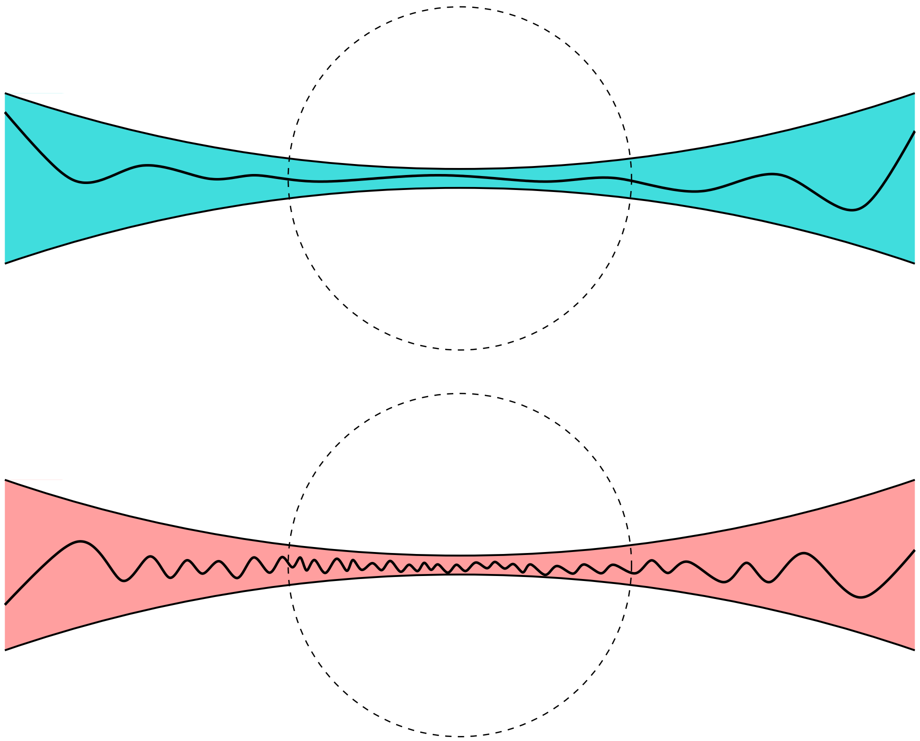
Theorem (Seppi, 2016): Principle curvatures satisfies $\kappa(z) = O(\delta(z))$.



$\sinh(\mathrm{dist}(z, P))$ satisfies $\Delta_S u - 2u = 0$. Use Schauder estimate $\|\nabla^2 u\|_\infty \leq C\|u\|_\infty$.

Theorem (Anderson, 1983): Every closed Jordan curve $\Gamma \subset \mathbb{R}^2$ bounds a minimal disk $S \subset \mathrm{CH}(\Gamma) \subset \mathbb{H}^3$.

Theorem (Seppi, 2016): Principle curvatures satisfies $\kappa(z) = O(\delta(z))$.

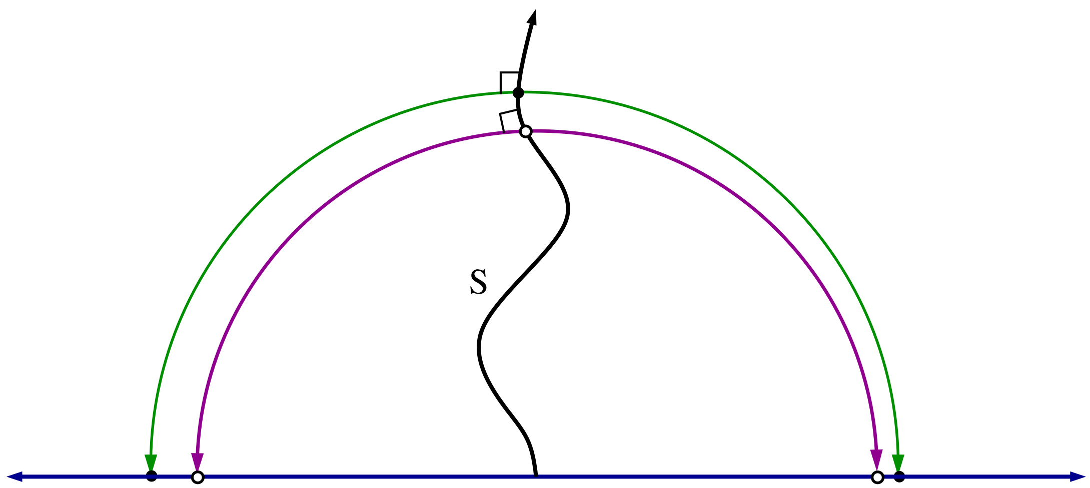
Theorem: Γ is WP iff it bounds a minimal disk with finite total curvature

$$\int_S |K+1| dA_\rho = \int_S \kappa^2(z) dA_\rho < \infty.$$

Cor: Boundary of finite total curvature surface need not be C^1.

Gauss map: follow normal geodesic from surface S to $\mathbb{R}^2 = \partial\mathbb{H}^3$.

Two directions. Defines reflection across Γ.



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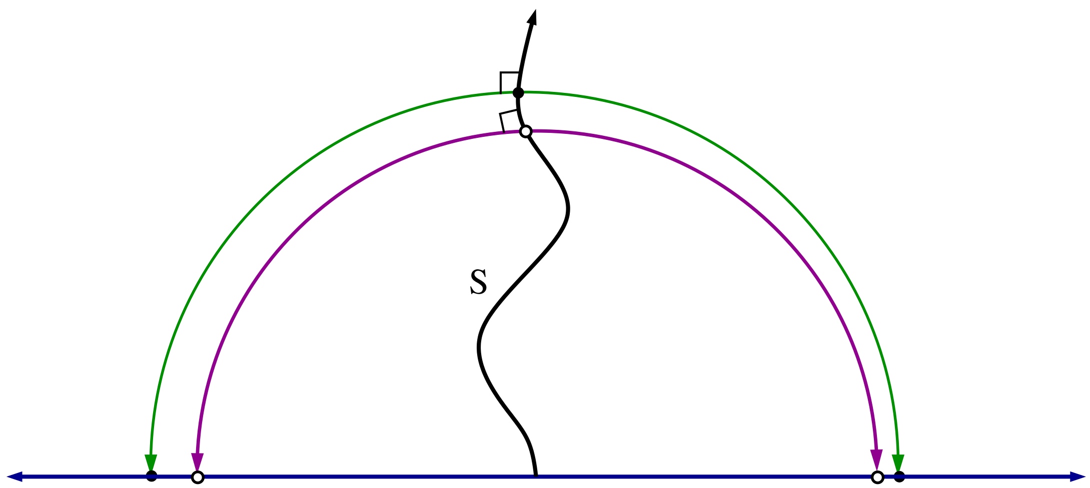
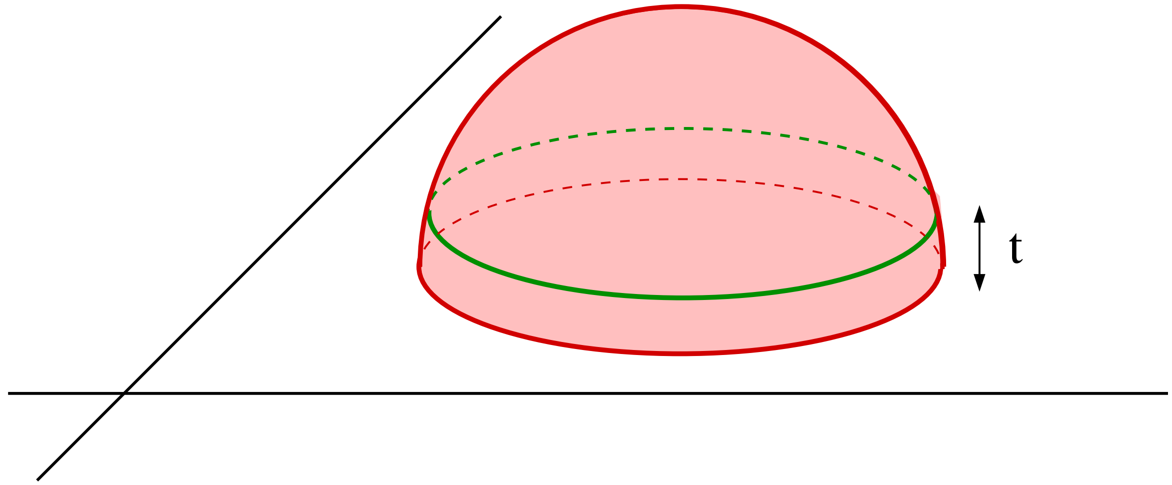


Theorem (C. Epstein, 1986): If S is a surface with $|K| < 1$, then the Gauss maps G_j, $j = 1, 2$ define a quasiconformal reflection across Γ. If S is has finite total curvature, then $\int_{\mathbb{C}\setminus\Gamma} |\mu|^2 d\mathrm{A}_\rho < \infty$.

