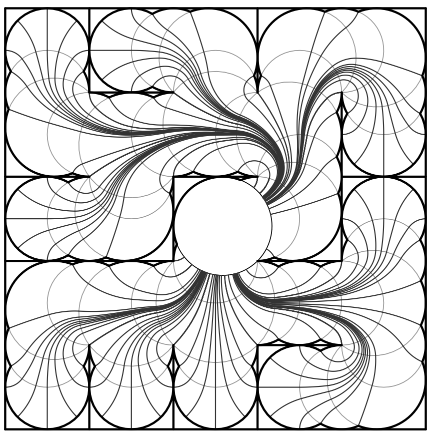
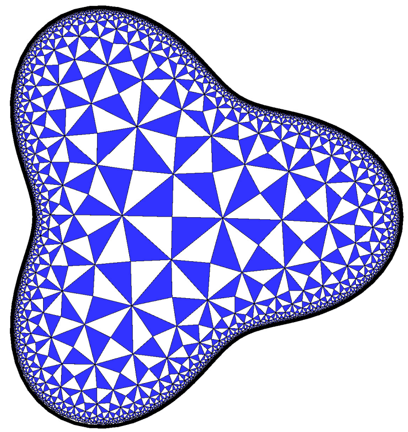
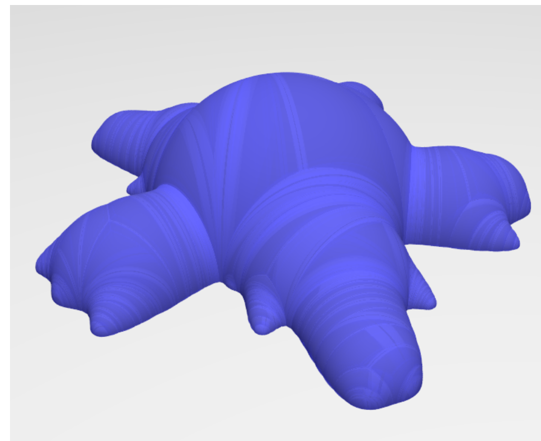
CONFORMAL MAPPING IN LINEAR TIME

Christopher Bishop, Stony Brook University

Texas A&M, Colloquium, Wednesday, October 12, 2022

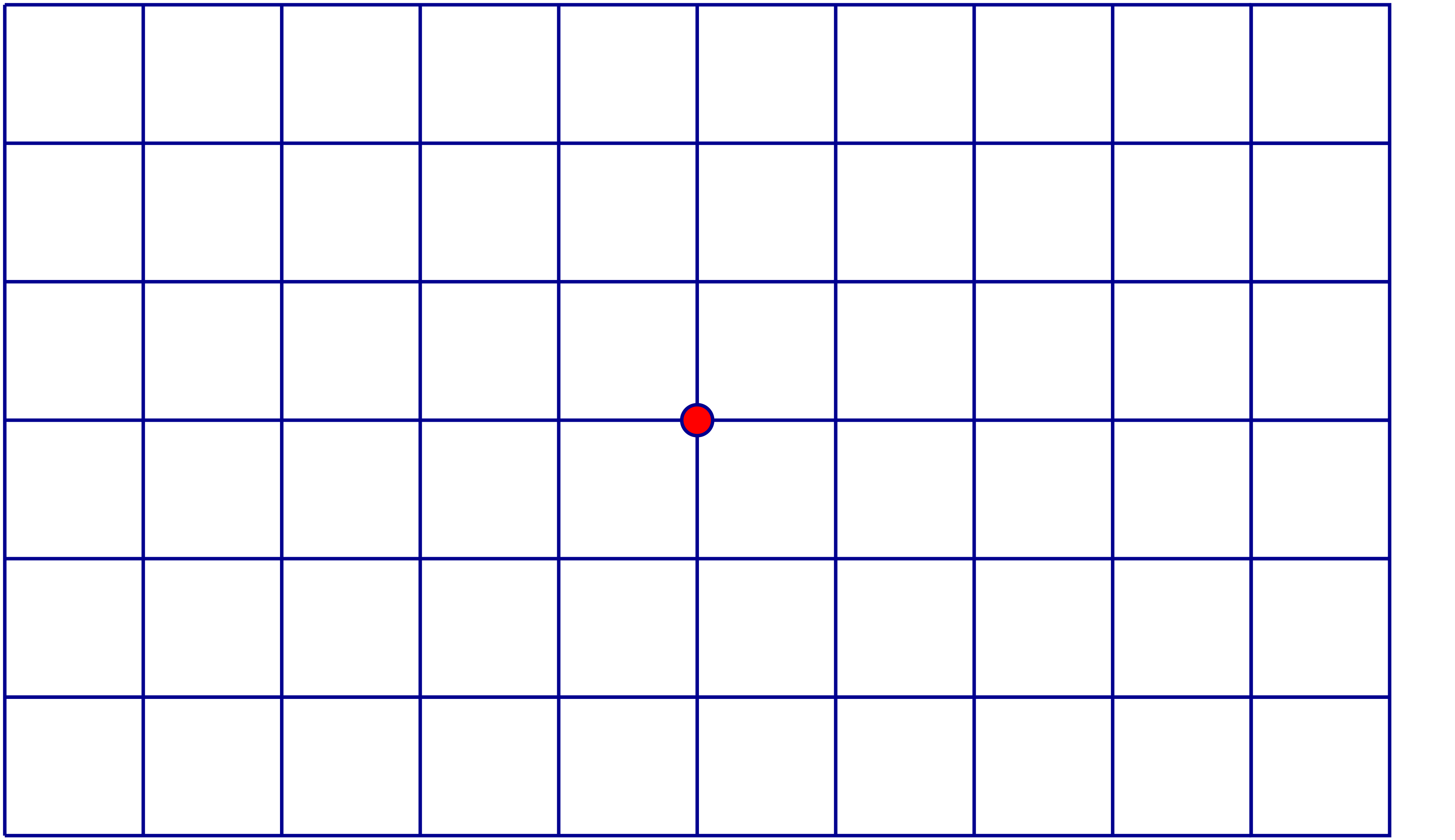


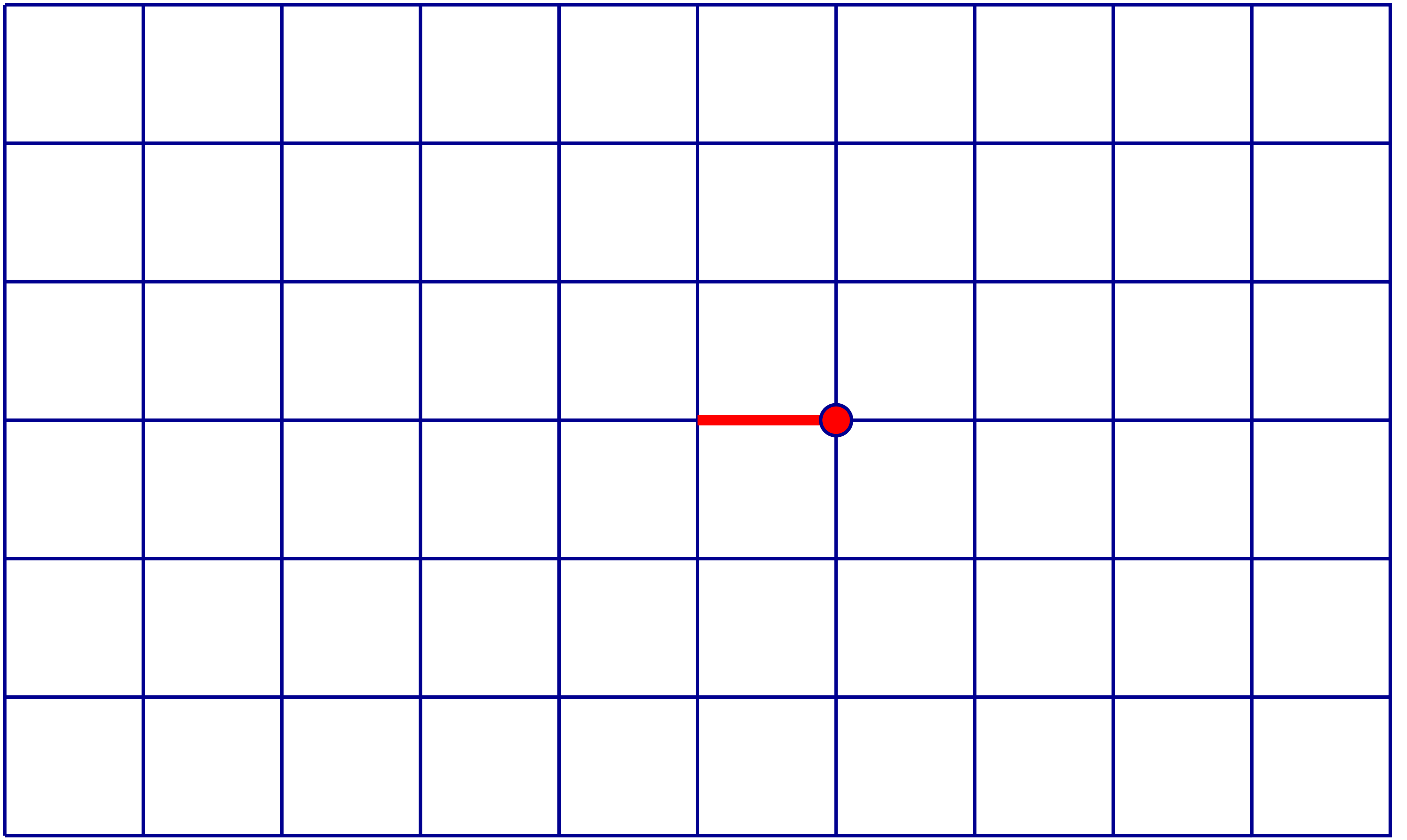


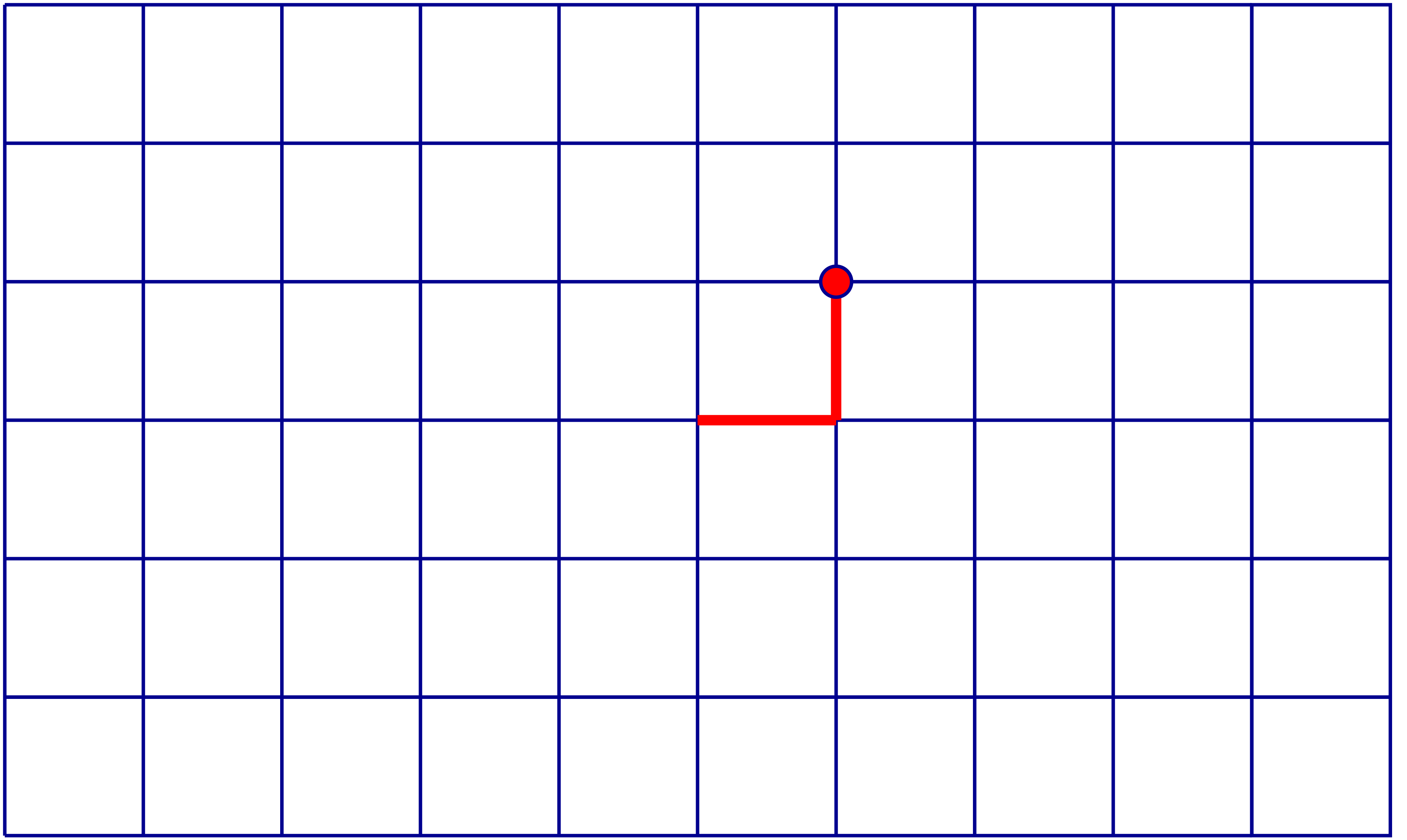


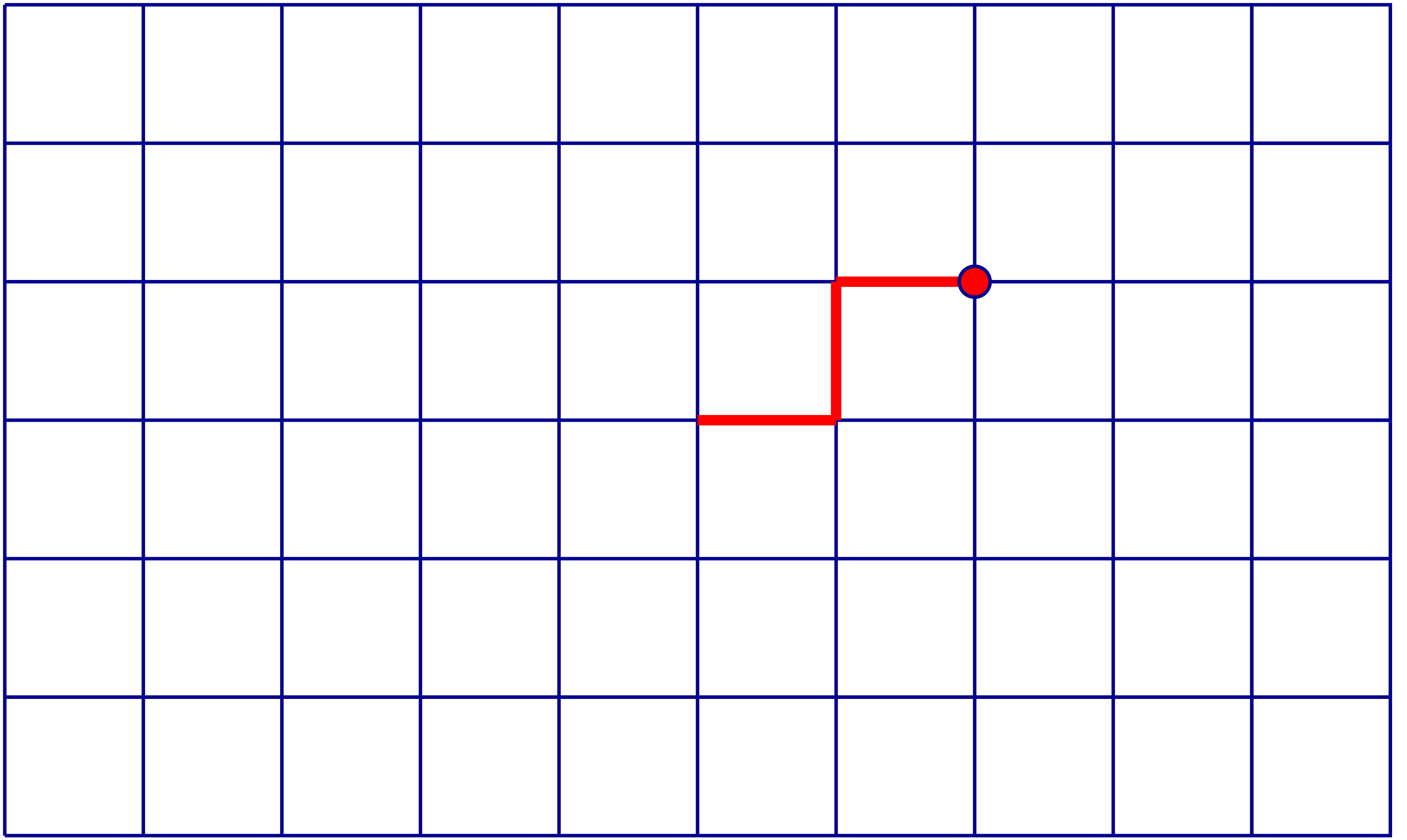
THE PLAN

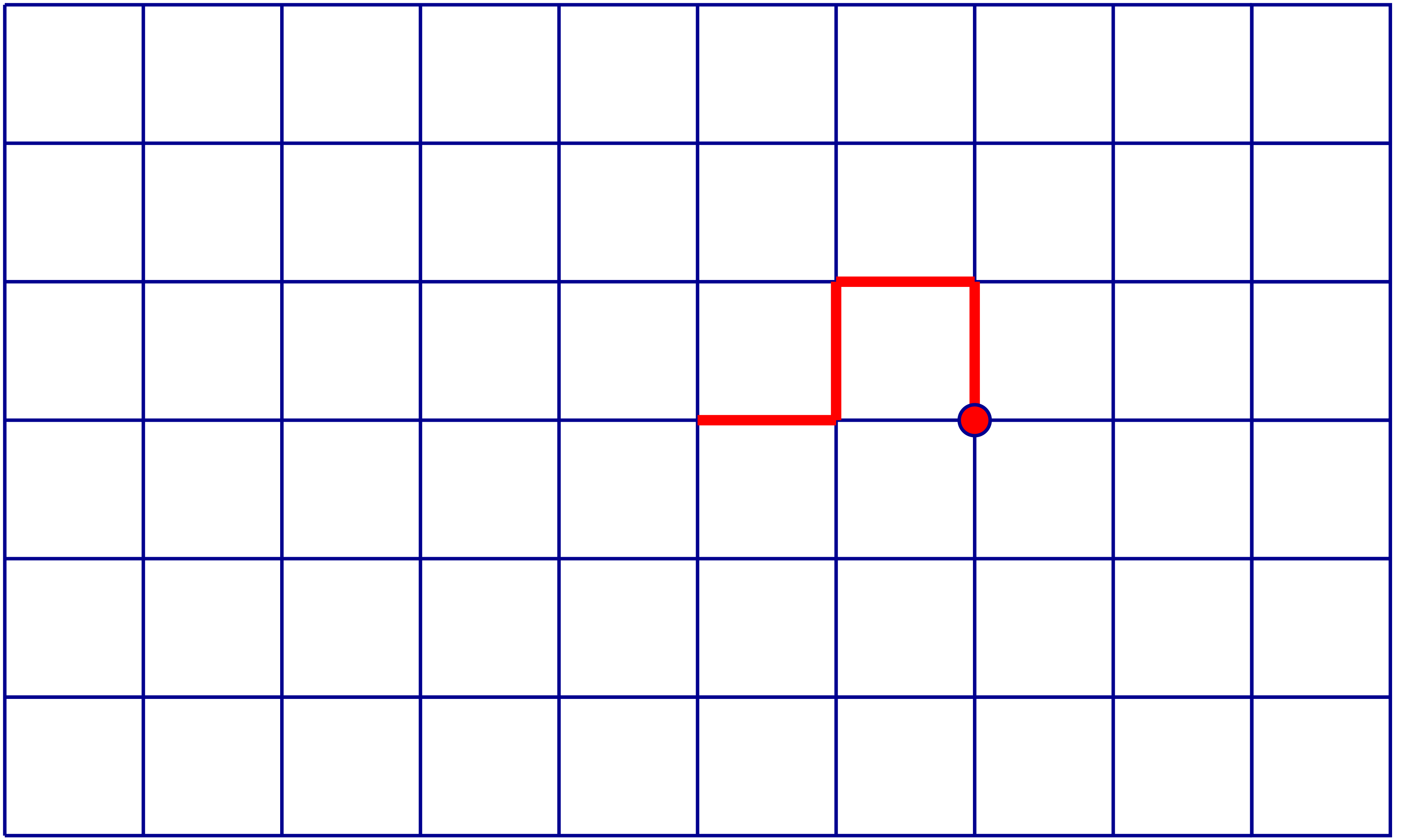
- Review: harmonic measure, conformal maps, QC maps
- Use discrete geometry (medial axis) to build fast approximation.
- Use hyperbolic geometry to show approximation is accurate.
- A few applications.

