Counting on Coincidences

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What is the probability that two people in the room have the same birthday (same month/day)?
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It’s easier to compute the probability that everyone has a different birthday.
Let $P(N)$ be the probability that $N$ random people have different birthdays.

\[
P(1) = 1
\]

\[
P(2) = 1 \cdot \frac{364}{365} \approx 0.99726
\]

\[
P(3) = 1 \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.991796
\]

\[
P(4) = 1 \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \approx 0.983644
\]

\[
P(N+1) = P(N) \cdot \frac{365-N}{365}
\]
Probability of different birthdays for $N$ people
Probability of different birthdays, $N \leq 50$

\[
\begin{align*}
P(22) &\approx 0.524 & P(50) &\approx 0.0296 \\
P(23) &\approx 0.492 & P(60) &\approx 0.00587 \\
P(30) &\approx 0.293 & P(70) &\approx 0.000840 \\
P(40) &\approx 0.108 & P(100) &\approx 0.000000307
\end{align*}
\]
\[ P(365) \approx 1.45495521563 \times 10^{-157} \]
How many people do we need to have a 50-50 chance of having $k$ people with same birthday?

<table>
<thead>
<tr>
<th>number with same birthday</th>
<th>size of group needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>23</td>
</tr>
<tr>
<td>3</td>
<td>88</td>
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<tr>
<td>4</td>
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<tr>
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<tr>
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<td>460</td>
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<tr>
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<td>11</td>
<td>1385</td>
</tr>
<tr>
<td>12</td>
<td>1596</td>
</tr>
<tr>
<td>13</td>
<td>1813</td>
</tr>
</tbody>
</table>
A lottery sells a million tickets and chooses one winner every day. What is the chance that someone wins the lottery twice in a year?

(a) about 1 in a trillion
(b) about 1 in a billion
(c) about 1 in a million
(d) about 1 in a thousand
(e) about 1 in a hundred
(f) about 1 in ten
(g) about 50-50

(Assume same million players every day.)
**Answer:** Probability of 365 different winners is

\[
1 \cdot \frac{999,999}{1,000,000} \cdots \frac{999635}{1000000} \approx .9353.
\]

About a 6.5% chance of a double winner.

45% chance of double winner in 3 year period.

99.87% chance of a double winner in 10 years.
Suppose we randomly put $N$ balls into $K$ boxes. What is the chance that no box has more than $M$ balls in it?

Call this probability $P(N, K, M)$.

The Birthday Problem is computing $P(N, 365, 1)$. 
MIDTERM: What is $P(14400, 9000, 7)$?

(a) $0.000000000000000000000000132$
(b) $0.095395$
(c) $0.664954$
(d) $0.999323$
(e) $0.9999999999999999999999845$

If we drop 14,000 balls into 9000 boxes, what is the chance no box has more than 7 balls in it?
MIDTERM: What is $P(14400, 9000, 7)$?

(a) .00000000000000000000000132
(b) .095395
(c) .664954
(d) .999323
(e) .99999999999999999999999845

If we drop 14,000 balls into 9000 boxes, what is the chance no box has more than 7 balls in it?

Why is this an important example?
In 1960 there were 14,400 cases of leukemia in US and 8 cases in Niles, IL, population 20,000. The average for a town this size would be 1.6 cases.

Is the cluster random?

Population of US in 1960 was

\[ 180,000,000 = 9,000 \times 20,000. \]

Divide US population into 9,000 “boxes” of 20,000 people each. Drop in 14,400 “cases”. What is the chance that one box has 8 cases?

**Answer** = \( 1 - P(14400, 9000, 7). \)
Calculation gives $P(14400, 9000, 7) = .095395$

<table>
<thead>
<tr>
<th>N</th>
<th>Probability biggest cluster $\leq N$</th>
<th>Probability biggest cluster $&gt; N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>.000005</td>
<td>.999995</td>
</tr>
<tr>
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<td>.095395</td>
<td>.904605</td>
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<td>.001212</td>
</tr>
<tr>
<td>12</td>
<td>.999852</td>
<td>.000148</td>
</tr>
</tbody>
</table>

A town of 20,000 with 8 or 9 cases is expected. More cases is very unlikely.
A bag contains $N$ numbered balls. You pull one out, look at it and put it back. Continue until you know how big $N$ is. How long does this take?
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**Answer:** Forever. You can never be sure whether or not you missed one.
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**Better question:** How long before you have a “good guess” how many balls are in the bag?
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**Better question:** How long before you have a “good guess” how many balls are in the bag?