$\Rightarrow$ Γ is fixed by a QC involution with $\mu \in L^2(dA_\rho)$ $\Rightarrow$ Weil-Petersson.

For surface in upper half-space with boundary on $\mathbb{R}^2$, we can form subdomains by cutting at a certain height.



Truncate $S \subset \mathbb{R}^3_+$ at a fixed height above the boundary, i.e.,

$$S_t = S \cap \{(x, y, s) \in \mathbb{R}^3_+ : s > t\}, \quad \partial S_t = S \cap \{(x, y, s) \in \mathbb{R}^3_+ : s = t\}$$

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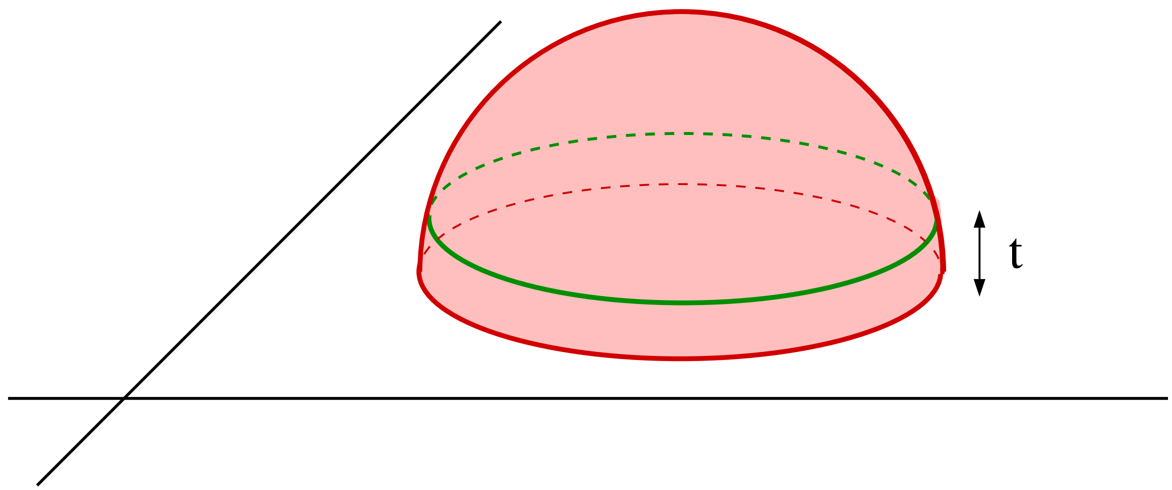

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Surfaces with curvature $K \leq -1$, such as minimal $S \subset \mathbb{H}^3$, satisfy

$$L(\partial S_t) \geq A(S_t) + 4\pi\chi(S).$$

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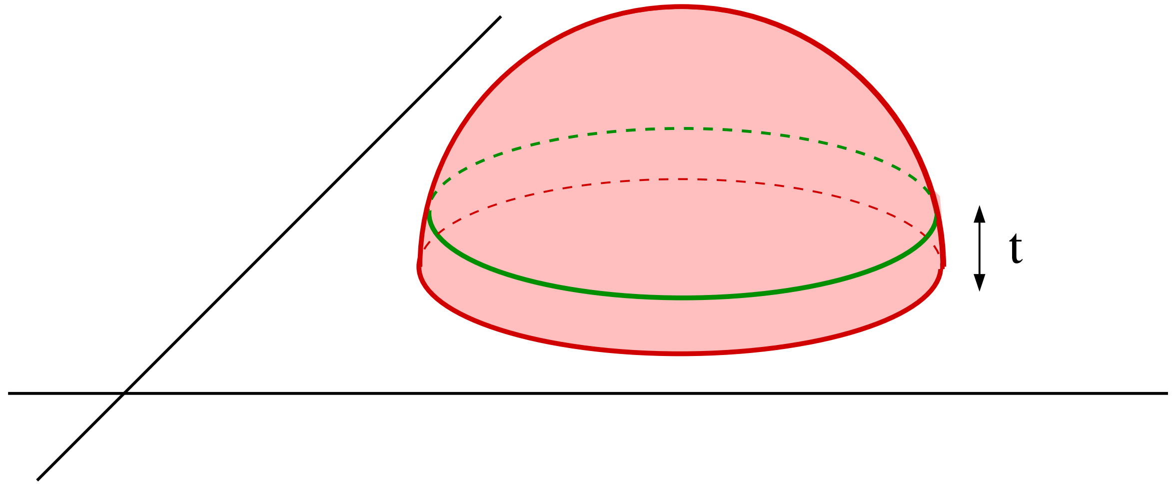

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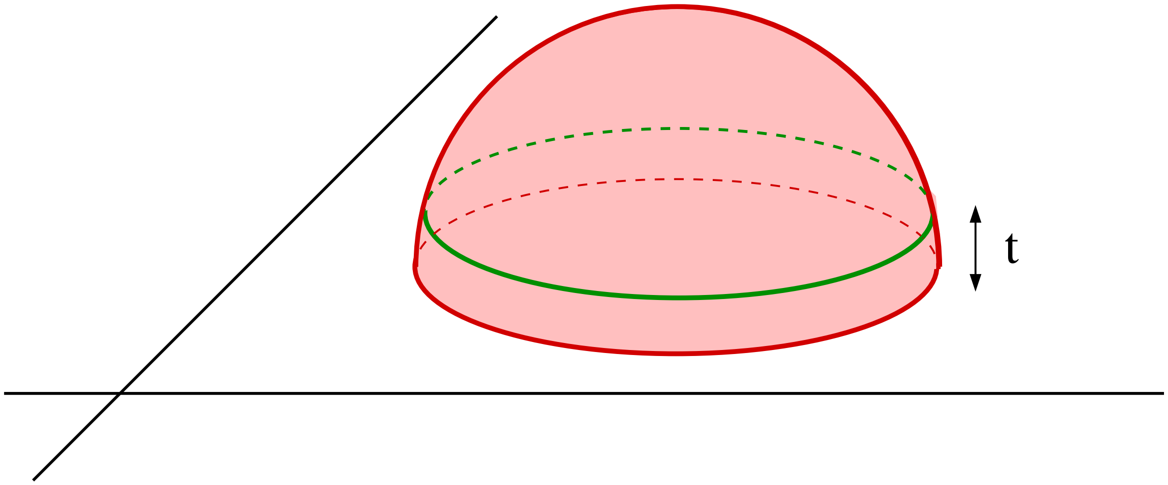
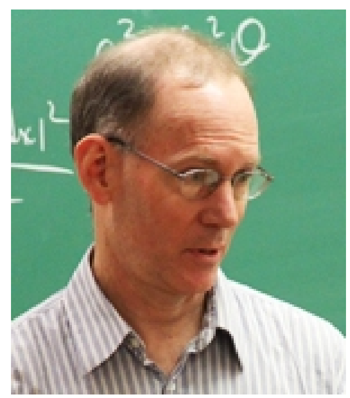
Additive upper bound? (Cheeger constant $= 1 + O(\frac{1}{A})$, asymptotically.)

For surface in upper half-space with boundary on $\mathbb{R}^2$, we can form sub-domains by cutting at a certain height.