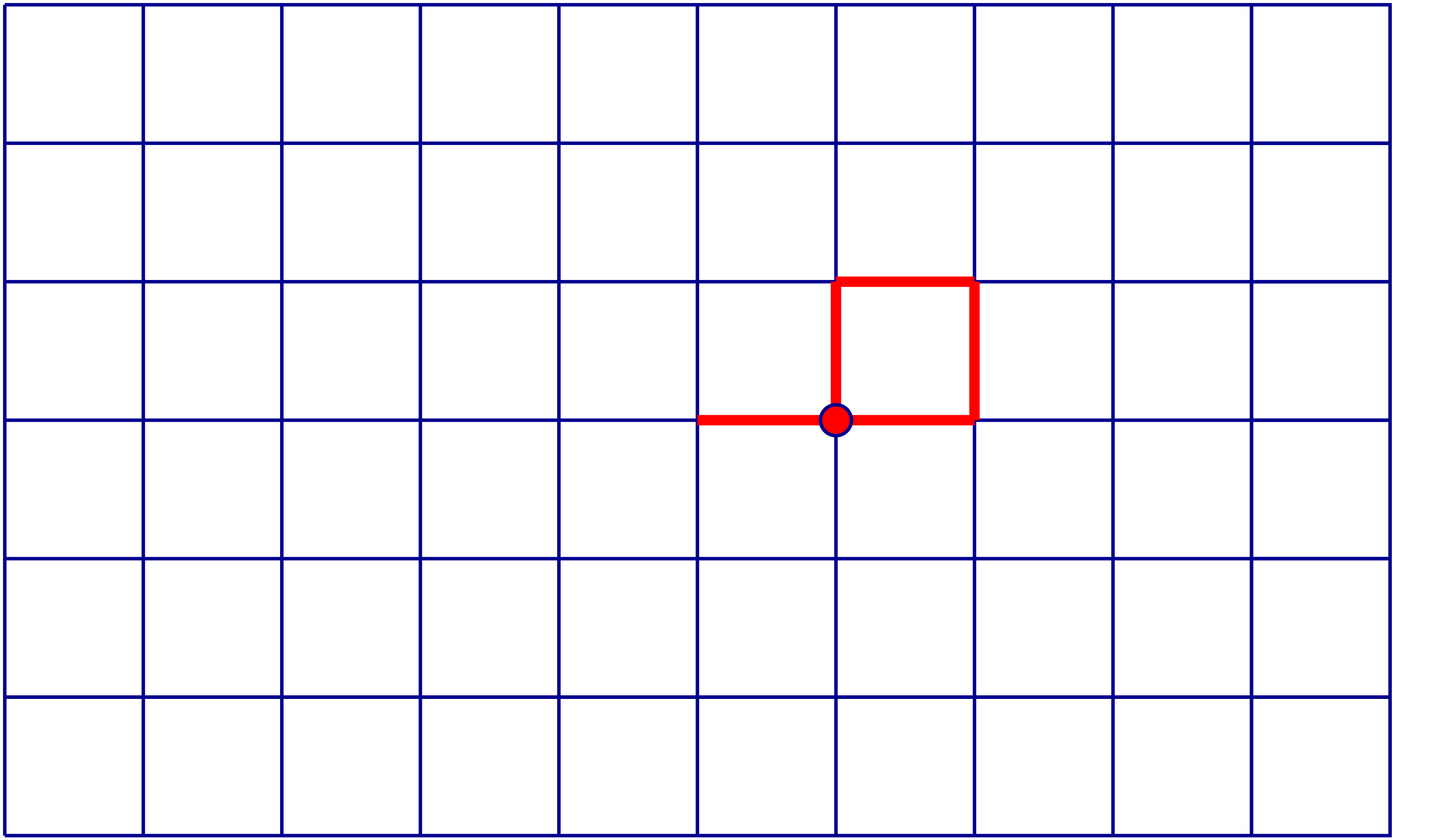


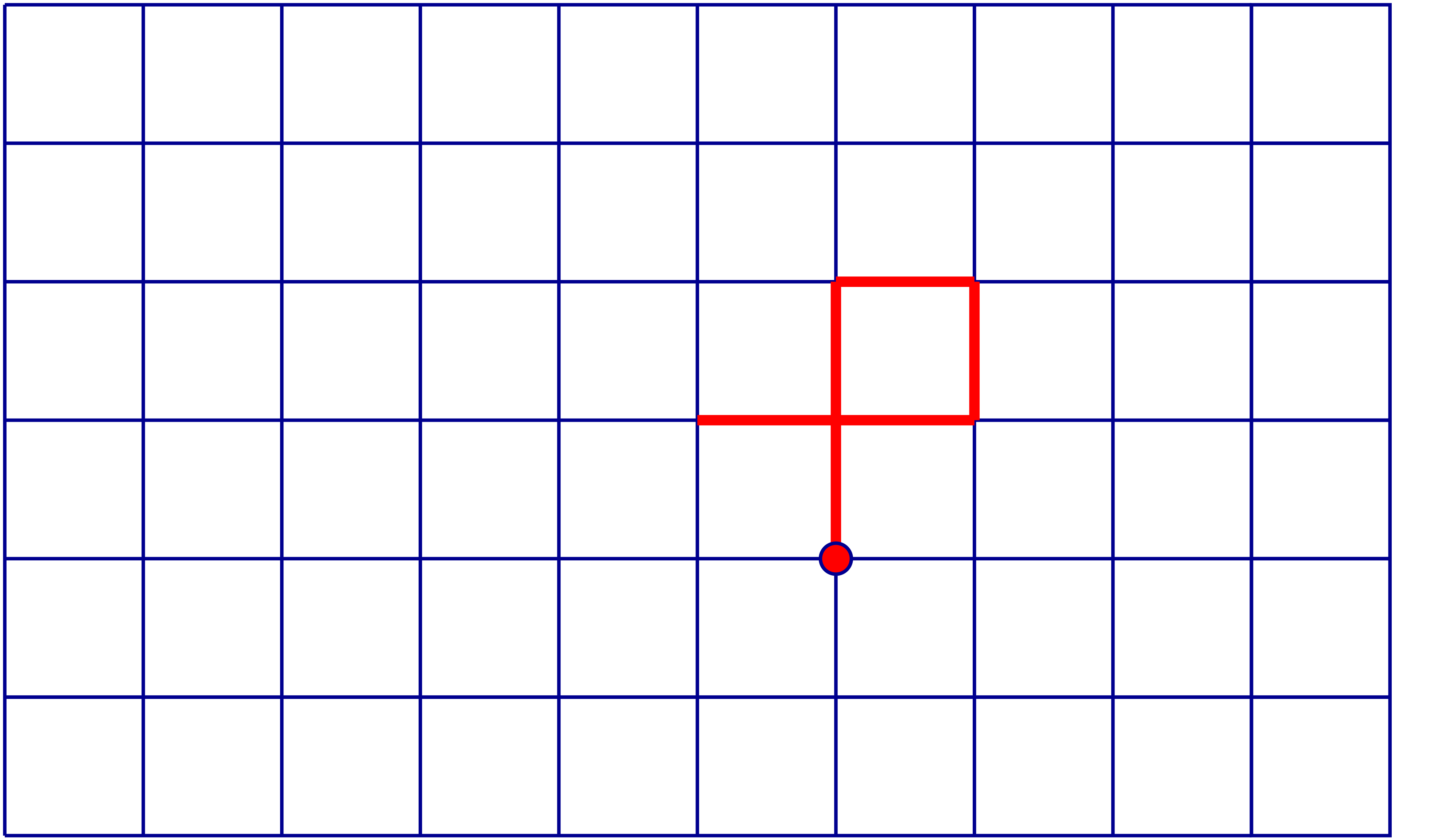


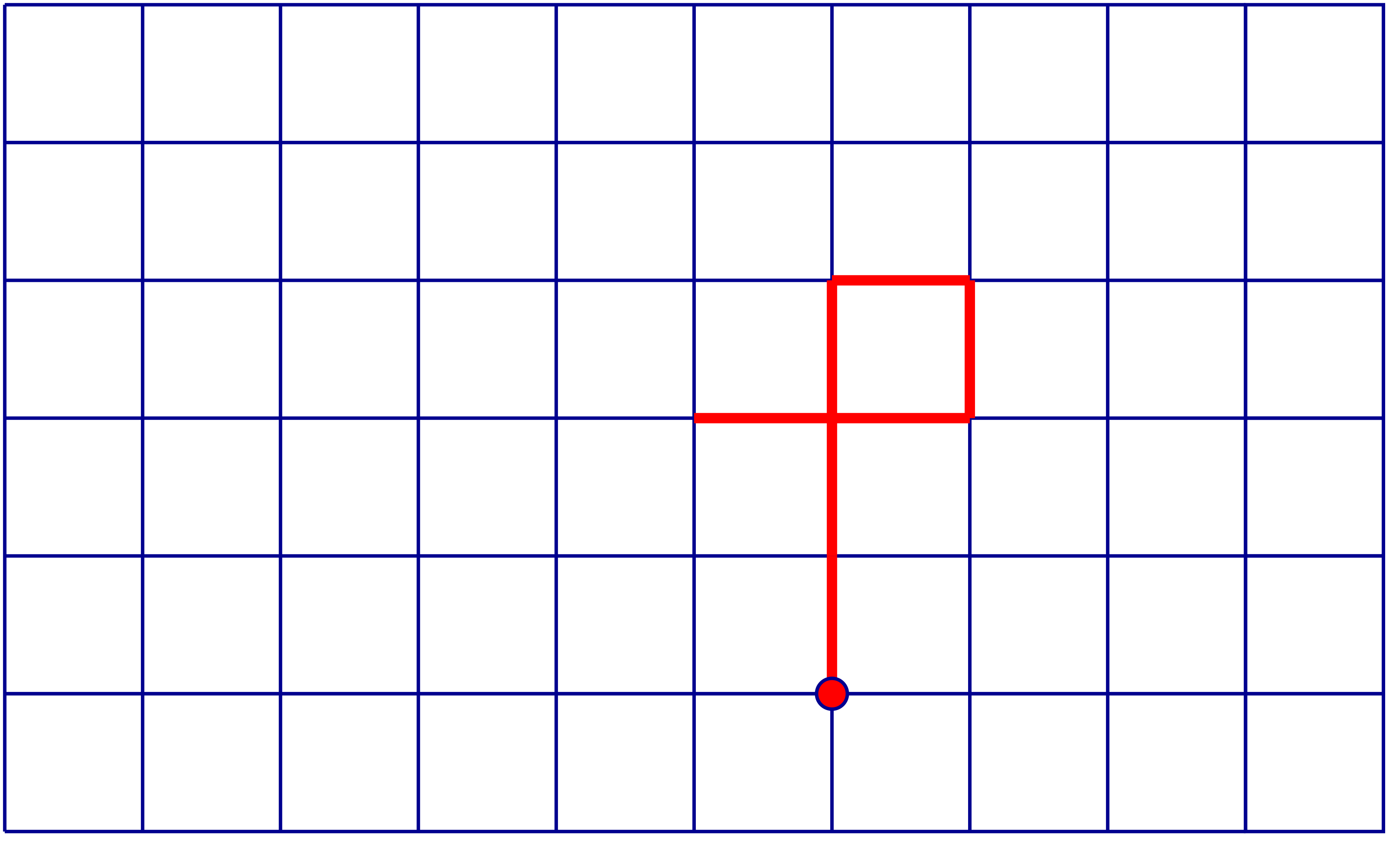


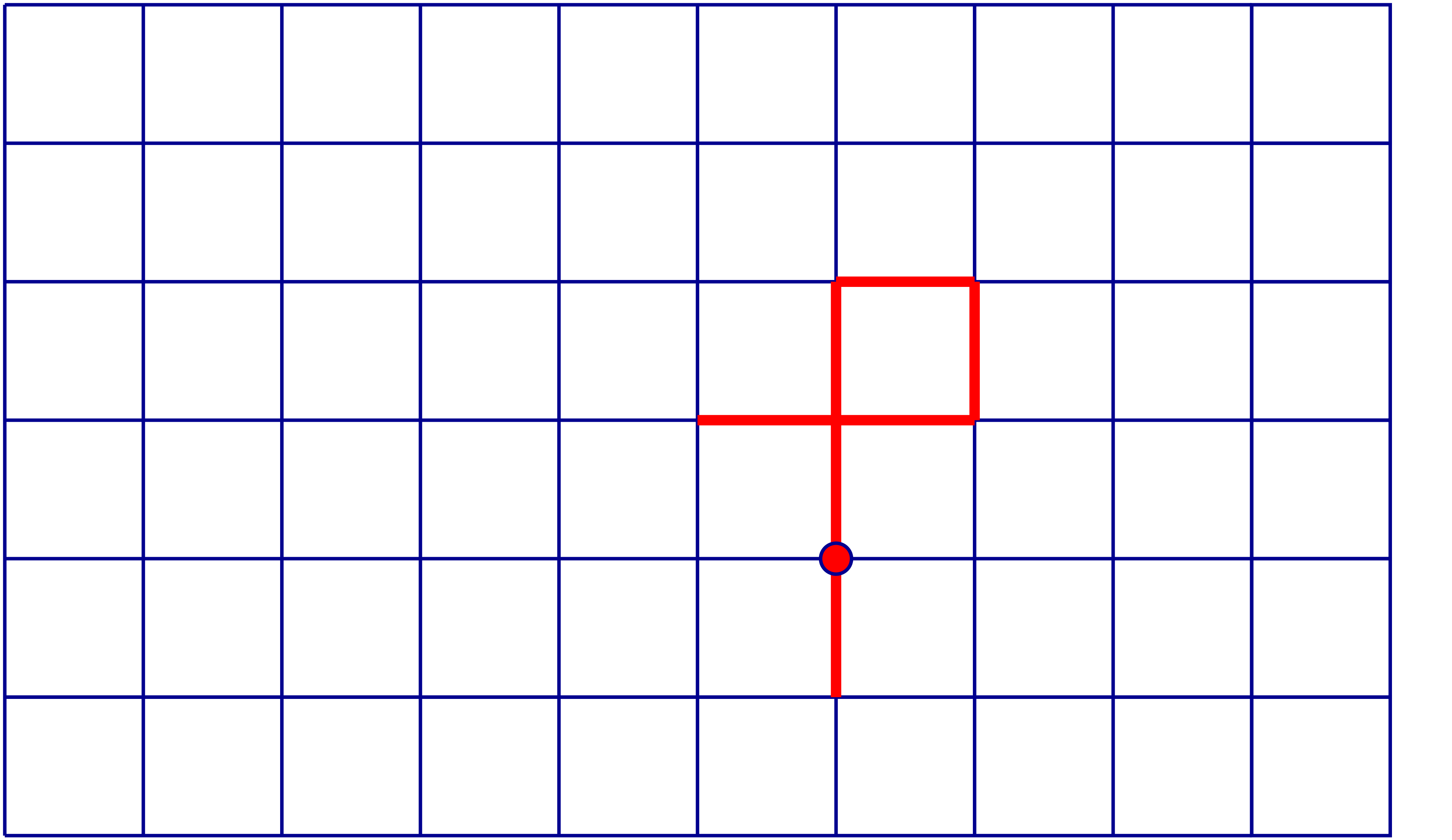


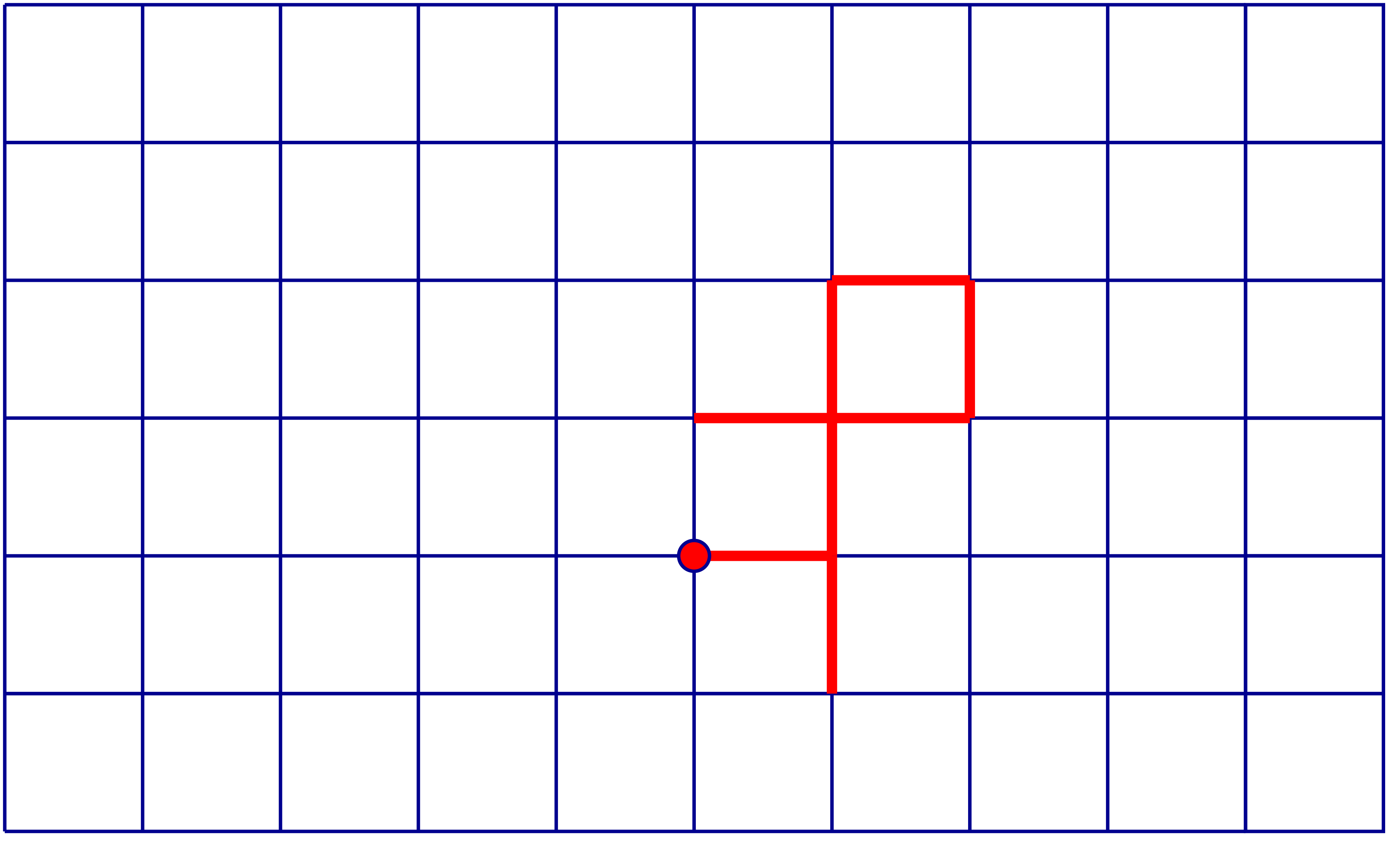


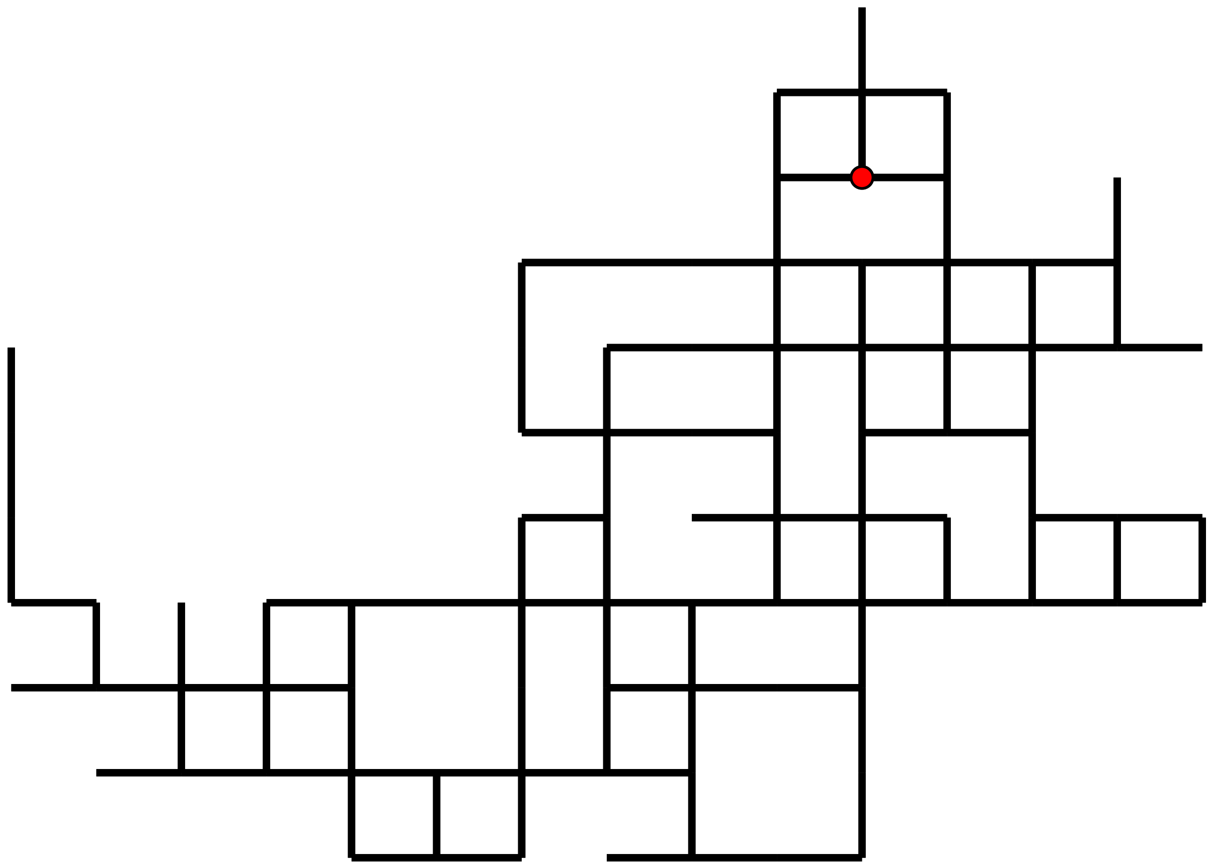




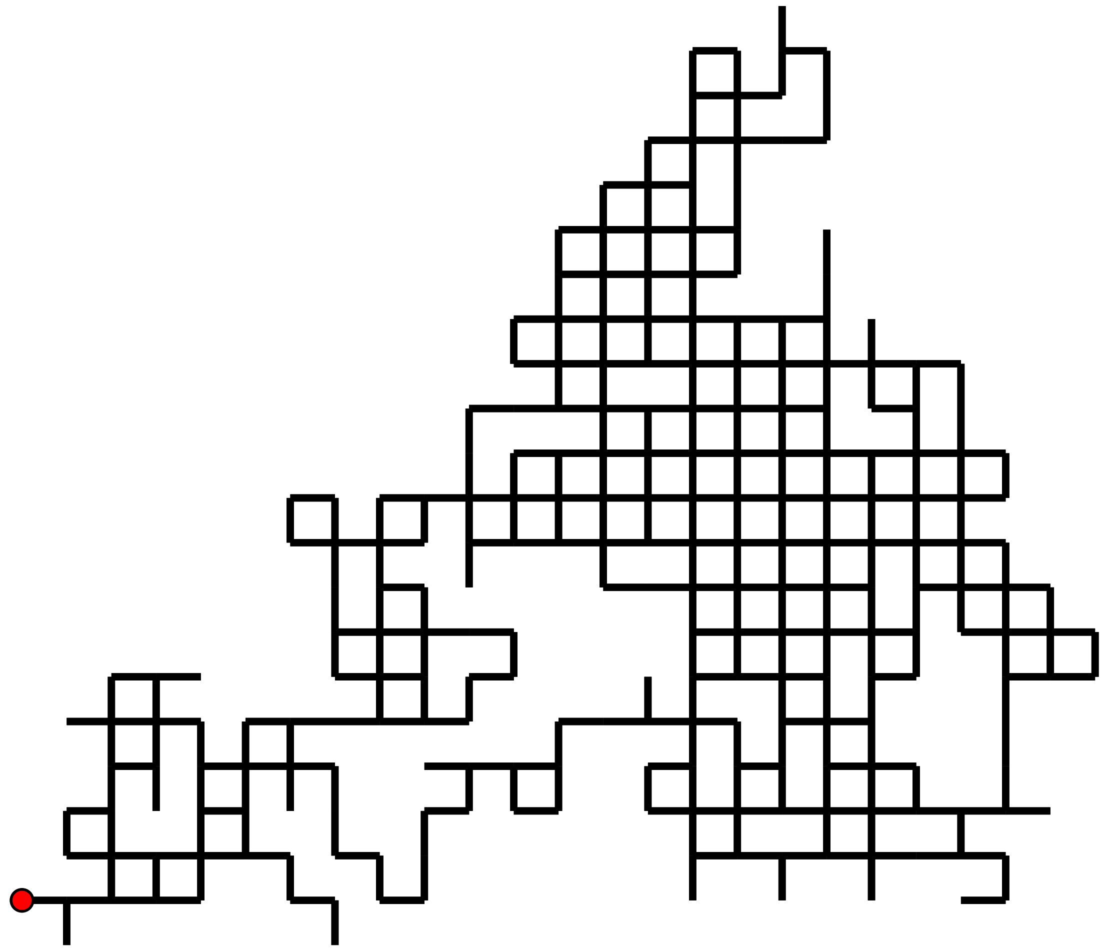




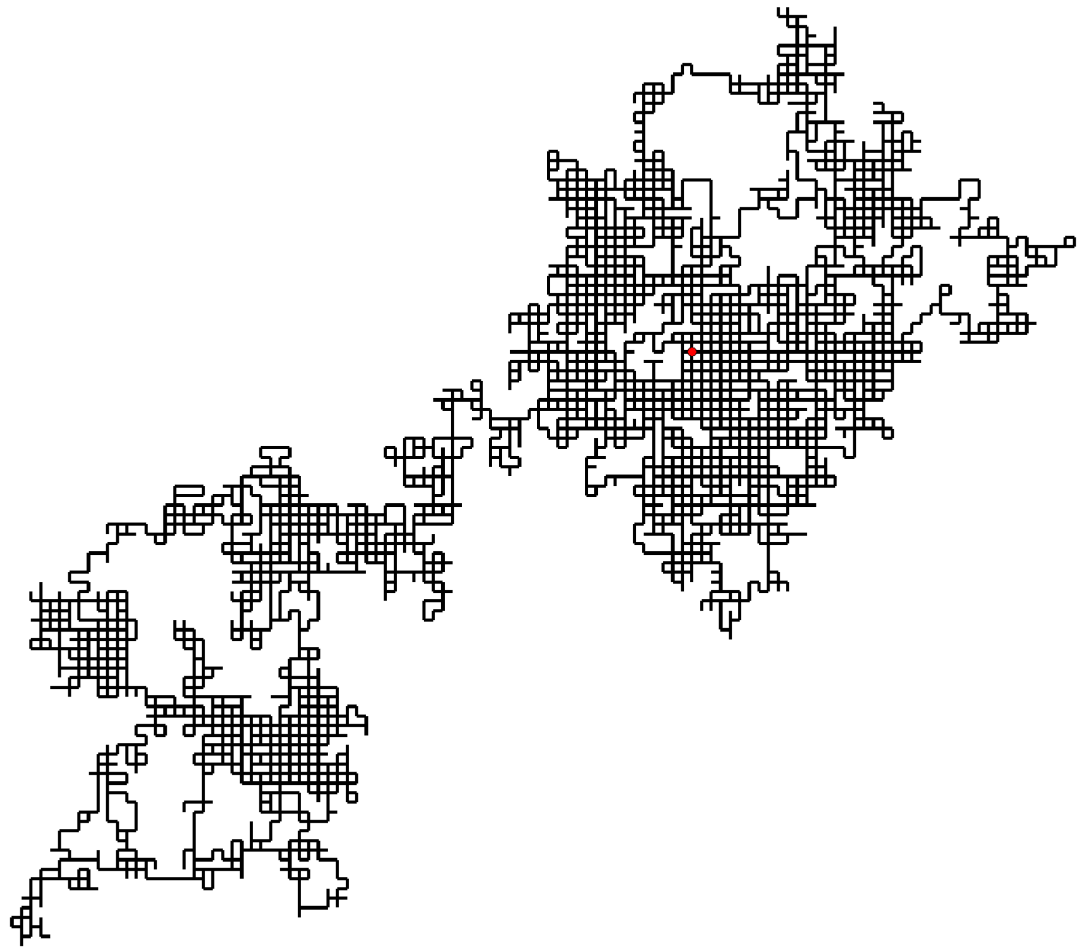




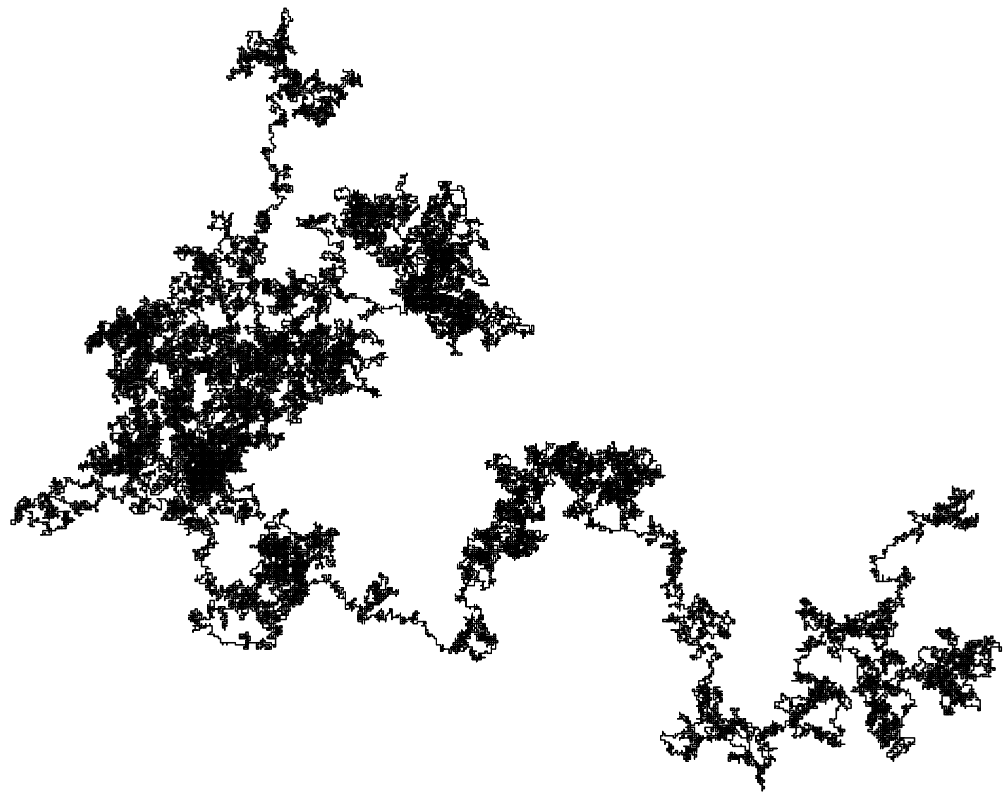
200 step random walk.



1000 step random walk.

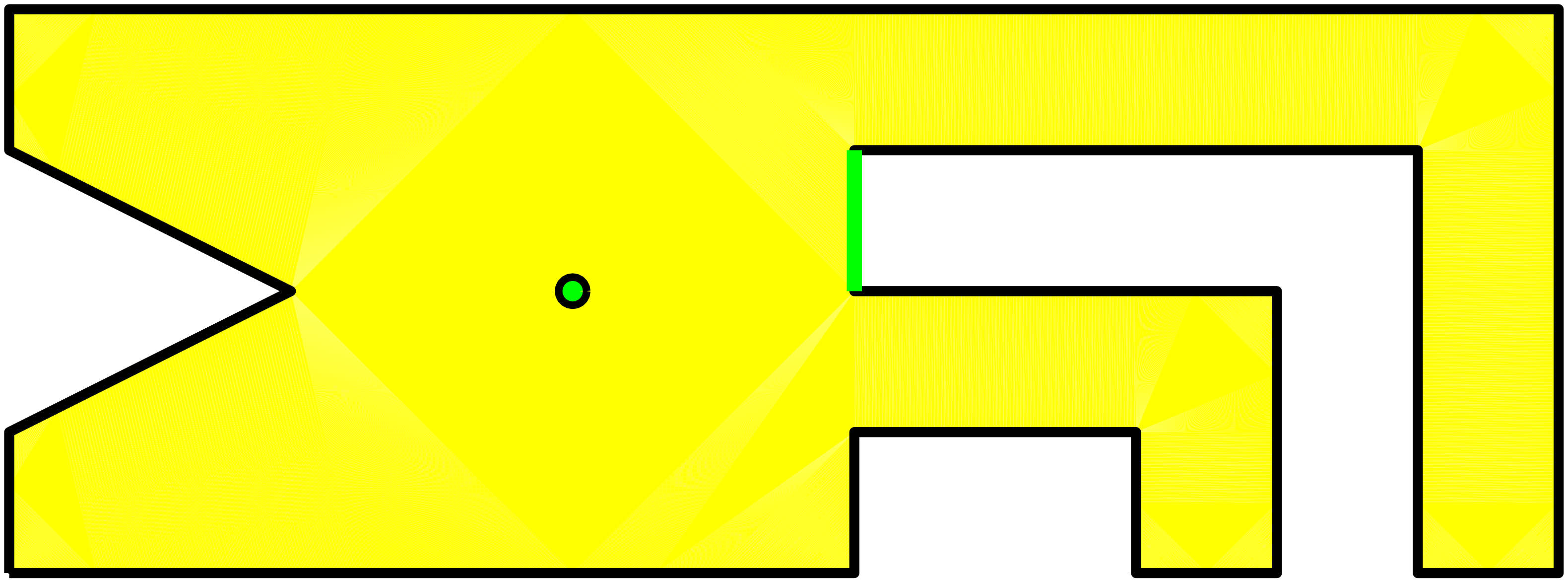


10,000 step random walk.