**Answer:** $\approx \sqrt{N}$
$N$ balls into $K$ boxes. Chance of “no repeats”:

**Exact Formula**

$$P(N) = 1 \cdot \frac{K - 1}{K} \cdots \frac{K - N + 1}{K}$$

**Approximate Formula**

$$P(N) \approx e^{-N^2/2 \cdot K}$$

Comparing exact and approximate formulas
How many balls to get a 50% chance of a repeat?

Must solve

\[ e^{-\frac{N^2}{2K}} = .5 \]

\[ \frac{-N^2}{2K} = \log .5 \]

\[ N = \sqrt{2K \log 2} \approx 1.17741\sqrt{K} \]

If we throw \( N \approx \sqrt{K} \) balls into \( K \) boxes we have a good chance to get two in the same box.

Note \( 1.177\sqrt{365} \approx 22.49. \)

\( 1.177\sqrt{1000000} = 1177\text{days} = 3.22\text{years}. \)
Counting balls in a bag problem

Rough guess:
If first repeat in on the $n$th draw from the bag, guess $K = (n/1.1774)^2$ as number of balls in bag.

Repeat and take average for better estimate.
<table>
<thead>
<tr>
<th>$n$</th>
<th>$(n/1.177)^2$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>14</td>
<td>143.181</td>
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<tr>
<td>15</td>
<td>164.366</td>
</tr>
</tbody>
</table>
Better solution:
Examine and return $m$ samples.
Let $t$ be total number of repeats.
Estimate $k = \frac{m(m-1)}{2t}$
Estimate is probably accurate if $m \gg \sqrt{K}$.

Example: I picked $K$ distinct 3 digit numbers, then drew 30 random samples:

231  903  979  705  153  231  402  231  540  716
930  836  386  836  659  671  284  114  814  716
317  588  386  358  143  660  392  588  979  553

What is $K$?
There are $m = 30$ samples and $t = 8$ pairs of repeated numbers, so our guess is

$$\frac{m(m - 1)}{2t} = \frac{30 \cdot 29}{16} \approx 54.375.$$ 

The true $K$ is 57.

The large $m$ is, the better the estimate.
Applications:
   Counting fish in lake
   Counting distinct users on internet
   Cryptography (security of digital signatures)
   Factoring integers (Pollard rho method)
   Many others
**FINAL EXAM**

Two thieves steal $N$ diamonds with random values between $\$1$ and $\$1,000,000$. Can they divide the loot into two piles of equal value?

Impossible if $N = 1$ and unlikely if $N = 2, 3, \ldots$?

There is a 50% chance even splitting if $N > ?$

(a) 11
(b) 25
(c) 78
(d) 979
(e) 10,122
Divide diamonds into two piles of size $N/2$.

Randomly divide first pile into two smaller sets (red and blue). Let $R_1, B_1$ be the value of each subset. Let $D_1 = R_1 - B_1$. This is a random number between

$$-rac{N}{2} \times 1,000,000 < D_1 < \frac{N}{2} \times 1,000,000$$

There are $2^{N/2}$ ways to choose $D_1$.

$$D_1 = R_1 - B_1$$

$$D_2 = R_2 - B_2$$
Do the same for the second pile. Get a random number $D_2 = R_2 - B_2$

If $D_1 = D_2$ then

$$R_1 - B_1 = R_2 - B_2,$$
$$R_1 + B_2 = R_2 + B_1,$$

so we get a division into two equal parts.

What is the chance $D_1 = D_2$?
By Birthday Problem the odds $\approx 50-50$ if
\[
\# \text{ random choices} \approx \sqrt{\# \text{ possible choices}}
\]
\[
2^{N/2} = \sqrt{N \times 1,000,000}
\]

**Answer:** An equal division is likely if $N \geq 25$.

(But finding the division can be very hard; what we call an NP-hard problem.)
References

Application to disease clusters:

Counting via random sampling:

Counting fish by statistics:

Dividing into equal piles:

Theory of coincidences: