An Introduction to Besicovitch-Kakeya Sets

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lecture slides available at
`www.math.sunysb.edu/~bishop/lectures`

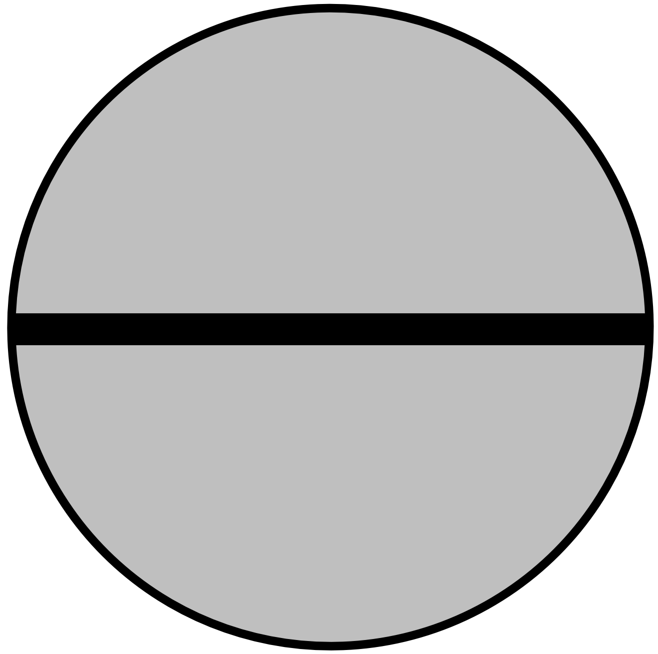
A **Kakeya** set is a set in which a unit segment can be continuously rotated 180°.

Such a set contains unit segments in all directions.

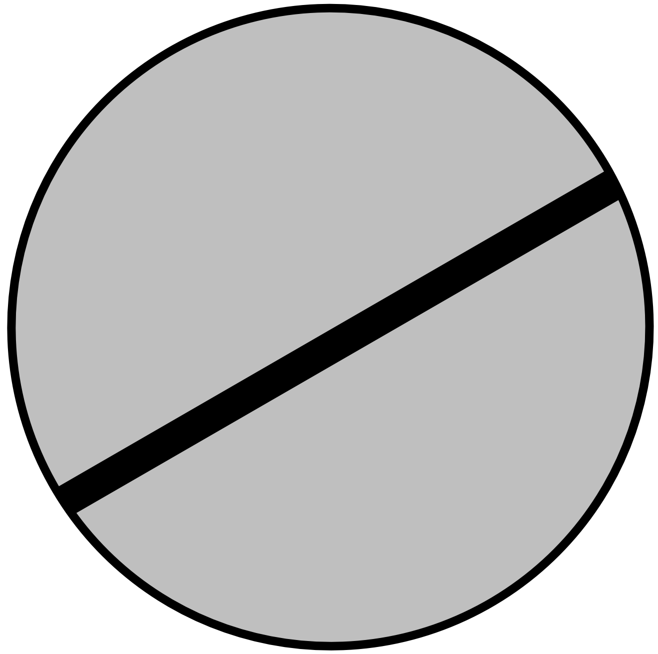
Such a set cannot have zero area.

Soichi Kakeya (1886-1947)

In 1917 Kakeya and Fujiwara asked what is smallest area that a unit needle can be continuously rotated $180°$.



$$\text{Area} = \frac{\pi}{4} \approx .785398$$



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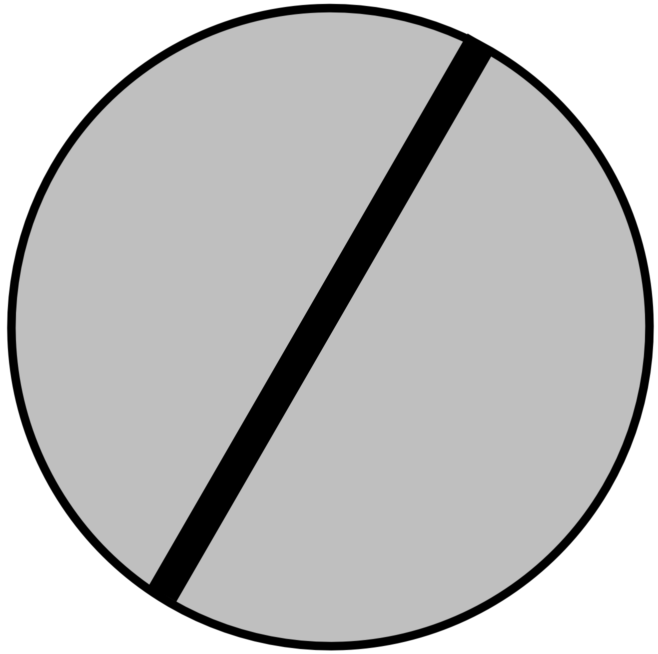

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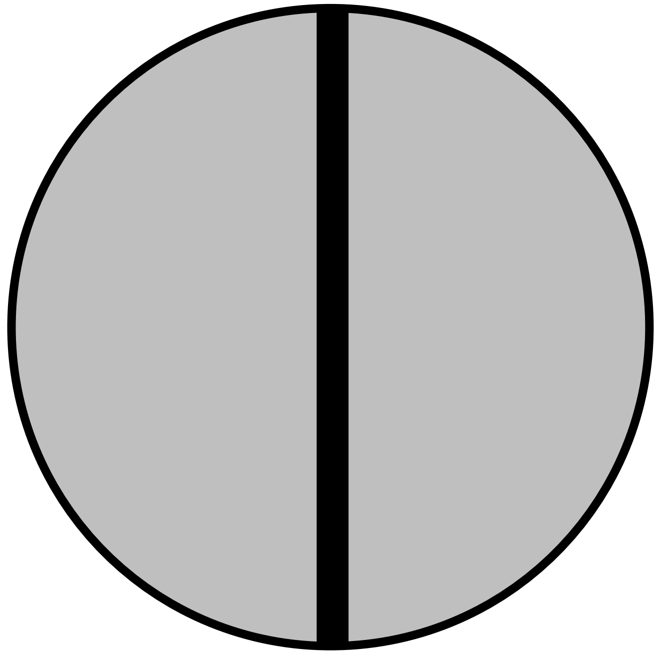

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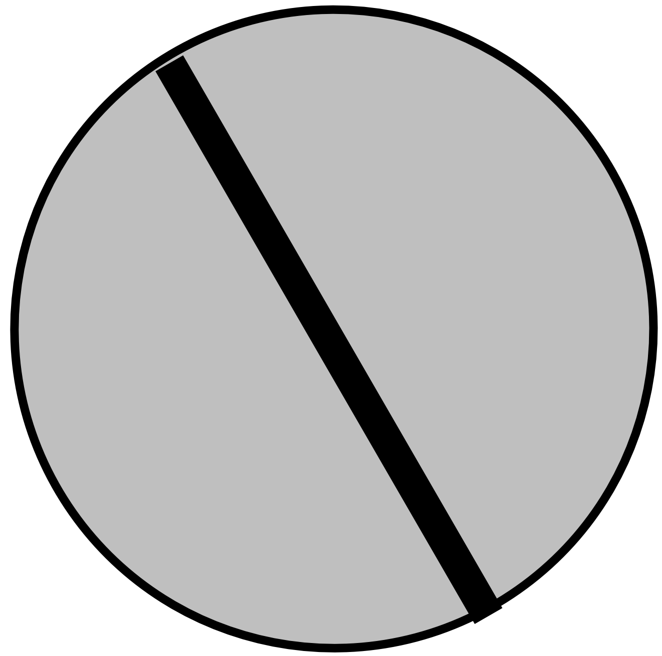

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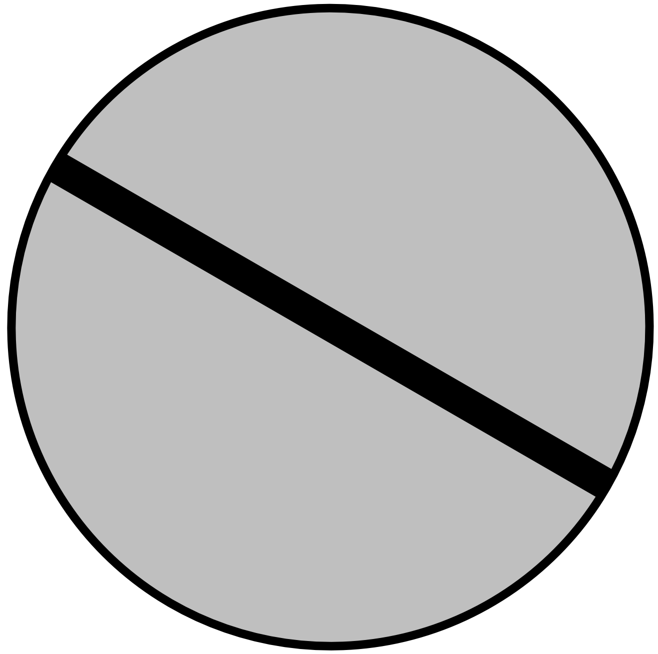

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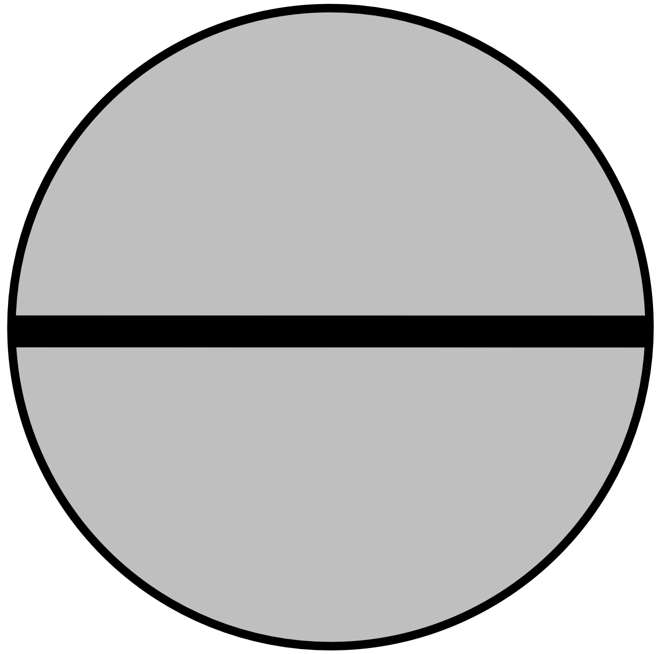

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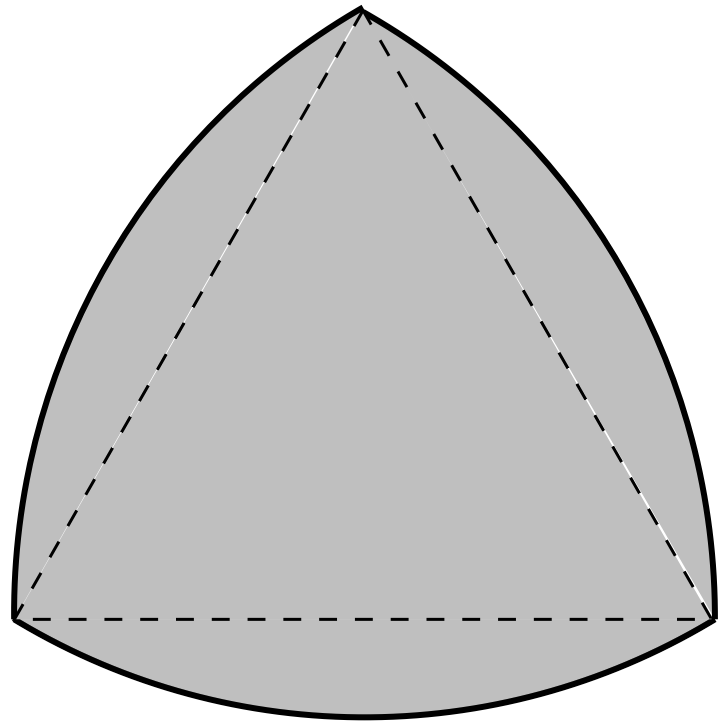
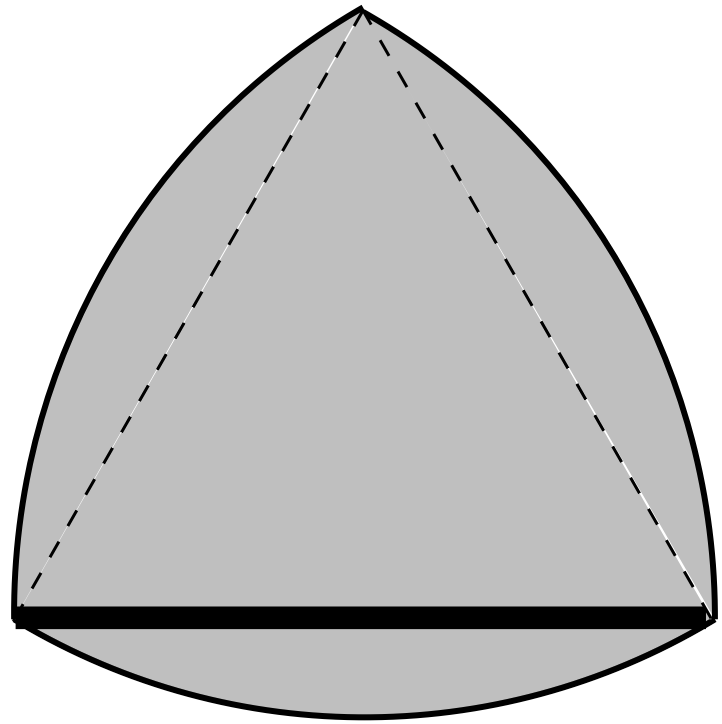


$$\text{Area} = 3\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + \frac{\sqrt{3}}{4} \approx .704771$$



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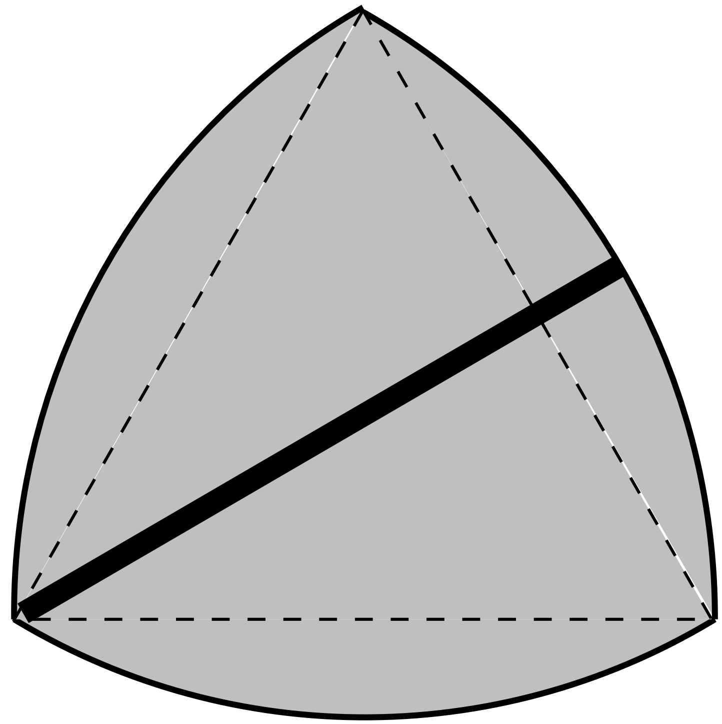

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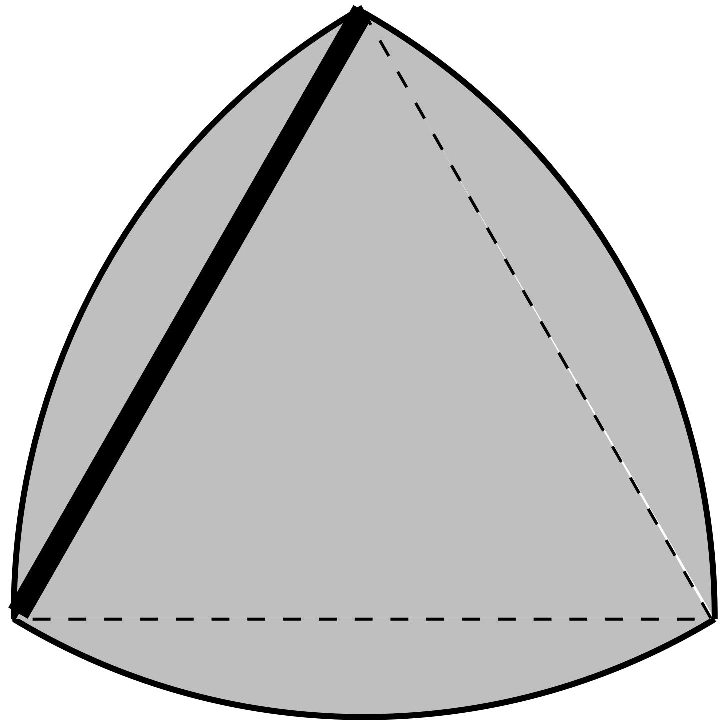

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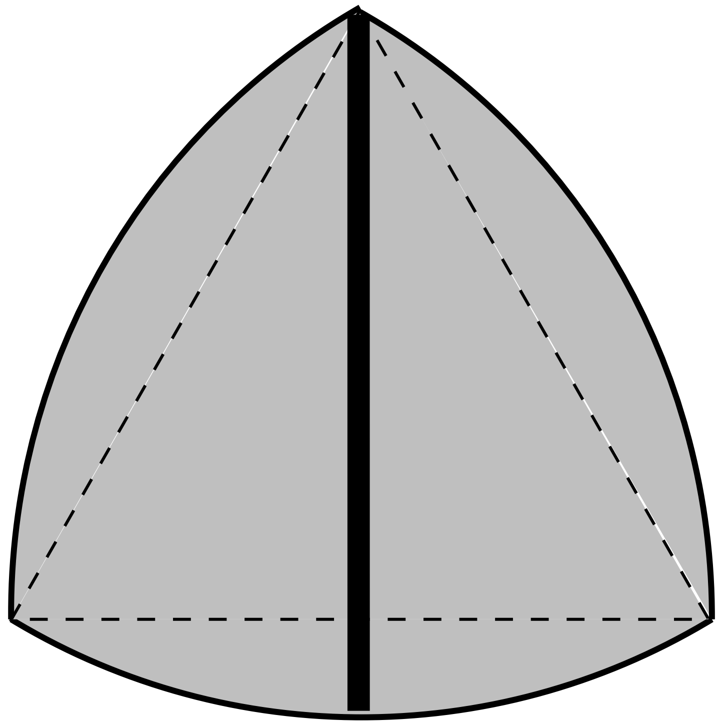

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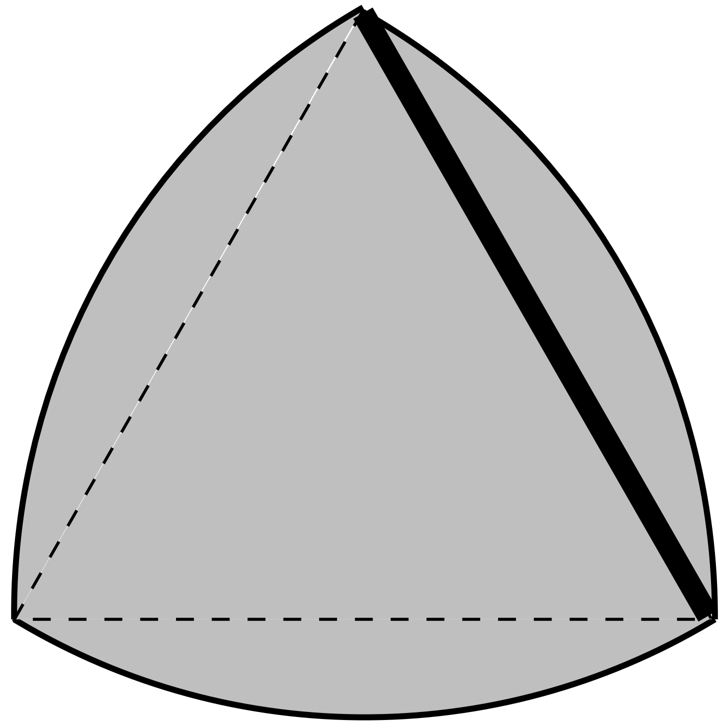

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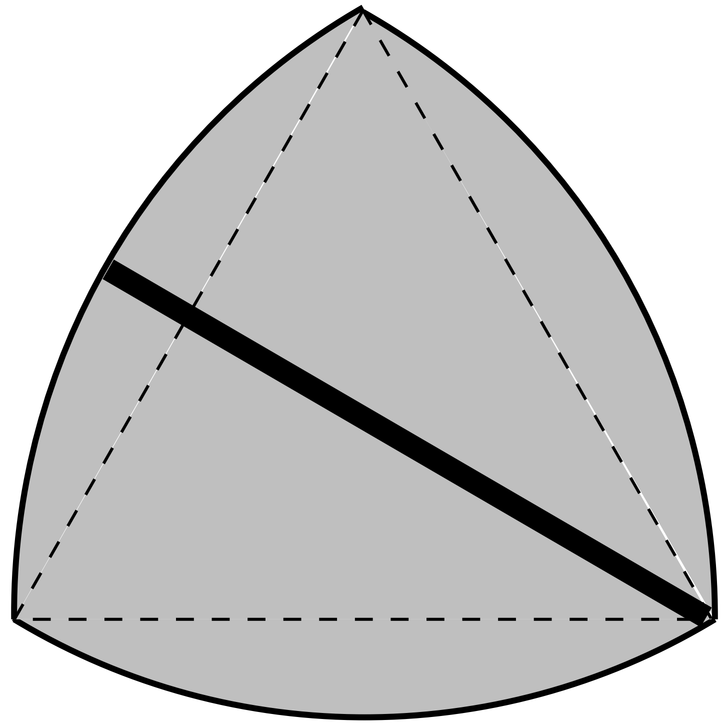

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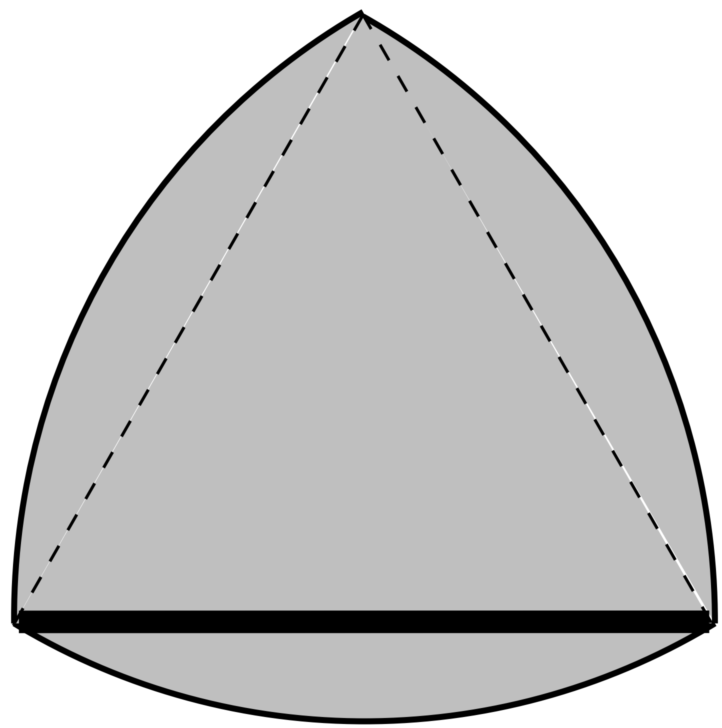

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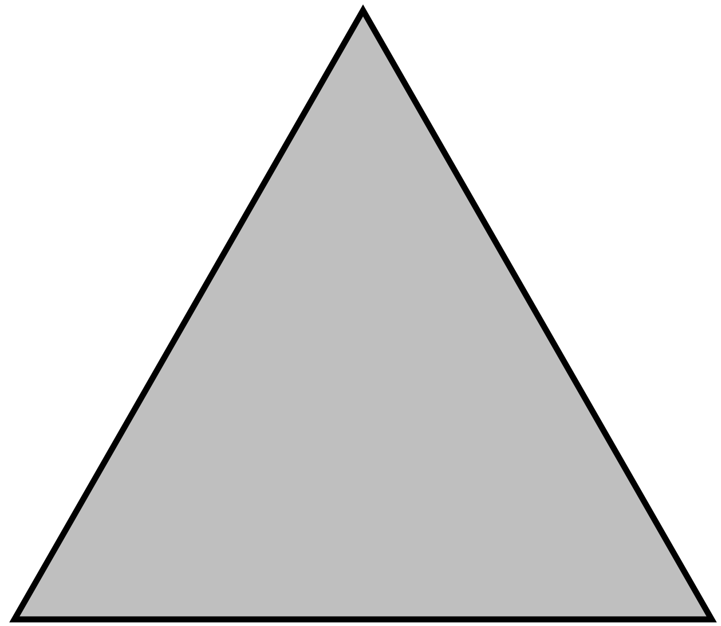

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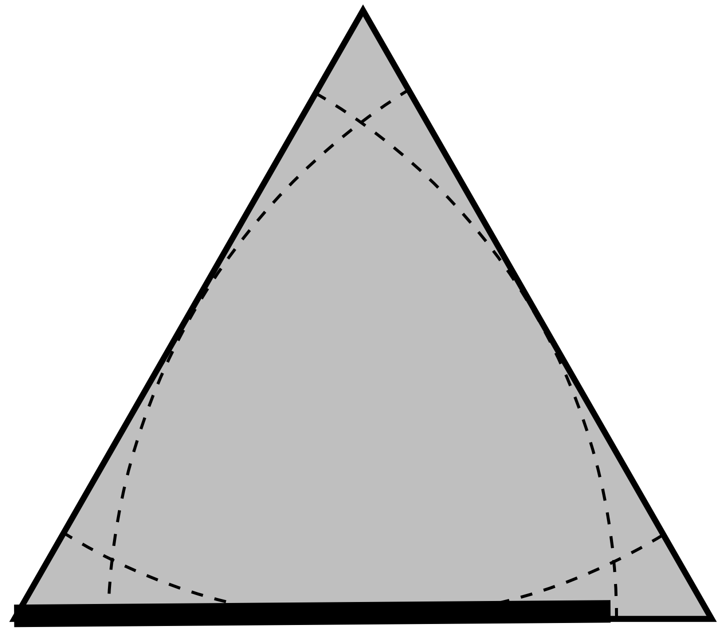
$$\text{Side Length} = 4/\sqrt{3}$$



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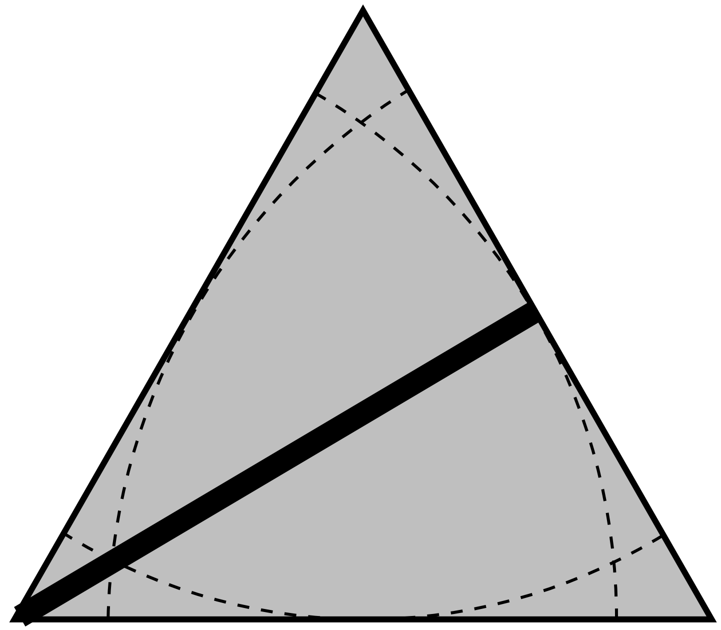

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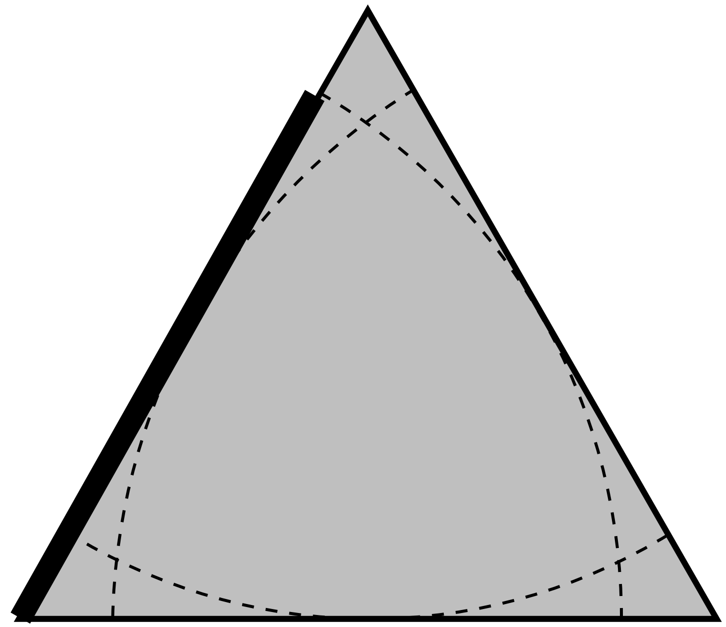

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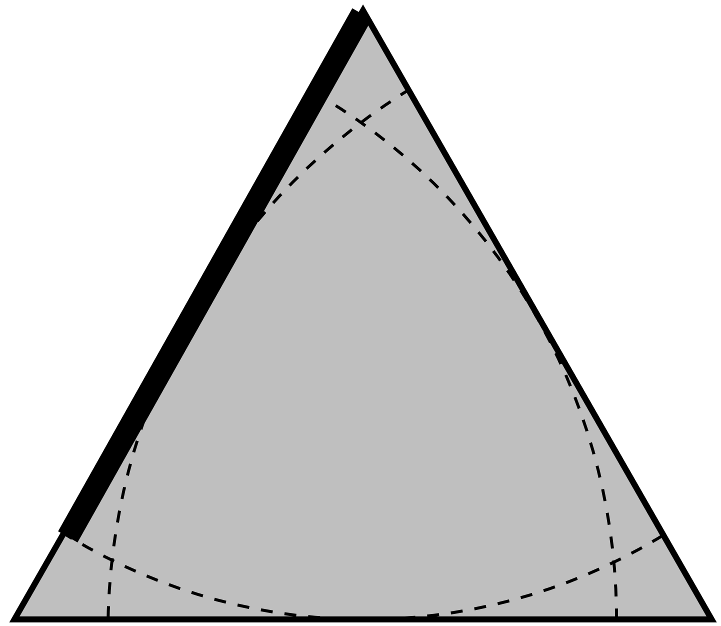

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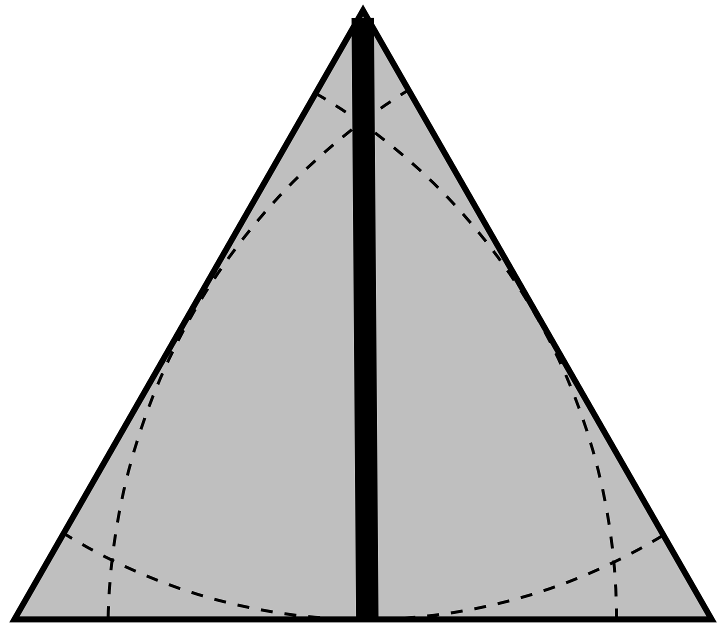

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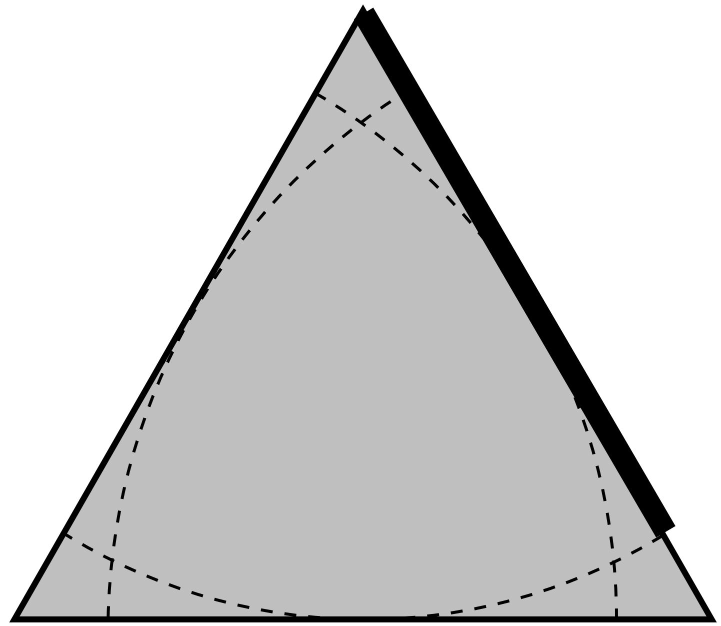

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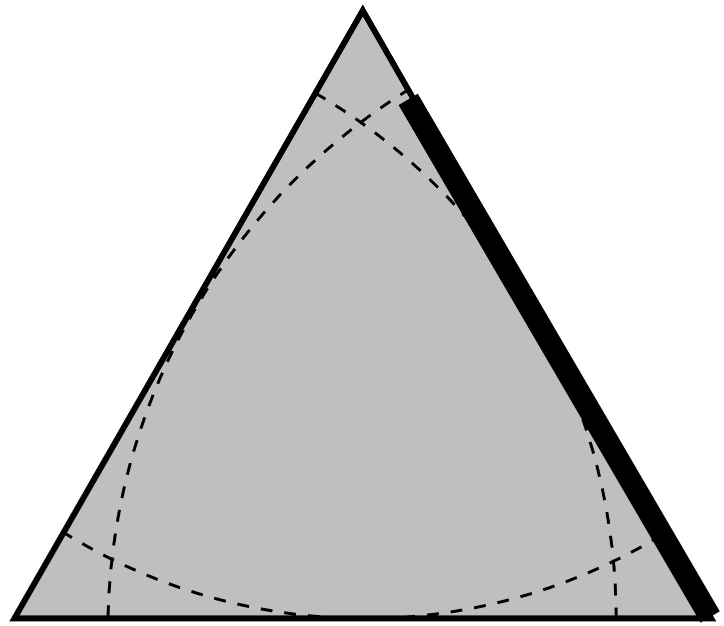

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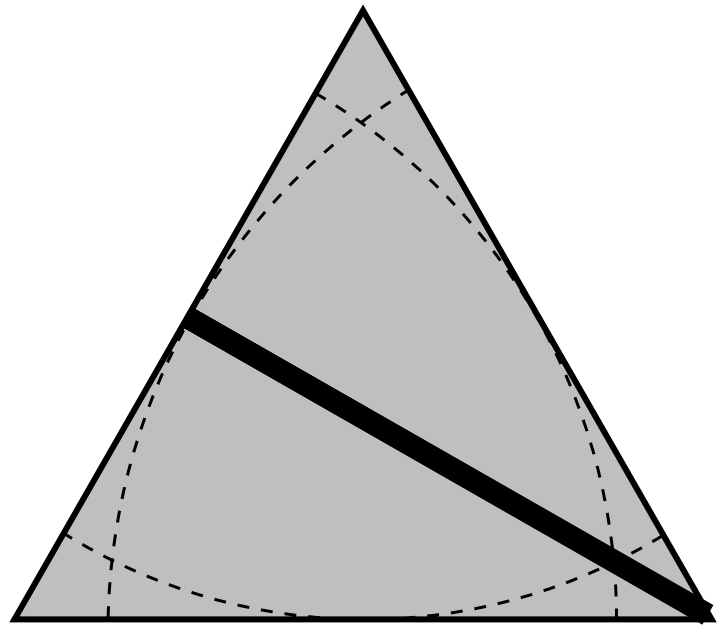

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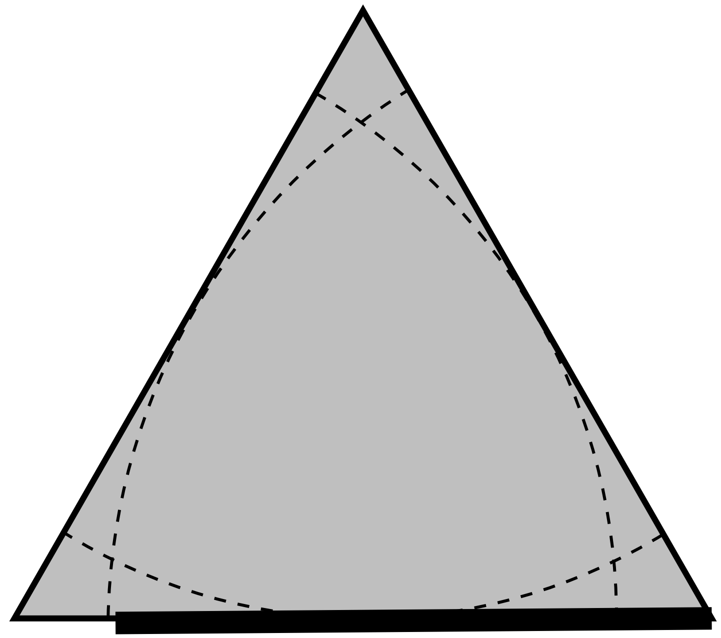

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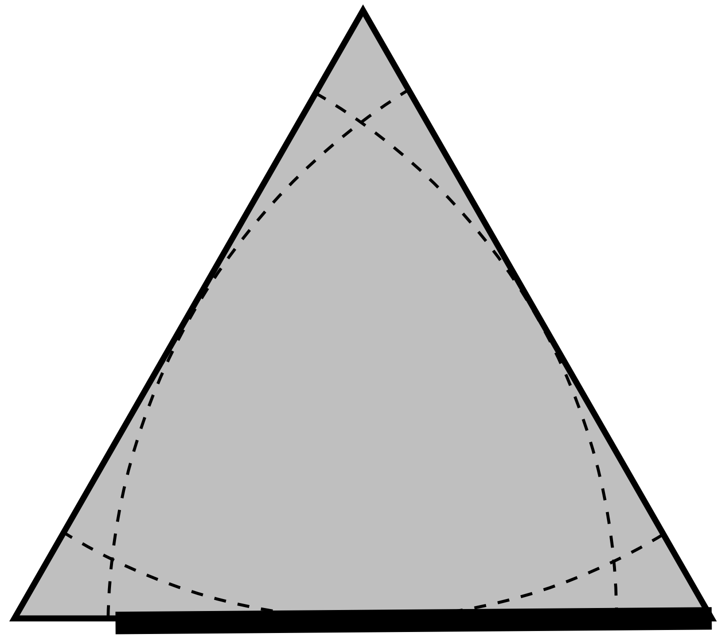
This is optimal among convex shapes (Pal, 1921)



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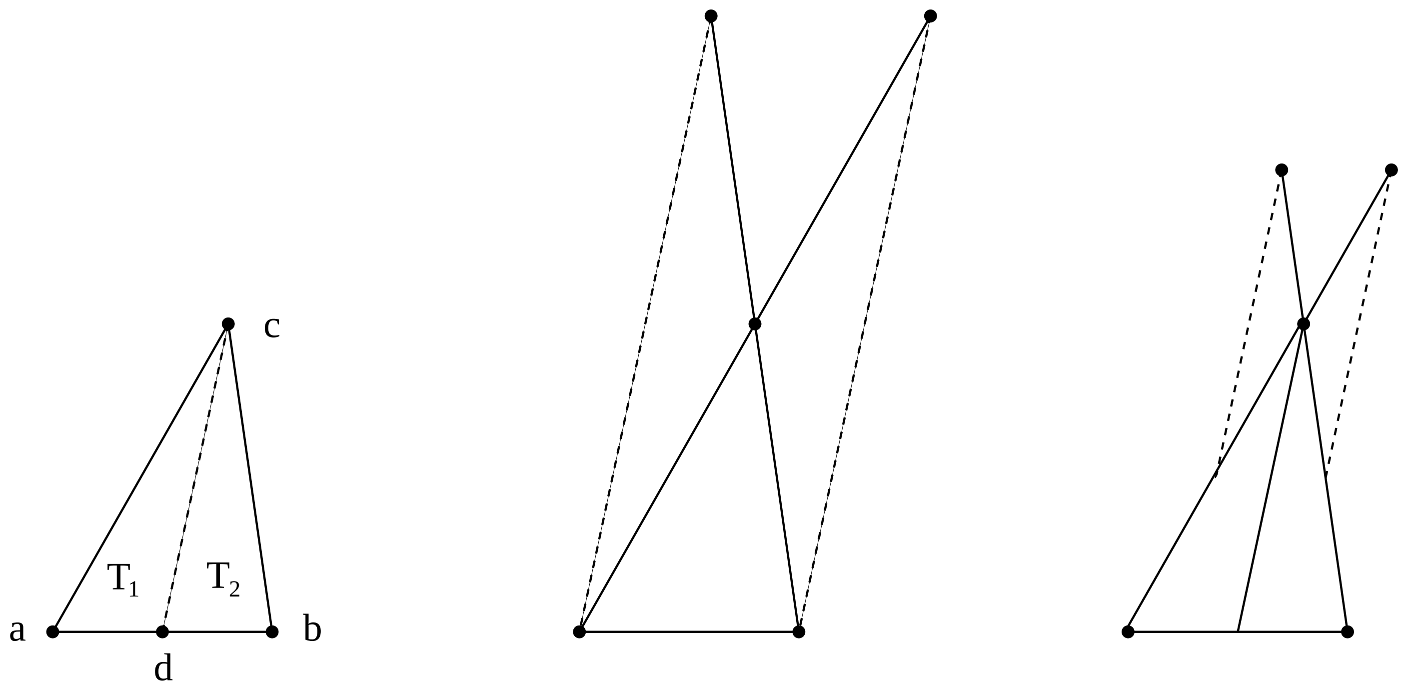
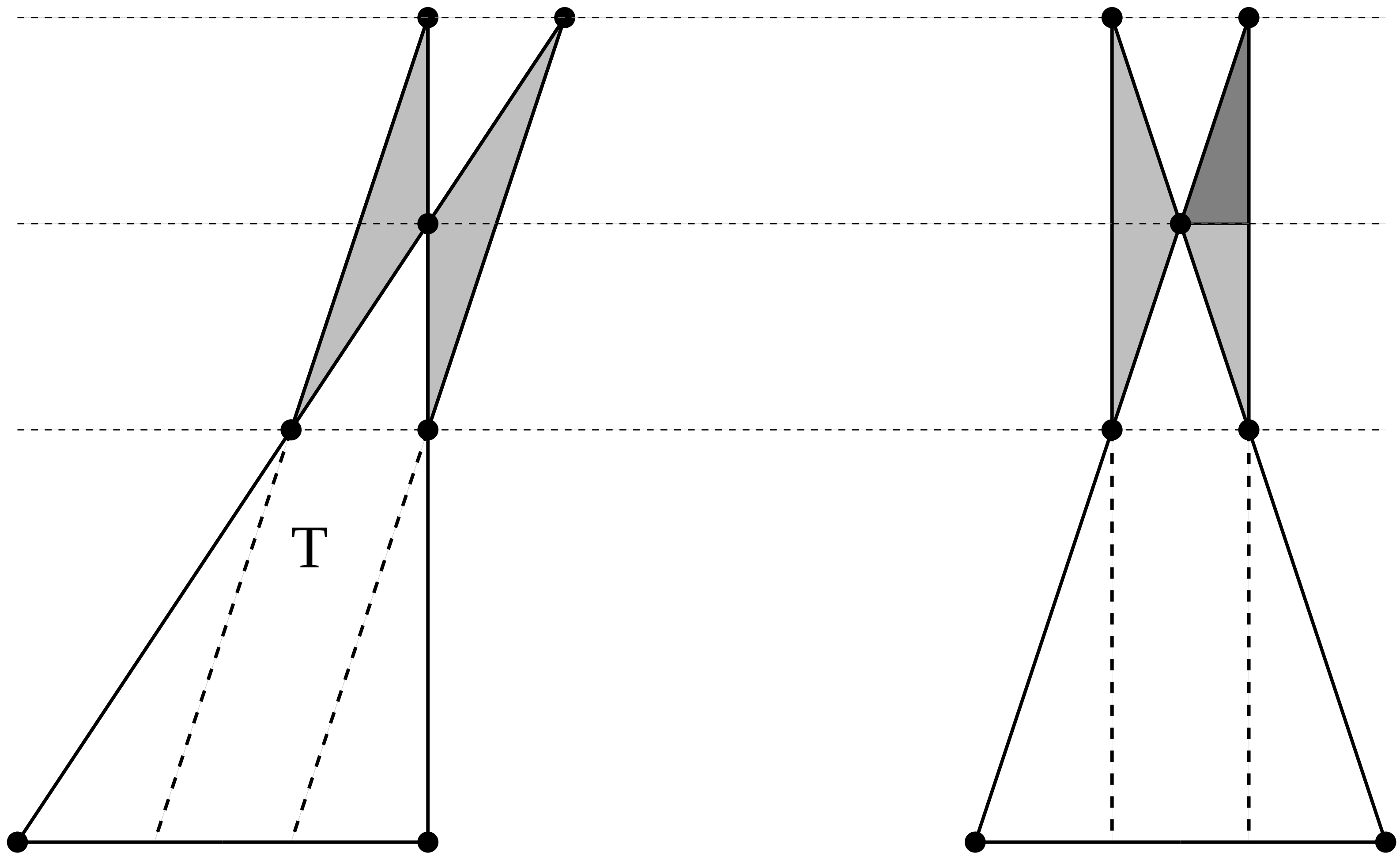
In general, arbitrarily small area is needed.

Perron triangles construction:

- Start with triangle with base on real line.
- Cut triangle from midpoint of base to top vertex.
- Expand subtriangle by $\frac{k+1}{k}$, fixing outer endpoint.
- Get 2^n triangles, base $(n+1)2^{-n}$, height $n+1$.



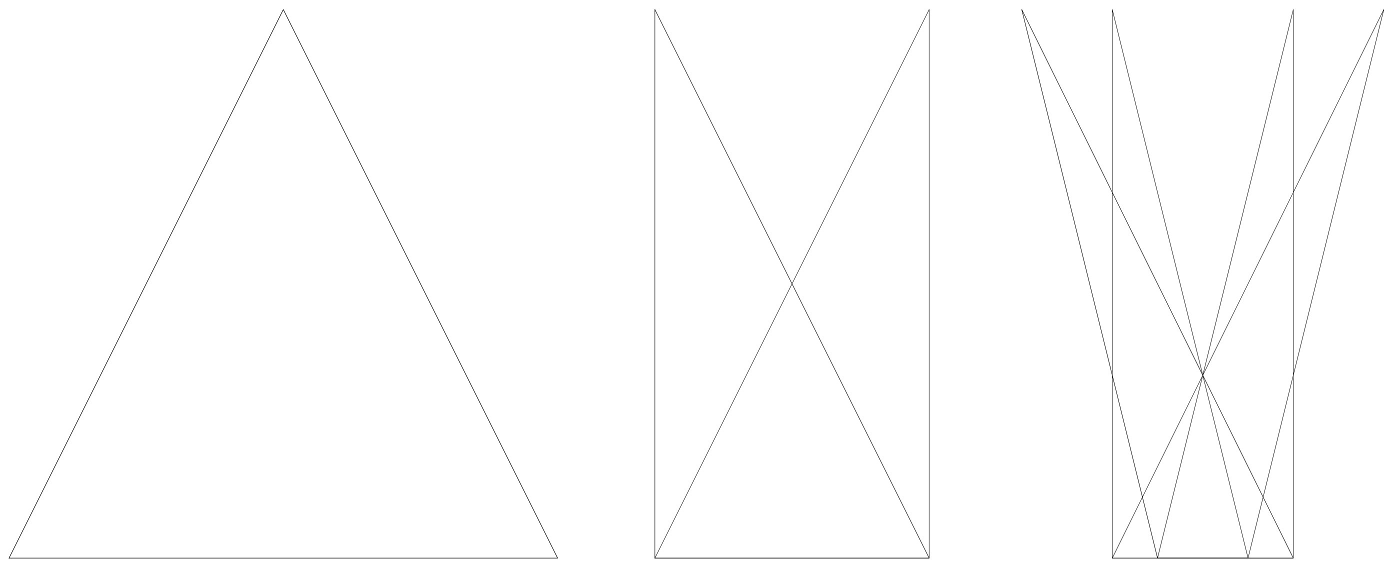


New area per triangle $(1/k)^2 \cdot (k2^{-k}) \cdot k \simeq 2^{-k}$

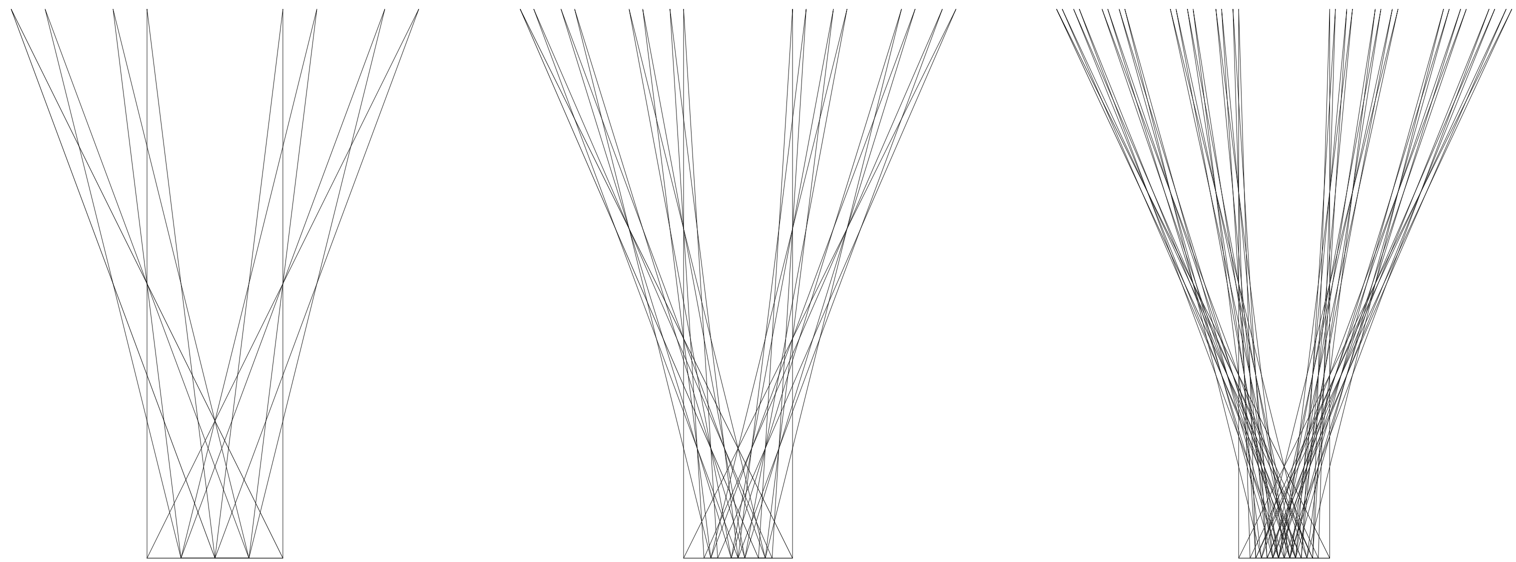
Total new area $\simeq 1.$

Rescale to height 1. New total area $\simeq k/k^2 = 1/k$.

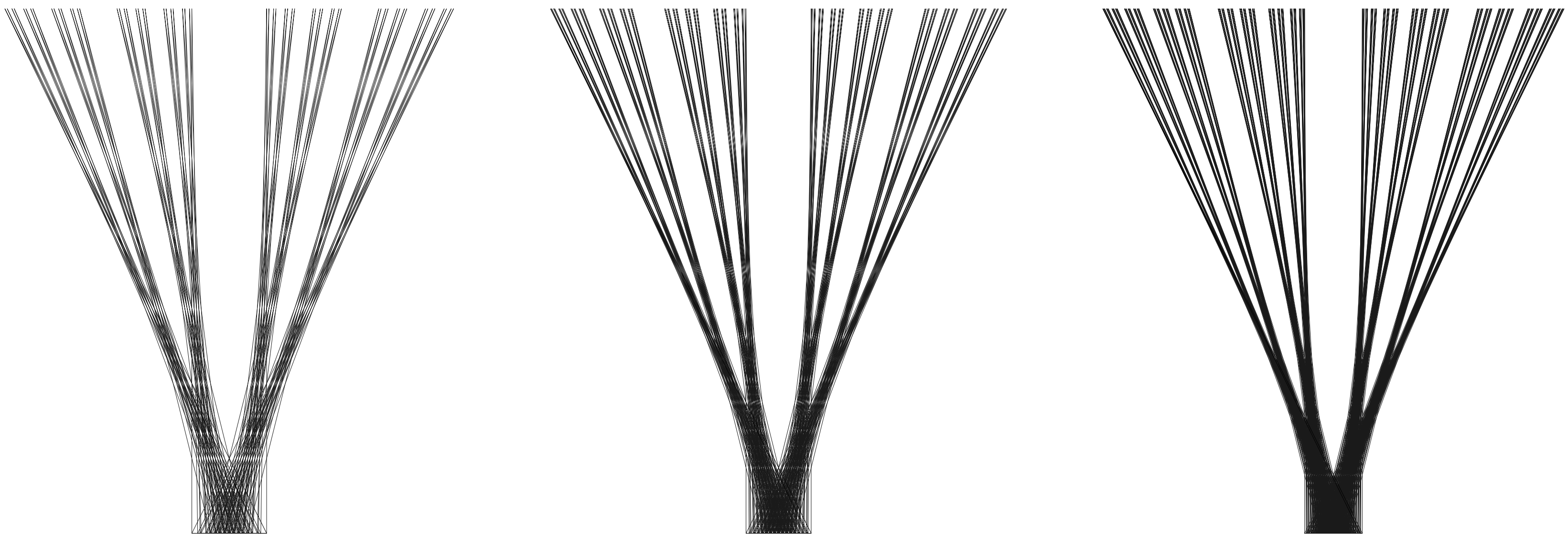
Stages 1,2,3:



Stages 3,4,5:



Stages 6,7,8:



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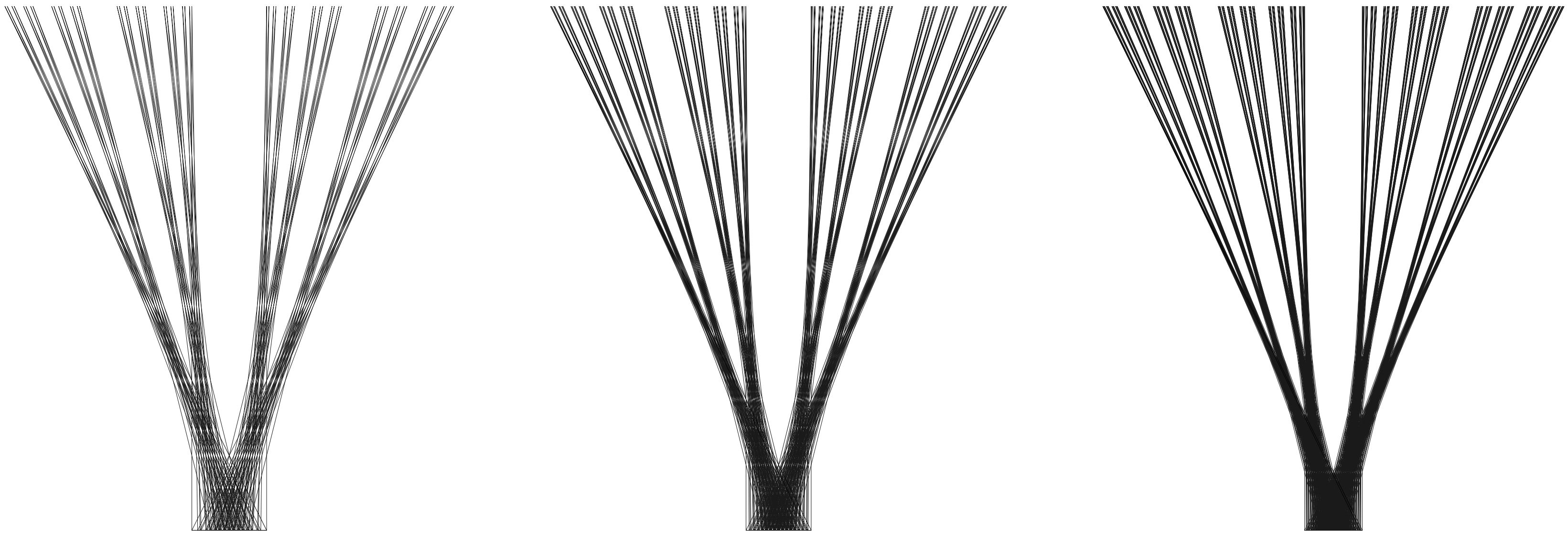
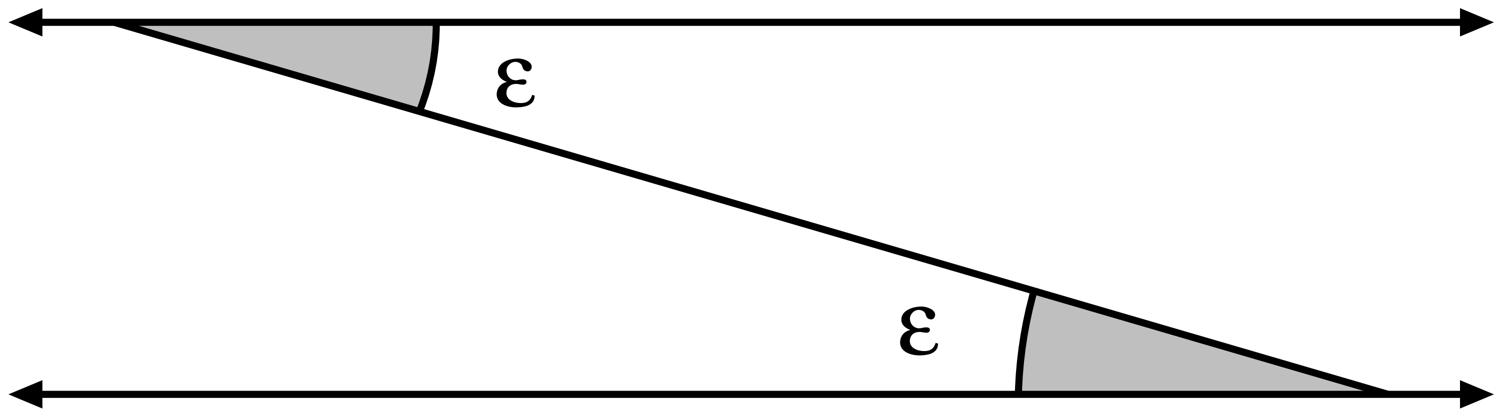


After k steps we have 2^k triangles, total area $1/k$.

Each triangle contains unit segments in angle arc $\simeq 2^{-k}$.

Can order triangles so T_j, T_{j+1} have a parallel edge.

Trick of Pal to move needle between parallel lines.



Works with any area $\epsilon > 0$.

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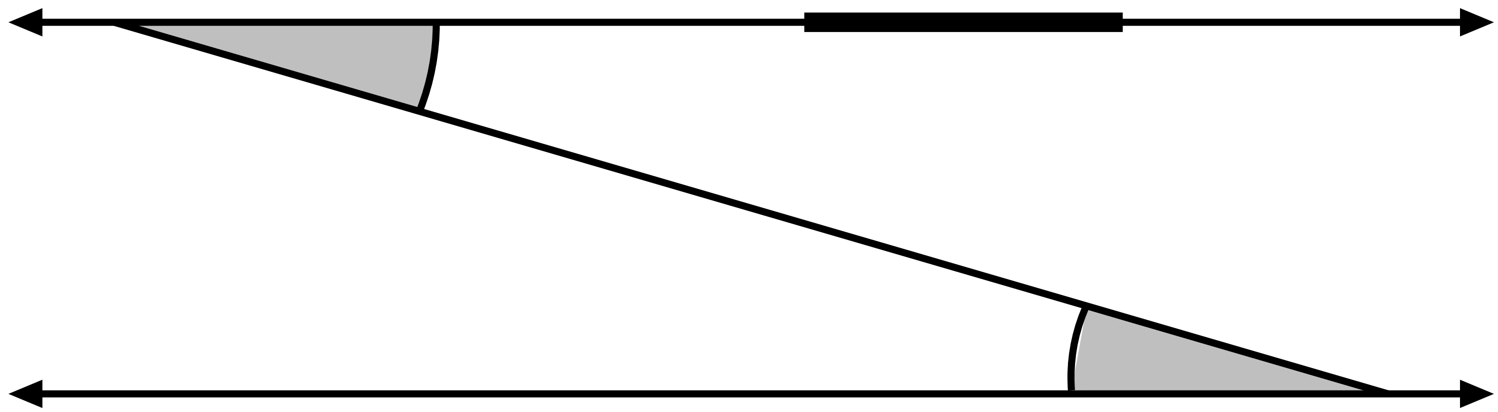

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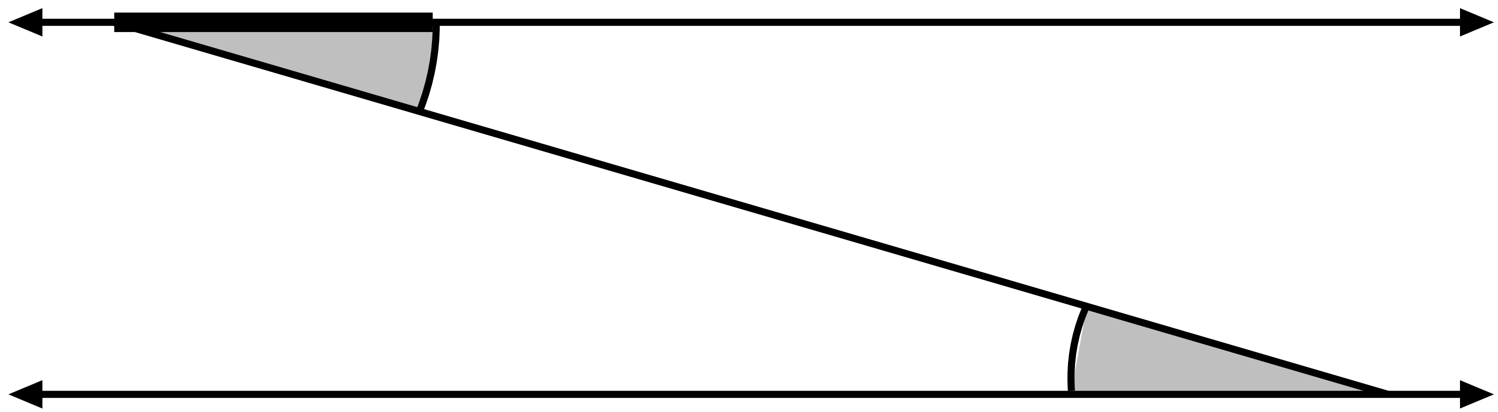

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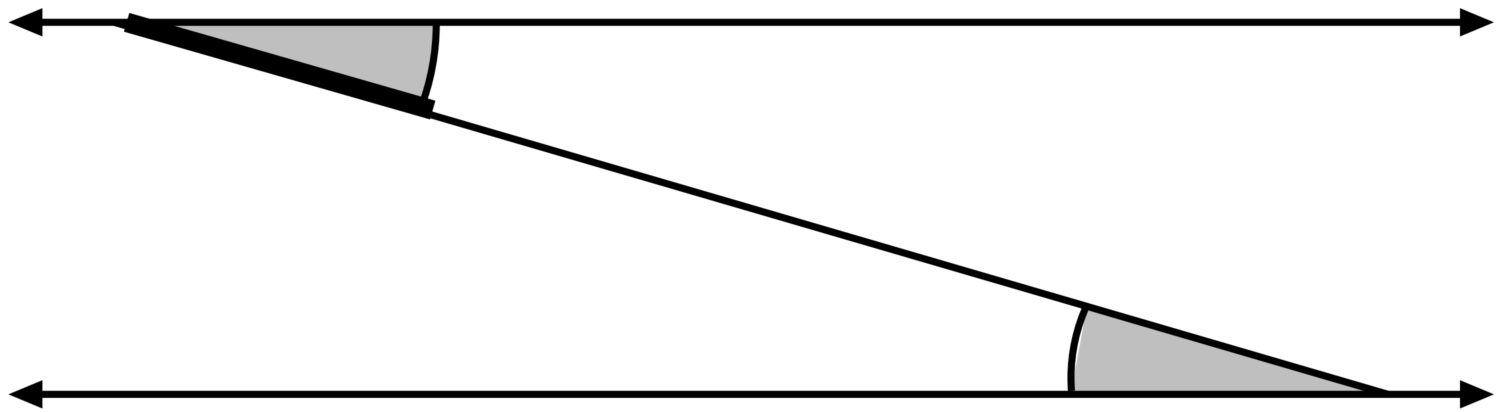

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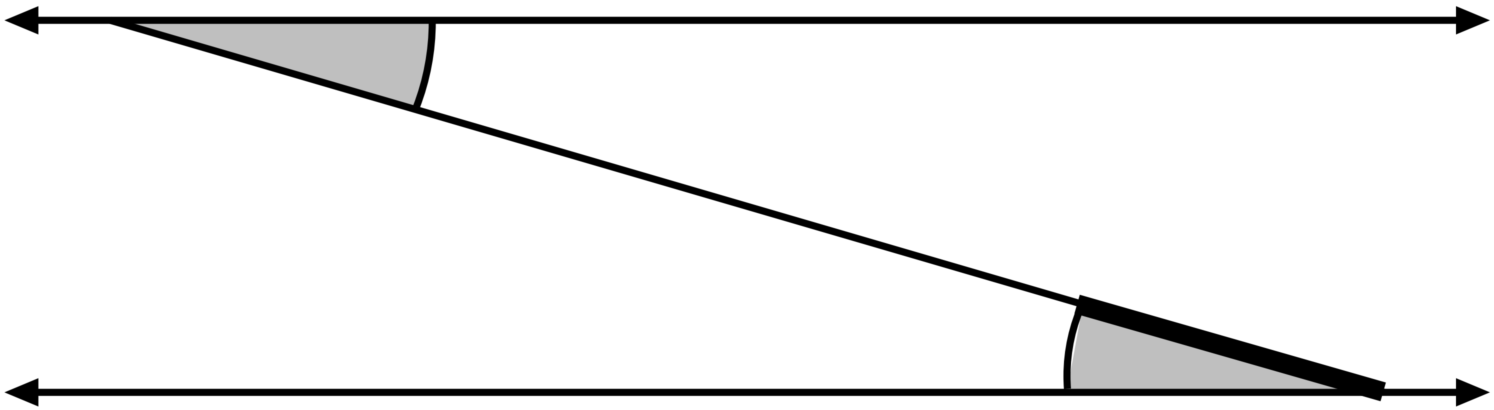

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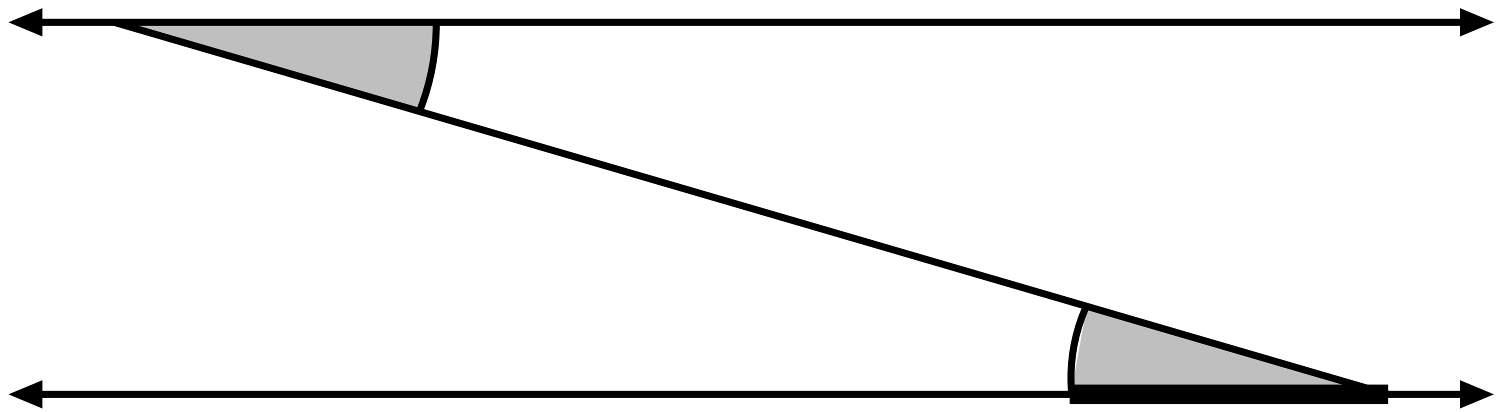

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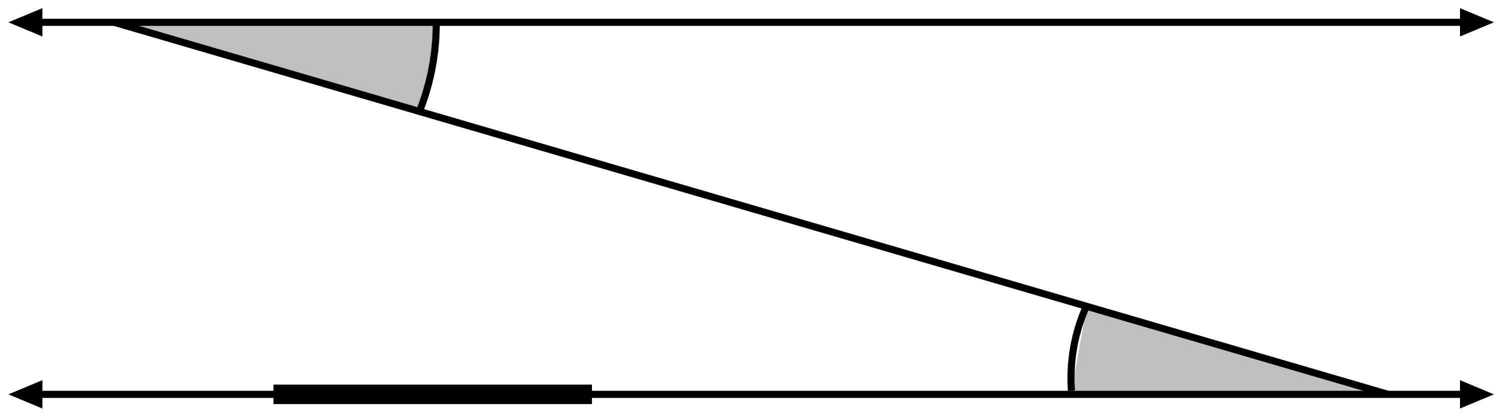
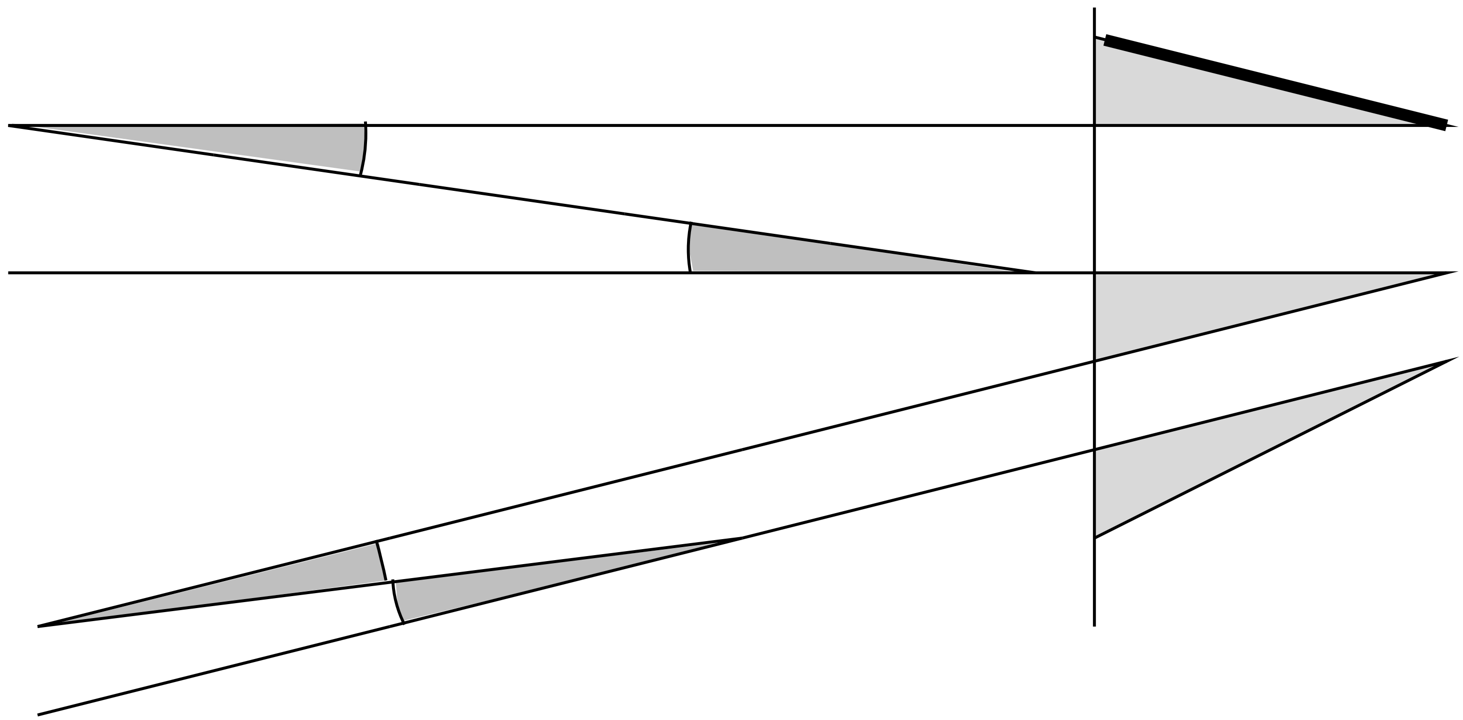


Use this to move needle between Perron triangles.



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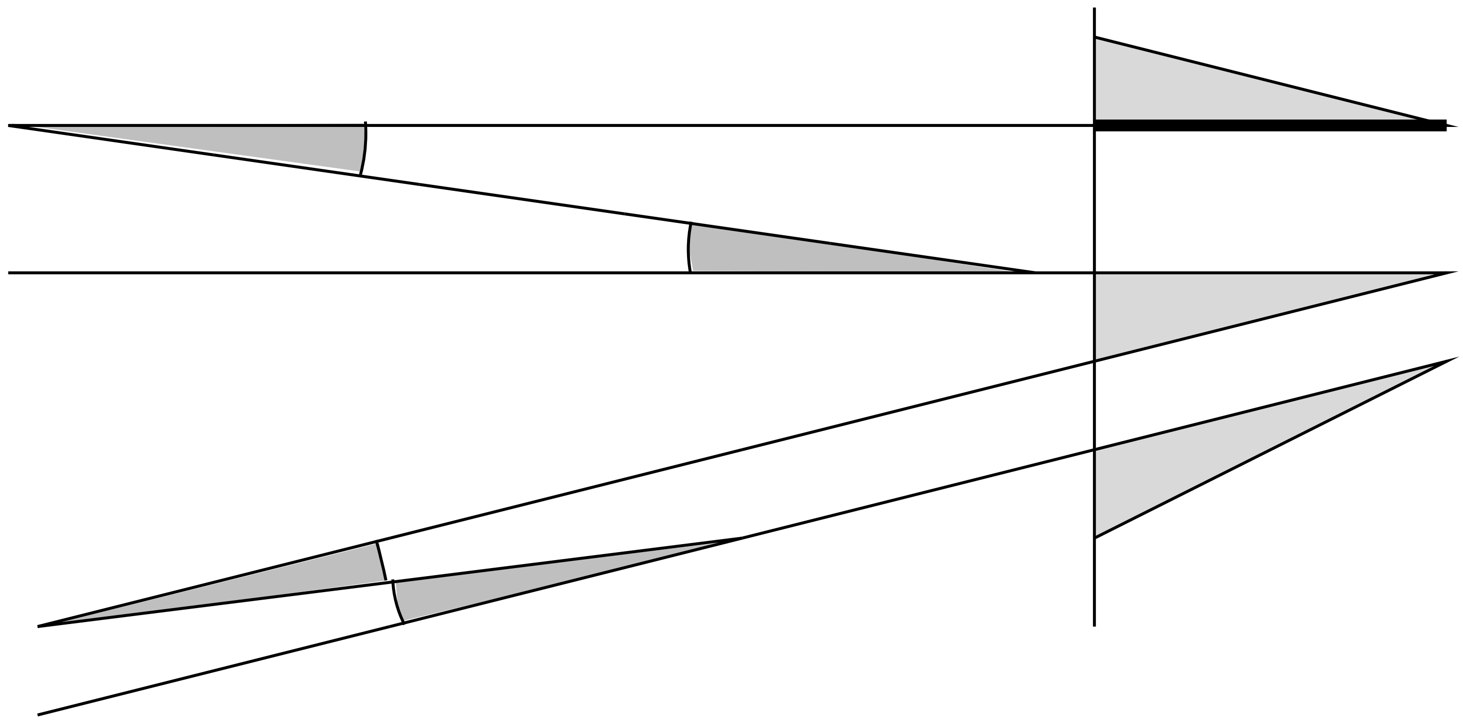

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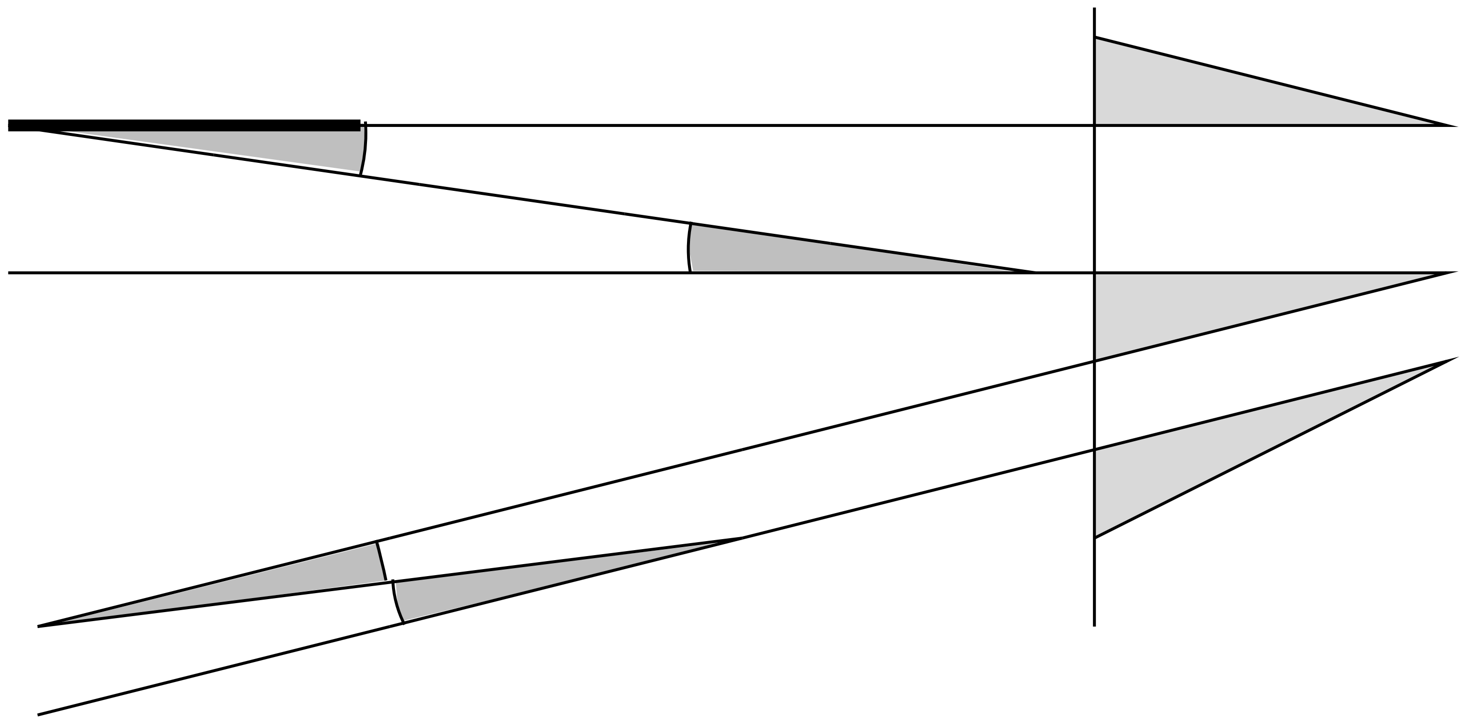

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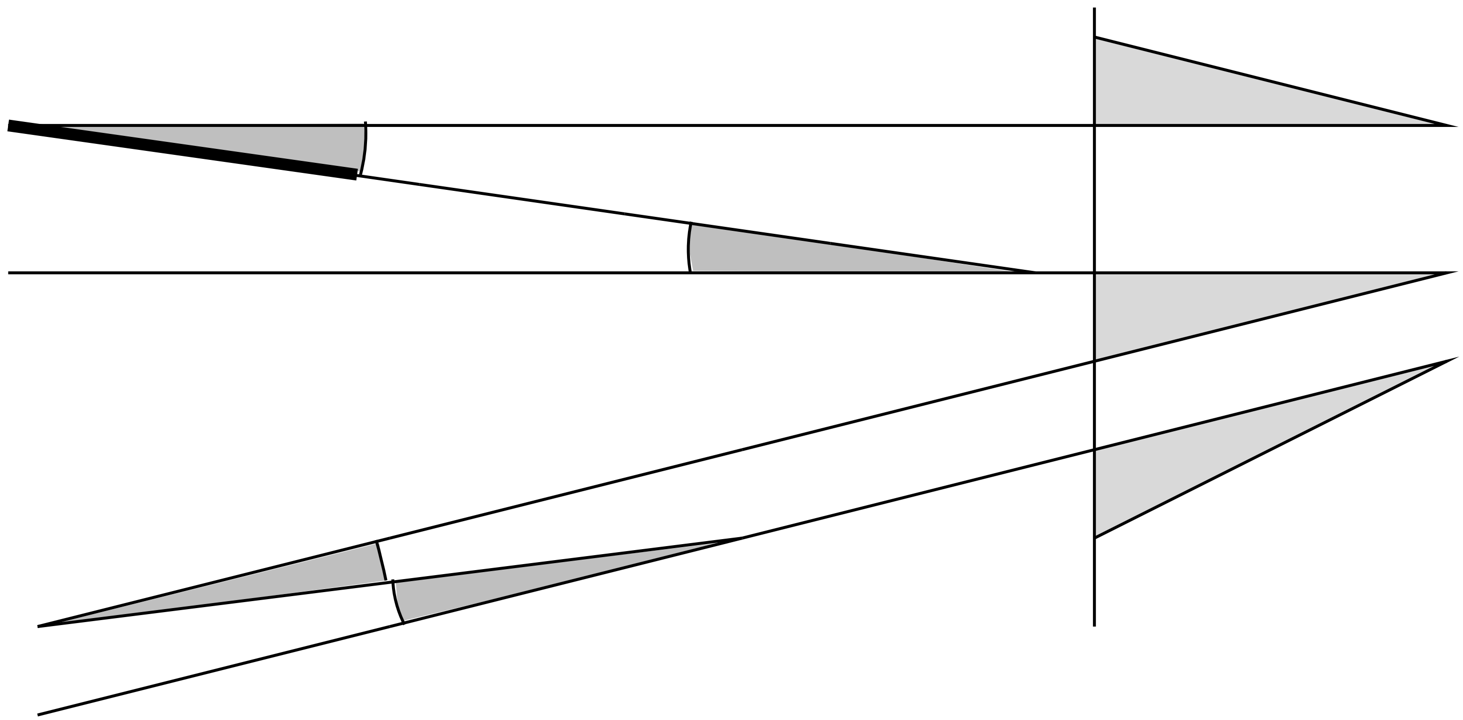

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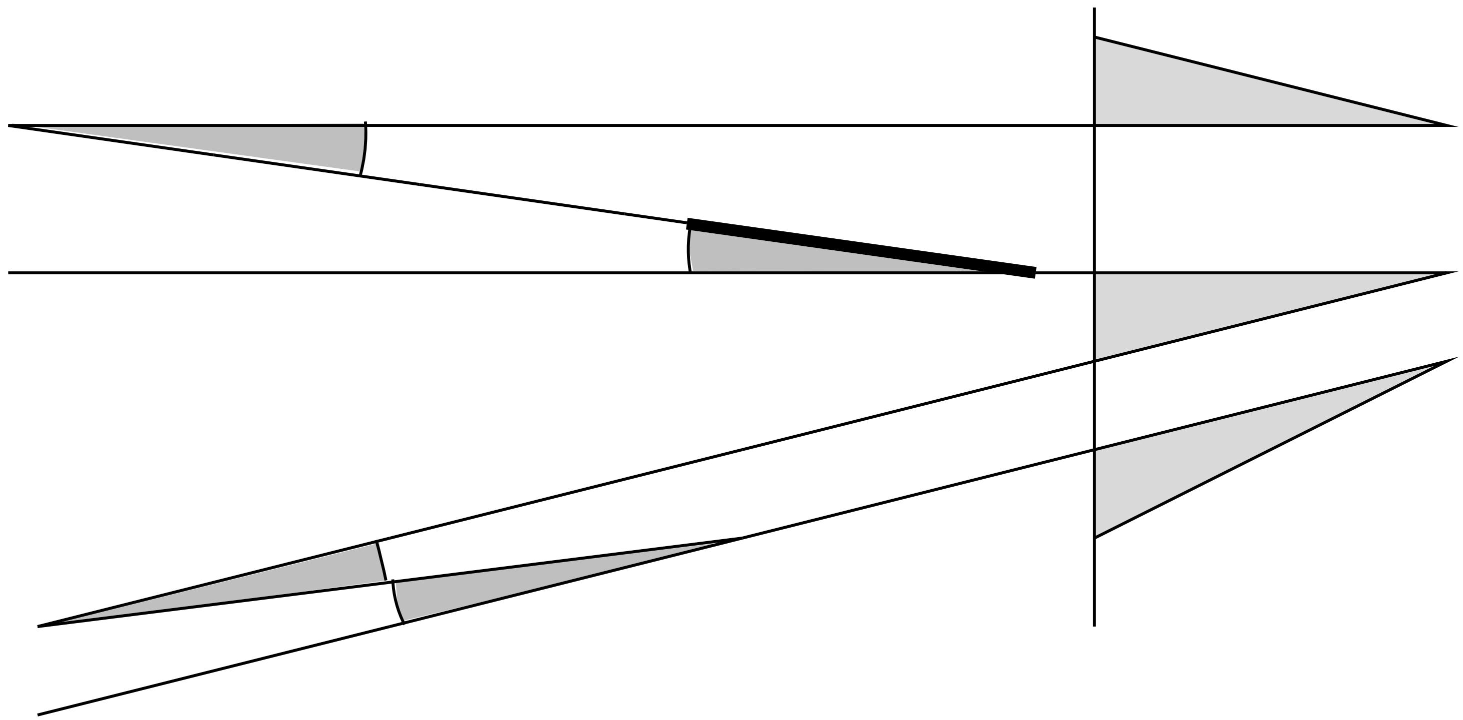

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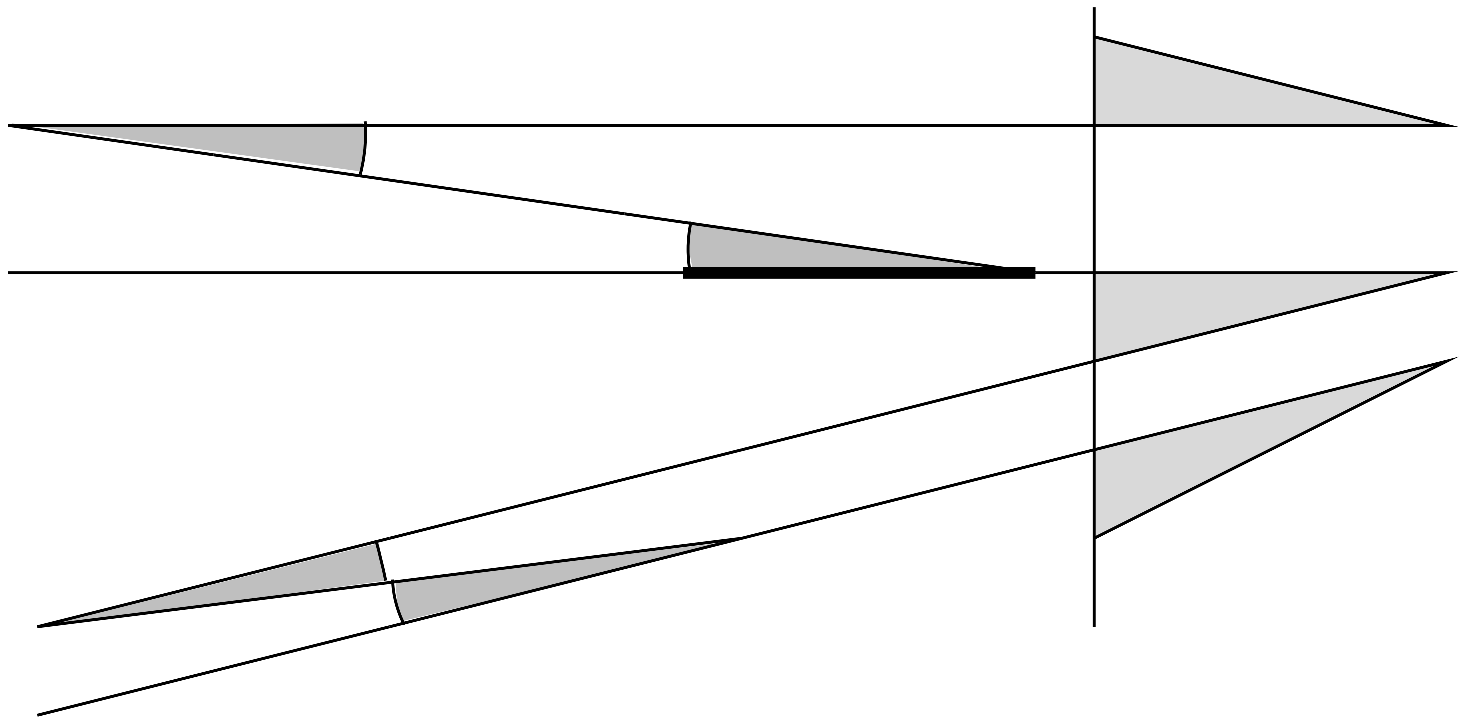

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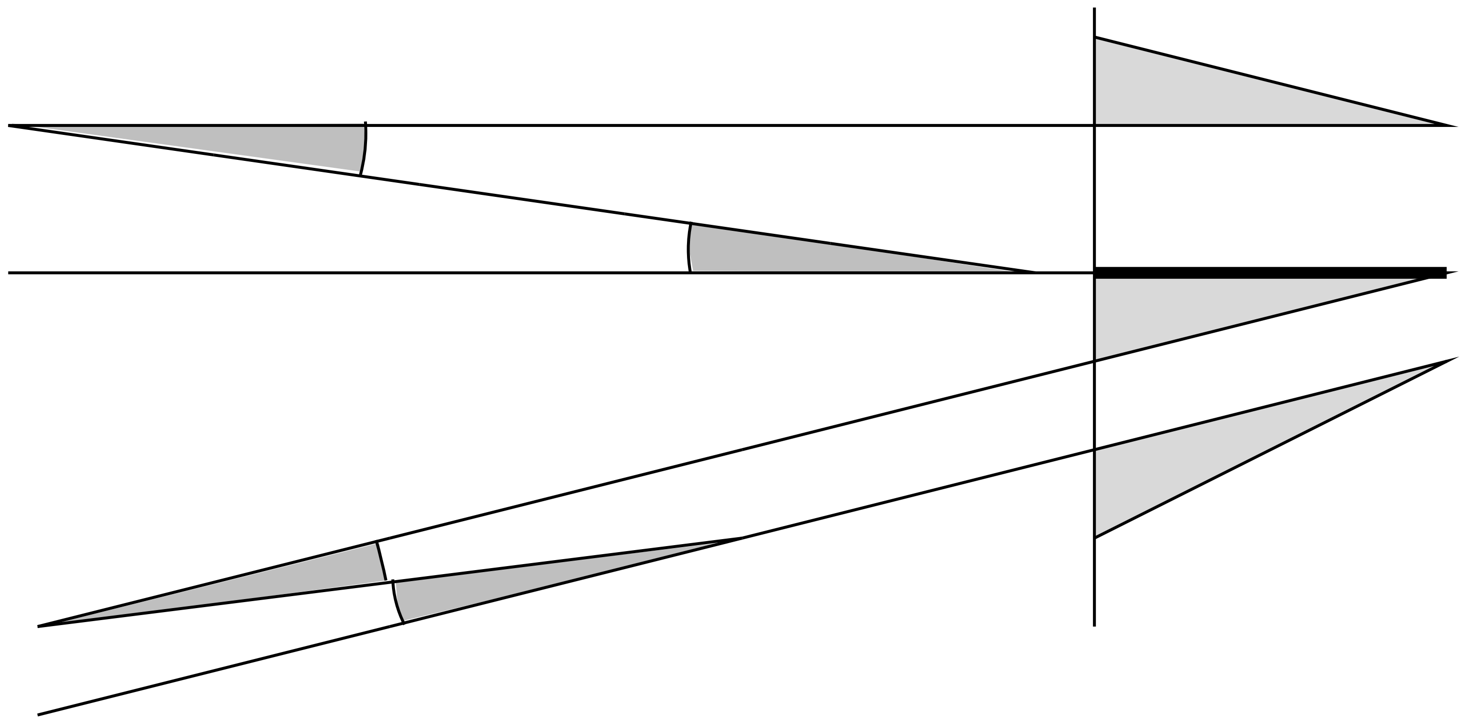

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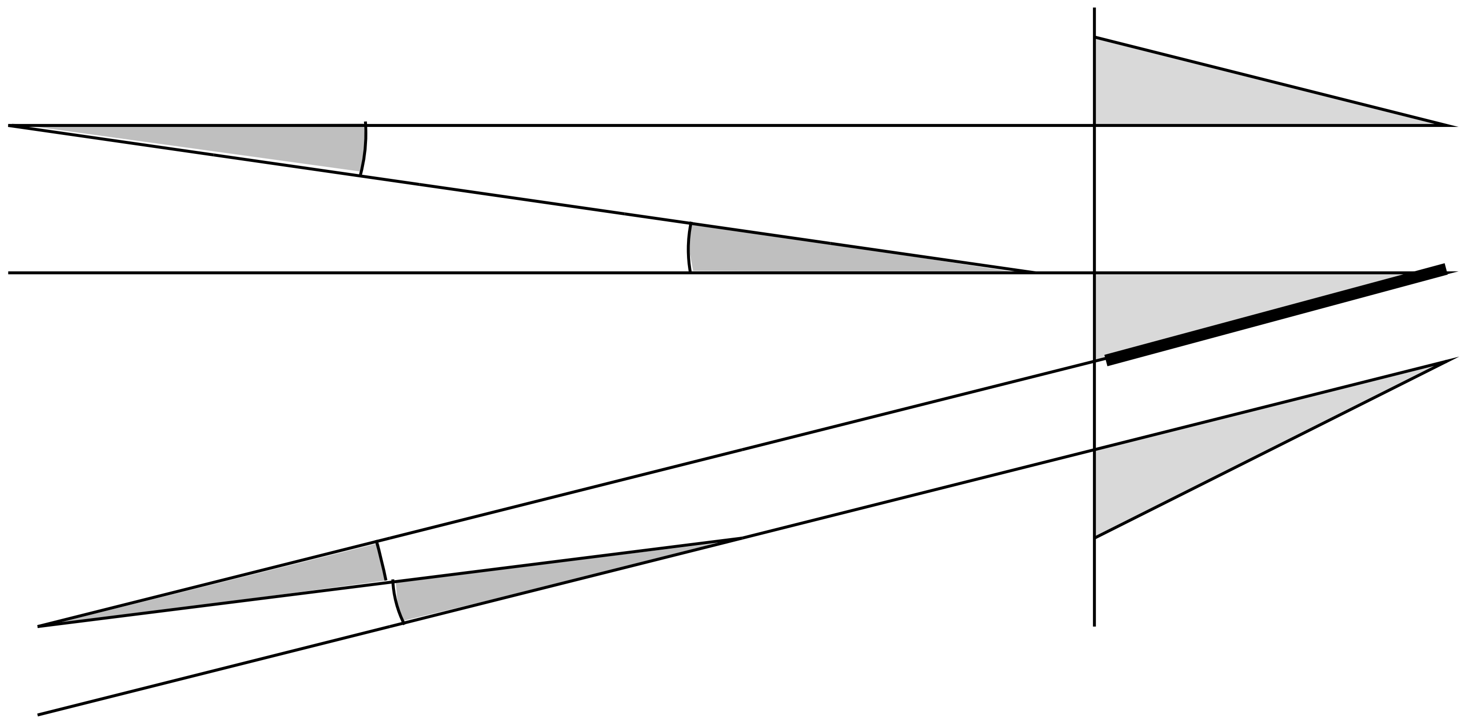

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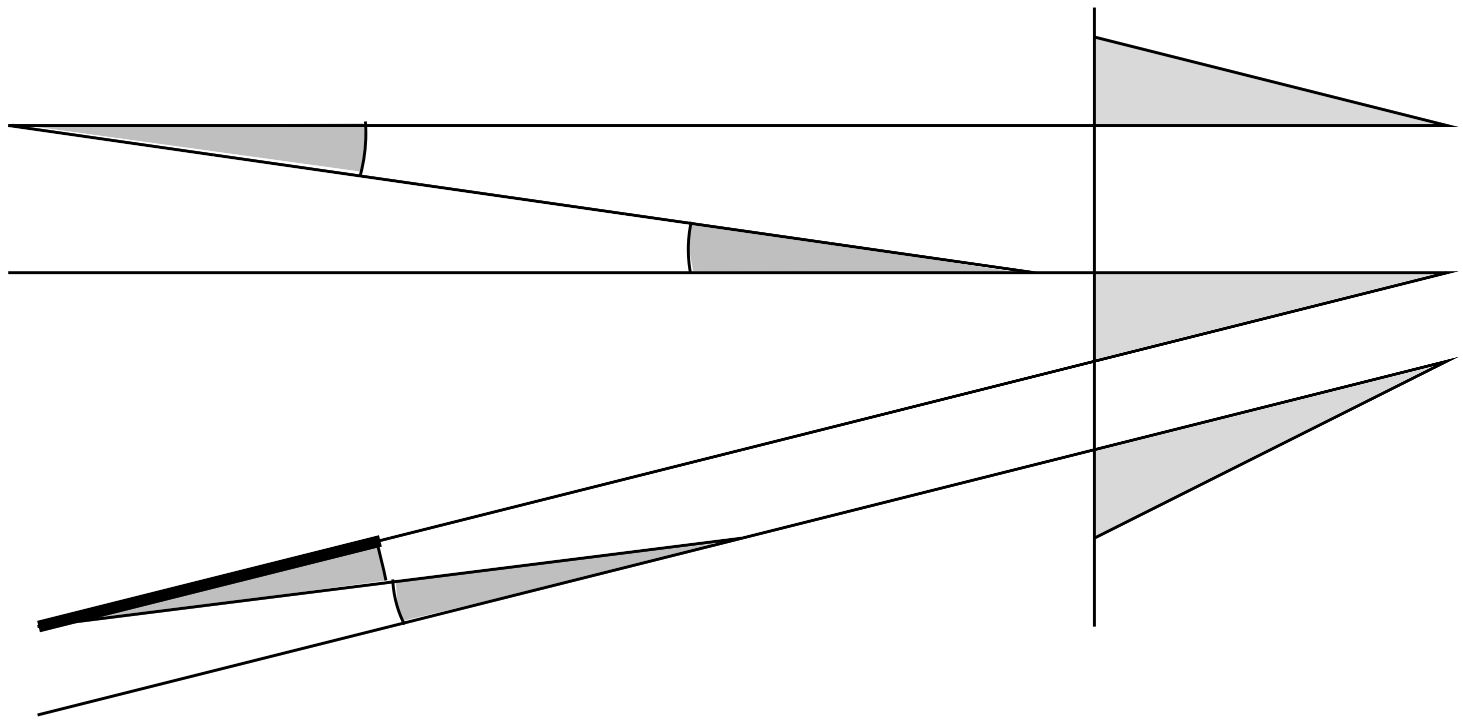

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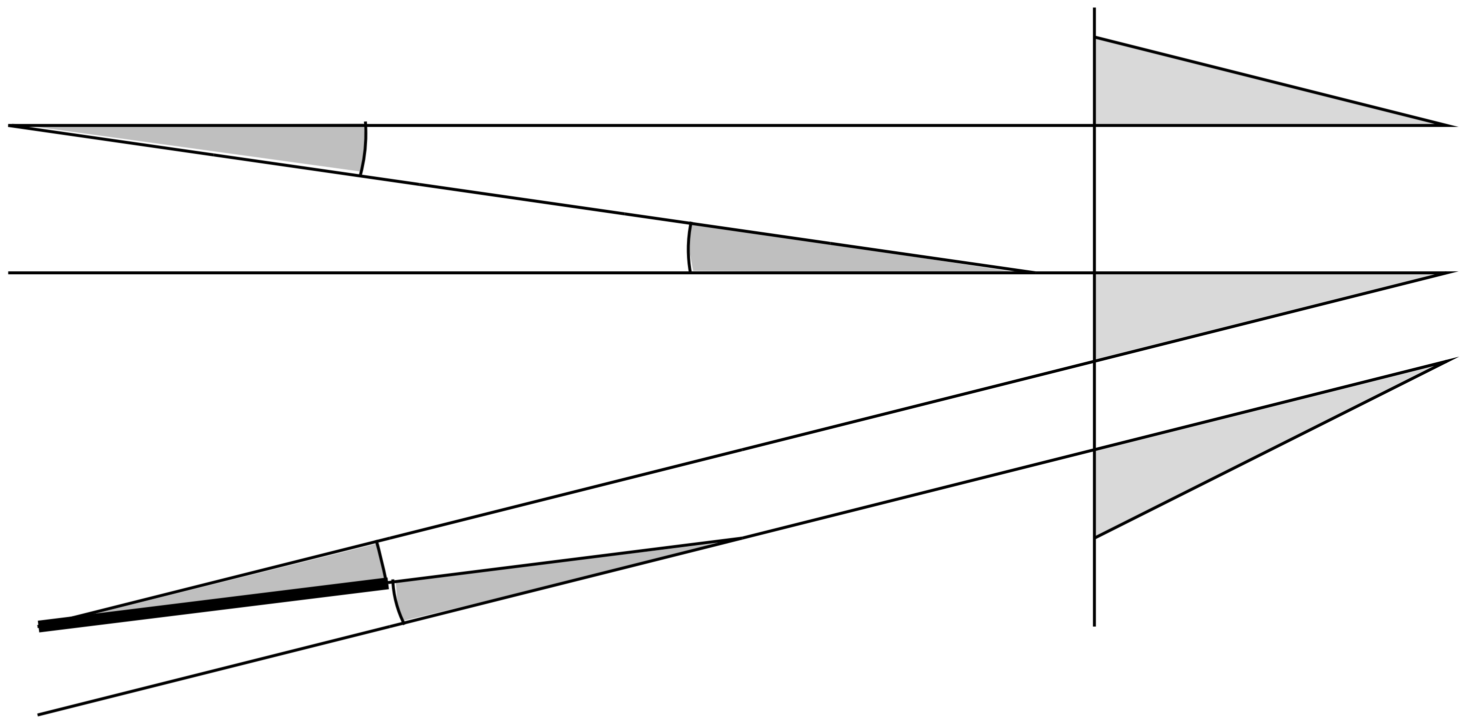

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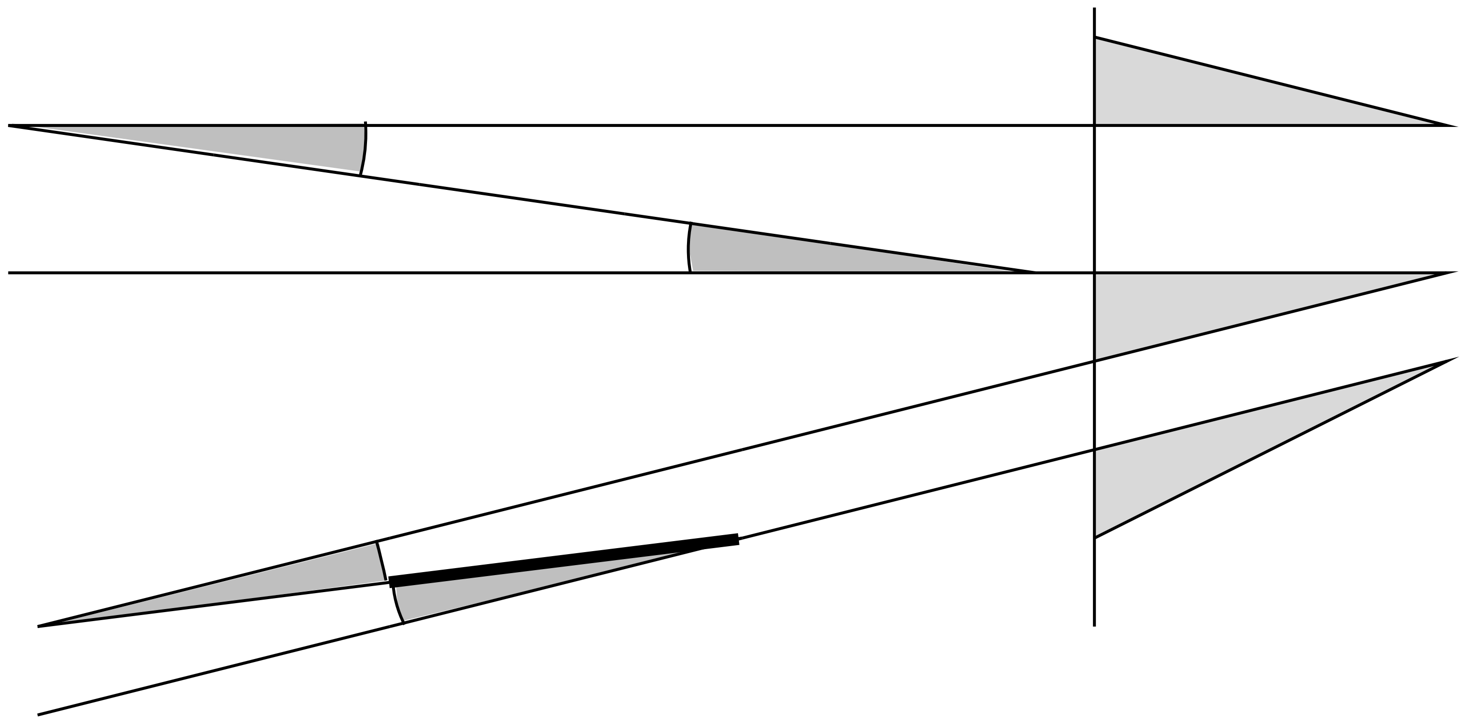

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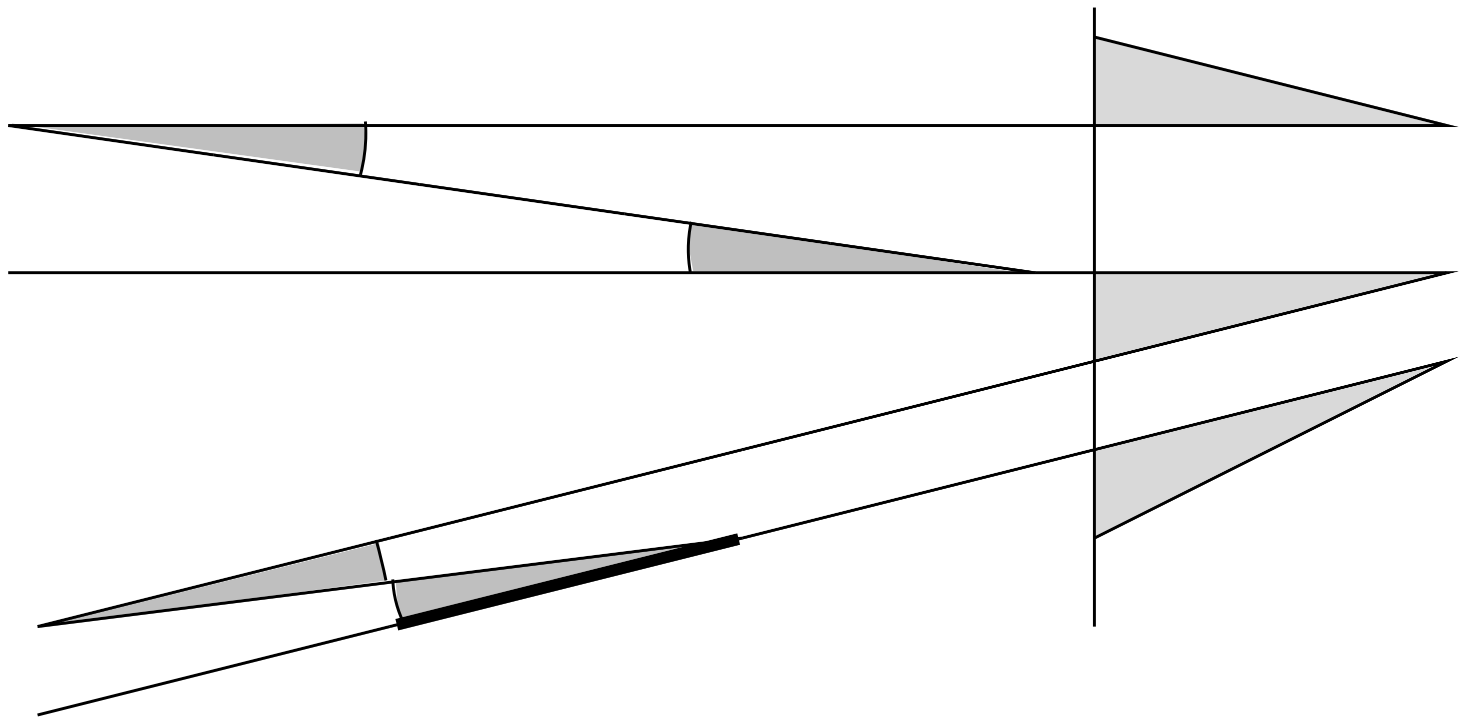

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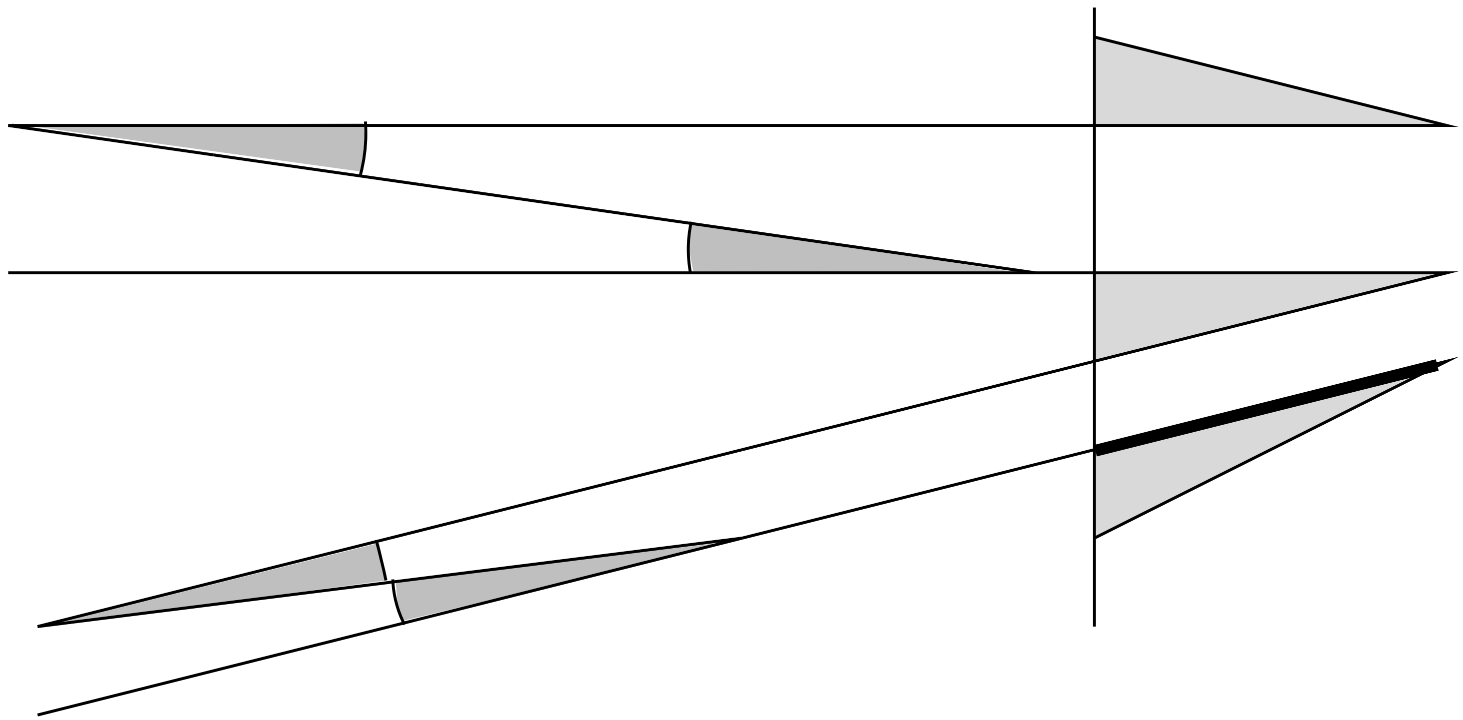

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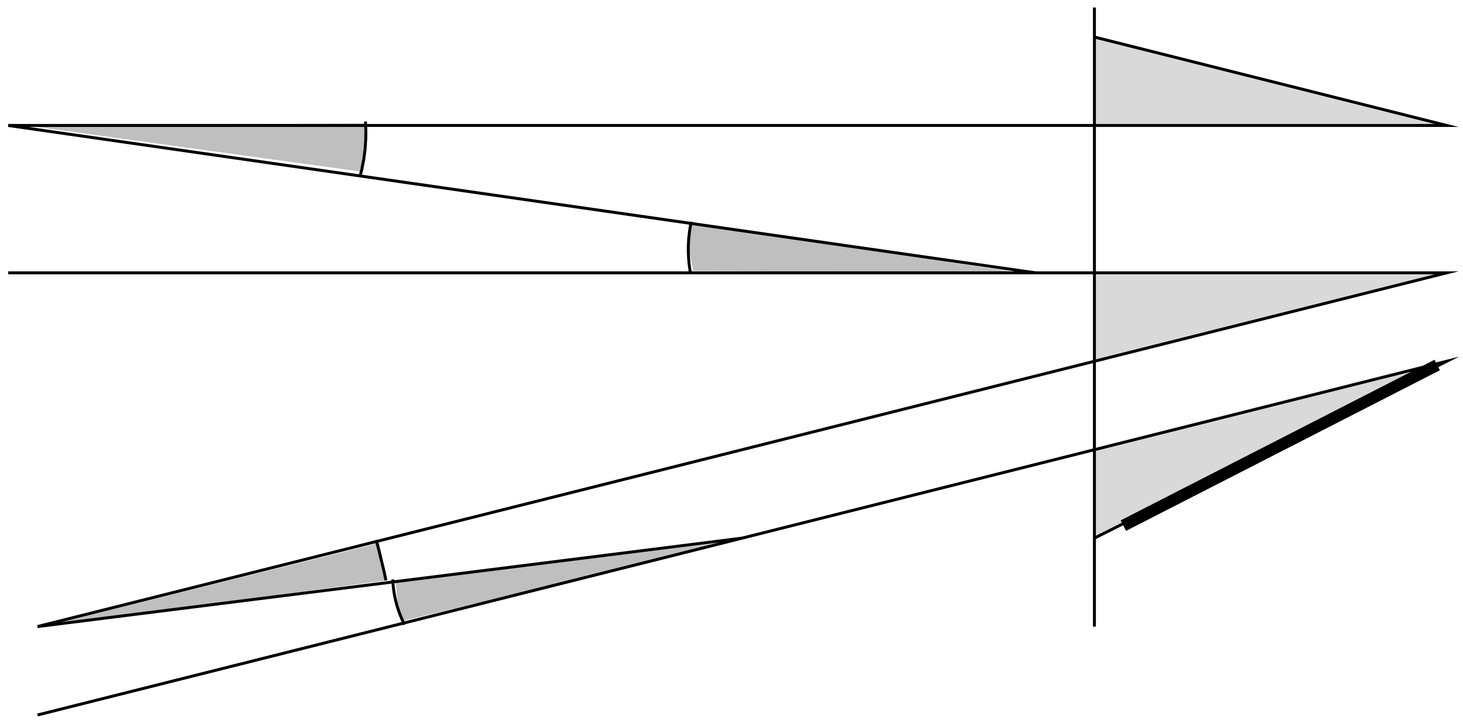

Thus arbitrarily small area suffices to rotate needle.

In 1941 Van Alphen showed there are small Kakeya sets inside a $D(0, 2+\epsilon)$ for any $\epsilon > 0$.

In 1971 Cunningham showed needle can be rotated in simply connected region of arbitrarily small area.

Exercise: positive area is needed to rotate needle.

A **Besicovitch set** is a compact set of zero area that contains a unit line segment in every direction.

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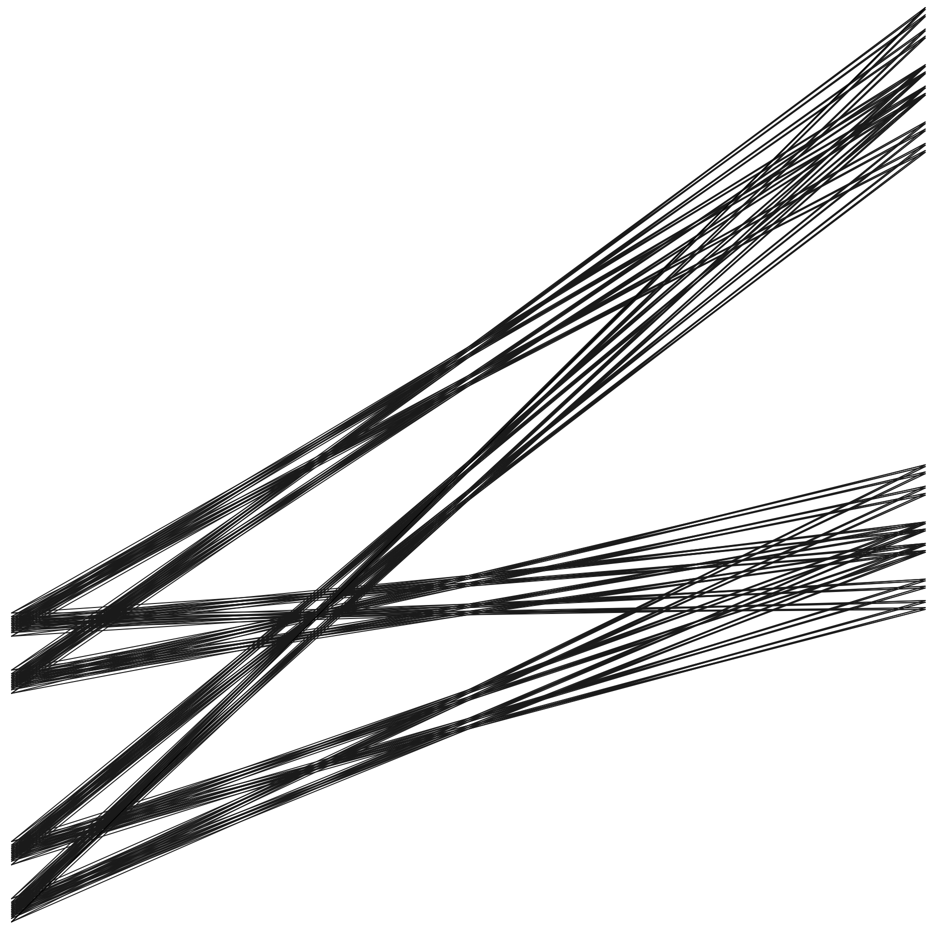

Perm 1917-1919

Petrograd 1919-1924

Copenhagen 1924-26

Liverpool 1926-27

Cambridge 1927-1970

Succeeded Littlewood in 1950
as Rouse Ball Chair of Math.

Abram Samoilovitch Besicovitch (1891-1970)

Question: If $f(x, y)$ is Riemann integrable, is

$$\int\int f(x, y)dxdy = \int[\int f(x, y)dx]dy?$$

If f is Riemann integrable on the plane, is it Riemann integrable on every horizontal line?

Recall f is Riemann integrable iff f is continuous except on a set of Lebesgue measure zero.

No.

No.

Let $f(x, y) = 1$ if $x \in \mathbb{Q}, y = 0$, and $f = 0$ otherwise.

This function is zero except on single line, so is Riemann integrable on the plane.

It is not Riemann integrable on the real line.

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It is Riemann integrable on all non-horizontal lines, so we can still evaluate integral by rotating coordinates.

Is there a function $f(x, y)$ that is Riemann integrable on the plane, but fails to be Riemann integrable on some line in every possible direction?

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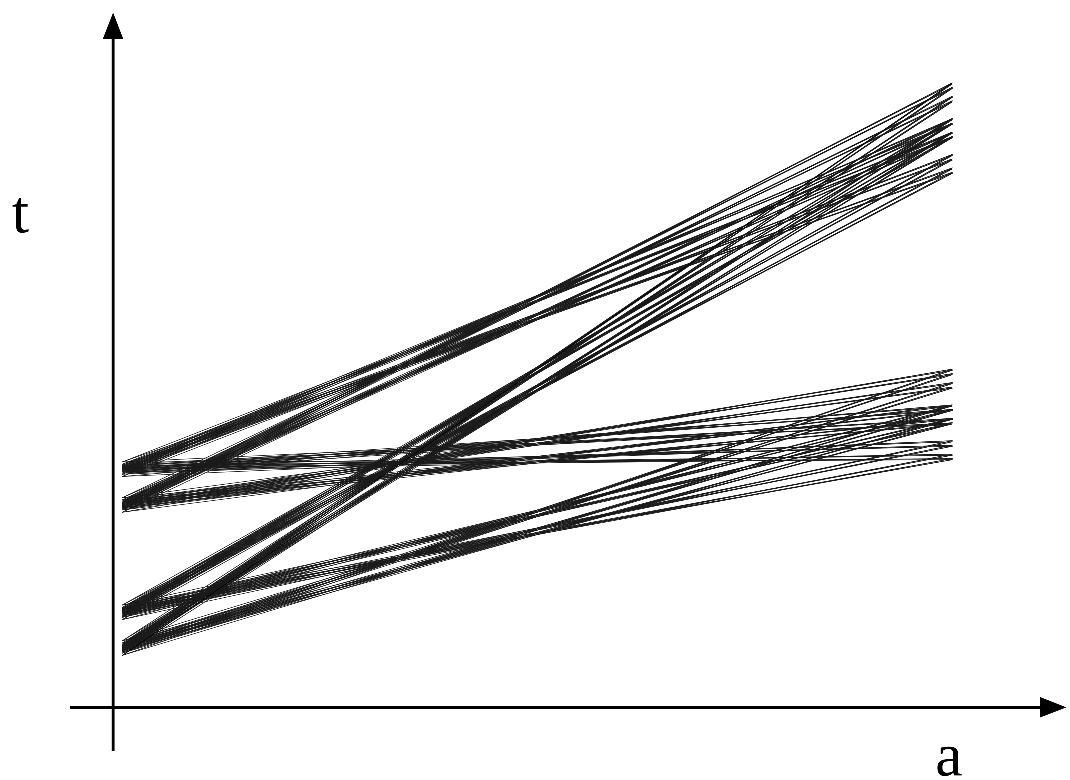
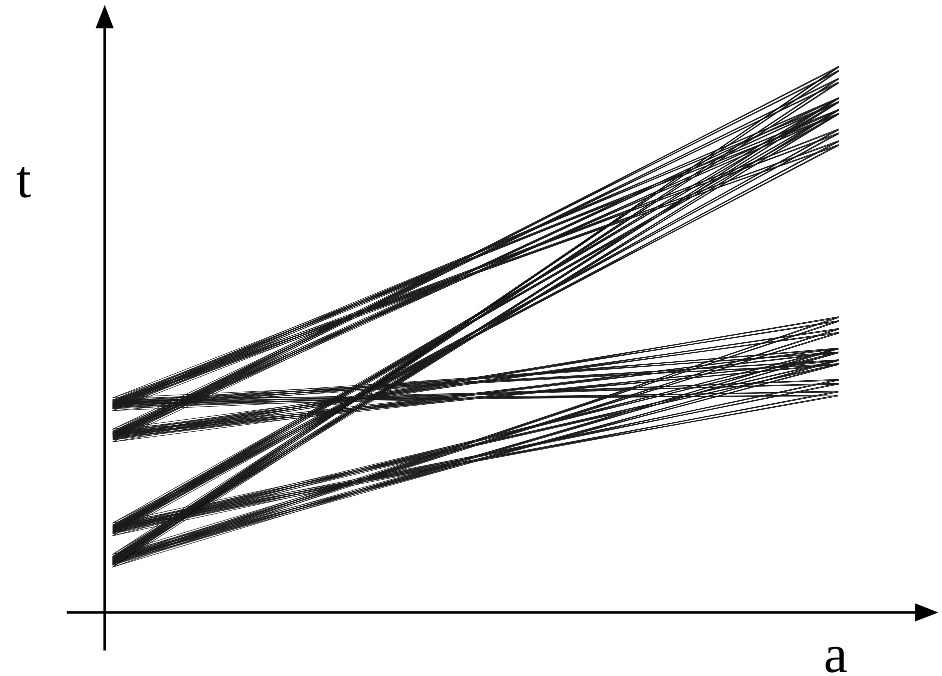
Yes, if a Besicovitch set K of zero area exists.

Construction of a Besicovitch set:

Build $f:[0,1]\to[0,1]$ so that $f_a(t)=f(t)+at$ maps $[0,1]$ to zero length for all a.

Given this, let $K=\{(a,at+f(t)):a,t\in[0,1]\}$.





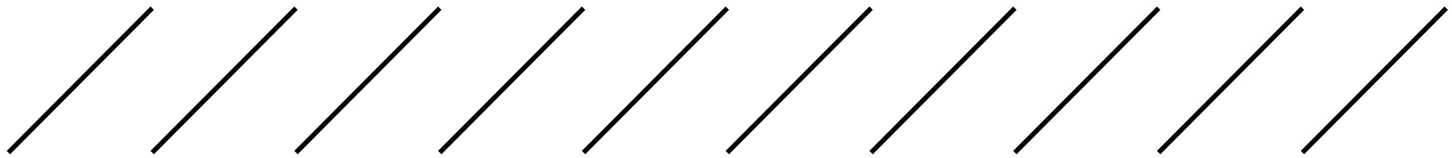
$$K = \{(a, at + f(t)) : a, t \in [0, 1]\}.$$

Fix t, vary $a \Rightarrow K$ contains a line of slope t.

Vertical slices of K are of the form $K_a = f_a([0, 1])$.

All vertical slices have zero length, so K has zero area.

Recall:



$$g(t) = t - \lfloor t \rfloor$$

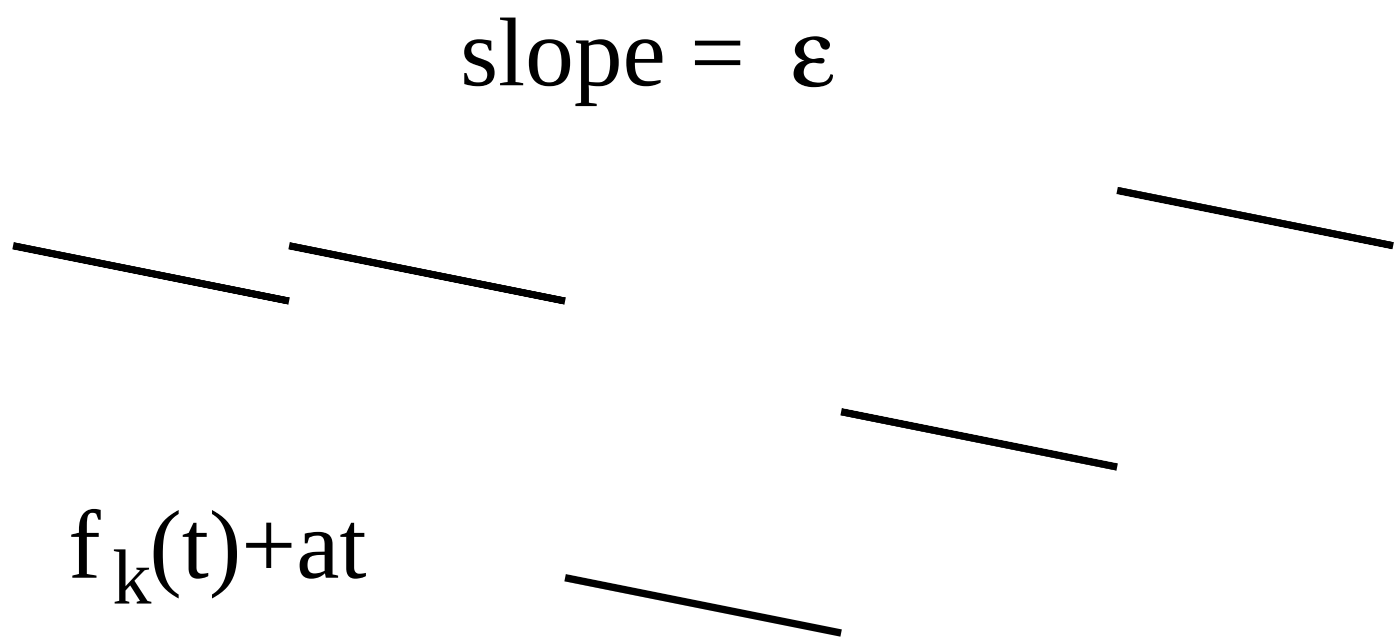
Fix $\{a_k\}$ dense in $[0,1]$ and $|a_{k+1} - a_k| \le \epsilon(k) \searrow 0$.

$$f_k(t) = \sum_{m=1}^{k} (a_{m-1} - a_m) \frac{g(2^m t)}{2^m},$$

If $a_0 = 0$, then by telescoping series $f_k'(t) = -a_k$ on each component I of $[0,1] \setminus 2^{-k}\mathbb{Z}$.

$$f(t) = \lim_{k \to \infty} f_k(t) = f_k(t) + r_k(t),$$

$$|r_k(t)| \le \epsilon(k) \cdot 2^{-k},$$

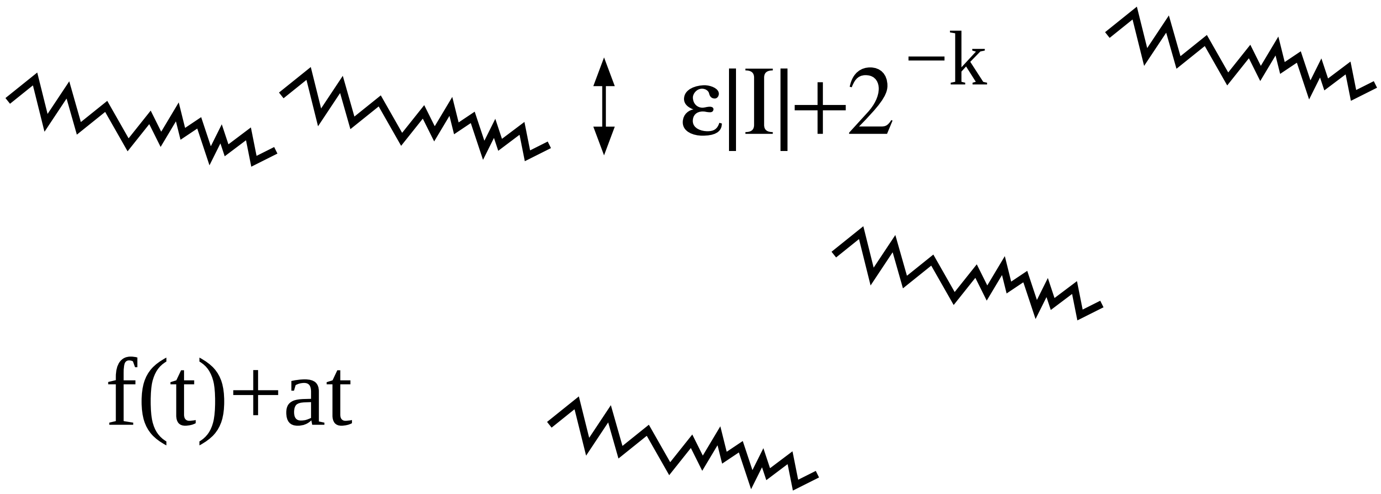


Given a choose k so $|a - a_k| < \epsilon(k)$.

$f_k(t) + at$ is piecewise linear with slopes $\leq \epsilon(k)$.

$f_k([0,1])$ is covered by 2^k intervals of length $\epsilon(k) \cdot 2^{-k}$.

Total image length $= \epsilon(k)$.



$f_a(t) = f(t) + at = (f_k(t) + at) + r_k(t)$

$f_a([0,1])$ covered by 2^k intervals of size $\epsilon(k)2^{-k}+\epsilon(k)2^{-k}$.

Thus f_a maps $[0,1]$ to zero length.

Sample paths of the Cauchy process have this property.
Used by Babichenko, Peres, Peretz, Sousi and Winkler to produce random Besicovitch sets.

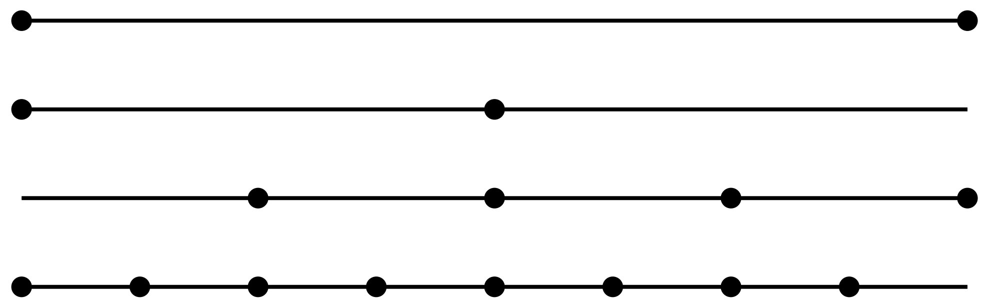
My example was inspired by these random examples.

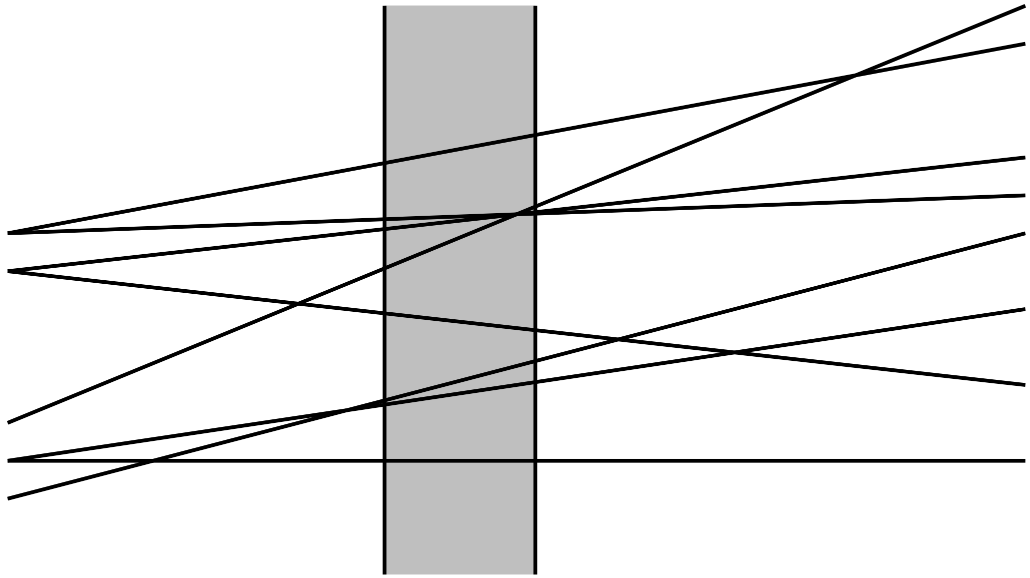
Special choice of $\{a_k\}$ gives explicit estimate.

Special choice of $\{a_k\}$ gives explicit estimate.

Take $\epsilon(k) \simeq 1/k$.

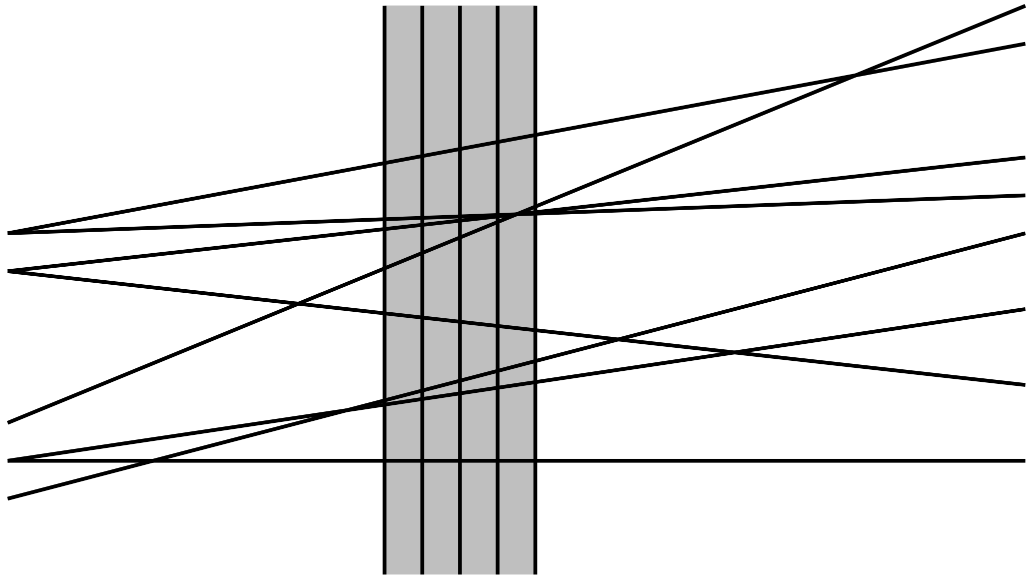
For example, take $\epsilon(k) = 2^{-n}, k \in [2^n, 2^{n+1})$.





Previously showed that $|a - a_k| < \epsilon(k)$ implies the vertical slice of K through a was covered by 2^k sets of diameter $d_k \simeq \epsilon(k)2^{-k} \simeq 2^{-k}/k$.

Consider vertical strip $S = \{(a,t) : |a - a_k| < \epsilon_k\}$.



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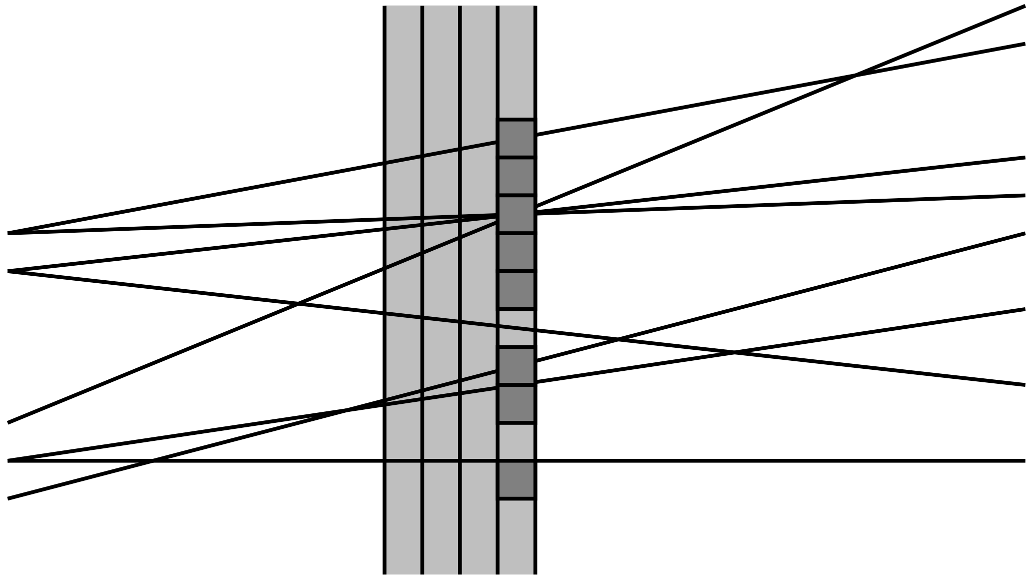
Consider vertical strip $S = \{(a, t) : |a - a_k| < \epsilon_k\}$.

Subdivide S into 2^k sub-strips T of width $d_k = \epsilon_k 2^{-k}$.

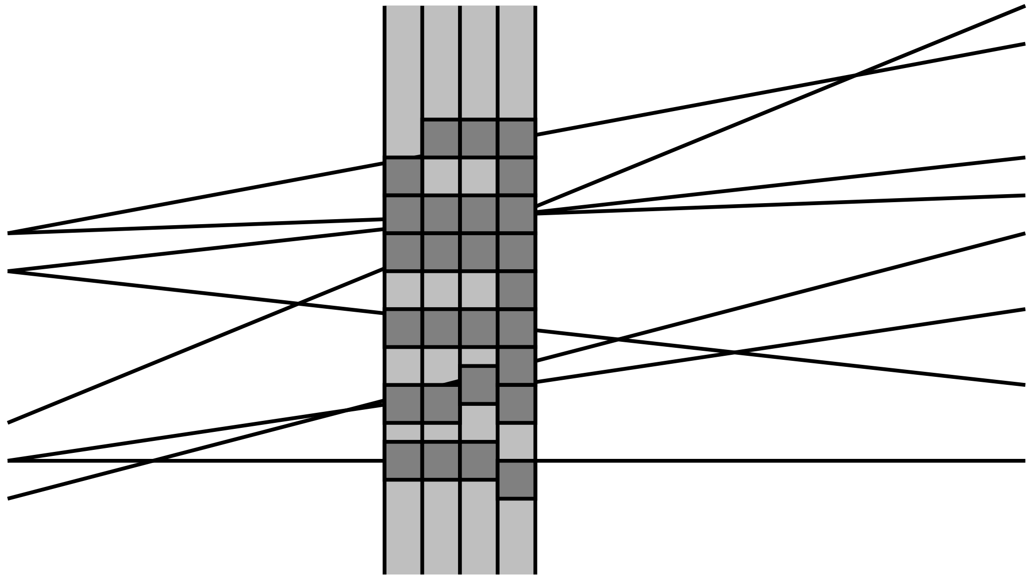


Each $K \cap T$ is covered by 2^k squares of size d_k.

Thus $K \cap S$ is covered by 2^{2k} squares of size d_k.

$$\operatorname{area}(S \cap K) \leq 2^{2k} \cdot (\epsilon_k 2^{-k})^2 \simeq \epsilon_k^2.$$

$$\operatorname{area}(K) \leq \epsilon_k \simeq \frac{1}{|\log d_k|}.$$



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This implies

$$\text{area}(K_\delta) = O(\frac{1}{\log(1/\delta)})$$

where

$$K(\delta) = \{z : \text{dist}(z, K) < \delta\}.$$

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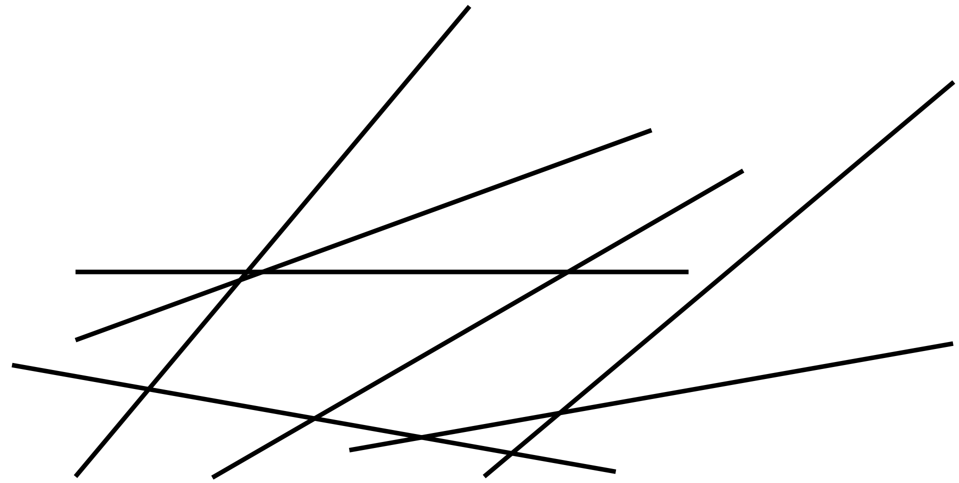
$$K(\delta) = \{z : \text{dist}(z, K) < \delta\}.$$

This is optimal (Antonio Cordoba)

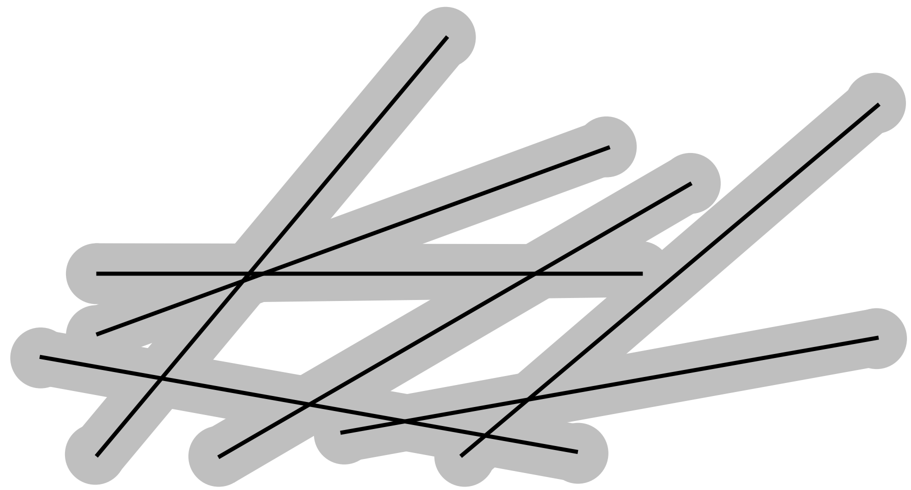
Lemma: Suppose $K \subset \mathbb{R}^2$ contains unit line segment in all directions. Then all ϵ small enough

$$\text{area}(K(\epsilon)) \geq \frac{1}{4\log(1/\epsilon)}.$$

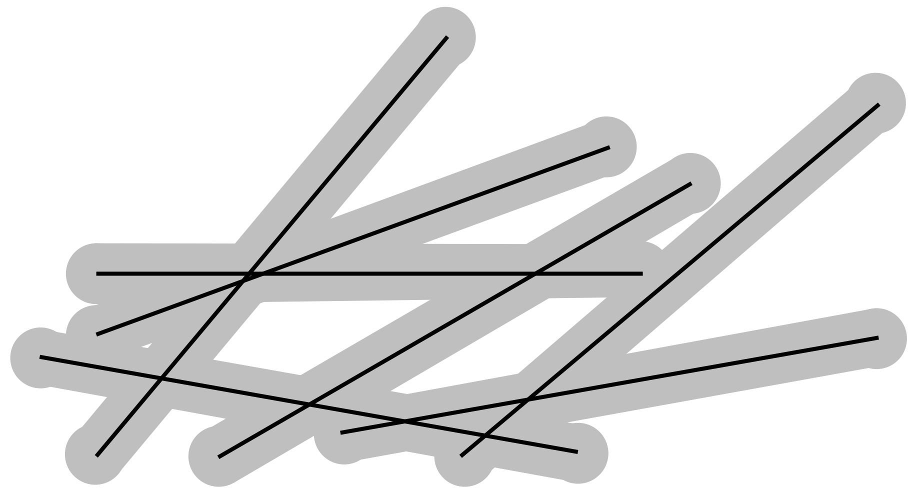


Let $\epsilon > 0$ and $n = \lfloor \epsilon^{-1} \rfloor$.

Suppose ℓ_i, $i = 1, \ldots, n$ are segments with angles $\geq \frac{\pi}{n}$.



Let $\ell(\epsilon)$ be ϵ-neighborhood of ℓ.



Let Ψ_i be the indicator function of $\ell_i(\epsilon)$ and $\Psi = \sum_{i=1}^{n} \Psi_i$.

Since $\mathrm{supp}(\Psi) \subset K(\epsilon)$, it suffices to show

$$\mathrm{area}(\{\Psi > 0\}) \geq \frac{C}{\log(1/\epsilon)}.$$

By Cauchy-Schwarz

$$(\int_{\mathbb{R}^2} \Psi(x)\, dx)^2 \leq \left(\int_{\mathbb{R}^2} \Psi^2(x)\, dx\right) \left(\int_{\Psi>0} 1^2\, dx\right)$$

which gives

$$\operatorname{area}(\{\Psi > 0\}) \geq \frac{\left(\int_{\mathbb{R}^2} \Psi(x)\, dx\right)^2}{\int_{\mathbb{R}^2} \Psi^2(x)\, dx}.$$

By the definition of Ψ,

$$\int_{\mathbb{R}^2} \Psi(x)\, dx = \sum_{i=1}^{n} \operatorname{area}(\ell_i(\epsilon)) \geq 2\epsilon n \geq 2.$$

and

$$\int_{\mathbb{R}^2} \Psi(x)\, dx = \sum_{i=1}^{n} \operatorname{area}(\ell_i(\epsilon)) \leq 2\epsilon n + \pi\epsilon^2 n = 2 + o(1).$$

Since $\Psi_i^2 = \Psi_i$ for all i, we have

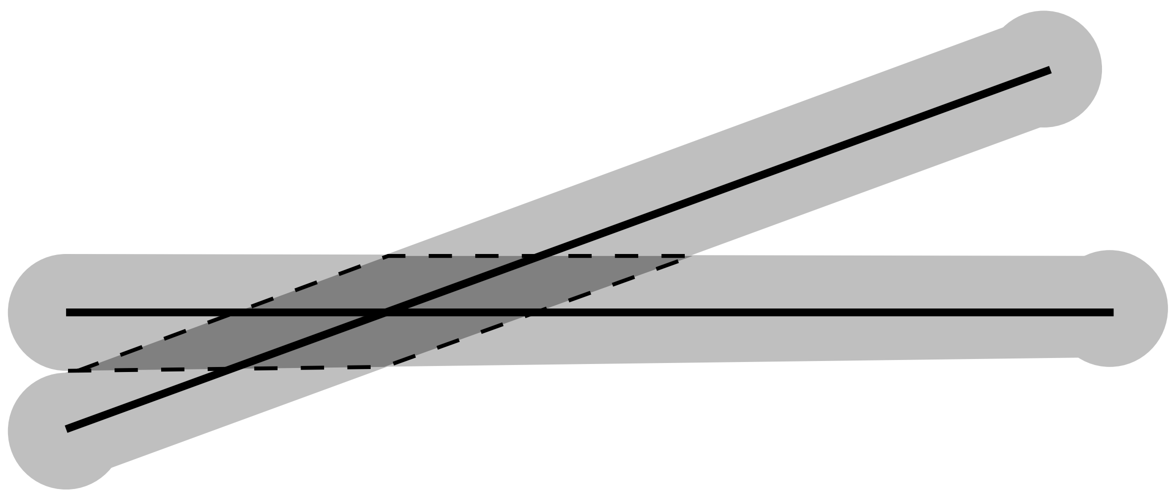
$$\begin{aligned}
\int_{\mathbb{R}^2} \Psi^2(x)\, dx &= \int_{\mathbb{R}^2} (\sum_{i=1}^{n} \Psi_i)(\sum_{k=1}^{n} \Psi_k) dx \\
&= \int_{\mathbb{R}^2} \Psi(x)\, dx + \sum_{i=1}^{n} \sum_{k \neq i} \mathrm{area}(\ell_i(\epsilon) \cap \ell_{i+k}(\epsilon)) \\
&\leq 2 + o(1) + \sum_{i=1}^{n} \sum_{k \neq i} \mathrm{area}(\ell_i(\epsilon) \cap \ell_{i+k}(\epsilon)).
\end{aligned}$$

The angle between the lines ℓ_i and ℓ_{i+k} is $k\pi/n$.

A simple calculation shows that if $k\pi/n \leq \pi/2$, then

$$\operatorname{area}(\ell_i(\epsilon) \cap \ell_{i+k}(\epsilon)) \leq \frac{4\epsilon^2}{\sin(k\pi/n)} \leq \frac{2\epsilon}{k}$$

with a similar estimate for $k\pi/n > \pi/2$.



Hence

$$\sum_{i=1}^{n} \sum_{k=1}^{n-1} \operatorname{area}(\ell_i(\epsilon) \cap \ell_{i+k}(\epsilon)) = 8 \sum \sum \frac{\epsilon}{k} = 8 \log n.$$

Thus

$$\operatorname{area}(\{\Psi > 0\}) \geq \frac{\left(\int_{\mathbb{R}^2} \Psi(x)\, dx\right)^2}{\int_{\mathbb{R}^2} \Psi^2(x)\, dx} \geq \frac{2}{8 \log n} \geq \frac{1}{4 \log(1/\epsilon)}.$$

If K is compact, let $N(K, \epsilon)$ be the minimal number of ϵ-balls that are needed to cover K.

The **upper Minkowski dimension** is

$$d = \limsup_{\epsilon \to 0} \frac{N(K, \epsilon)}{\log 1/\epsilon}.$$

Says ϵ^{-d} balls are needed.

Also called **box counting dimension**.

Lower Minkowski dimension defined using liminf.

Hausdorff dimension allows different sized balls.

H-dim $\leq$ lower-M-dim $\leq$ upper-M-dim

Besicovitch sets have Minkowski dimension 2.

They also have Hausdorff dimension 2.

Sharp guage is unknown.

Kakeya Conjecture: A Besicovitch set in $\mathbb{R}^d$ has Hausdorff dimension d.

Lemma: If K is a Besicovitch set in $\mathbb{R}^d$, then the Minkowski dimension is $\geq (d+1)/2$.

Sketch:

- If M-dim $< \alpha$ then $\mathrm{vol}(K(\delta)) \ll \delta^{d-\alpha}$.
- Consider δ^{1-d} tubes with angles $\geq \delta$.
- Each tube has volume δ^{d-1}; sum of volumes $\simeq 1$.
- Some point is in $\geq = 1/\mathrm{vol}(K(\delta)) = \delta^{\alpha-d}$ tubes.
- Tubes hitting this point have roughly disjoint volumes.
- Thus

$$\delta^{d-\alpha} \geq \mathrm{vol}(K(\delta)) \geq \mathrm{vol}(\cup_k T_k) \geq N \cdot \delta^{d-1} = \delta^{\alpha-1}.$$

$$\alpha - 1 \geq d - \alpha,$$

$$\alpha \geq (d+1)/2.$$

Similar, but more involved argument by Tom Wolff gives Minkowski dimension $\geq (d+2)/2$.

Katz and Tao proved $\geq (2-\sqrt{2})(d-4)+3$.

Can pass from Minkowski to Hausdorff dimension using maximal functions.

Bourgain's maximal function on $\mathbb{R}^d$:

$$M_{\mathrm{B}}^{\delta} f(u) = \sup_{R} \int_R |f(y)| dx dy,$$

where u is a unit vector in $\mathbb{R}^d$ and the supremum is over all δ-neighborhoods of unit line segments parallel to u.

Jean Bourgain conjectured:

$$\|M^{\delta}_{\mathrm{B}} f(u)\|_{L^p(S^{d-1})} = O(\delta^{(d/p)-1+\epsilon})\|f\|_{L^p(\mathbb{R}^d)},$$

for all $\epsilon > 0$ and $1 \leq p \leq d$. a

If true for some $p \in [1, d]$, then any Kakeya set K in $\mathbb{R}^d$ has Hausdorff dimension at least p.

Sketch:

- Cover K by open dyadic boxes let B_j be the (finite) union of boxes of size 2^{-j}.

- If $\delta = 2^{-j_0-1}$ is small enough,

$$\sum_{j \geq j_0} M_{\mathrm{B}}^{\delta} \mathbf{1}_{B_j}(u) \geq c > 0$$

- Hence

$$\sum_j \int_{S^{n-1}} M_{\mathrm{B}}^{\delta} \mathbf{1}_{B_j}(u) \geq \int_{S^{n-1}} M_{\mathrm{B}}^{\delta} \sum_k \mathbf{1}_{B_j}(u) \geq A > 0.$$

- Choose $j < j_0$ so that $\int_D M^\delta_{\mathrm{B}} \mathbf{1}_{B_j}(u) \geq \frac{A}{100} j^{-2}$.

  - Bourgain's conjecture implies
  $$\frac{A}{100} j^{-2} \leq \|M^\delta_{\mathrm{B}} \mathbf{1}_{B_j}\|_p \lesssim 2^{-j((d/p)-1+\epsilon} \mathrm{vol}(B_j)^{1/p}.$$

- This implies the number of cubes in B_j is
$$\geq 2^{j(p-\epsilon)} j^{-2p}.$$

- Hence $\mathrm{Hdim}(K) \geq p$.

Kakeya conjecture is related to a number of other famous conjectures:

- Fourier restriction conjecture
- Bochner-Riesz conjecture
- Local smoothing conjecture

Also to problems in finite fields and arithmetic geometry.

If $f \in L^2$ then $\hat{f} \in L^2$ is not defined on sets of measure zero.

But if $f \in L^1$, then $\hat{f}$ is continuous; it makes sense to restrict to any set (and is bounded there).

If $f \in L^p$ and E has measure zero does $\hat{f}|_E$ makes sense? Is it in a L^q space?

Restriction Conj: If E is the unit sphere in $\mathbb{R}^n$ and $f \in L^p$ then $\hat{f} \in L^q(E)$ whenever $p < 2n/(n+1)$ and $q > 2n/(n-1)$.

Implies Bourgain's conjecture, Kakeya conjecture. Partially reverses.

Charlie Fefferman used Besicovitch sets to disprove a famous conjecture about the Fourier transform.

Disk multiplier: define a operator T by taking the Fourier transform, multiplying by the indicator function of the disk, and taking the inverse Fourier transform.

Clearly L^2 bounded in all dimensions.

L^p bounded on $\mathbb{R}^1$ for $1 < p < \infty$.

Fefferman (1971): if $n \geq 2$, not bounded on L^p, $p \neq 2$.

Fefferman's proof uses four ideas:

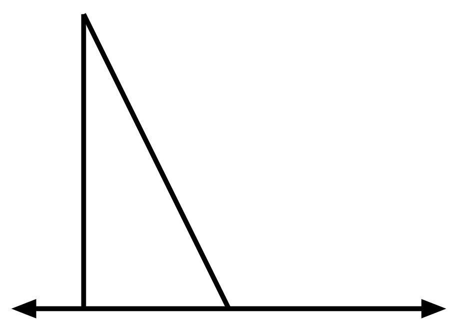
- Construction of a Besicovitch set
- Fatou's lemma
- A simple estimate of the Hilbert transform
- Randomization

• **Geometric fact from Besicovitch sets:**

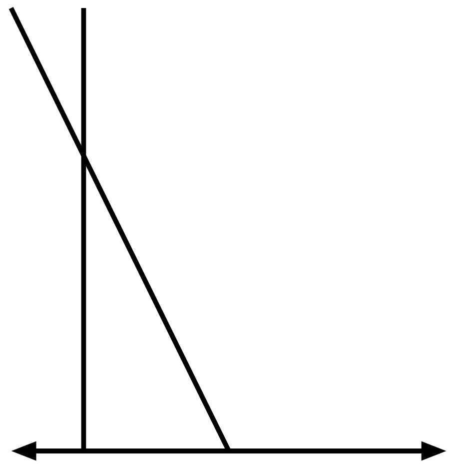
Lemma: there are k disjoint $1 \times \frac{1}{k}$ rectangles so that translating each by distance 1 in "long" direction gives set of area $O(1/\log k)$.

Proof is by modified Perron triangles method.

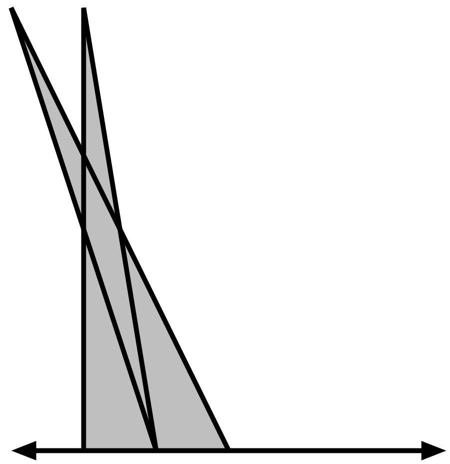
Start with a triangle with base on real line.



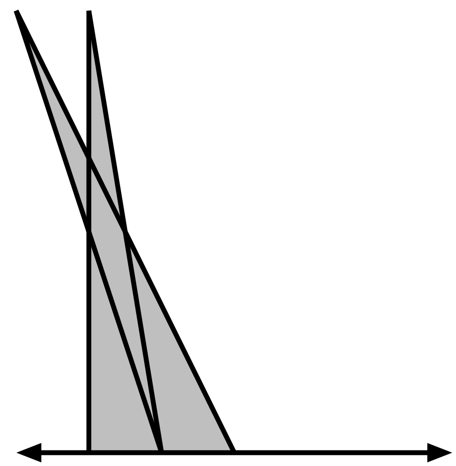
Extend sides by factor of $1 + 1/k$.



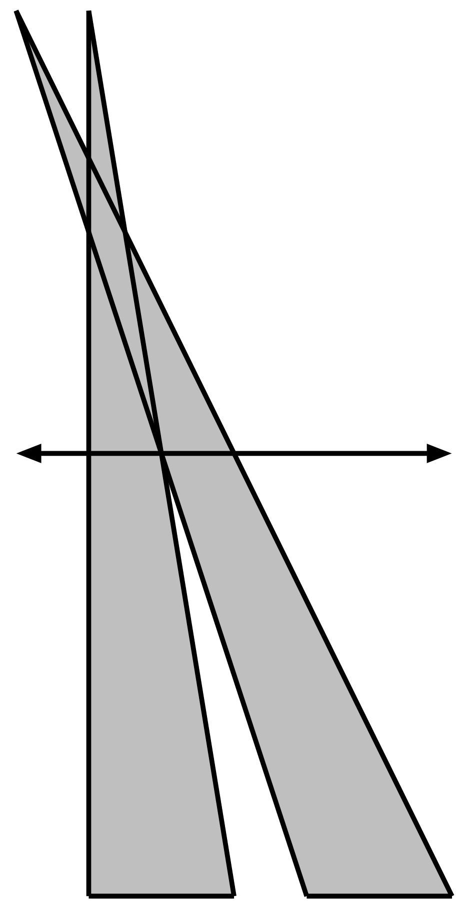
Connect new endpoints to midpoint of base.



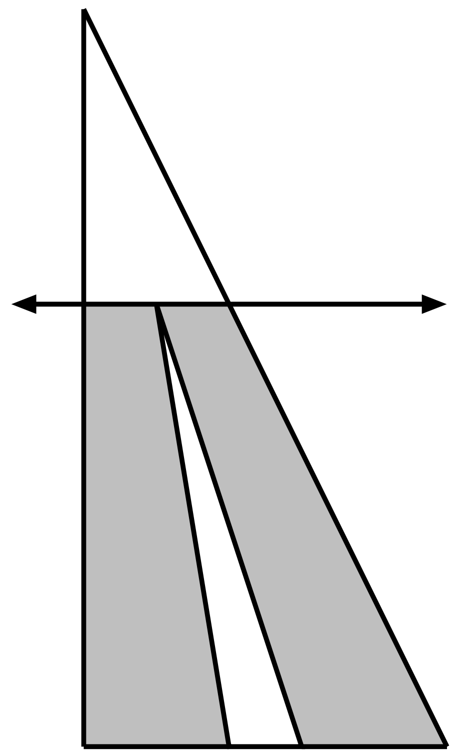
Sub-triangles of earlier construction. Thus smaller area.



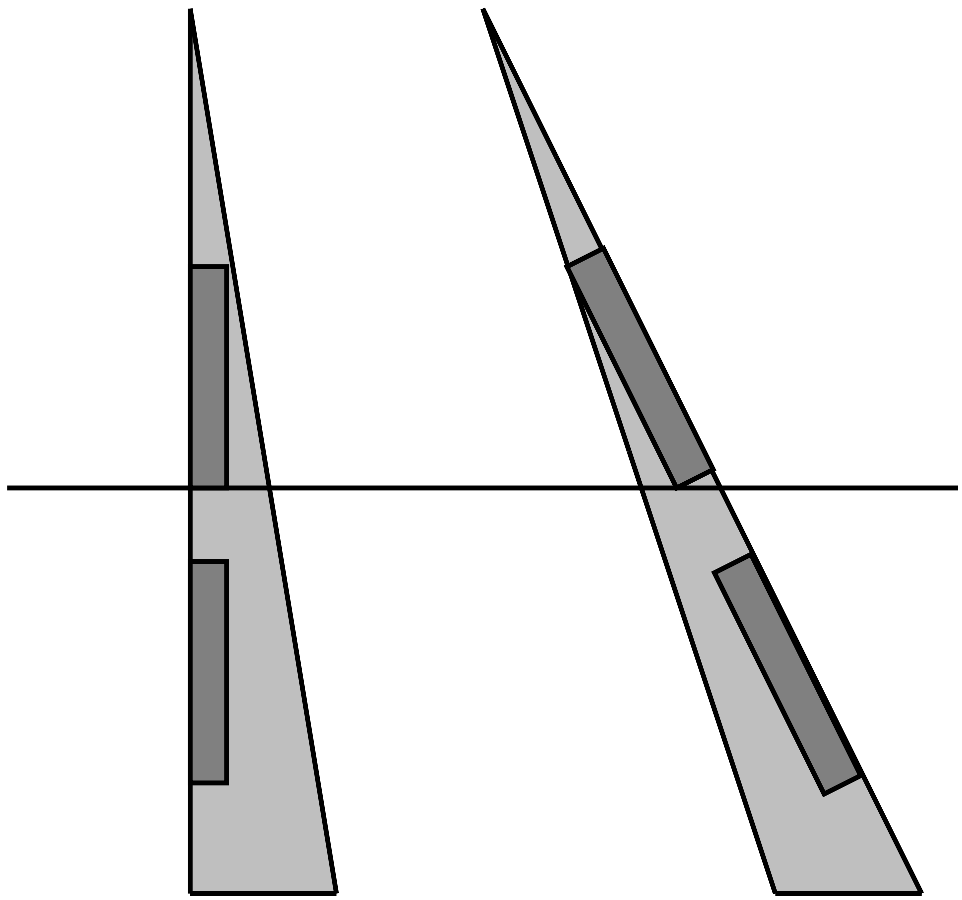
Extend triangles below real line. Gives disjoint regions.

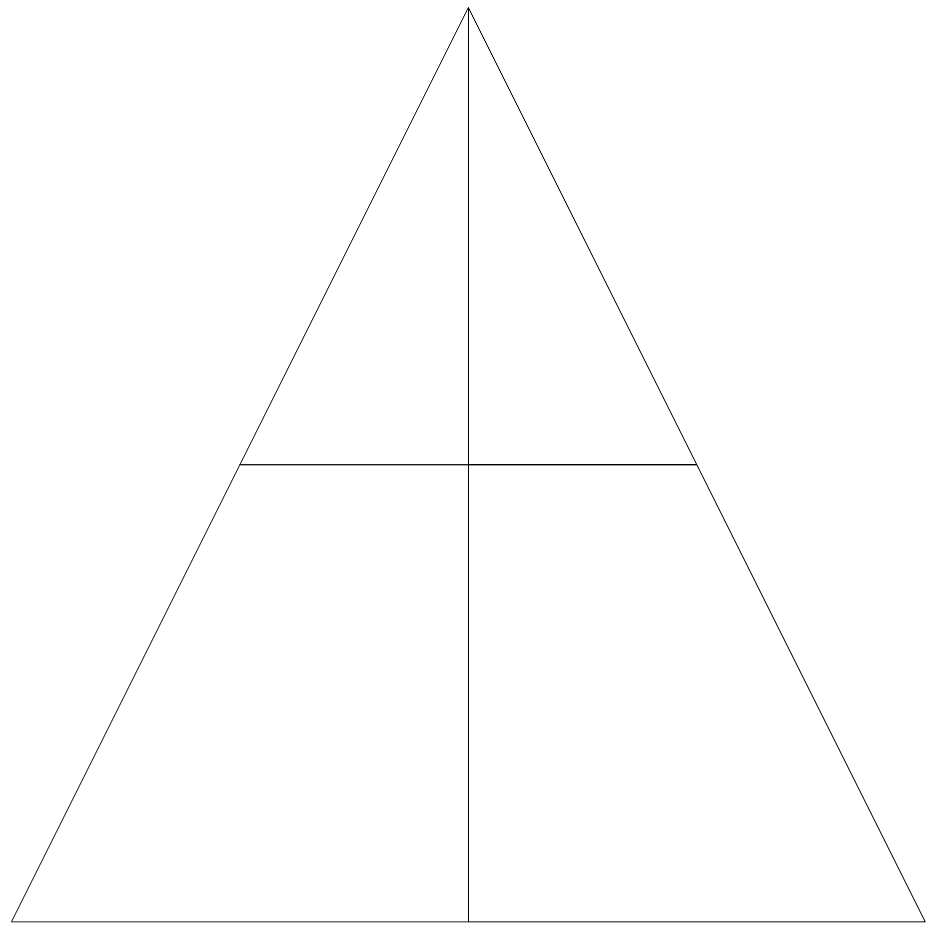


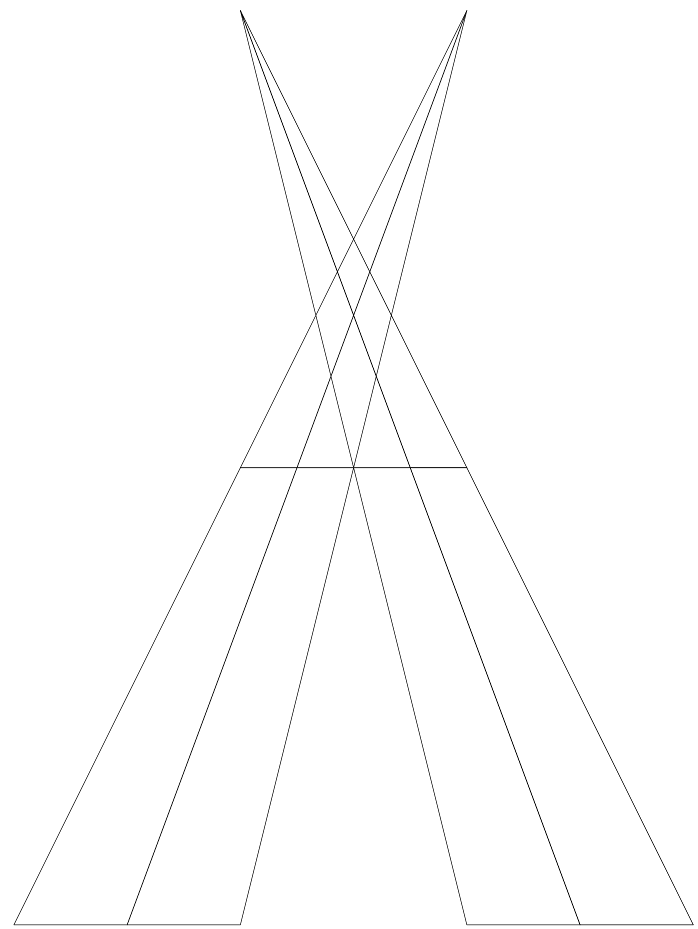
Extensions are subset of extension of original triangle.

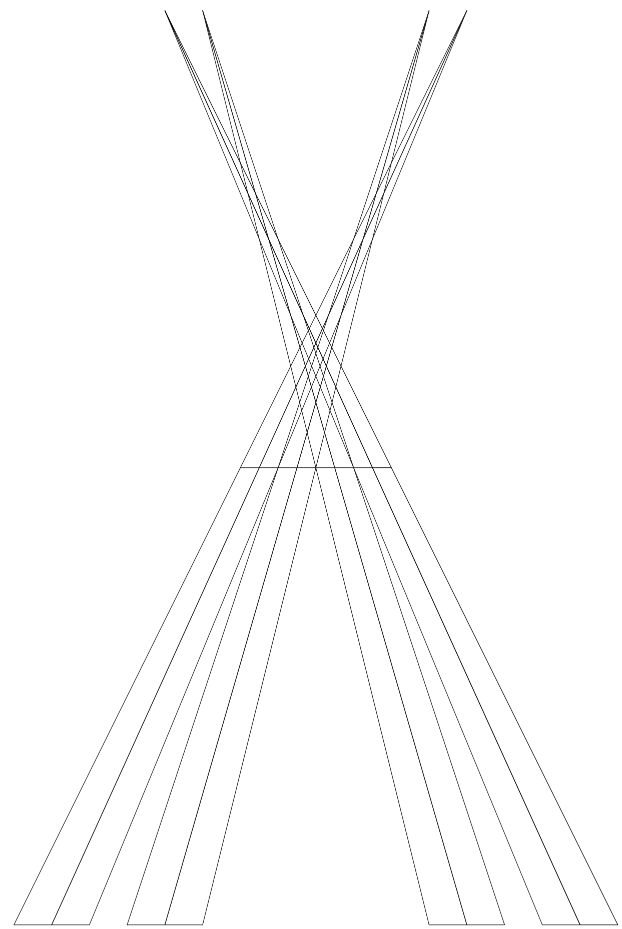


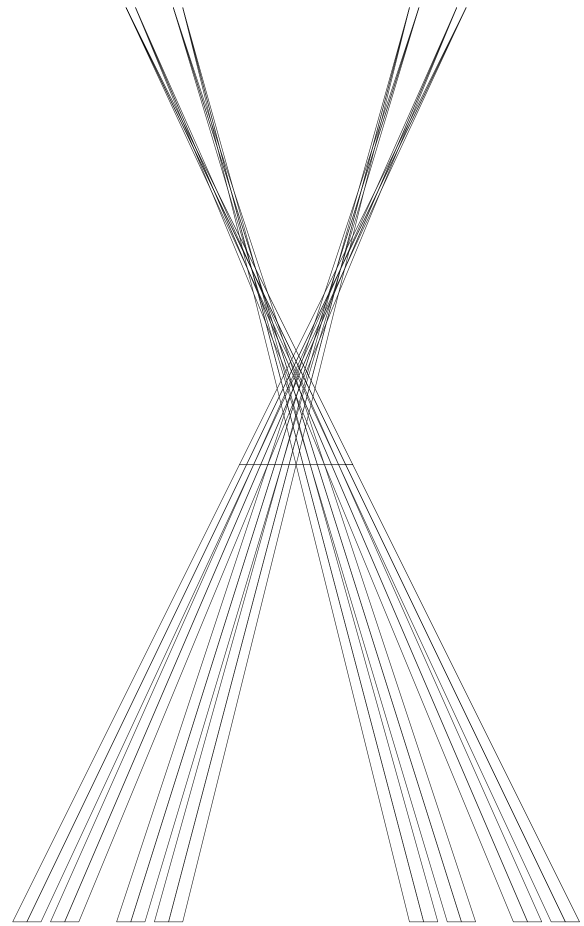
There are rectangles in each triangle and its extension that are translates parallel to longer side.