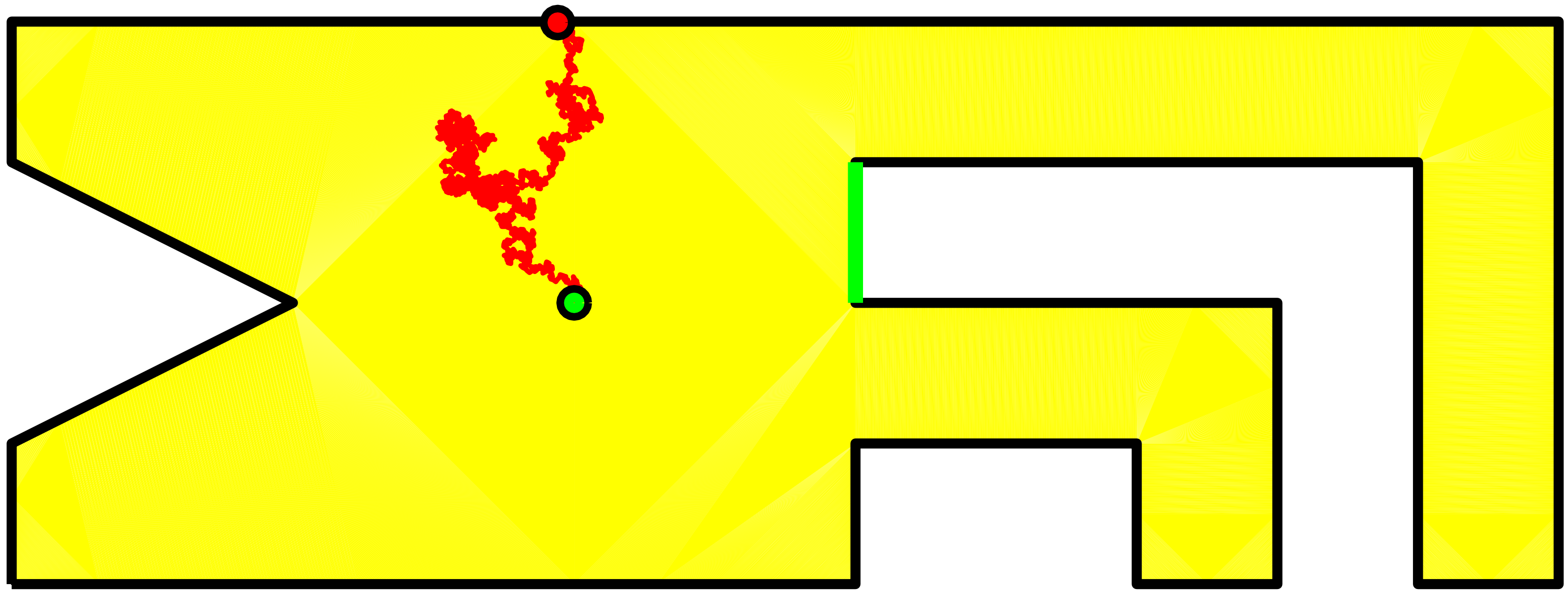
100,000 step random walk.

Harmonic measure = hitting distribution of Brownian motion



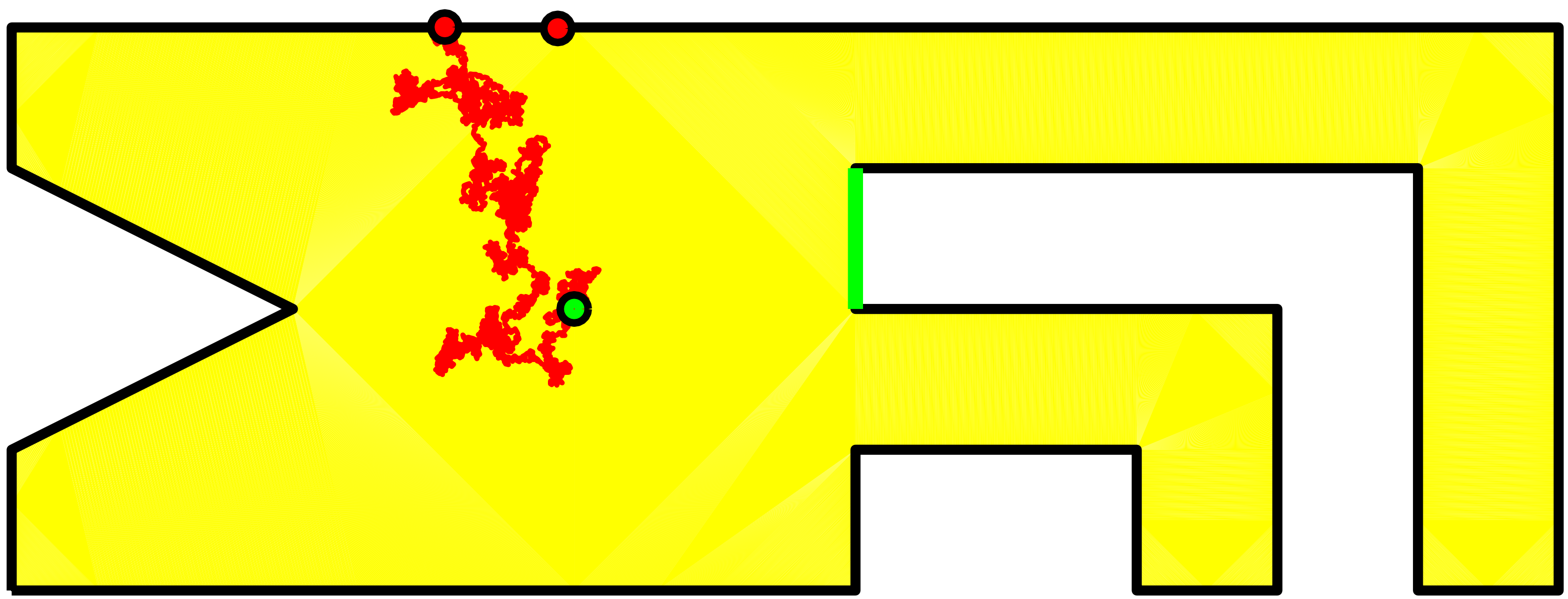
Suppose Ω is a planar Jordan domain.
Let E be a subset of the boundary, $\partial\Omega$.
Choose an interior point $z \in \Omega$.

Harmonic measure = hitting distribution of Brownian motion



$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



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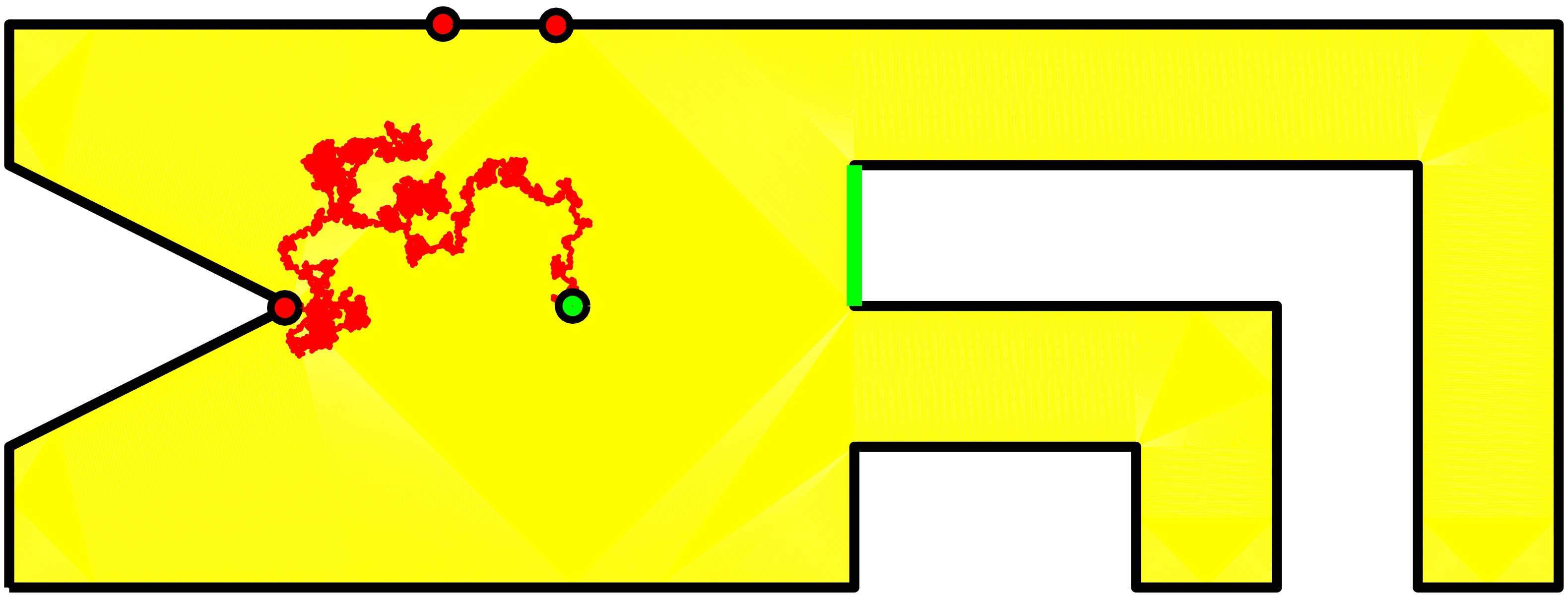

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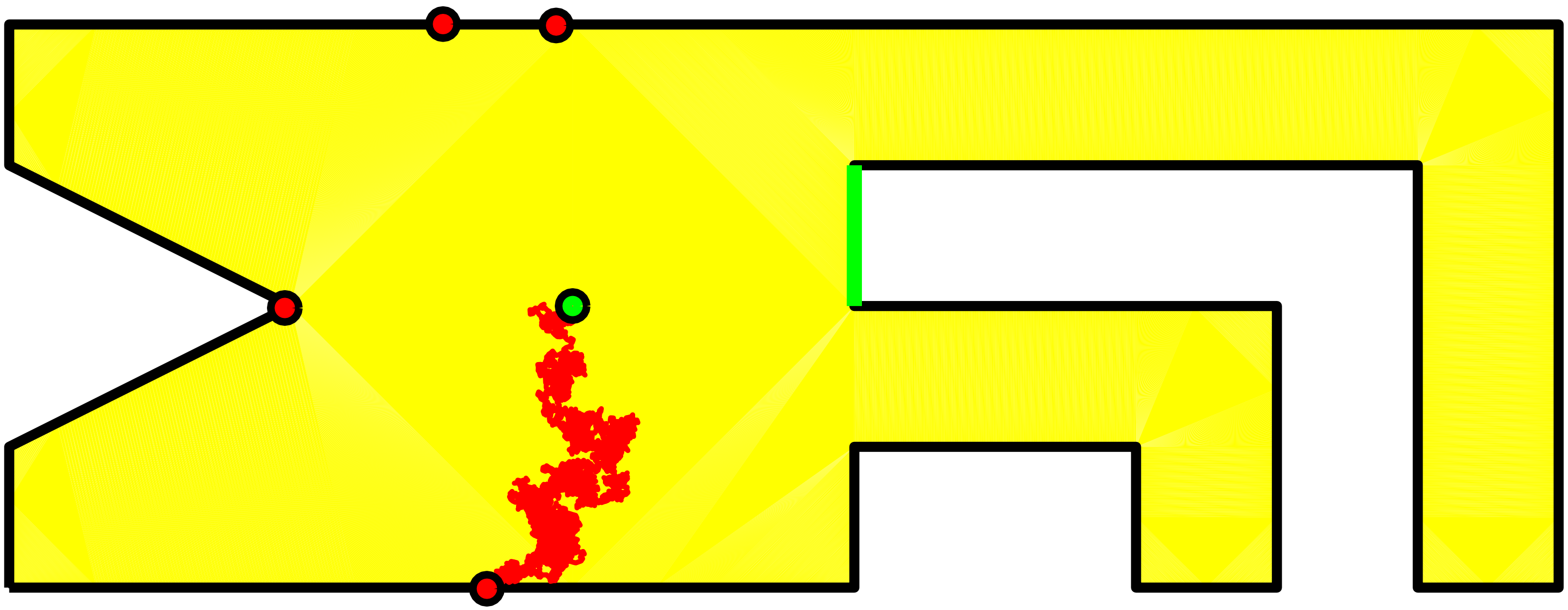

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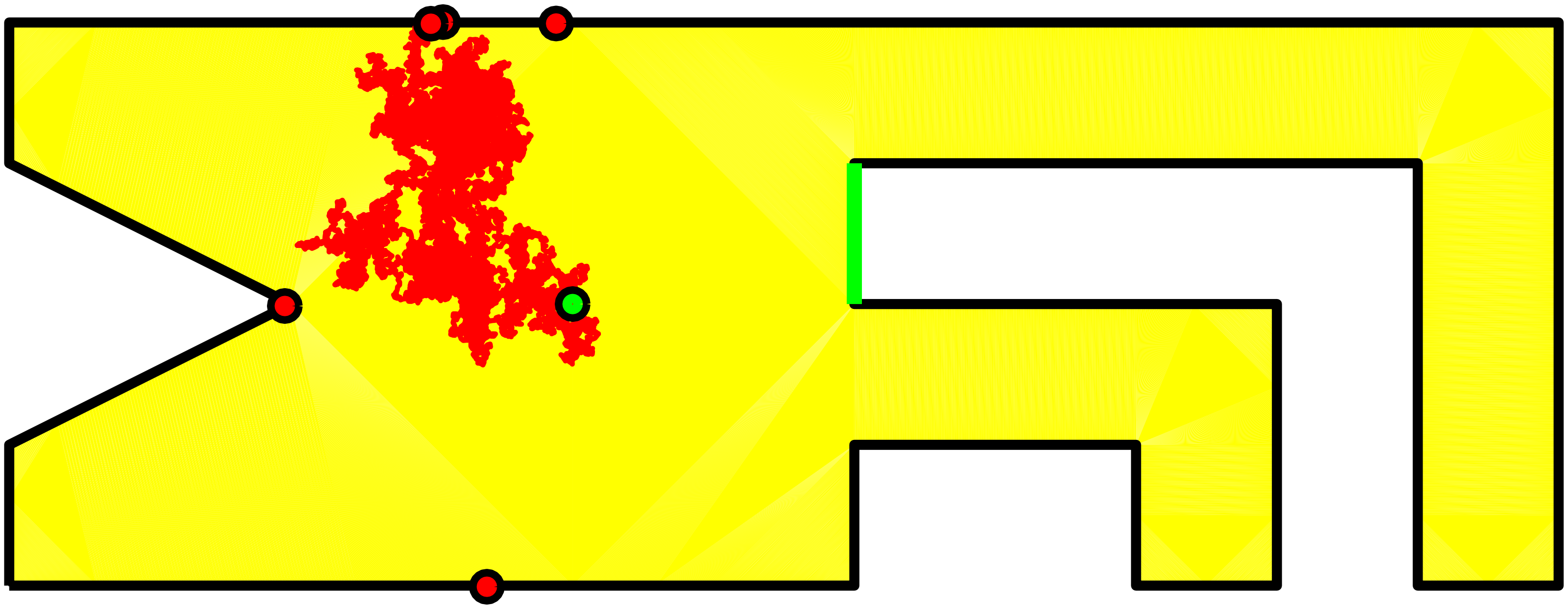

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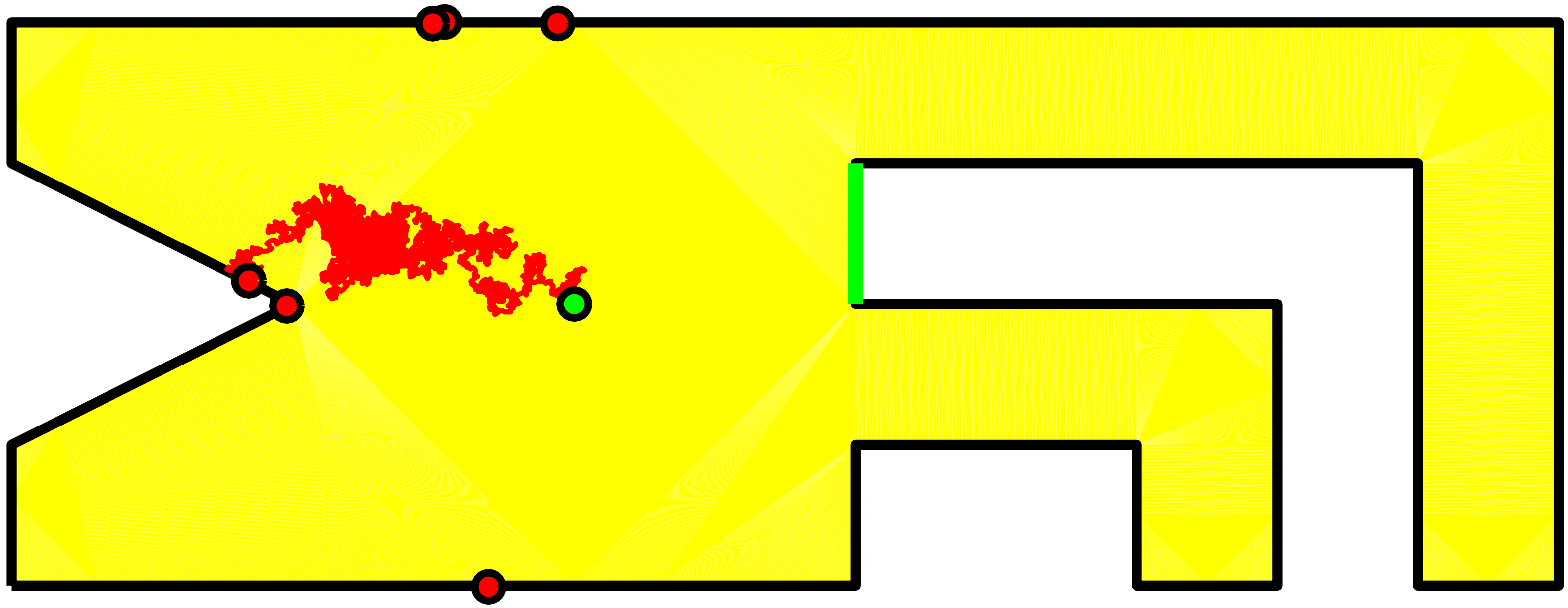

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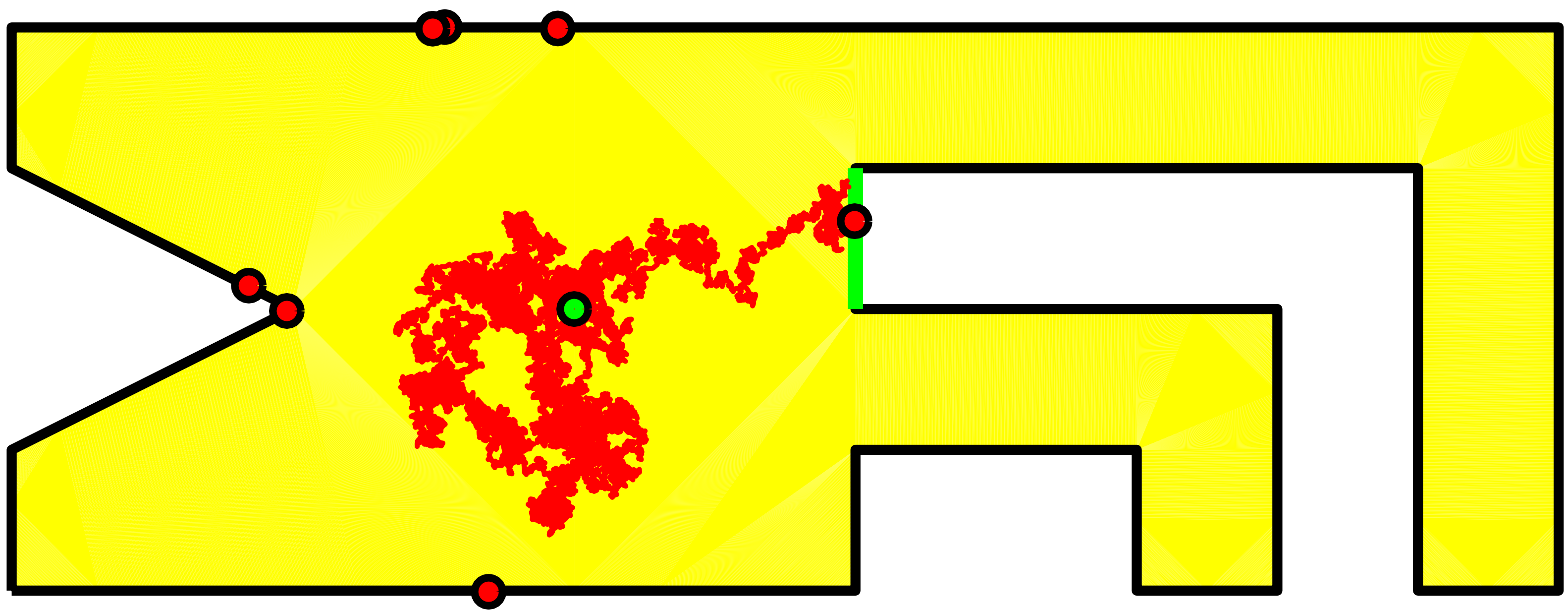

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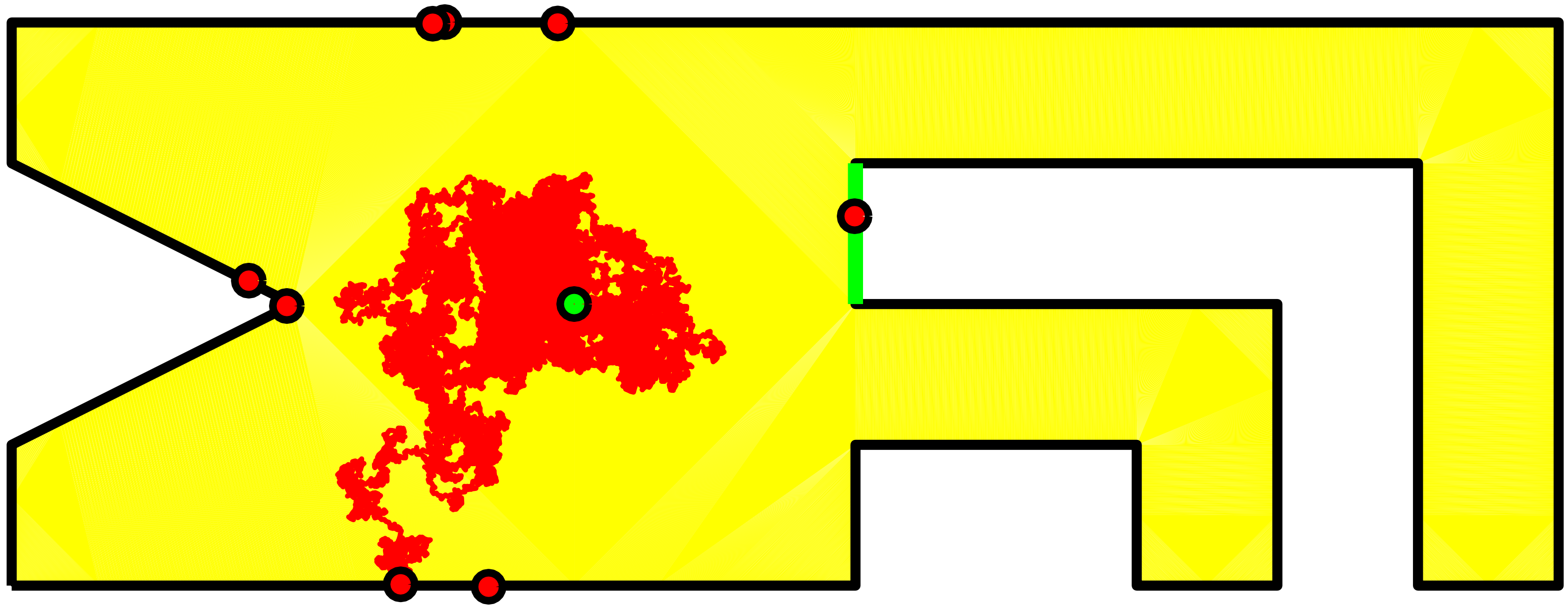

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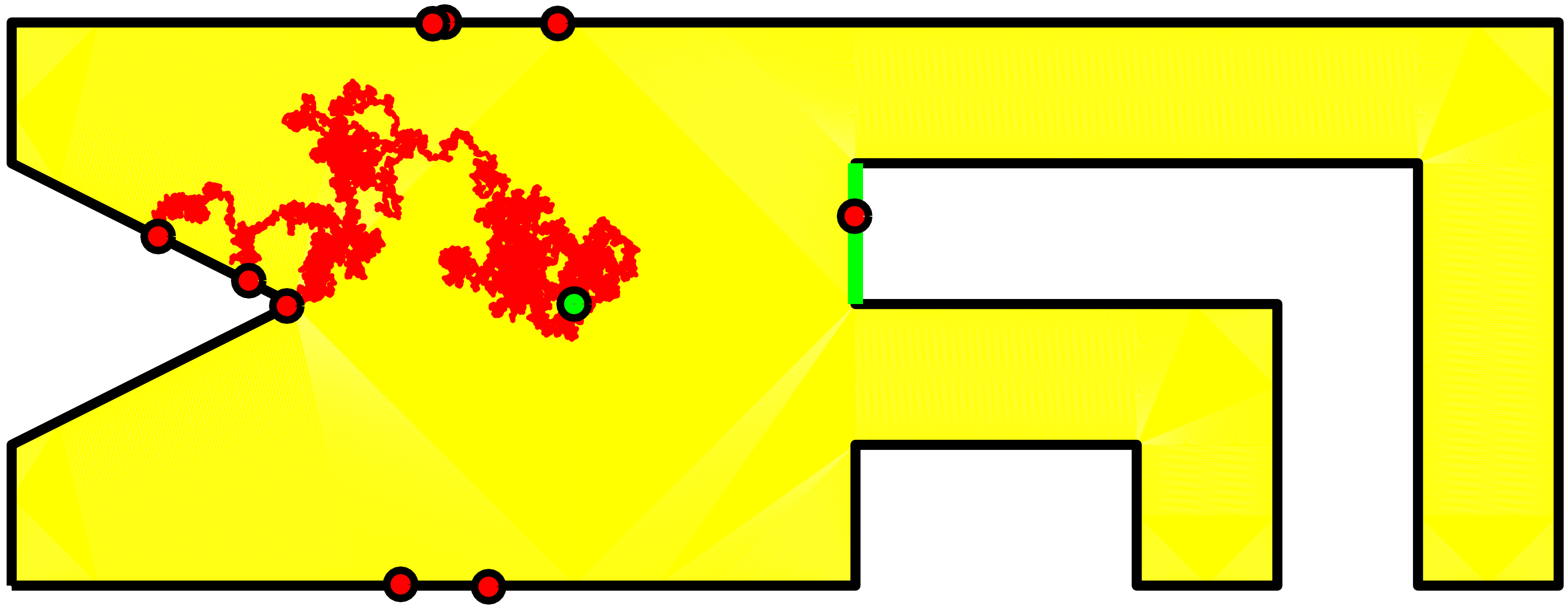

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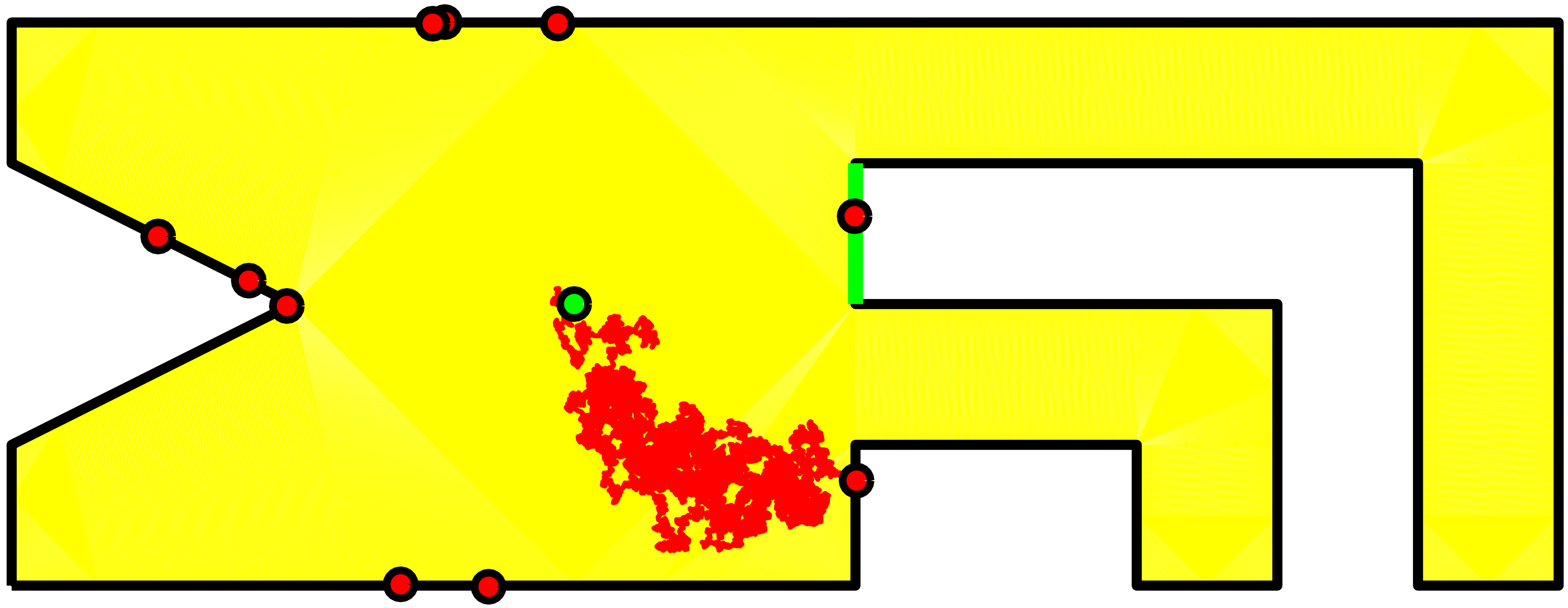
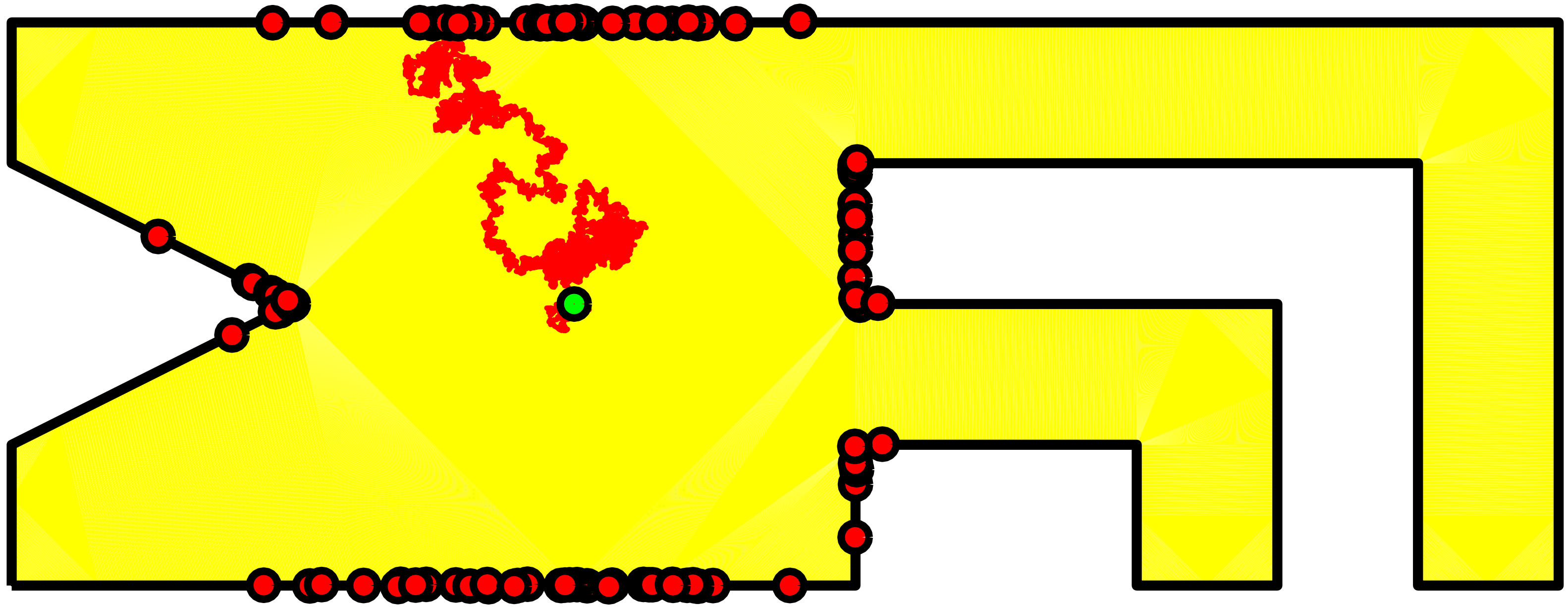


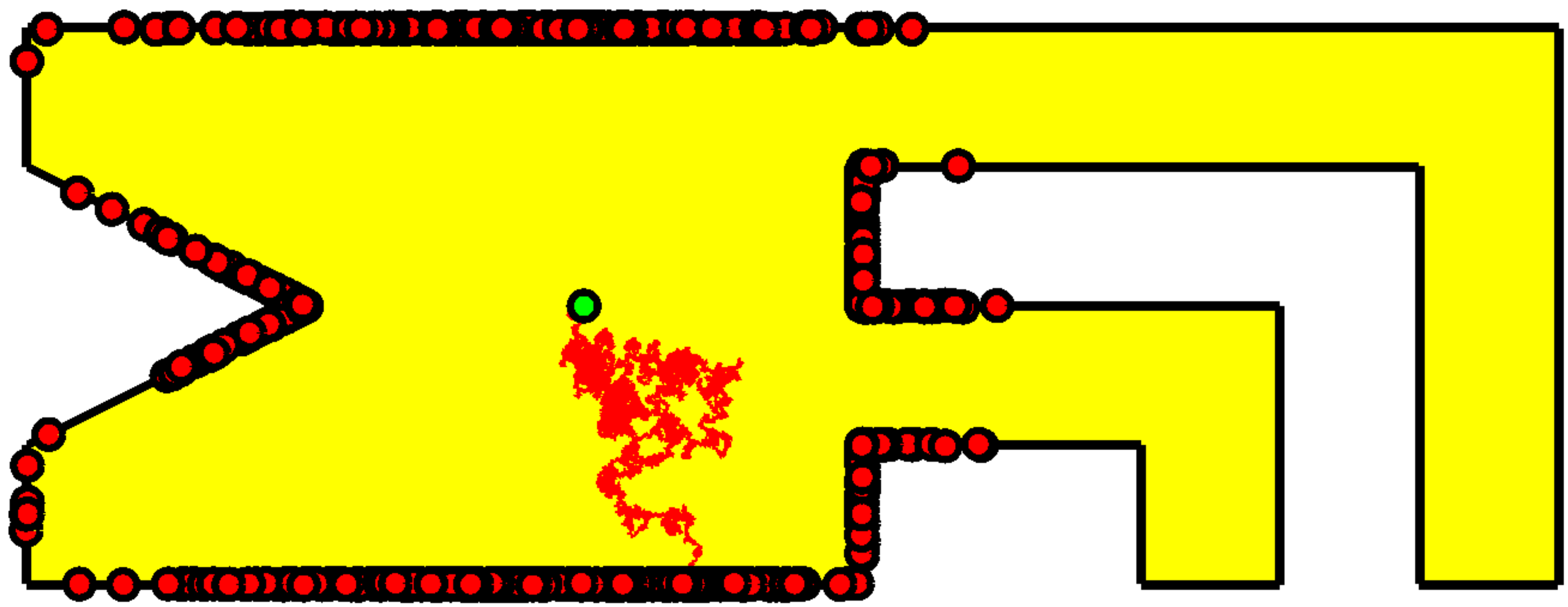
$$\omega(z, E, \Omega) \approx 1/10.$$

Harmonic measure = hitting distribution of Brownian motion



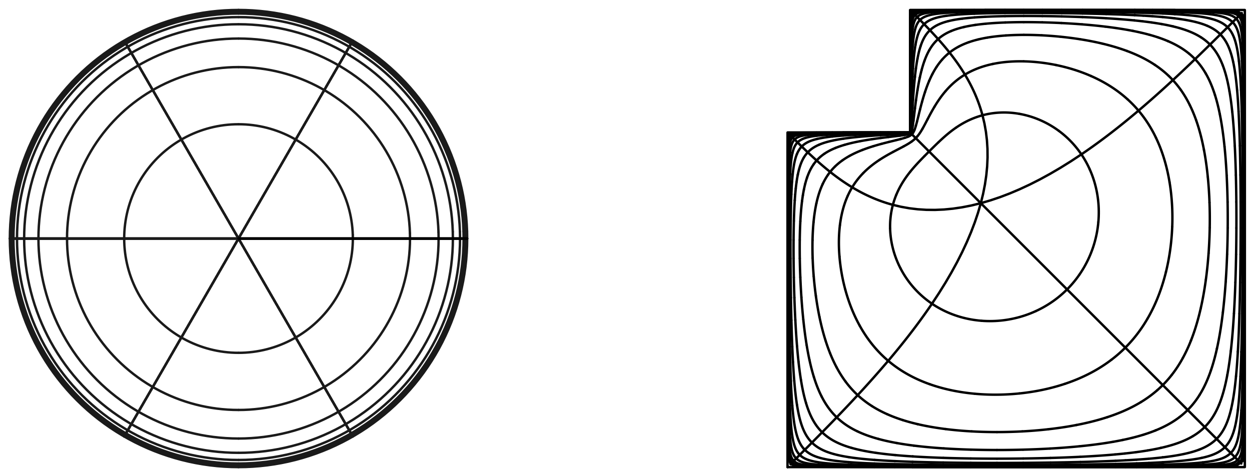
$$\omega(z, E, \Omega) \approx 13/100.$$

Harmonic measure = hitting distribution of Brownian motion

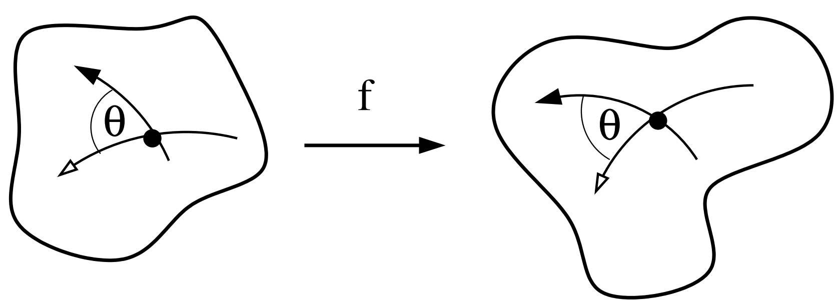


$$\omega(z, E, \Omega) \approx 126/1000.$$

Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected, then there is a conformal map $f : \mathbb{D} \to \Omega$.

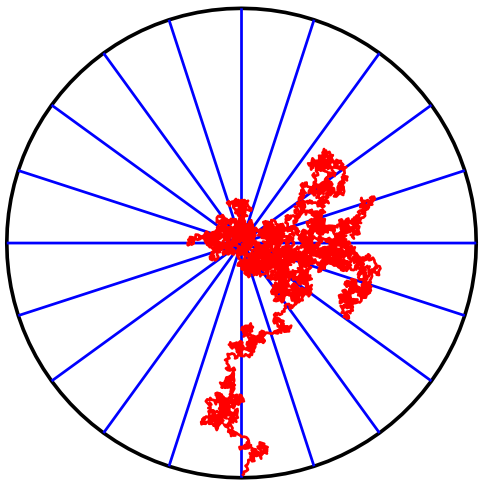


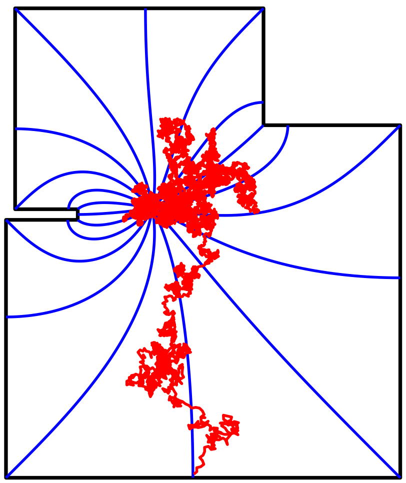
Conformal = 1-1 holomorphic = angle and orientation preserving

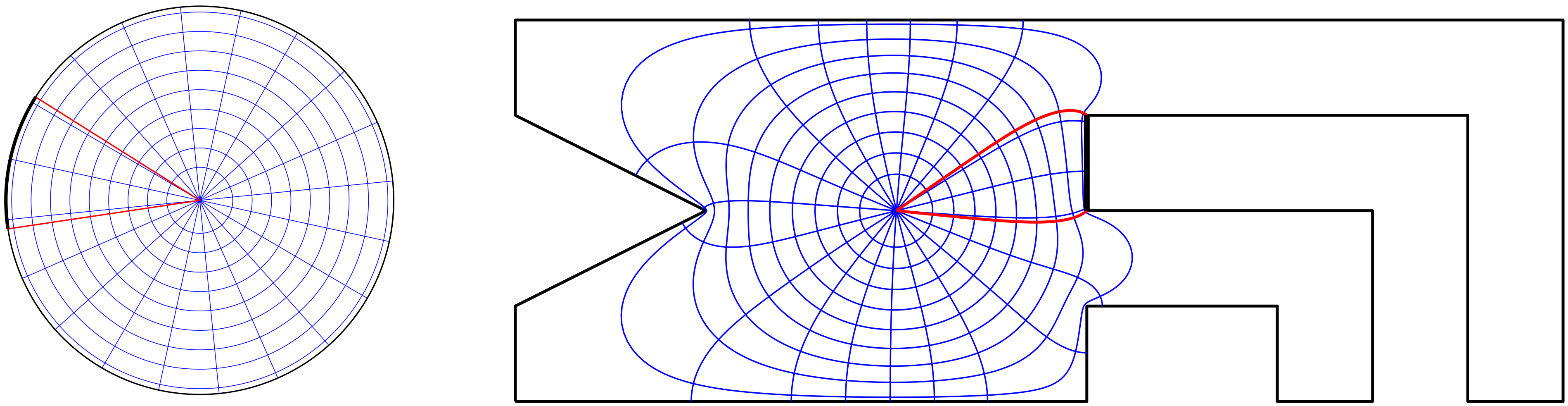


Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected, then there is a conformal map $f : \mathbb{D} \to \Omega$.

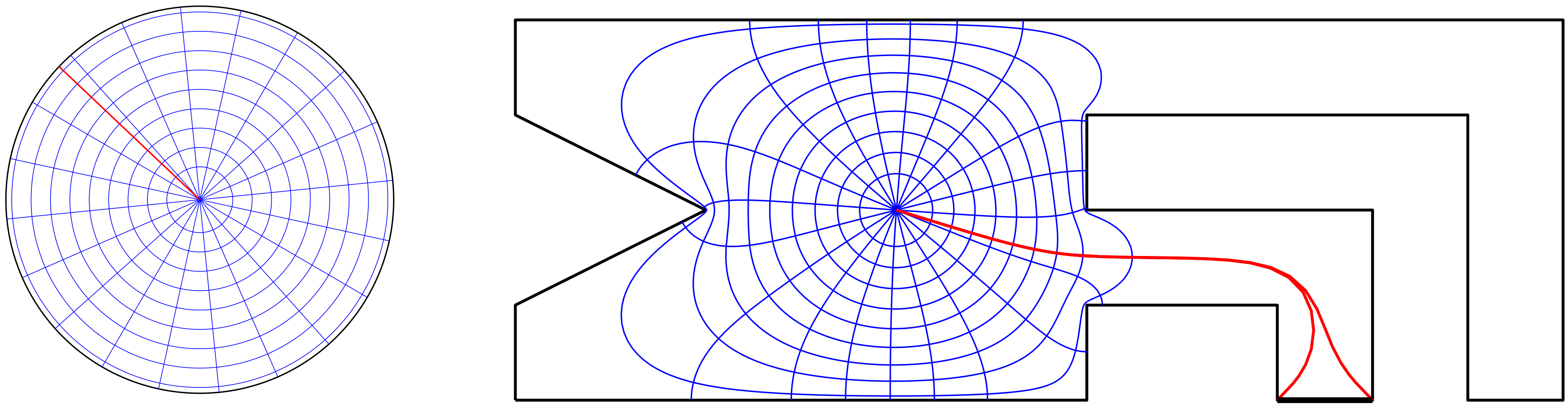
Brownian motion is conformally invariant, so normalized length measure maps to harmonic measure. Fastest way to compute harmonic measure.





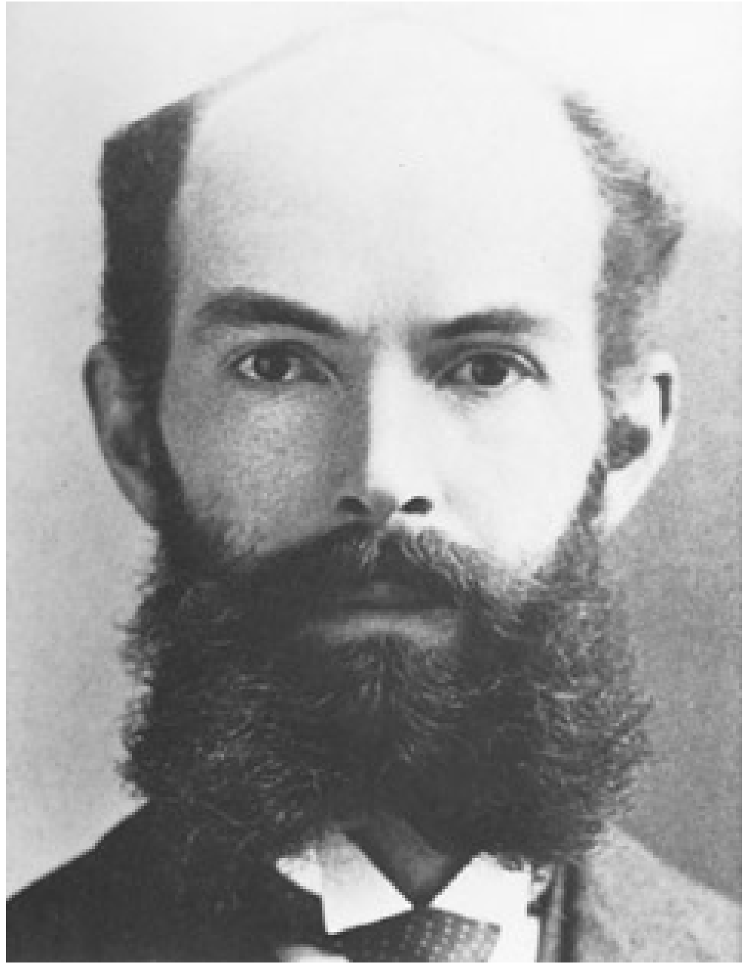


harmonic measure ≈ 0.1128027



harmonic measure $\approx 1.22155 \times 10^{-6}$

Georg Friedrich Bernhard Riemann
Stated RMT in 1851



William Fogg Osgood
First proof of RMT, Trans. AMS, vol. 1, 1900

Schwarz-Christoffel formula for maps to polygons (1867):

$$f(z) = A + C \int^{z} \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw,$$

Christoffel

Schwarz

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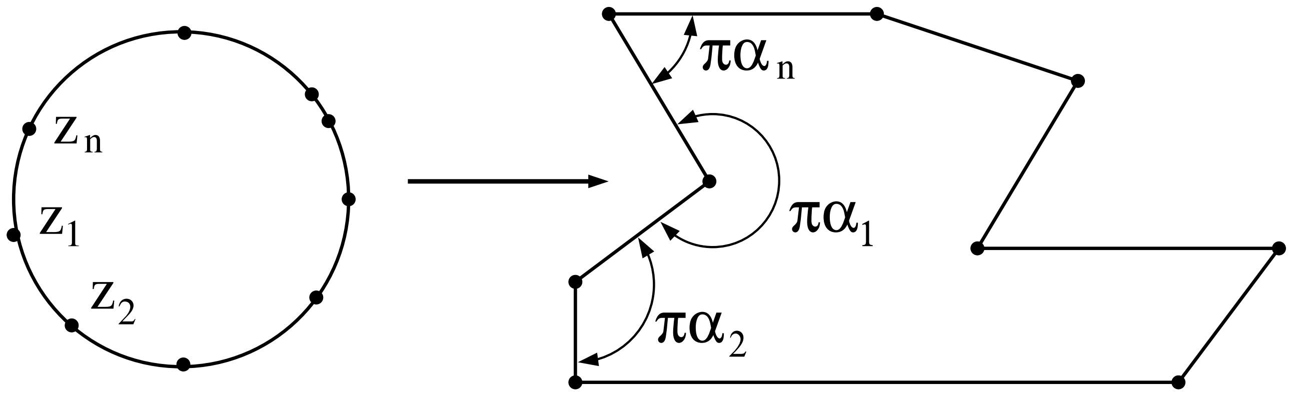

The α_k are known from polygon.

The z_k are unknown (= **SC-parameters** = **pre-vertices**)

Finding SC-parameters = Finding harmonic measure of edges

Numerical conformal mapping:

- Koebe
- Theodorsen
- Fornberg
- Wegman
- Gaier
- Symm
- Kerzman-Stein
- Integral equations via fast multipole, Rokhlin
- Circle packing, Sullivan, Rodin, Stephenson
- `CRDT`, Driscoll and Vavasis
- `SCToolbox`, Trefethen, Driscoll
- `ZIPPER`, Marshall

Problem: given n-gon, how fast can we compute the SC-parameters?

Theorem: Can compute ϵ-conformal map onto n-gon in time $C_\epsilon \cdot n$.

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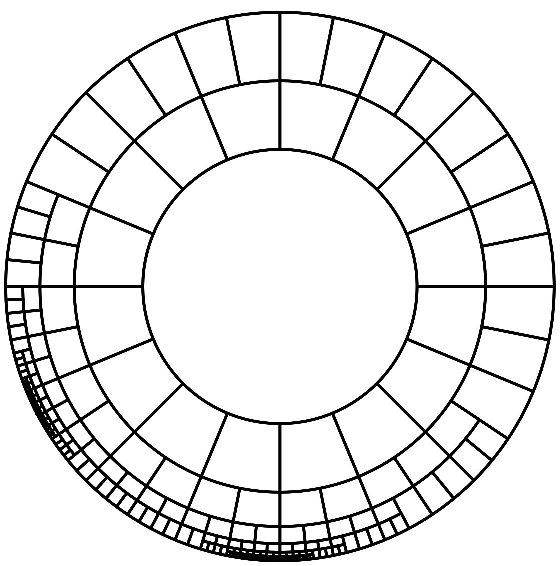
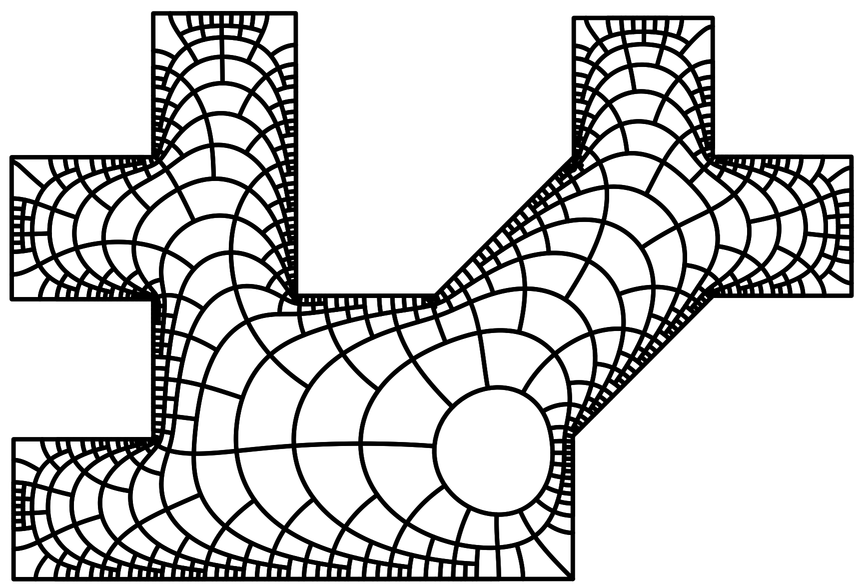
ϵ-conformal $= 1+\epsilon$ quasiconformal.

$C_\epsilon = O(\log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

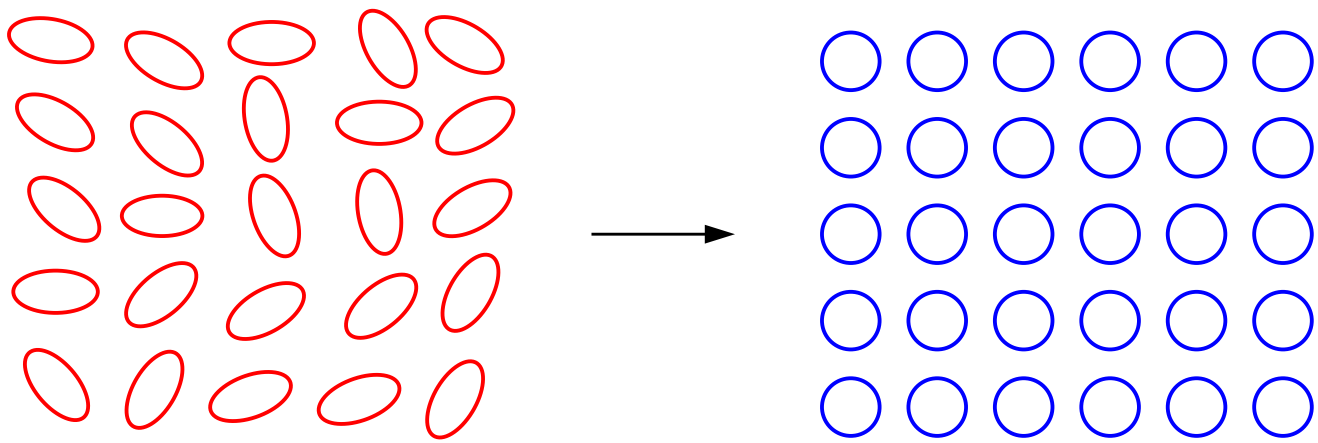
Data held as $O(n)$ Laurent series of length $p = \log \frac{1}{\epsilon}$.

Bottleneck is doing $O(1)$ FFTs per vertex of polygon.





Quasiconformal (QC) maps are homeomorphisms that are differentiable a.e. and send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

For K-QC maps, ellipses have eccentricity $\leq K$

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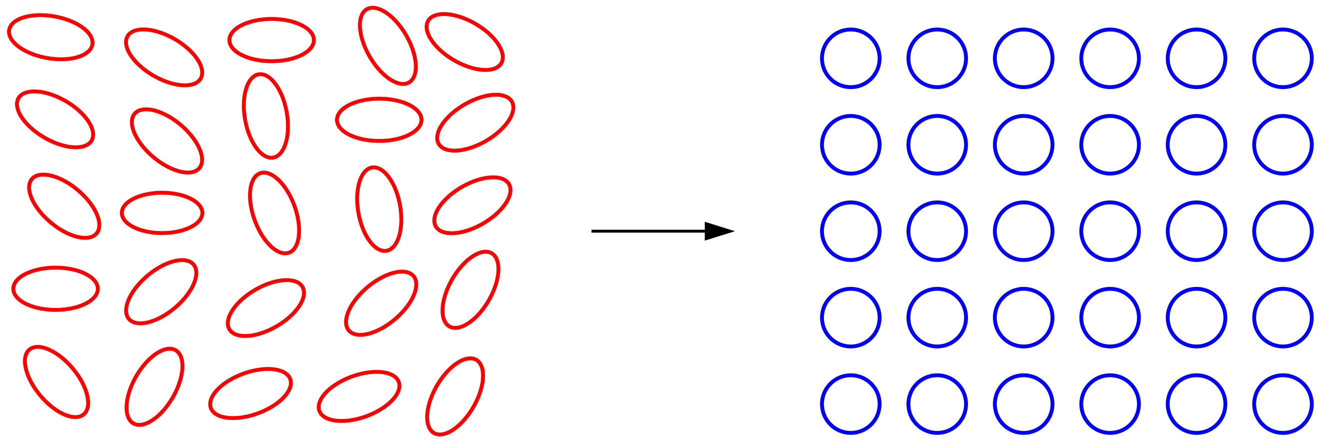

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For K-QC maps, ellipses have eccentricity $\leq K$

Ellipses determined a.e. by measurable dilatation

$$\mu = f_{\overline{z}}/f_z, \qquad f_{\overline{z}} = \mu \cdot f_z, \quad \text{with } |\mu| \leq \frac{K-1}{K+1} < 1.$$

Here $f_z = f_x - i f_y$ and $f_{\overline{z}} = f_x + i f_y$.