Renormalized area: $\mathcal{A}_R(S) = \lim_{t \searrow 0} \left[\mathrm{A}_\rho(S_t) - \ell_\rho(\partial S_t) \right]$.

Graham and Witten proved well defined.



Related to quantum entanglement, AdS/CFT correspondence.

Theorem: S has finite renormalized area iff Γ is Weil-Petersson.

Harder direction: isoperimetric inequalities for negatively curved surfaces to show

$$\mathcal{A}_R(S) < \infty \quad \Rightarrow \quad \int_S \kappa^2 dA_\rho < \infty. \quad \Rightarrow \quad \mathrm{WP}$$

Two proofs of Weil-Petersson $\Rightarrow \mathcal{A}_R(S) < \infty$.

- Use Gauss-Bonnet, Seppi's estimate and $\int \delta^2 < \infty$.
- Use "dyadic cylinder", a discrete version of minimal surface S.

Using the Gauss-Bonnet theorem

$$\begin{aligned}
\mathrm{A}_\rho(S_t) - \ell_\rho(\partial S_t) &= \int_{S_t} 1 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= \int_{S_t} (1+\kappa^2) d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= -\int_{S_t} K d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= -2\pi\chi(S_t) + \int_{\partial S_t} \kappa_g d\ell_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 d\ell_\rho \\
&= -2\pi\chi(S_t) - \int_{S_t} \kappa^2 d\mathrm{A}_\rho + \int_{\partial S_t} (\kappa_g - 1) d\ell_\rho
\end{aligned}$$

Prove $\kappa_g(z) = 1 + O(\delta^2(z))$. Then WP implies last term $\to 0$.

Theorem: For any closed curve $\Gamma \subset \mathbb{R}^2$ and for any minimal surface $S \subset \mathbb{R}^3_+$ with finite Euler characteristic and asymptotic boundary Γ,

$$\mathcal{A}_R(S) = -2\pi\chi(S) - \int_S \kappa^2(z) d\mathrm{A}_\rho.$$

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Due to Alexakis and Mazzeo (2010) assuming that Γ is $C^{3,\alpha}$.

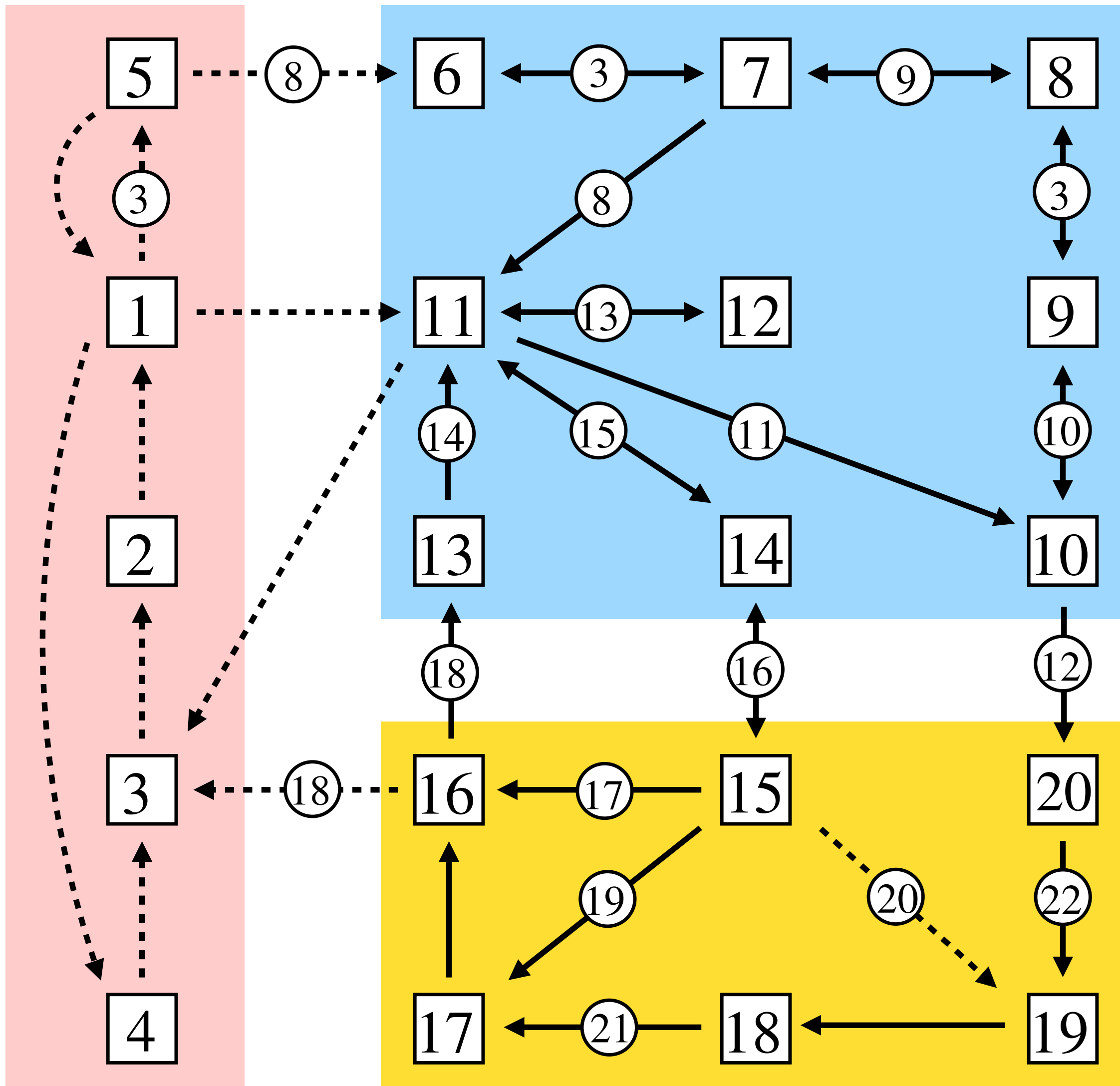
Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions
14	between disjoint disks
15	thickness of convex hull
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder

Weil-Petersson curves

André Weil

Hans Petersson

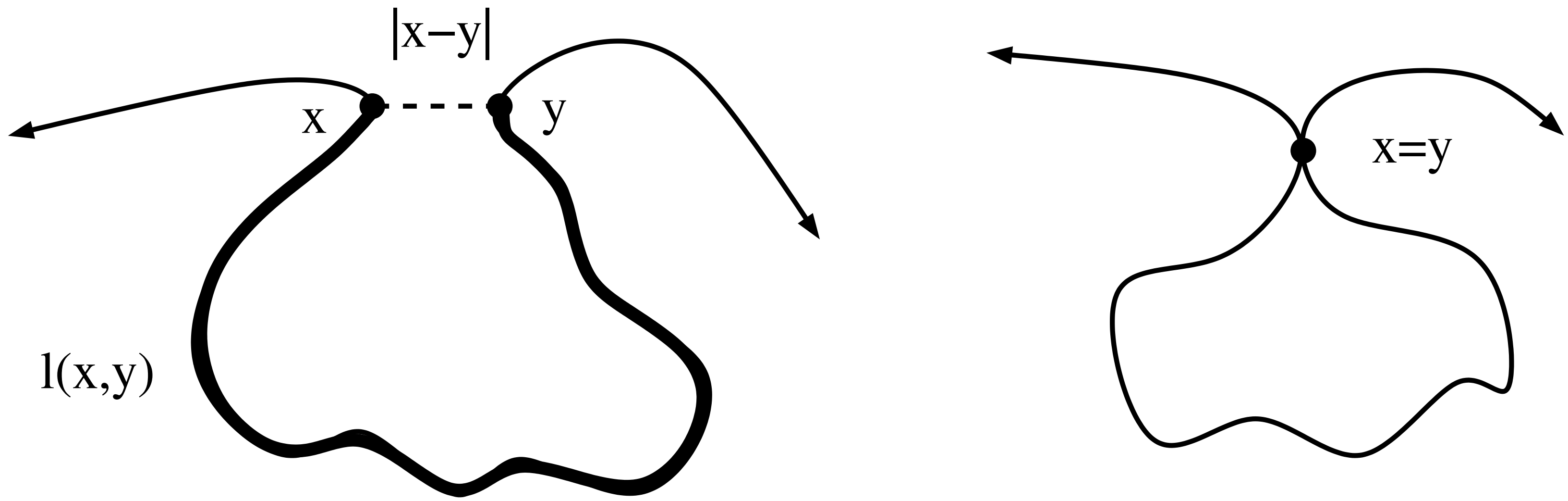
THE THEOREM



THANKS FOR LISTENING. QUESTIONS?

Recall that Γ is Weil-Petersson iff arclength parameterzation is in $H^{3/2}$.

$H^{3/2}$ curves arise naturally in other areas, e.g., knot theory.



The Möbius energy of a curve $\Gamma \in \mathbb{R}^n$ is

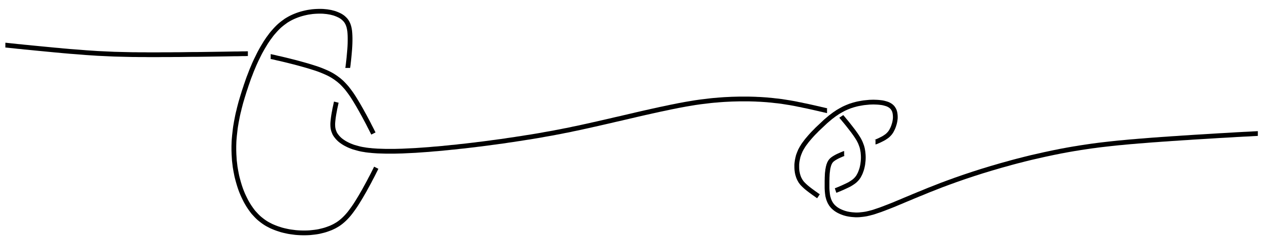
$$\text{Möb}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|x-y|^2} - \frac{1}{\ell(x,y)^2} \right) dx dy.$$

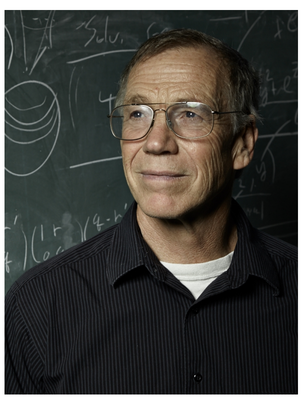
Blows up if curve self-intersects. Is “renormalized” inverse-cube force.

Hadamard renormalization of divergent integral.

Möbius energy is one of several "knot energies" due to Jun O'Hara.

Studied by Freedman, He and Wang. They showed:
- Möb(Γ) is Möbius invariant (hence the name),
- that finite energy curves are chord-arc,
- and in $\mathbb{R}^3$ they are topologically tame.



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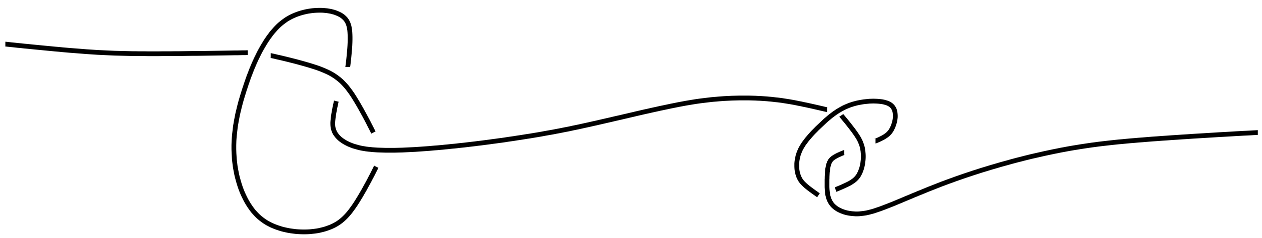


Theorem (Blatt): $\text{Möb}(\Gamma) < \infty$ iff arclength parameterization is $H^{3/2}$.

Thus WP curve = finite Möbius energy.

Connects to SLE?

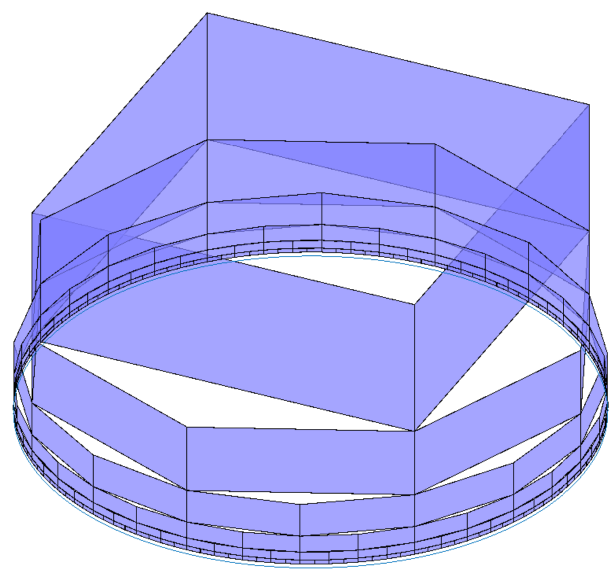
An idea connecting Euclidean and hyperbolic results.

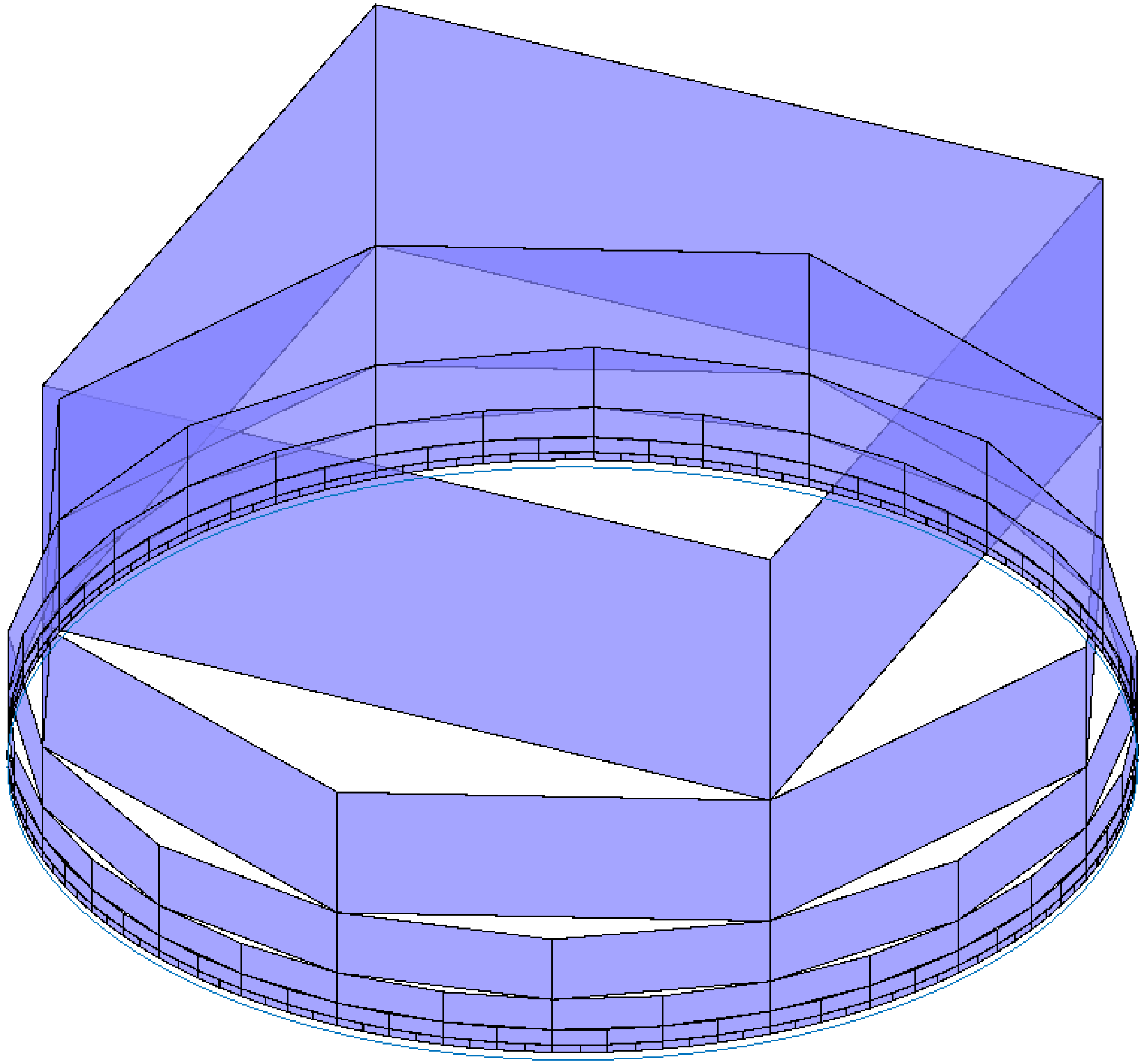
Define a dyadic cylinder in the upper half-space:

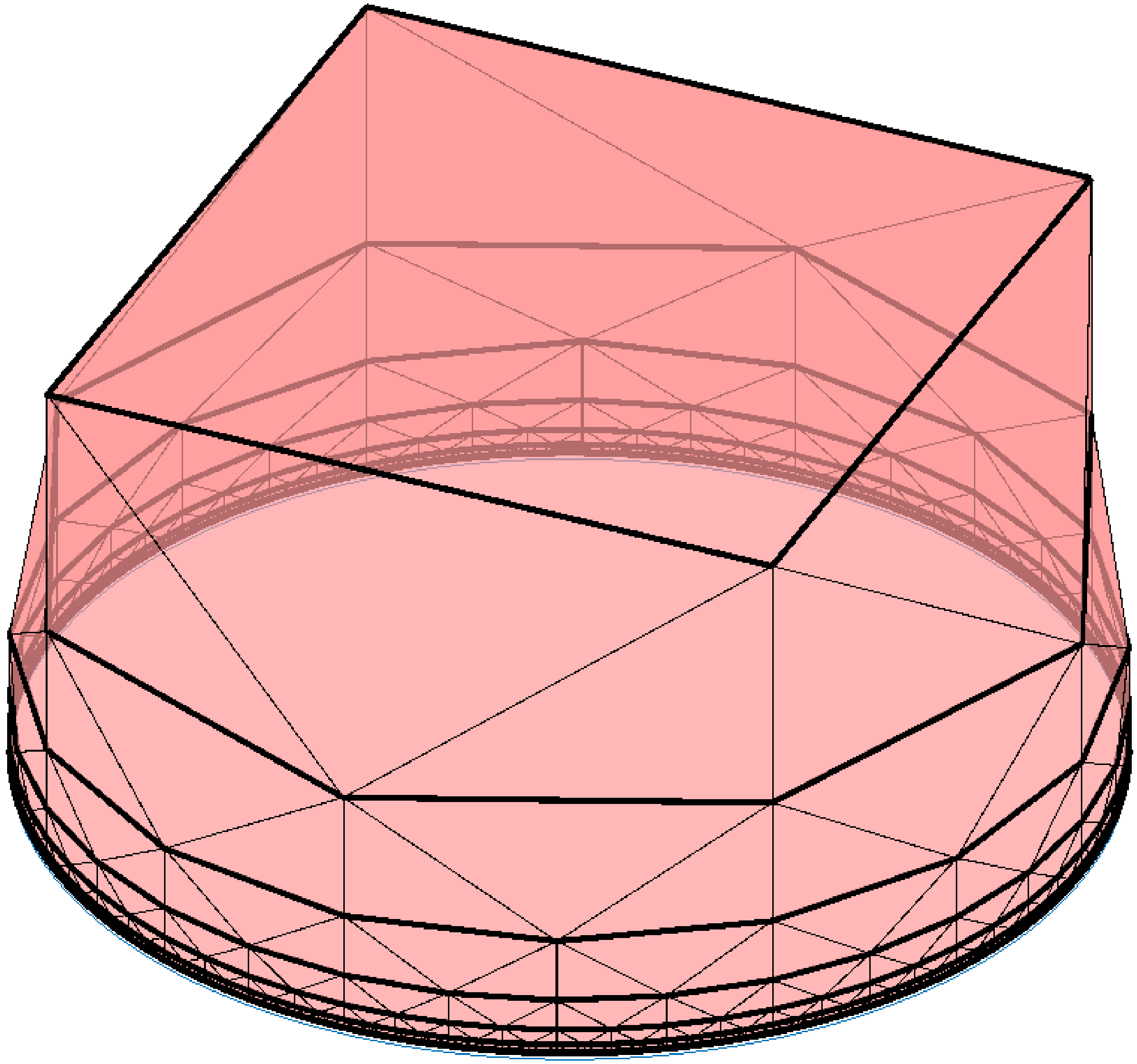
$$X = \bigcup_{n=1}^{\infty} \Gamma_n \times [2^{-n}, 2^{-n+1}),$$

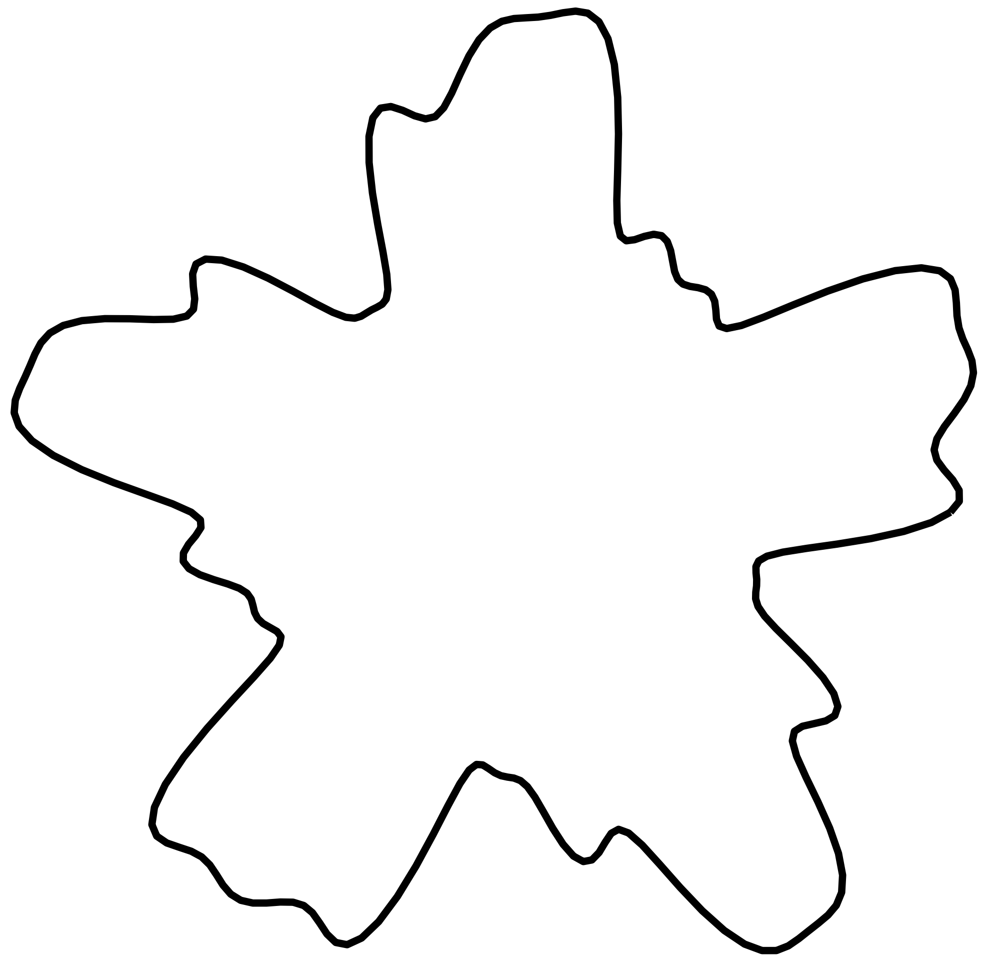
where $\{\Gamma_n\}$ are inscribed dyadic polygons in Γ.

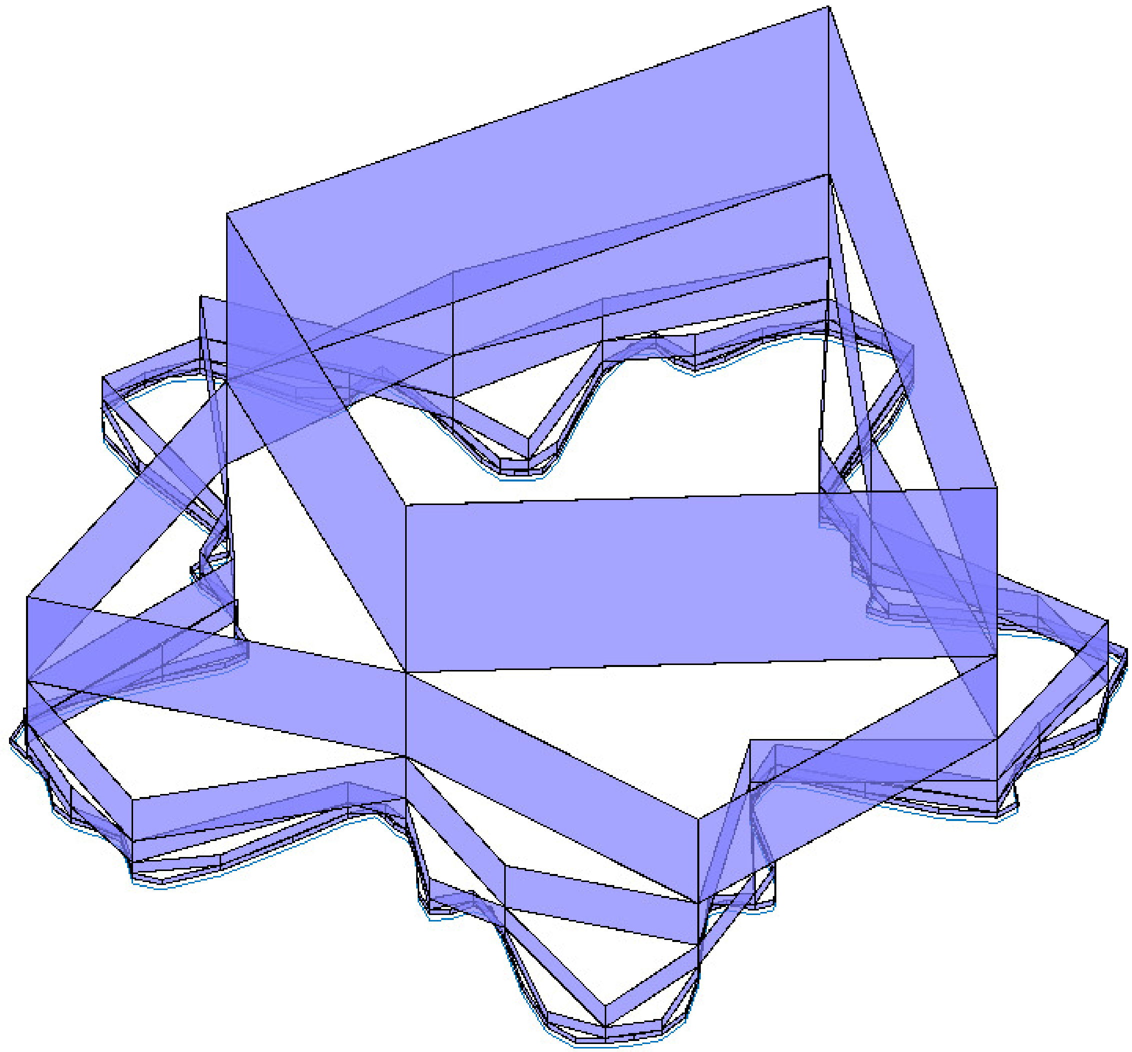
Discrete analog of minimal surface with boundary Γ.

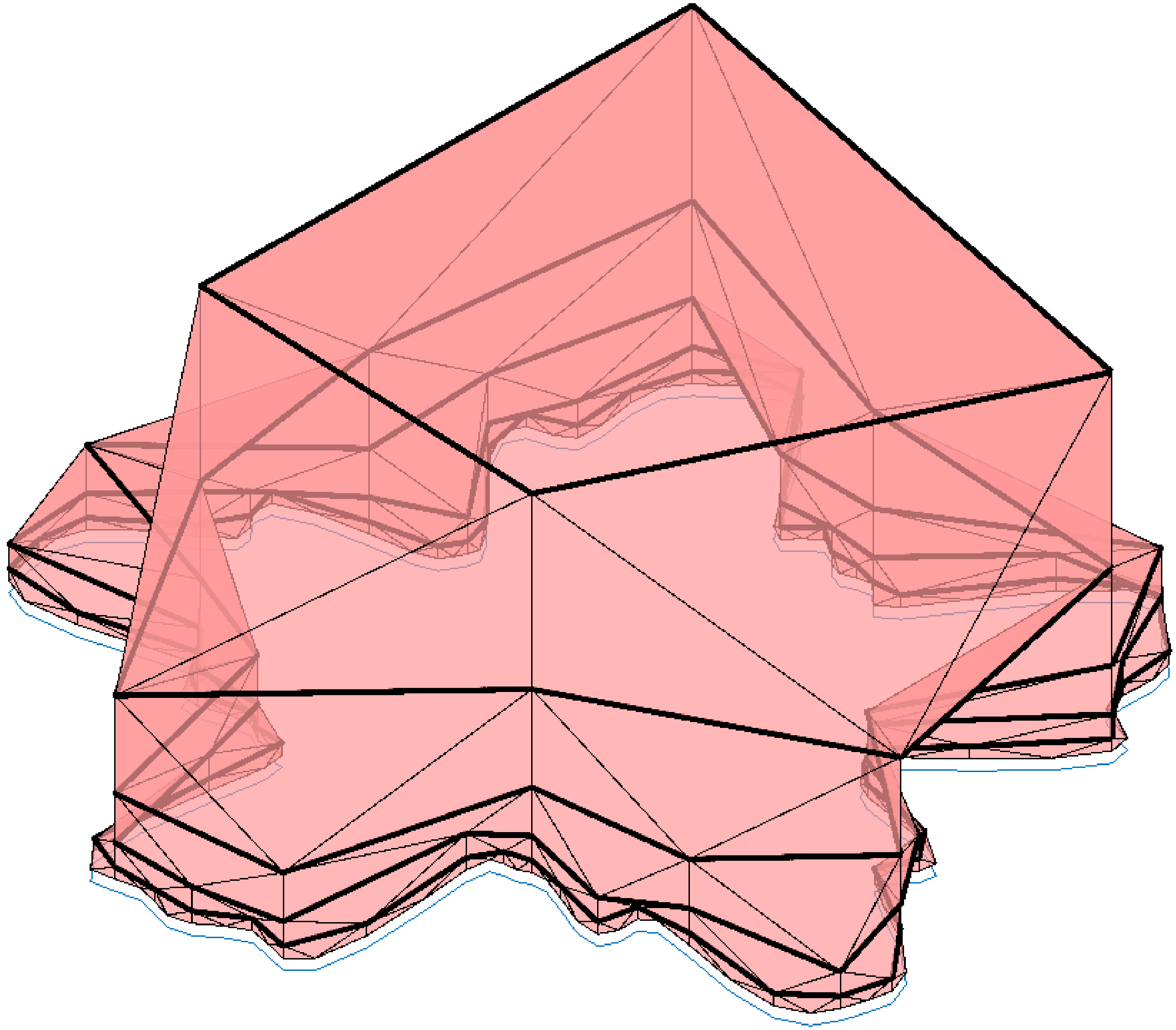












Our earlier estimate

$$\sum_n 2^n(\ell(\Gamma) - \ell(\Gamma_n)) < \infty$$

is equivalent to the dyadic cylinder having finite renormalized area.

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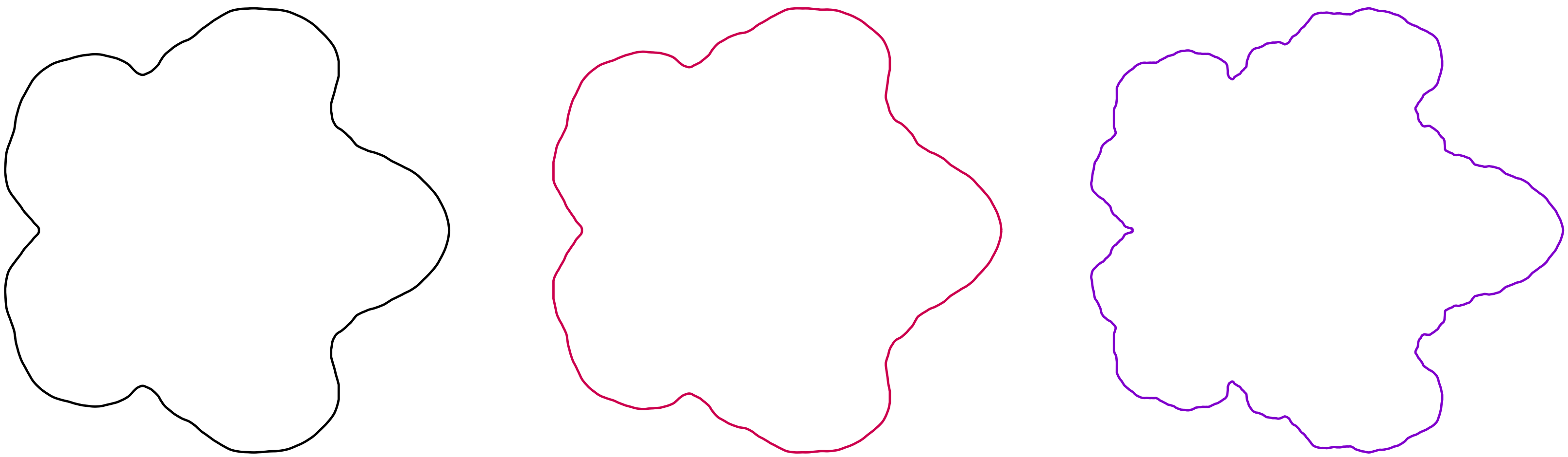
is equivalent to the dyadic cylinder having finite renormalized area.

Obvious "normal projection" from the dyadic cylinder to minimal surface, distorts length and area each by a bounded additive error.

We can deduce finite renormalized area for the minimal surface from the same result for the dyadic cylinder.

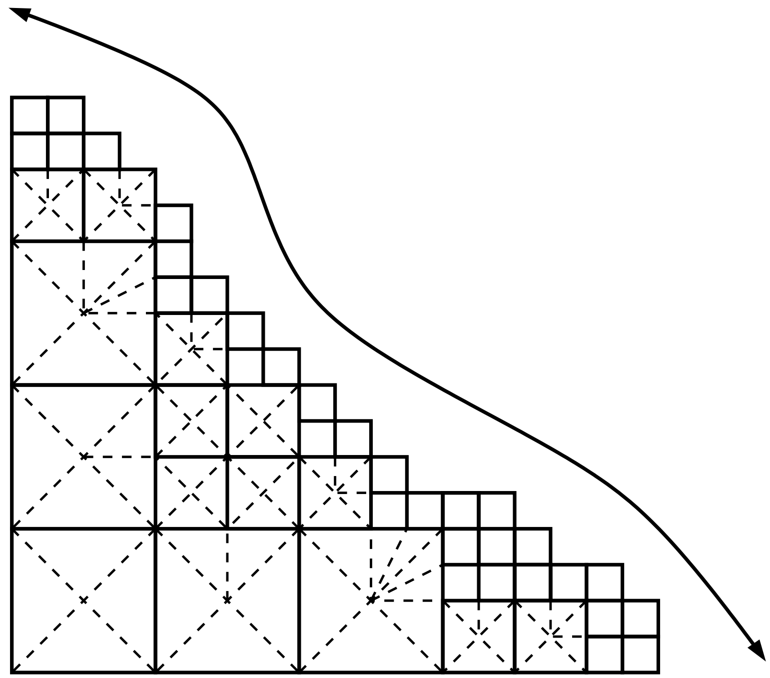
$F(z) = \sum_1^\infty a_n z^n$ is Dirichlet class iff $\sum n|a_n|^2 < \infty$.

If $\log f' = \sum \sqrt{\frac{b_k}{\lambda^k}}\, z^{\lambda^k}$ then $\Gamma = f(\mathbb{T})$ is WP iff $\sum b_k < \infty$.



$\sum \frac{1}{k \log^2 k} < \infty$ $\quad$ $\sum \frac{1}{k \log k} = \infty$ $\quad$ $\sum 1 = \infty$

Easy to see $\sum \beta^2 < \infty$ implies Weil-Petersson.



- Triangulate one side of Γ (e.g., triangulate Whitney squares).

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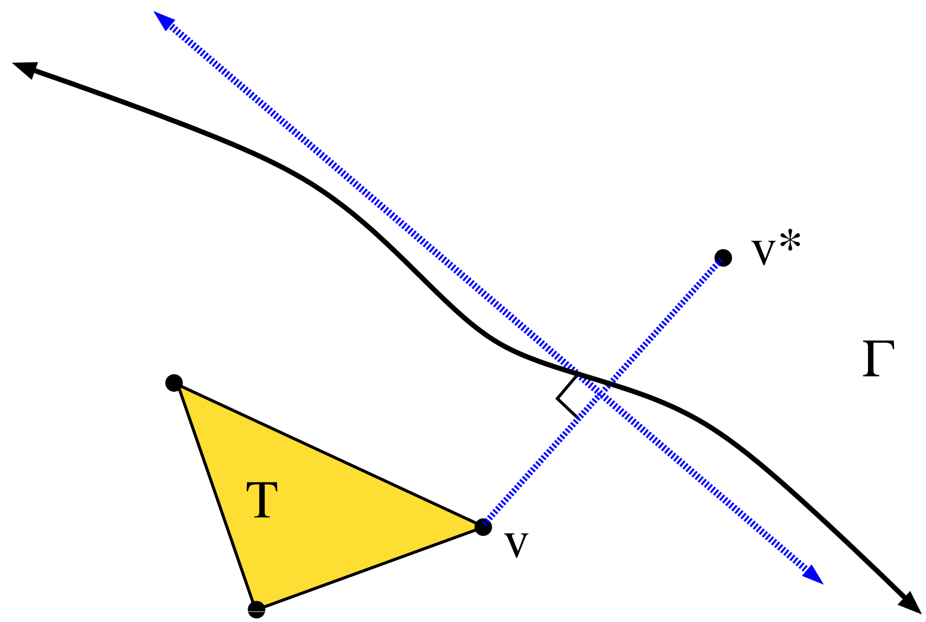

- Triangulate one side of Γ (e.g., triangulate Whitney squares).
- Use approximating lines to reflect vertices.