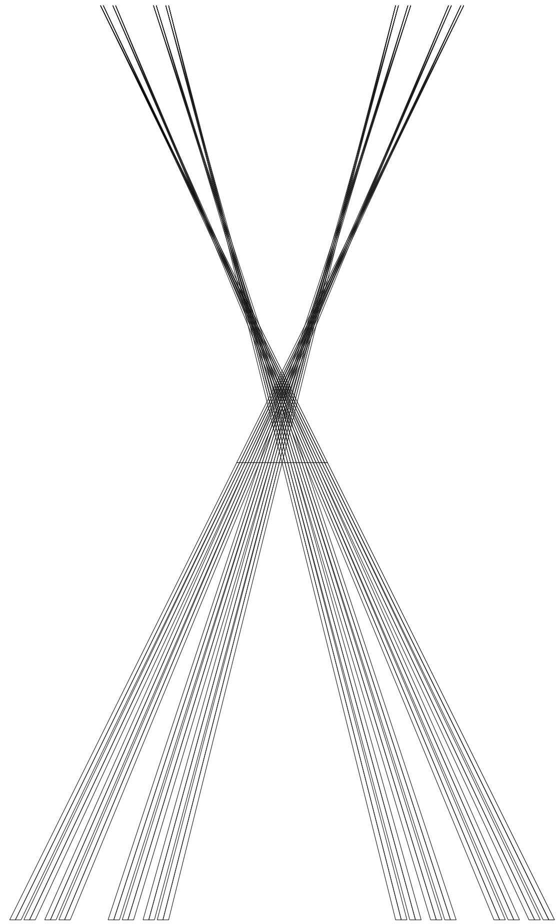


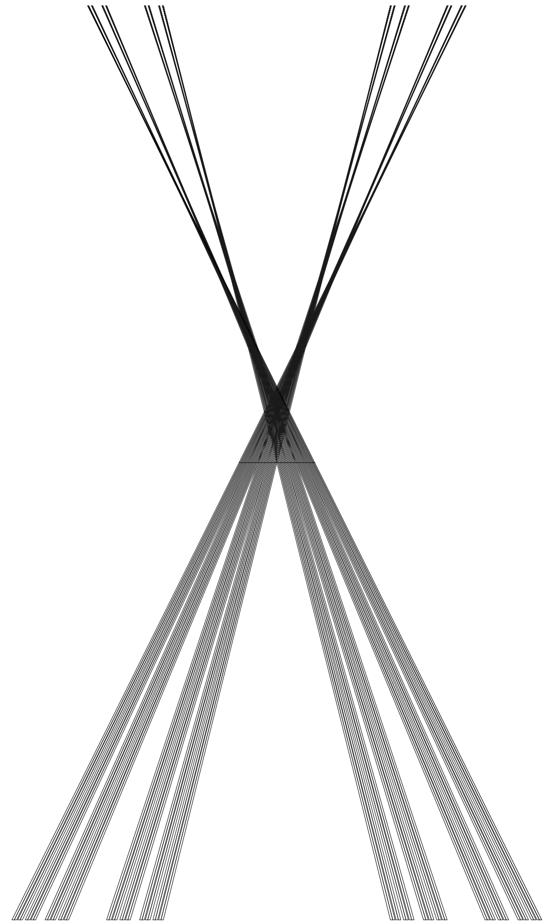


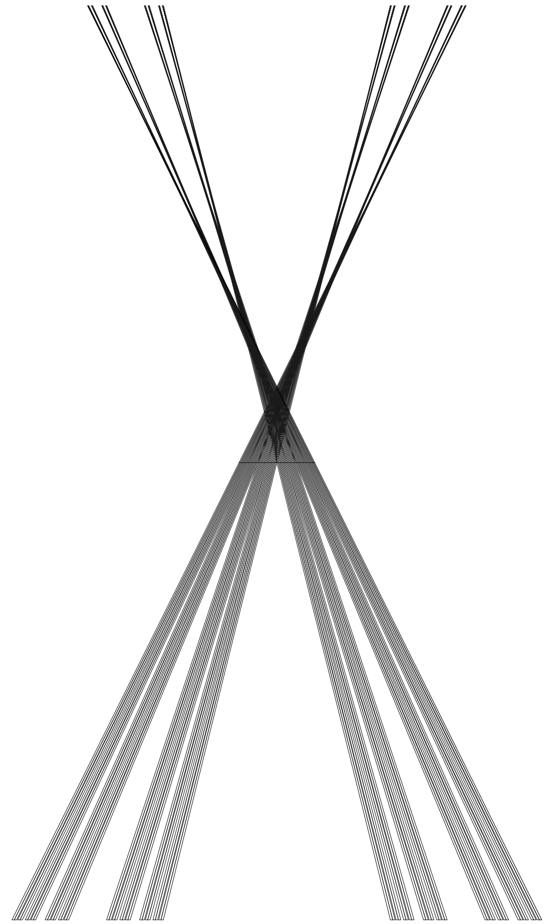








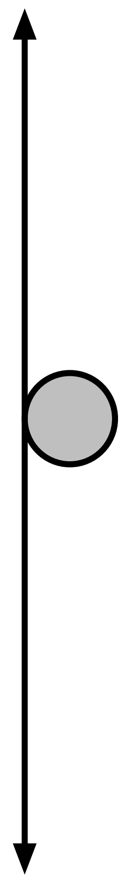




Top half tends to zero area; bottom has area ≥ 1.

- **Fatou's lemma:**

A half-plane is a limit of an expanding sequence of disks.



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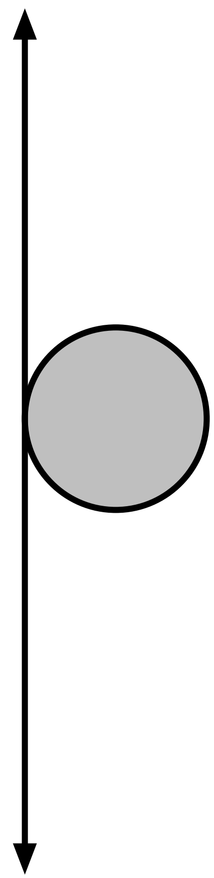

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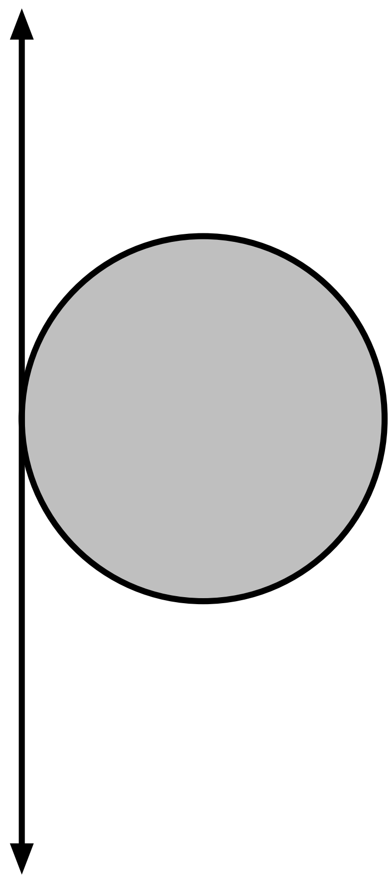

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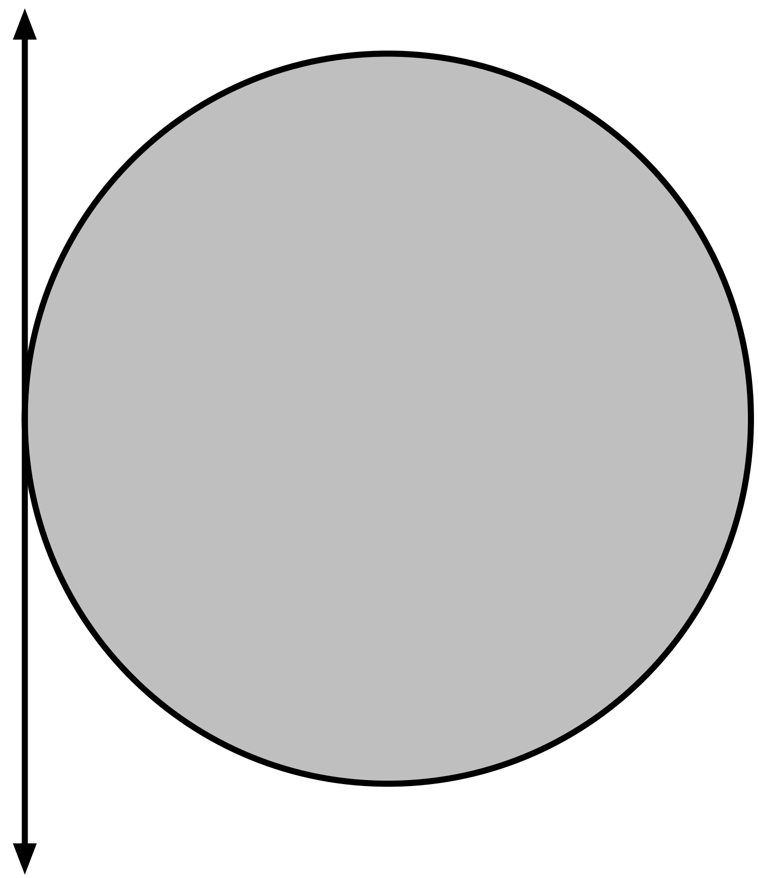

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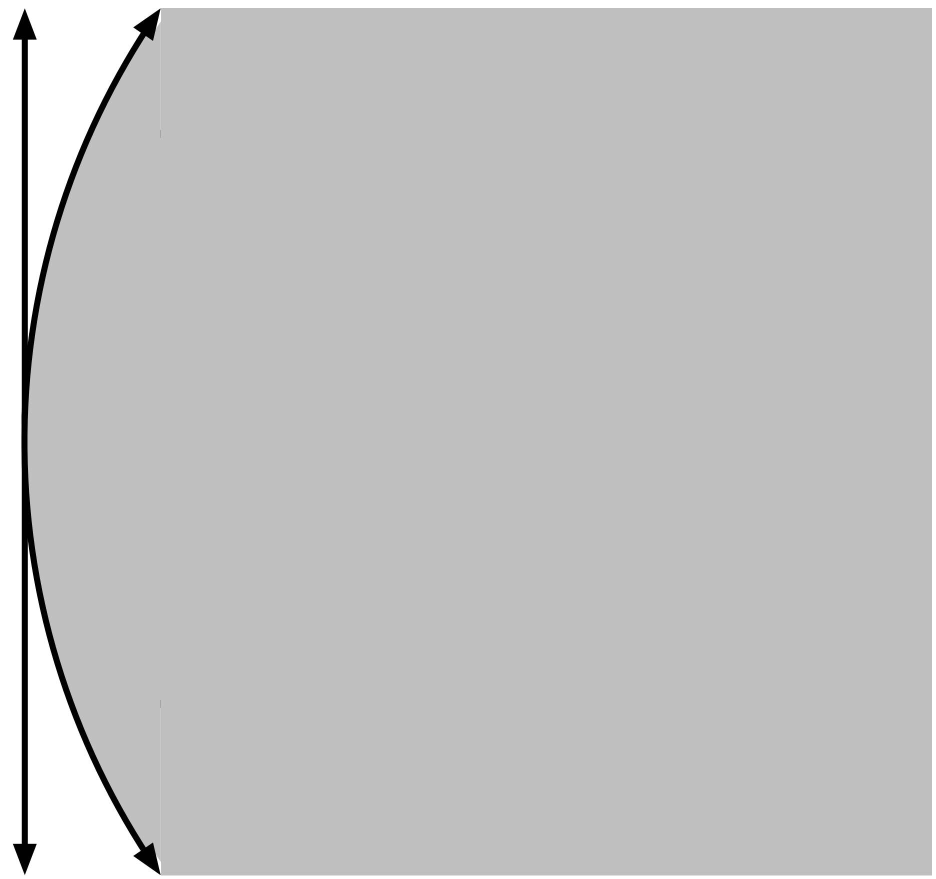


• **Fatou's lemma:**

A half-plane is a limit of an expanding sequence of disks.

Fatou's lemma implies L^p boundedness of half-plane multiplier from boundedness of the disk multiplier.

Half-plane multiplier is essentially 1-dimensional Hilbert transform in coordinate perpendicular to half-plane.

• **A simple estimate of the Hilbert transform:**

If f is indicator of $[-1,1]$, then $|Hf| > c > 0$ on $[2,3]$.

If f is indicator of R then $|T_H f| > c$ on $\tilde{R}$.

• **Lemma:** if $\|Tf\|_p \le A\|f\|_p$, then
$$\|(\sum |Tf_j|^2)^{1/2}\|_p \le \|(\sum |f_j|^2)^{1/2}|\|_p.$$
This standard fact. Follows from randomization method, such as Kinchine's inequality.

L^p boundedness of disk multiplier implies that if $\{T_j\}$ are half-plane multipliers, then

$$\|(\sum |T_j f_j|^2)^{1/2}\|_p \le \|(\sum |f_j|^2)^{1/2}|\|_p.$$

$R_j, \tilde{R}_j$ as above: $\sum \operatorname{area}(R_j) = 1, \sum \operatorname{area}(\tilde{R}_j) \to 0.$

Let f_j be indicator of R_j.

Let T_j the half-plane multiplier parallel R_j.

Since $|T_j f_j| > c > 0$ on $\tilde{R}_j$.

$$
\begin{aligned}
\int_E (\sum_j |T_j f_j|^2) dx &= \sum_j \int_E |T_j f_j(x)|^2 dx \\
&\geq c^2 \sum_j \operatorname{area}(\tilde{R}_j) \\
&\geq c^2 \sum_j \operatorname{area}(R_j) \\
&\geq c^2
\end{aligned}
$$

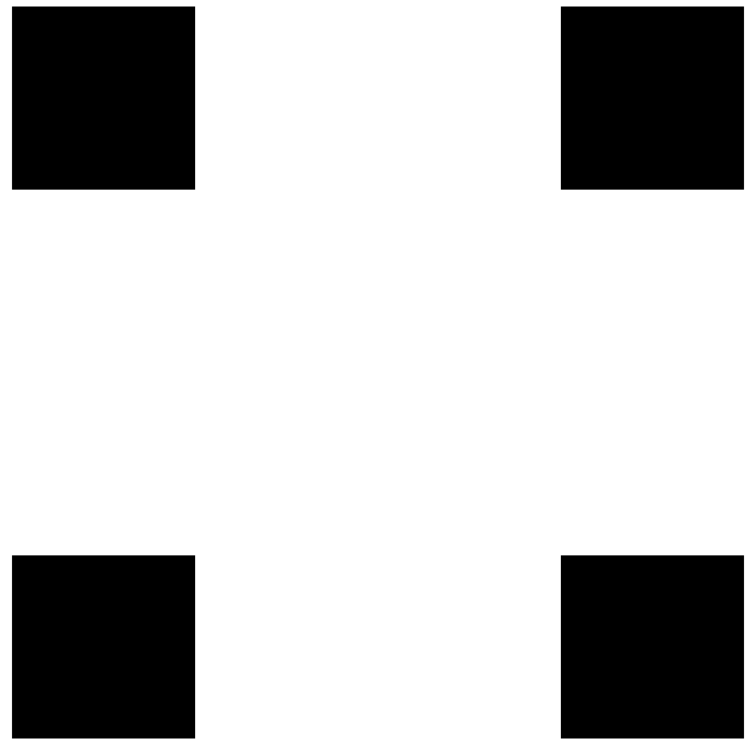
By Hölder with $p/2$ and $q = p/(p-2)$,

$$\begin{aligned}
&\int_E (\sum_j |T_j f_j|^2) dx \\
&\qquad \leq \operatorname{area}(E)^{(p-2)/p} \|(\sum_j |T_j f_j(x)|^2)^{1/2}\|_p^2 \\
&\qquad \leq B \cdot \operatorname{area}(E)^{(p-2)/p} \|(\sum_j |f_j(x)|^2)^{1/2}\|_p^2 \\
&\qquad \leq B \cdot \operatorname{area}(E)^{(p-2)/p} (\sum_j \operatorname{area}(R))^{2/p} \\
&\qquad \leq B \cdot (1/\log n)^{(p-2)/p} \to 0
\end{aligned}$$

Contradiction $\Rightarrow$ disk multiplier is not L^p bounded.

We present another construction of a Besicovitch set, due to Kahane, based on random projections of a self-similar Cantor set.

The "Four Corner" Cantor set:



Is product $K \times K$ where K is "middle-half" Cantor set.

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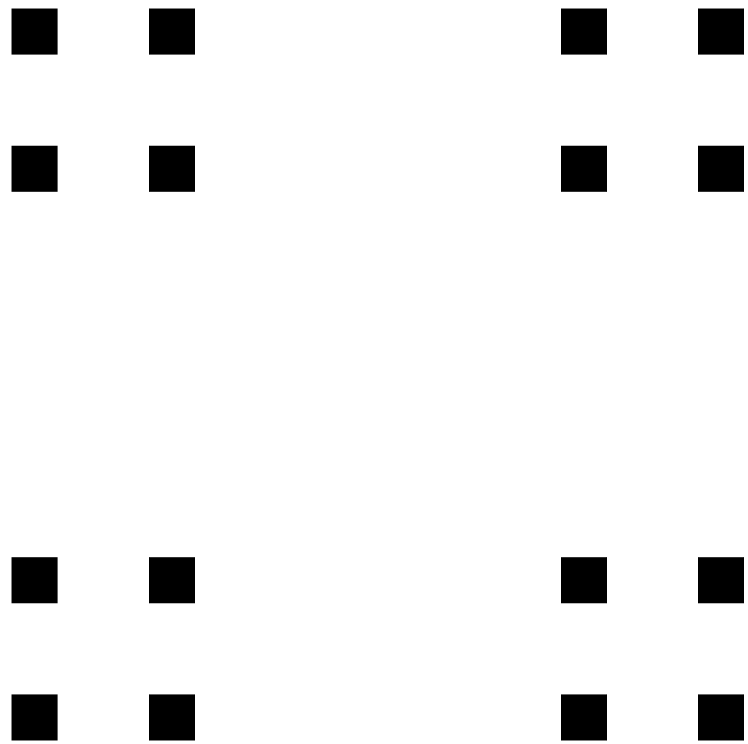

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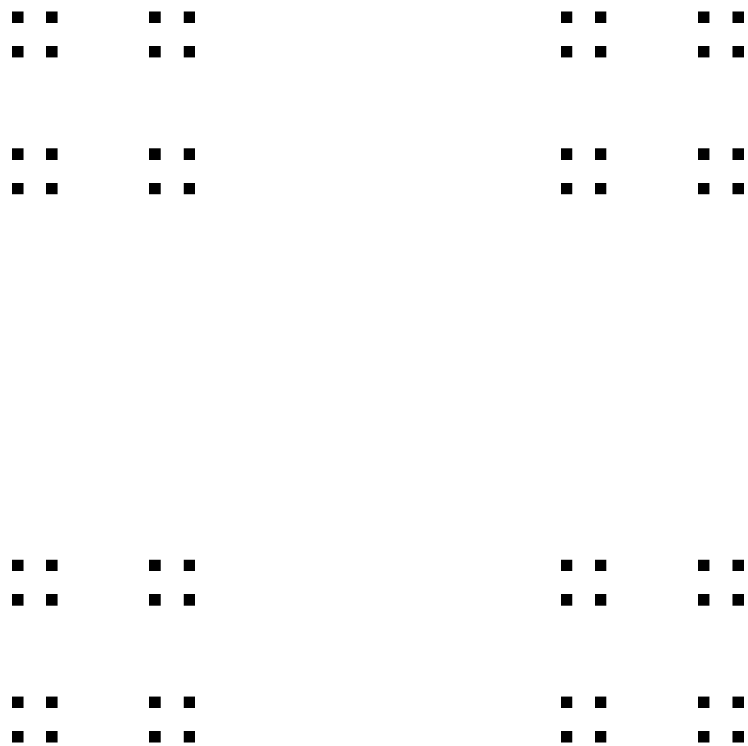
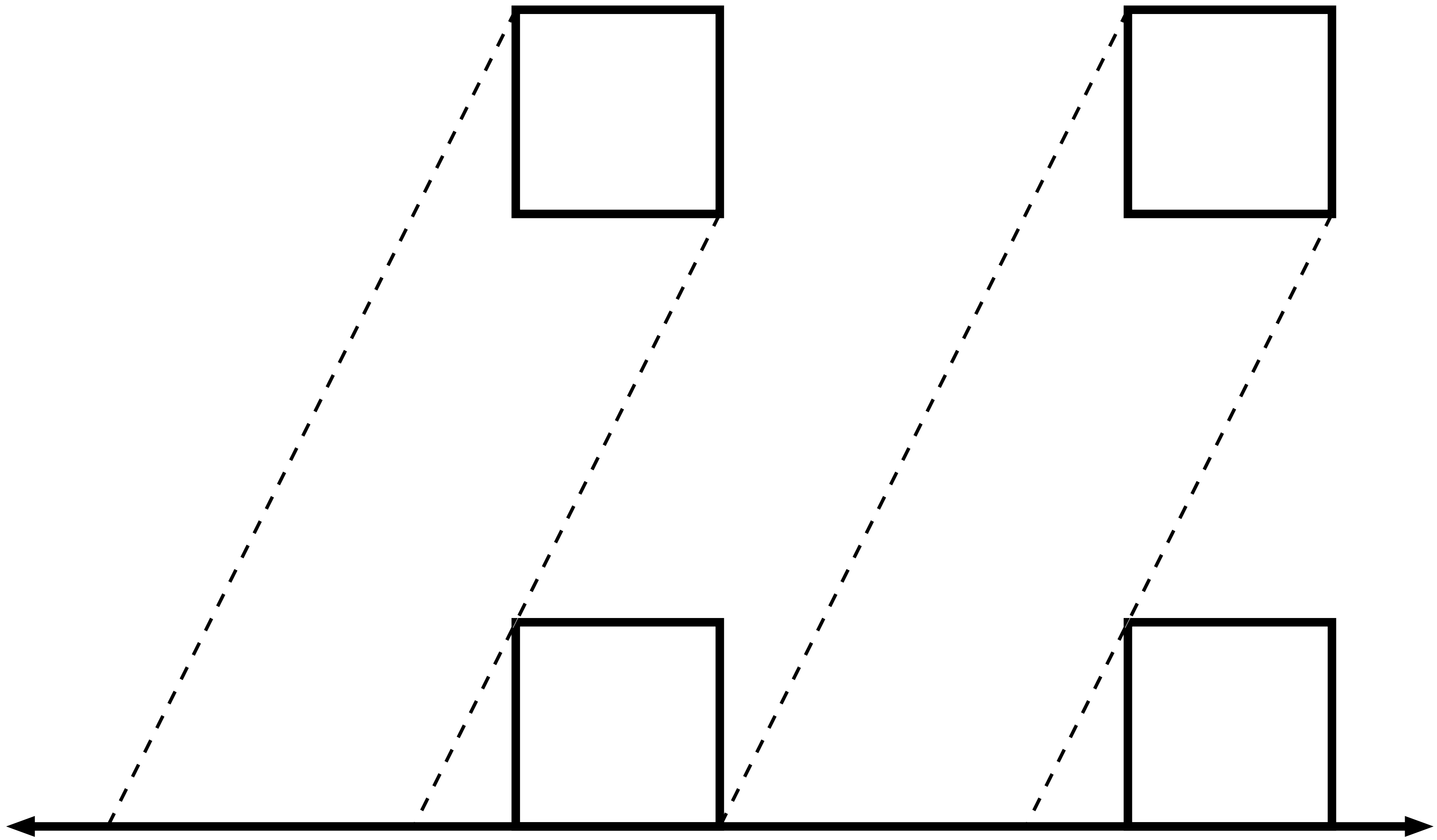
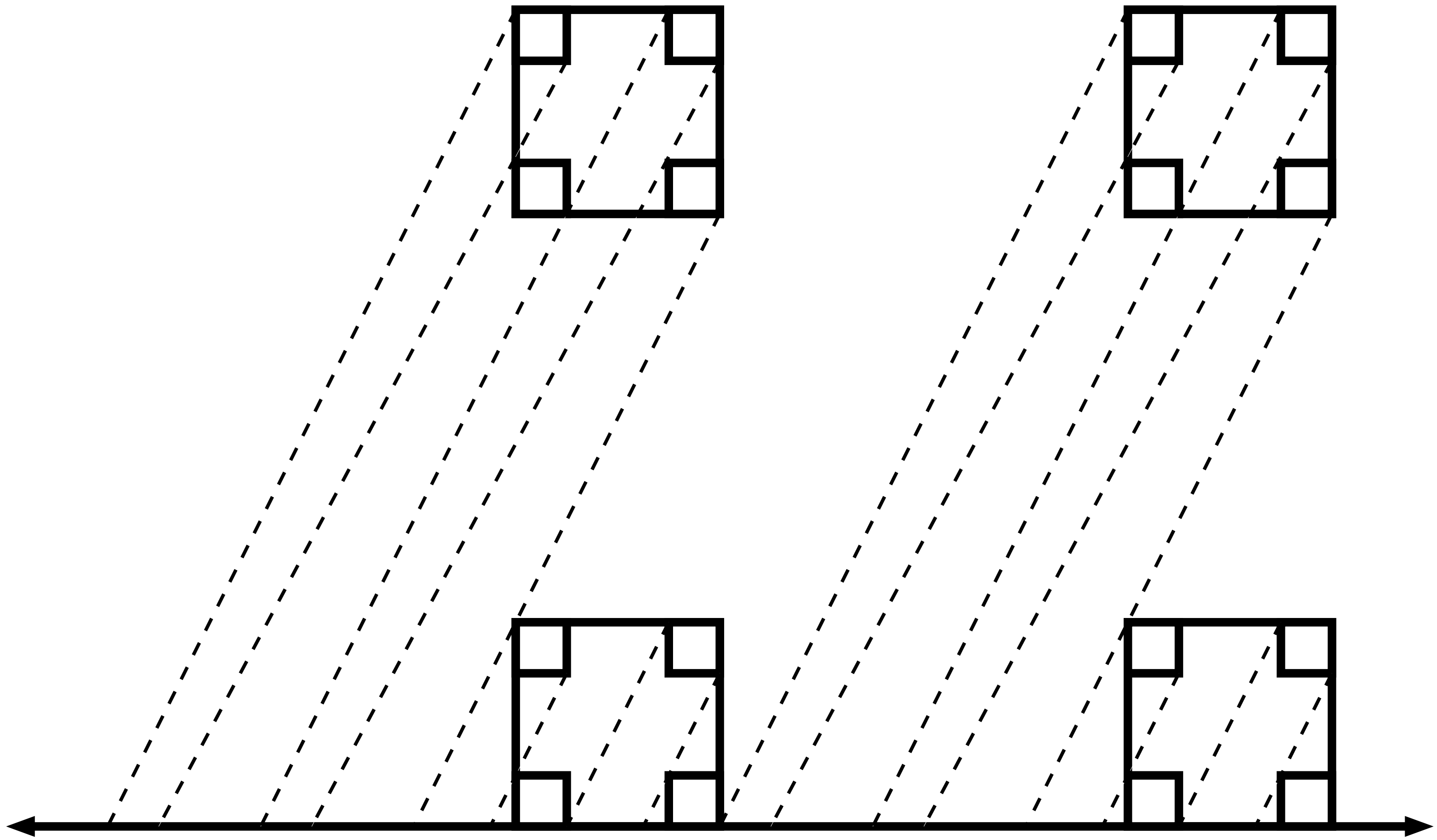


Is product $K \times K$ where K is “middle-half” Cantor set.



Fact 1: Projection along slope 2 covers interval.



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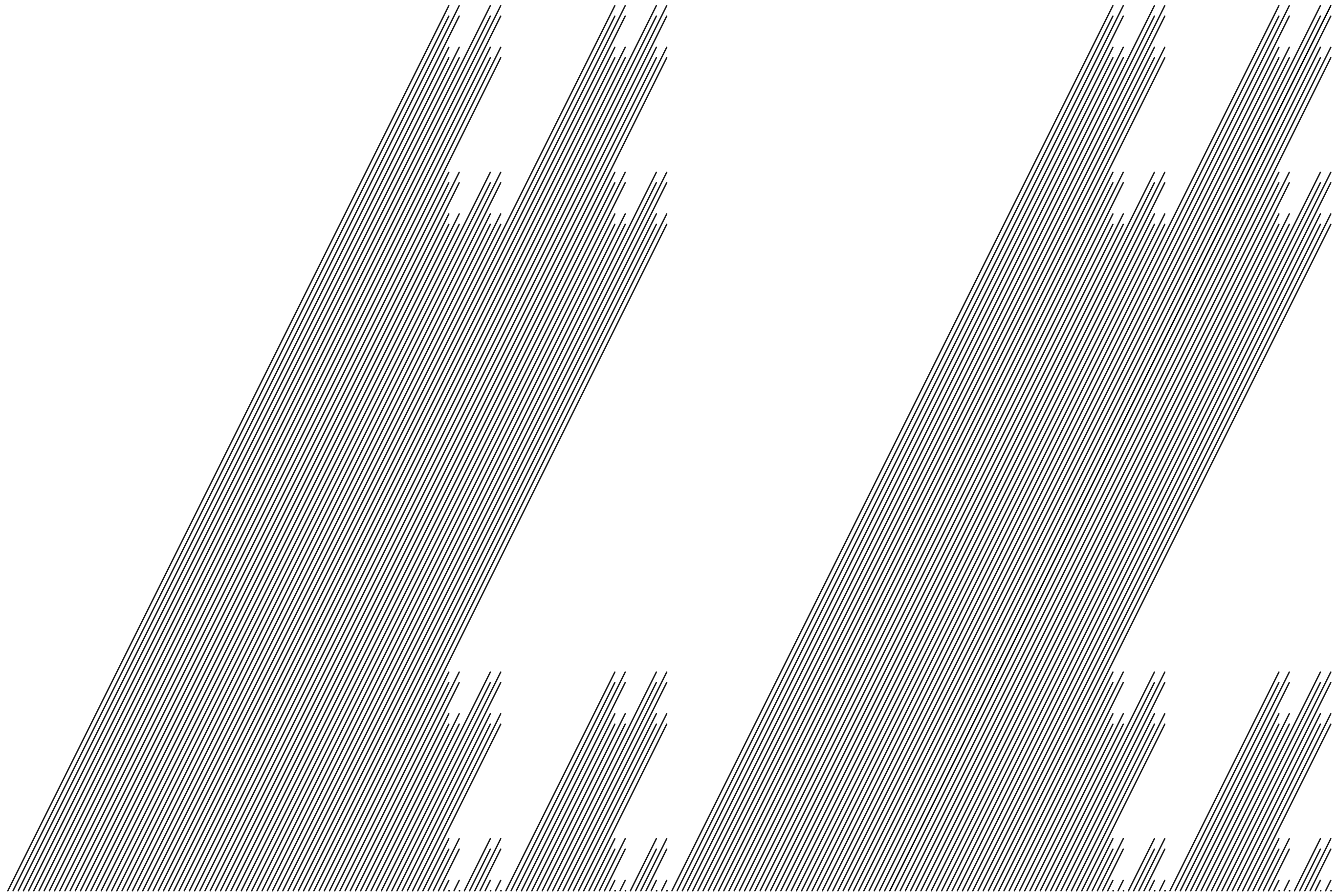

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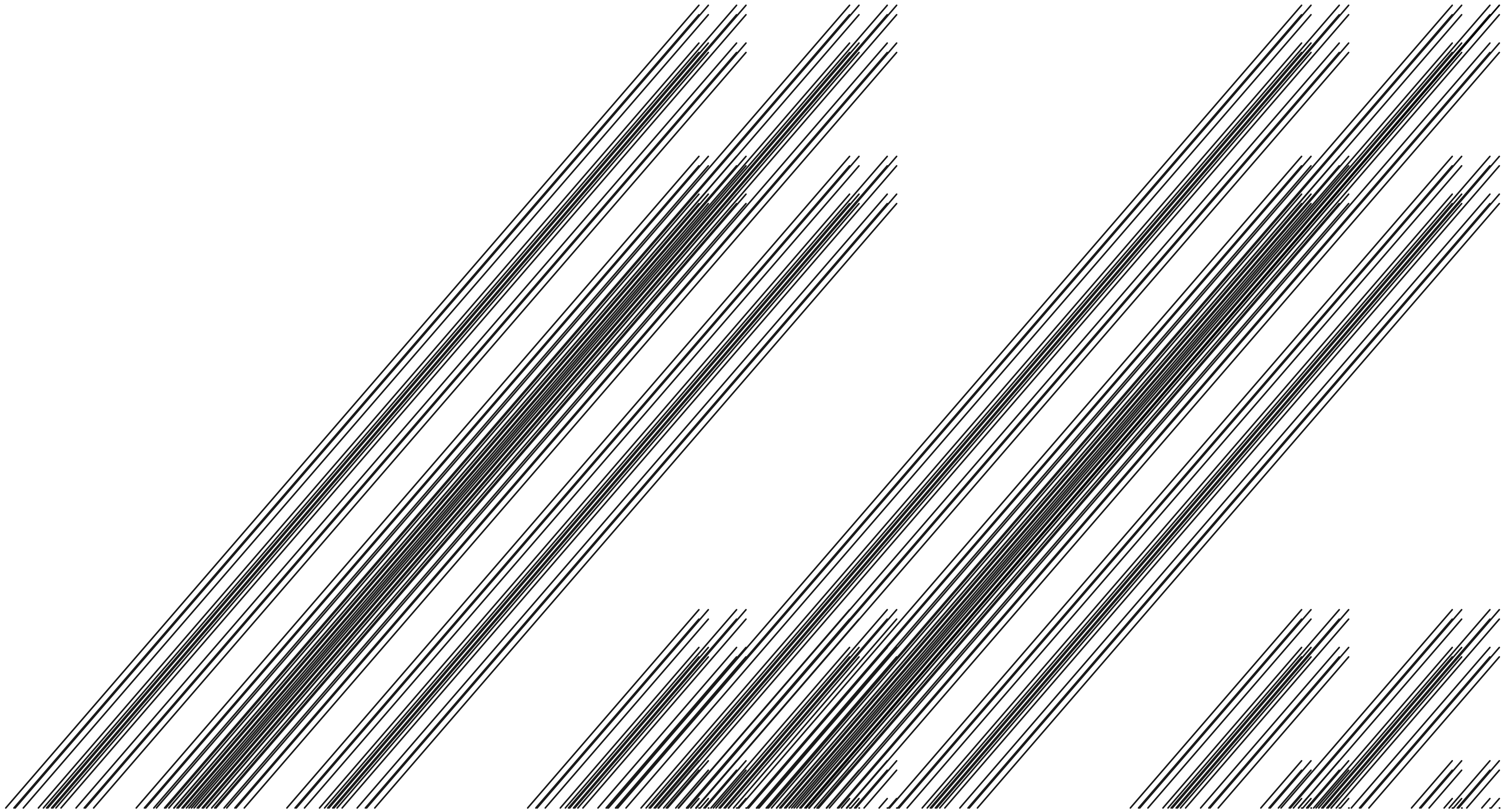
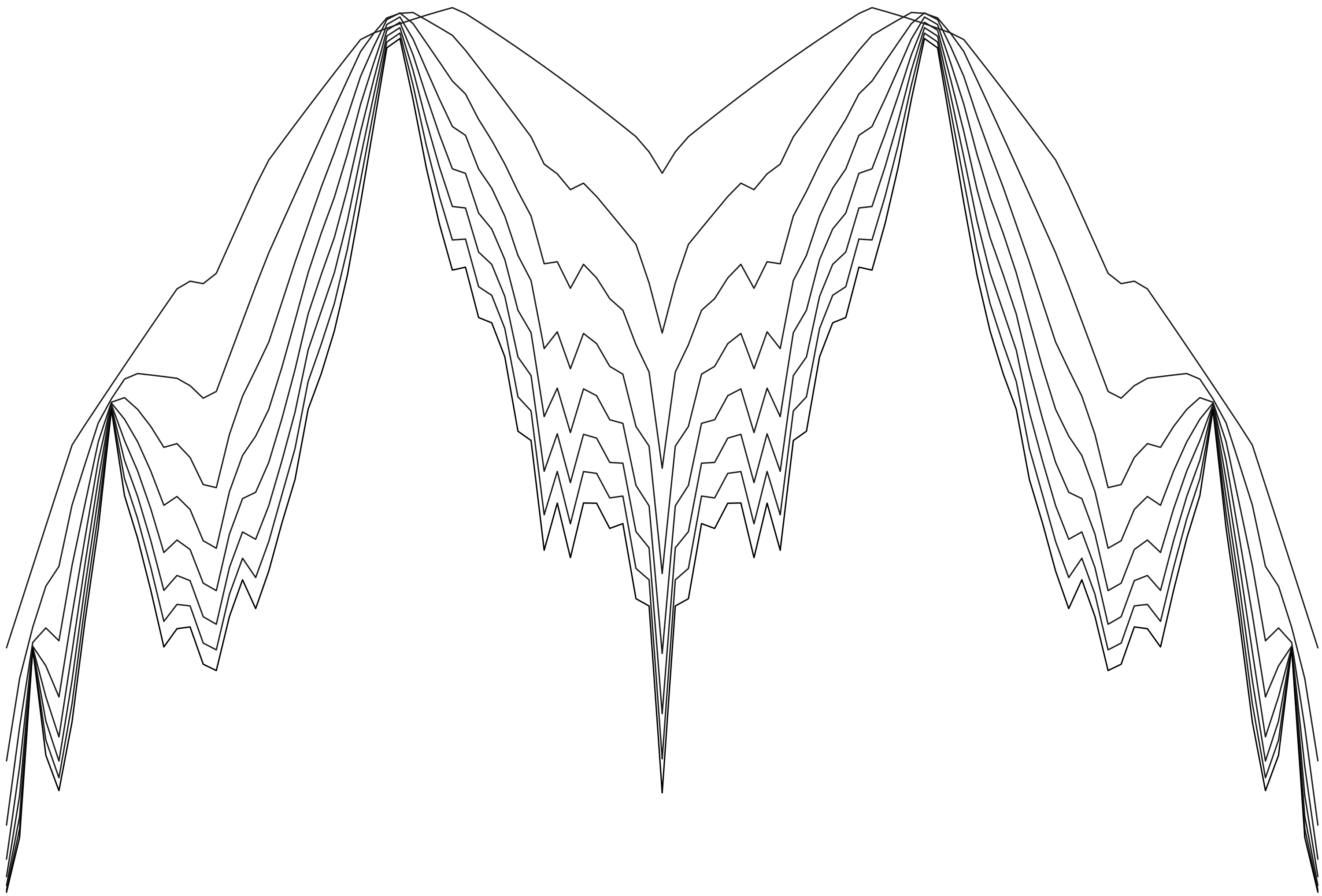
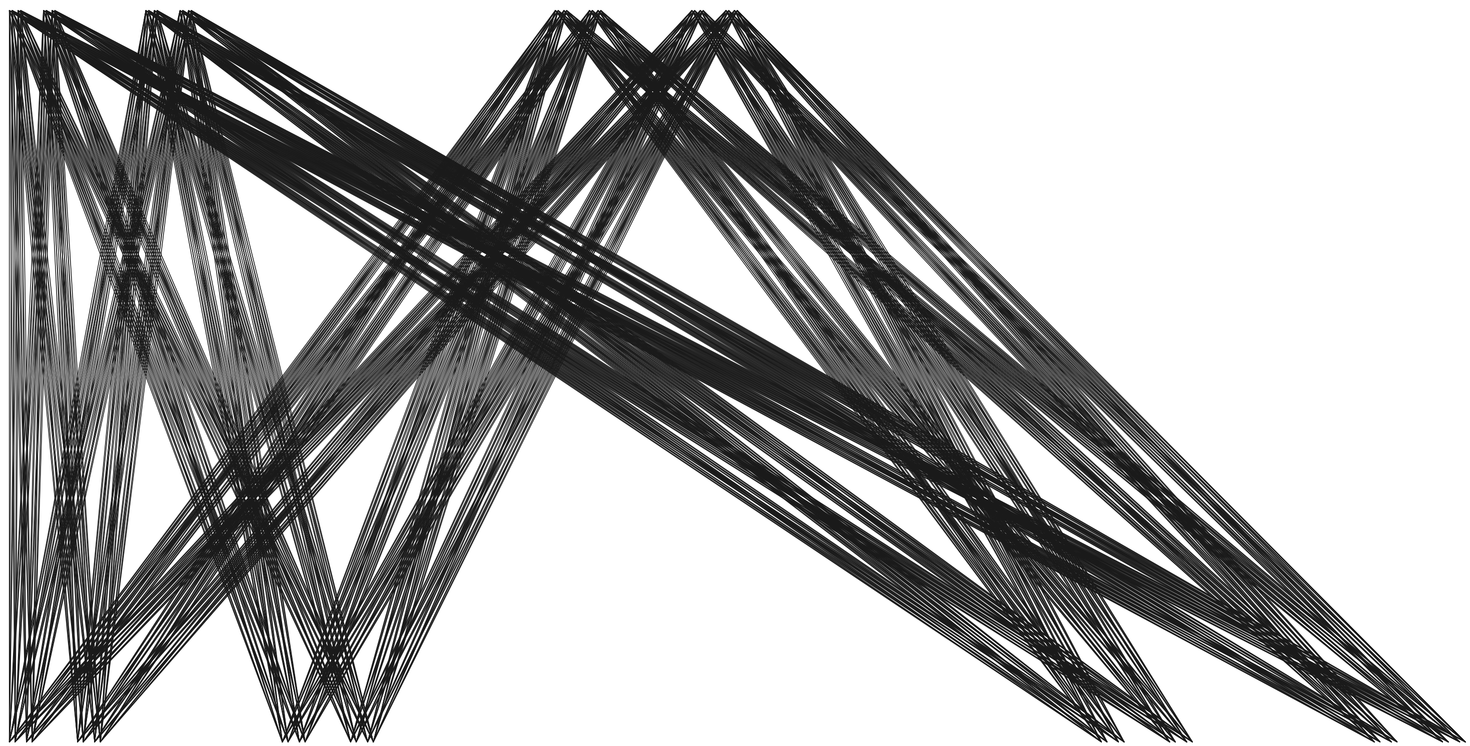


Fact 2: A.e. projection has length zero

Cantor set is Besicovitch irregular.



Length of projection of nth generation.

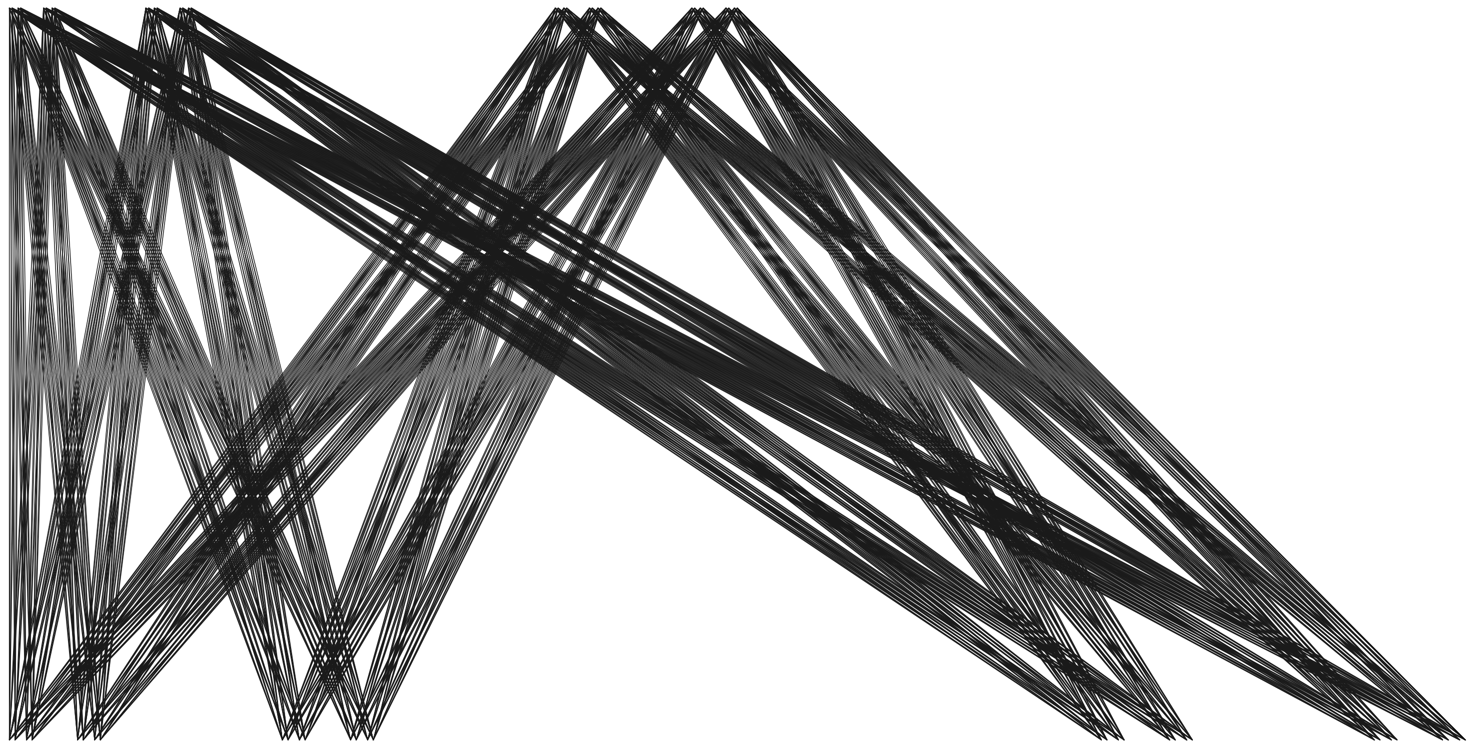


$K =$ middle-half Cantor set.

Place copy of K in $[0, 1]$.

Place copy of $\frac{1}{2}K$ in $[i, i + 1/2]$.

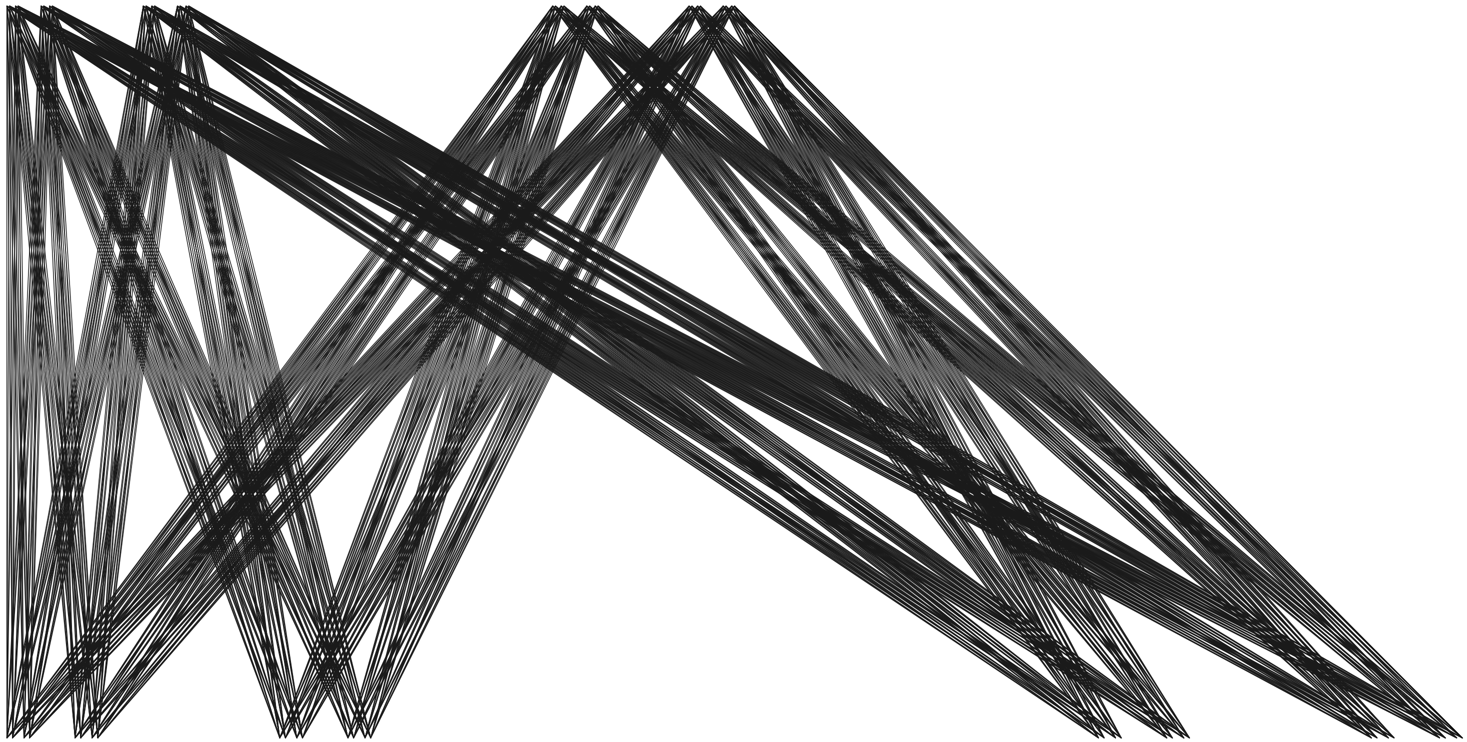
Make all straight connections.



Horizontal slice height t is $(1-t)K + (t/2)K$.

$= K \times K$ projected along lines $(1-t)x + (t/2)y = c$.

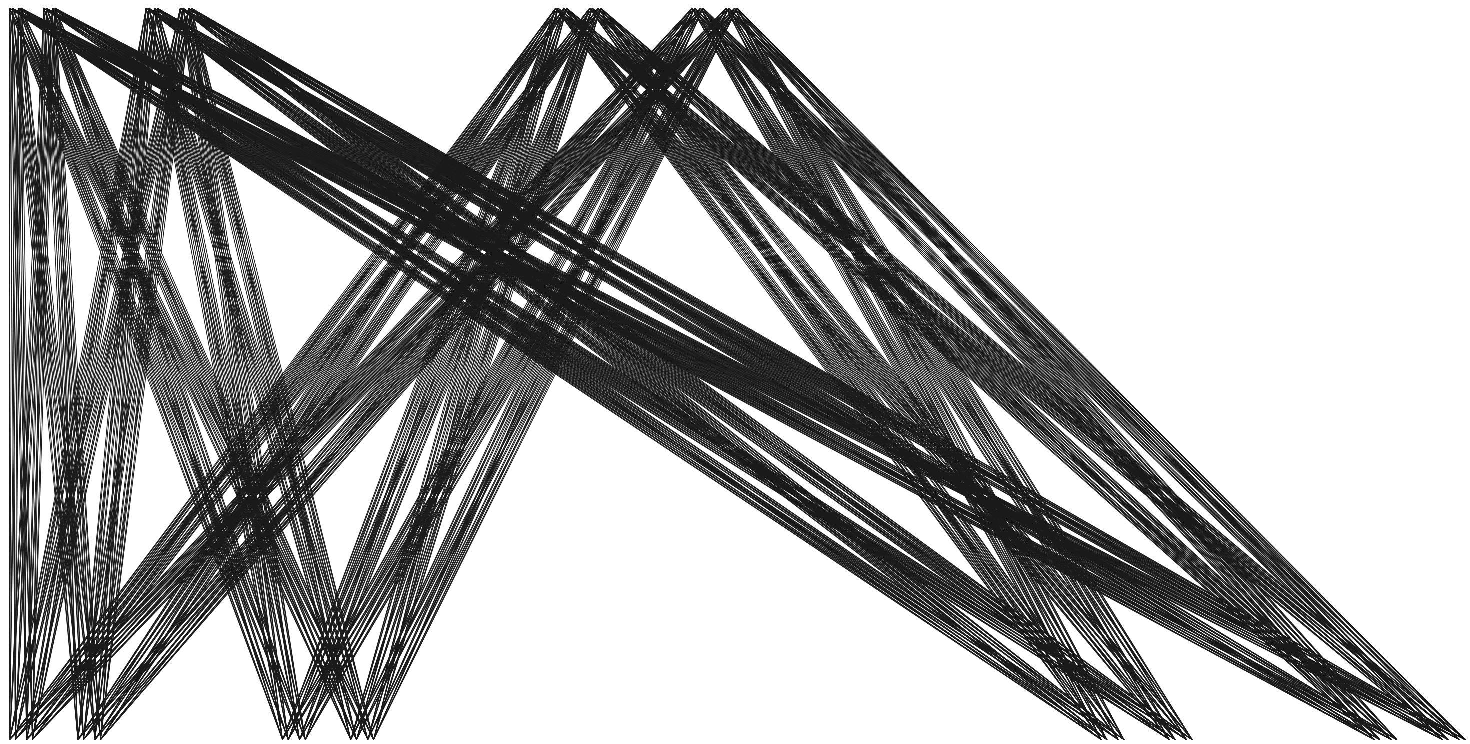
So a.e. horizontal slice has zero length.



For $x, y \in K$, set contains segment from $[(0, x), (1, y/2)]$,

This segment has slope $1/(x - y/2)$.

reciprocals of projection of $K \times K$ along slope 2.



So interval of slopes occurs in E.

Hence union of rotates gives Besicovitch set.

A Nikodym set K has full measure in $[0,1]^2$ and for every $x \in K$ there is a line L so that $L \cap K = \{x\}$.

Cor: There is a union of open half-rays that has zero area, but union of endpoints has full measure.

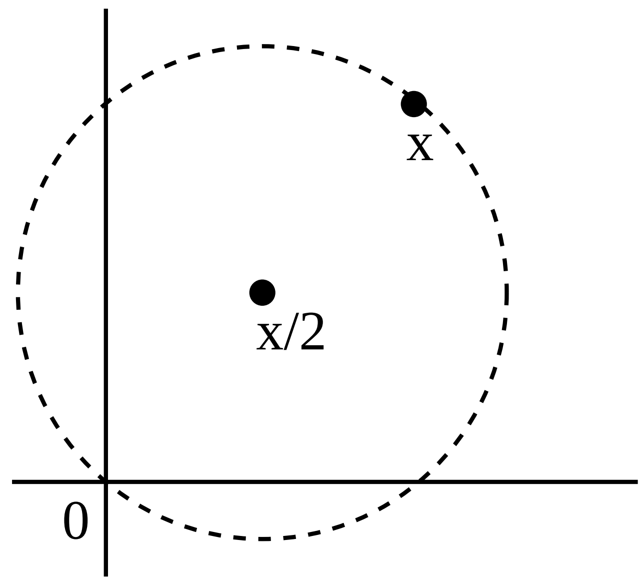
Otton Marcin Nikodym (1887-1974)

Let Π_θ be orthogonal projection onto direction θ.

Lemma: Given any $\epsilon > 0$, segment I in the plane and disjoint closed arcs J_1, J_2 of directions, there is a finite union of segments E so that

- E approximates I in the Hausdorff metric,
- $\Pi_\theta I \subset \Pi_\theta E$ for $\theta \in J_1$.
- $|\Pi_\theta E| < \epsilon$ for $\theta \in J_2$.

For $x \in \mathbb{R}^2$, let $C(x) = \{z : |z - x/2| = |x|/2\}$.



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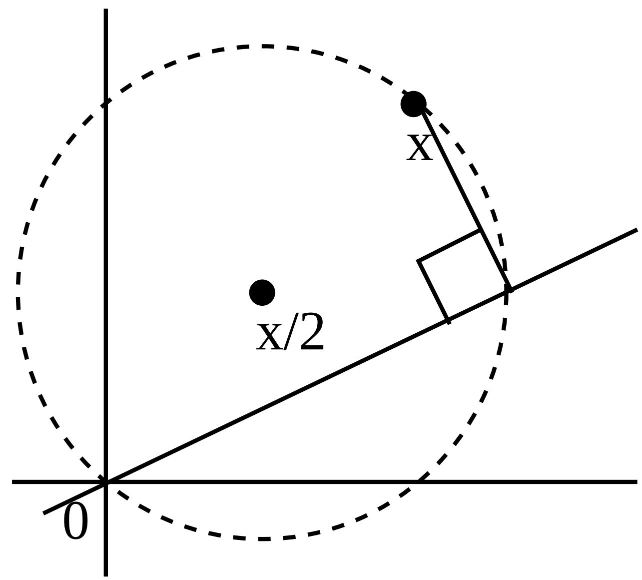
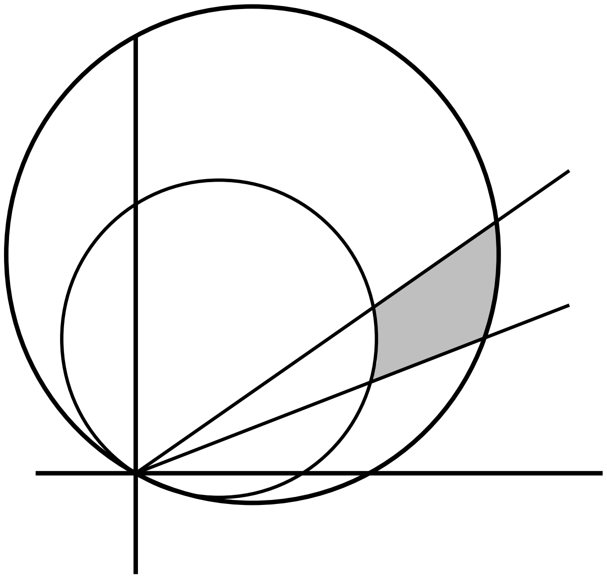
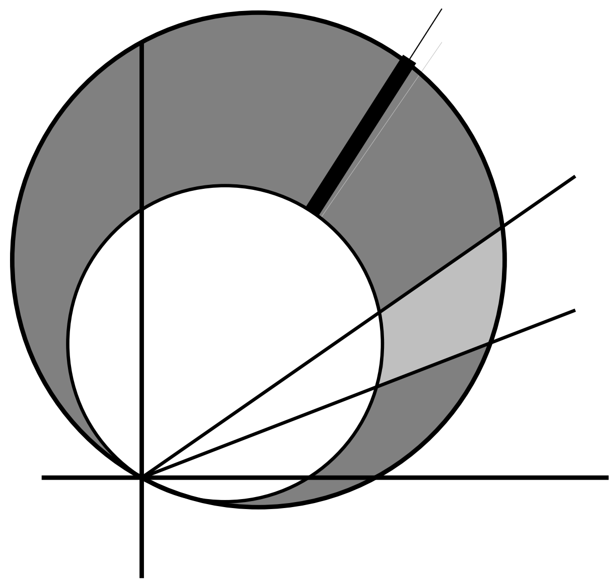


$C(x)$ is projection of x onto lines through the origin.

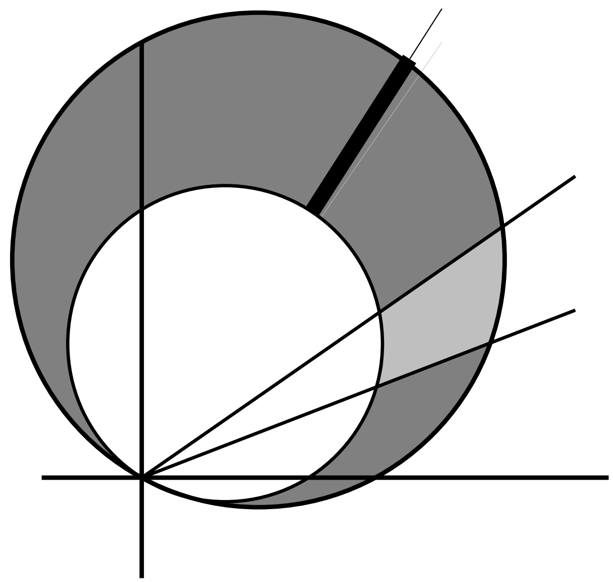
Let $C(E)$ denote union of all $C(x)$, $x \in E$.



Define base of open neighborhoods in $\mathbb{R}^2 \setminus \{0\}$ using radial lines and circles $C(x), C(rx)$.



Each neighborhood is projection of a segment in $\mathbb{R}^2$ onto an arc of lines through the origin.



Lemma: For each neighborhood set N and $\epsilon > 0$, there is finite union E of segments so that $C(E)$ covers N and $C(E) \setminus N$ has area $< \epsilon$.

If V is open and $N \subset C(V)$ we can choose $E \subset V$.

Lemma: There is full measure $K \subset \mathbb{R}^2$ so that $z \notin K$ implies there is $x \in K$ so that $C(x) \cap K = z$.

To get Nikodym set, invert K around unit circle. Circles through 0 become lines.

Sketch of Lemma:

- For $j = 1, 2, \ldots$ write $\mathbb{R}^2 \setminus \{0\} = \cup_k N_k^j$ so for each k $N_k^{j+1} \subset N_p^j$ for some p.

- Choose $\{E_k^j\}$ so that

$$N_k^j \subset C(E_k^j),$$

$$\text{area}(C(E_k^j) \setminus N_k^j) \leq 2^{-k-j}$$

- Note that $\cup_k C(E_k^j) = \mathbb{R}^2$.

- Can choose E_k^{j+1} as union of disks so each disk is contained in a component disk of some E_p^j.

- Let $K_j = \cup_k(\overline{C(E^j_k)} \setminus N^j_k)$ and $X = \cup_n \cap_{j>n} K_j$.
- Since area$(K_j) \to 0$, we have area$(X) = 0$.
- For any $z \neq 0$ choose $N^{j_1}_{k_1} \supset N^{j_2}_{k_2} \supset \cdots \supset \{z\}$.
- Then $z \in C(x)$ some $x \cap_j K_j$.
- Also, $C(x) \setminus N^{j_m}_{k_m} \subset K_m$.
- Thus $C(x) \setminus \{z\} \subset X$, as desired.