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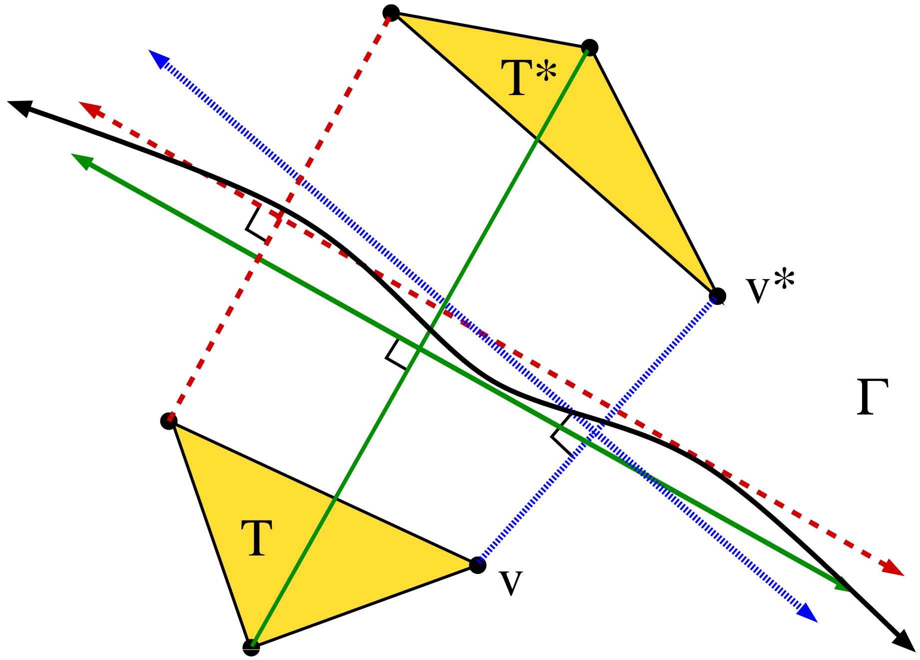

- Triangulate one side of Γ (e.g., triangulate Whitney squares).
- Use approximating lines to reflect vertices.
- Define piecewise linear map.
- $|\mu| = O(\beta)$.
- Get involution fixing Γ with $|\mu| \in L^2(dA_\rho) \Rightarrow$ Weil-Petersson.

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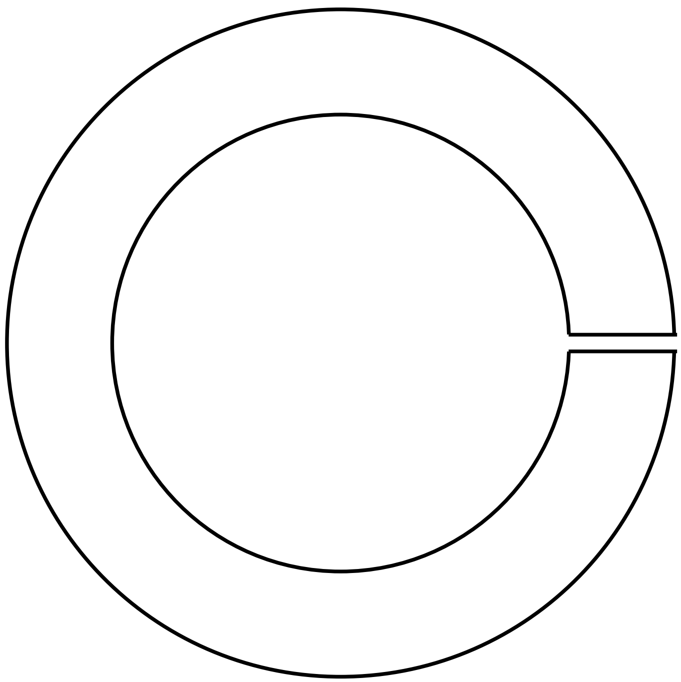

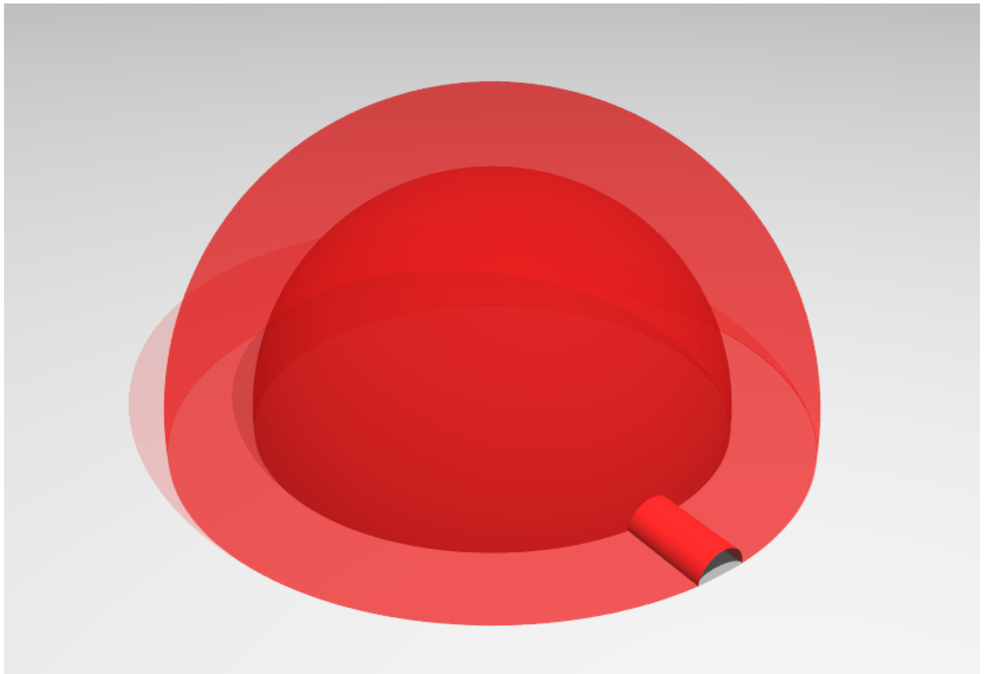

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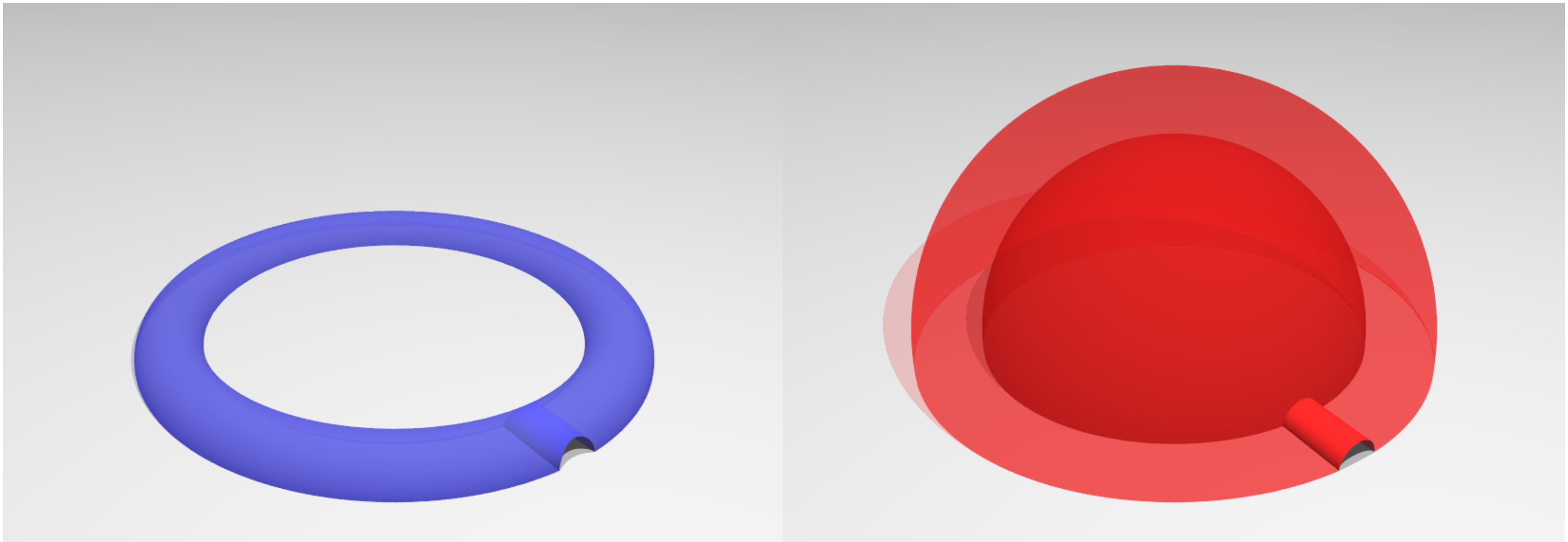

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