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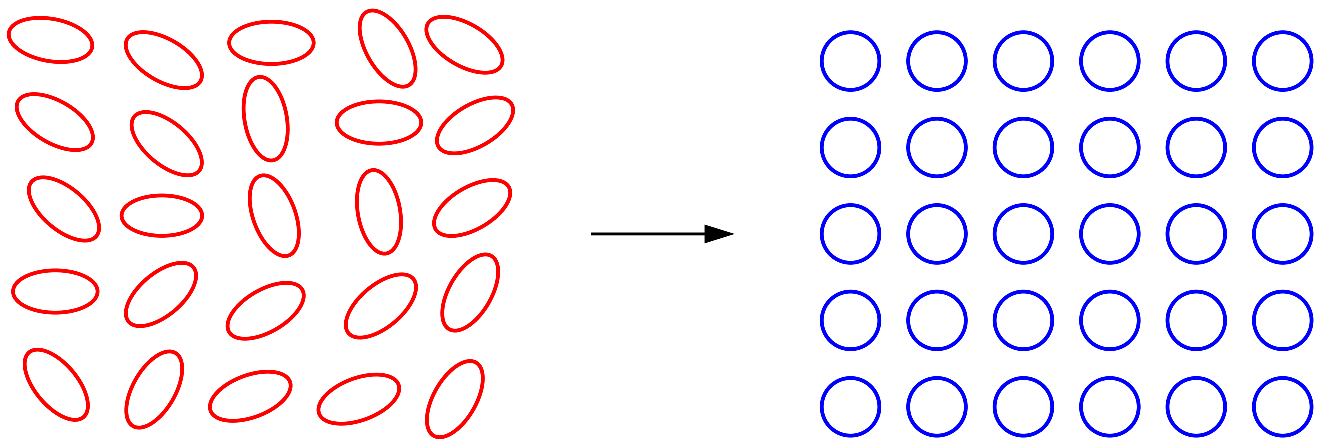
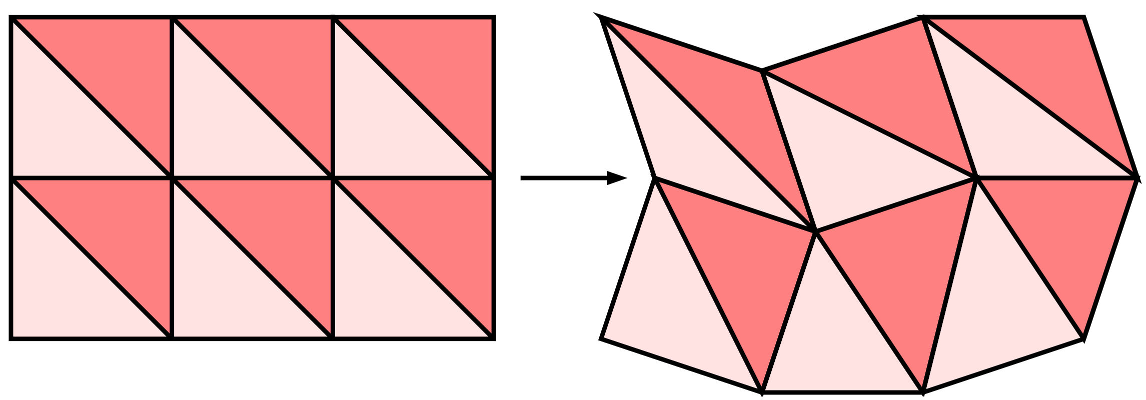
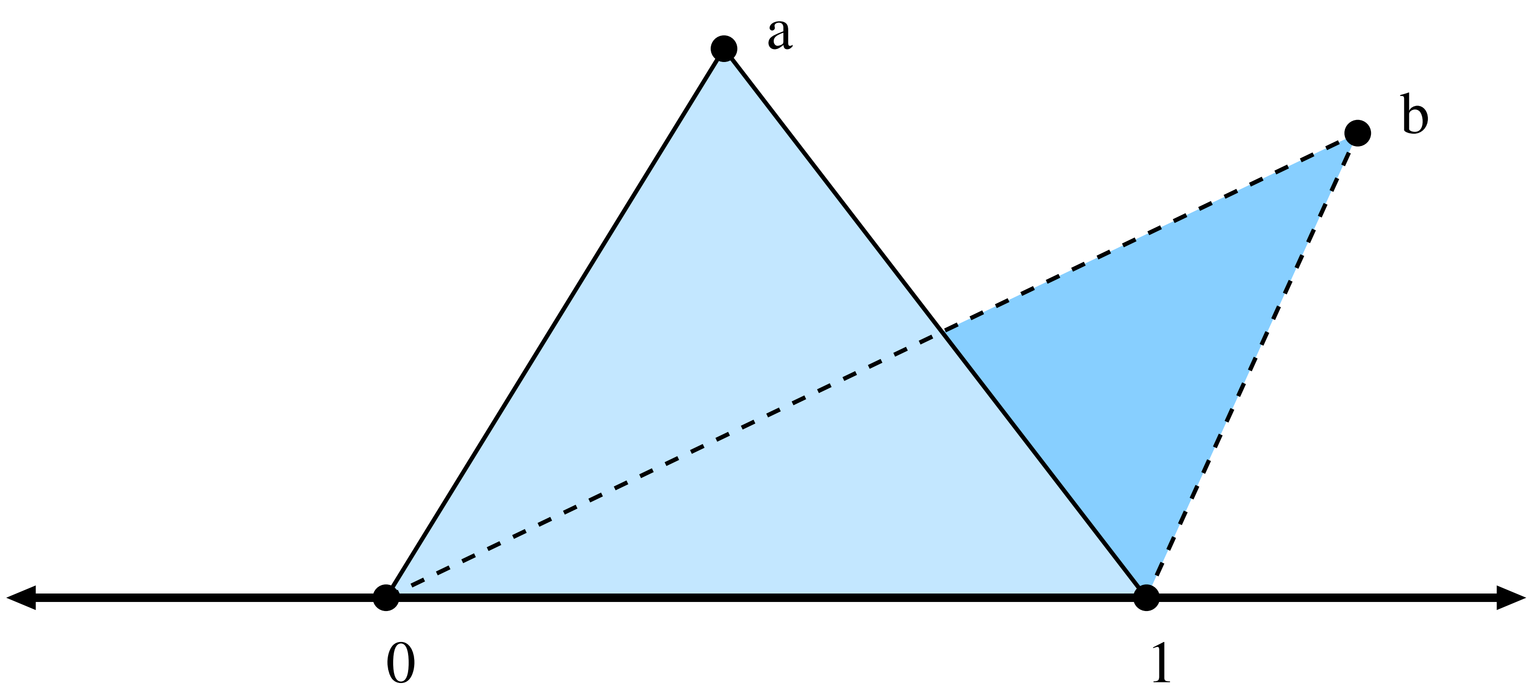


Example: piecewise affine maps between triangulations.



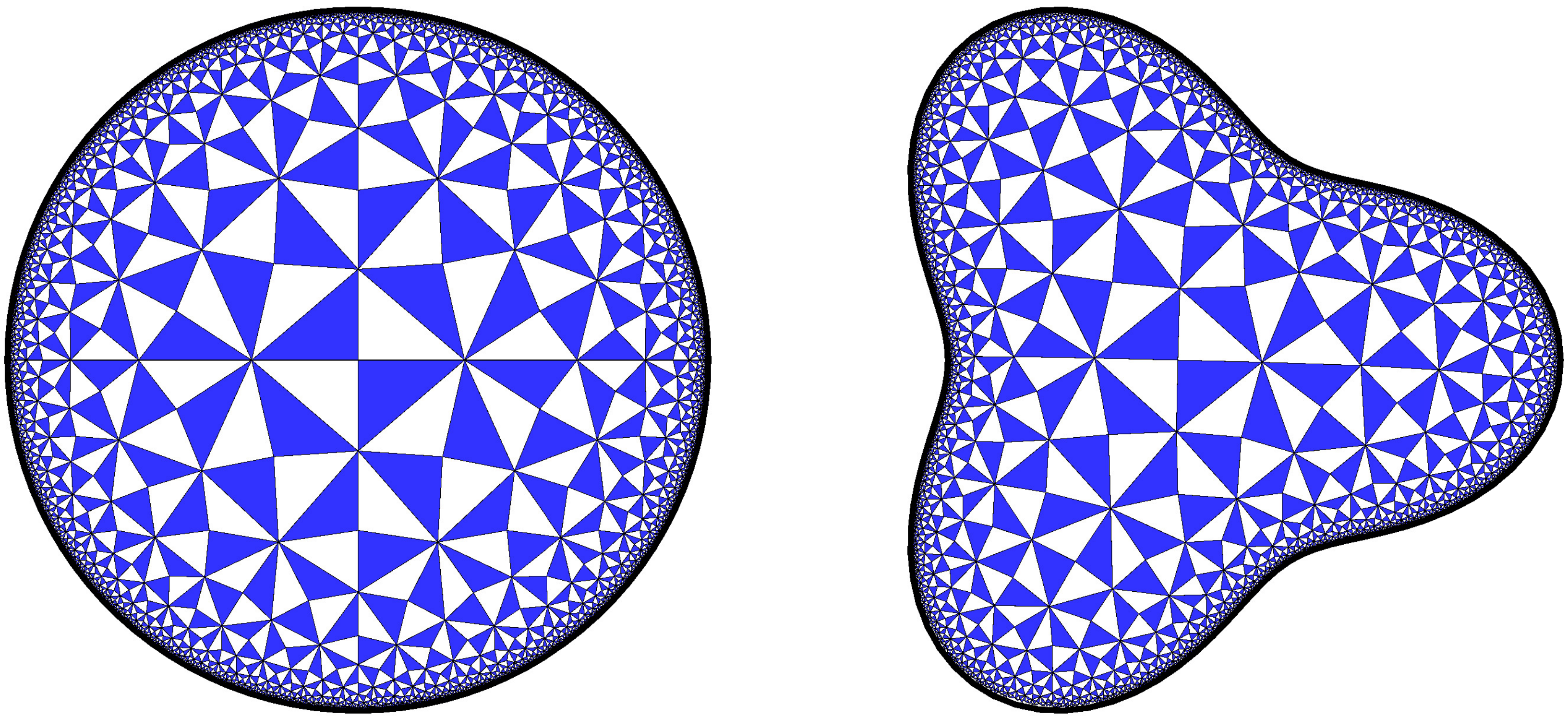
Map is QC if all angles bounded above and below.



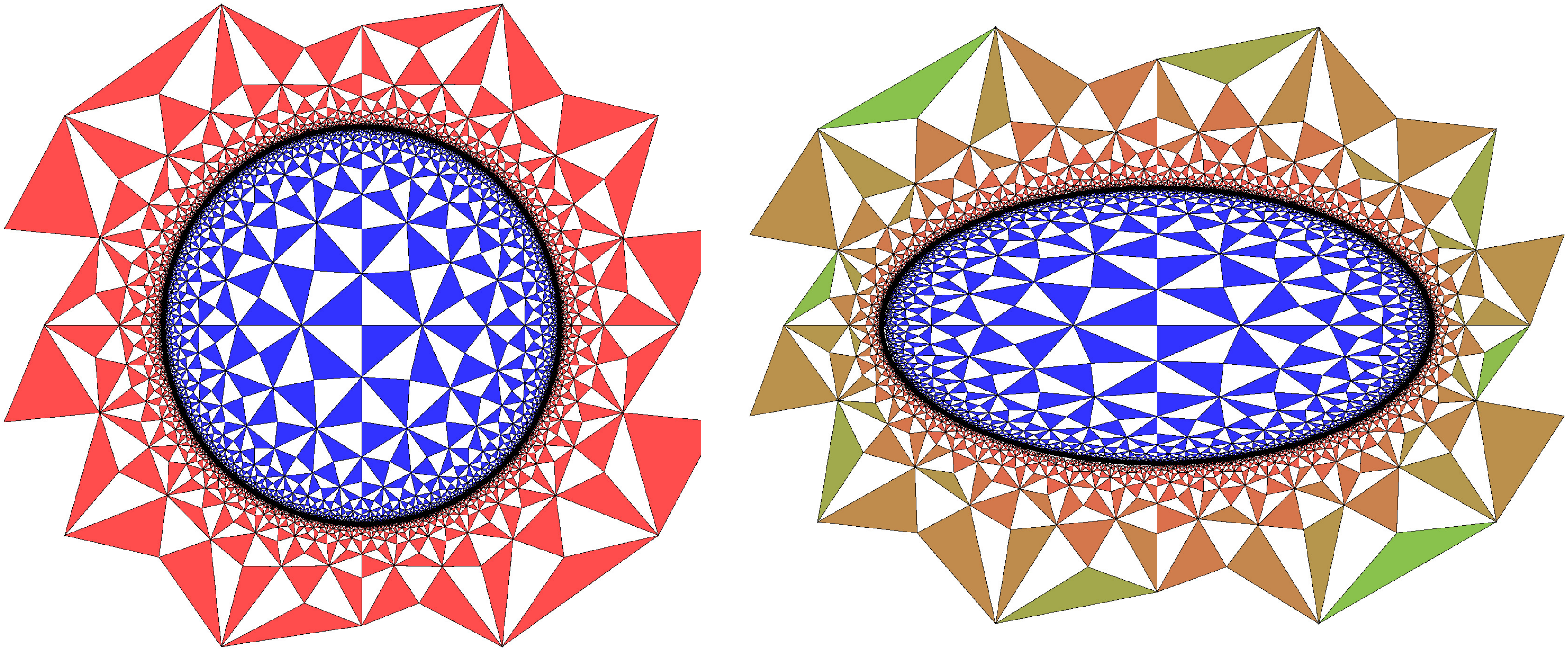
Affine map between triangles $\{0, 1, a\}$ and $\{0, 1, b\}$ has constant dilatation

$$\mu = \frac{b-a}{b-\overline{a}}$$

(For experts, this is pseudo-hyperbolic distance in upper half-plane.)



Conformal map (almost) preserves shapes of triangles



Conformal inside, quasiconformal outside.

Distortion of triangles indicated by change in color.

Quasiconformal = bounded angle distortion.

Every quasiconformal map f has a dilatation μ with $\|\mu\| < 1$.

Amazingly, the converse is also true:

Measurable Riemann Mapping Theorem:
Given a measurable μ on $\mathbb{D}$ with $\|\mu\|_\infty < 1$, there is a quasiconformal $f : \mathbb{D} \to \mathbb{D}$ so that $\mu = f_{\overline{z}}/f_z$ (solves Beltrami equation).

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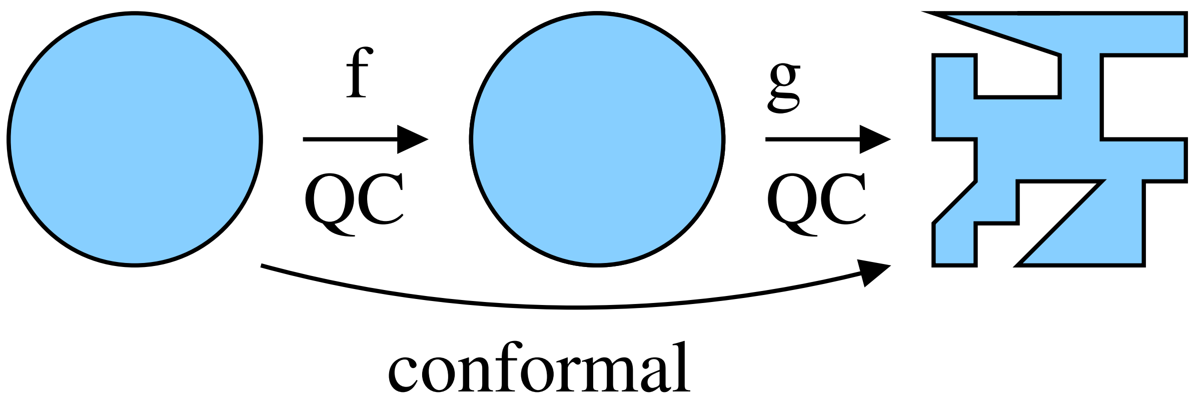
- Exact solution by power series of singular integral operators.
- Linearization can be solved by convolution with $1/z$.
- Newton's method: solve linear approximation, compute new μ, repeat.
- Converges if $\|\mu\|_\infty \le \epsilon_0$.

Other methods for solving too, e.g., Daripa 1993.

Corollary:

Given QC $g : \mathbb{D} \to \Omega$, there is $f : \mathbb{D} \to \mathbb{D}$ so that $g \circ f$ is conformal.



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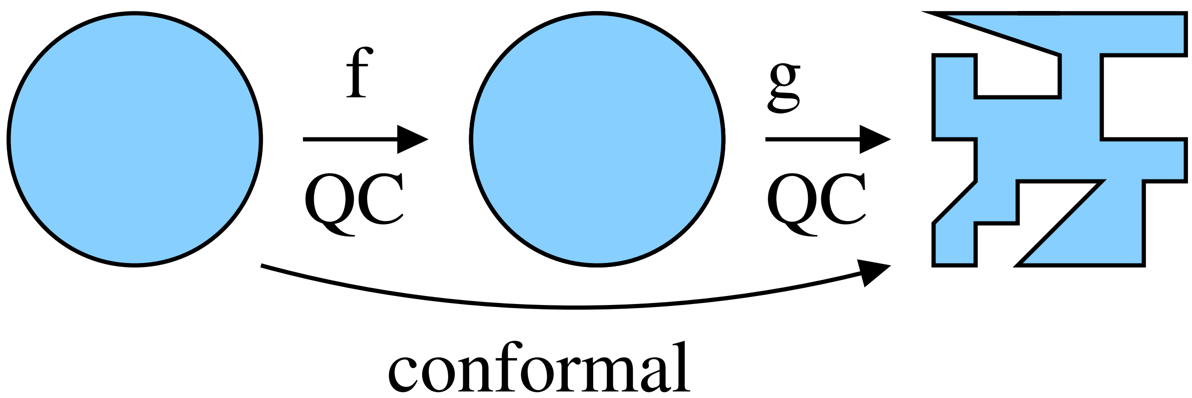

Fast mapping theorem reduces to two steps:

- Find initial QC map g to polygon.

- Solve $f_z \cdot \mu_g = f_{\overline{z}}$ to get $f : \mathbb{D} \to \mathbb{D}$.

We ignore 2nd part; just find good g.

A “good” g is **fast** to compute and guaranteed **close** to correct answer.

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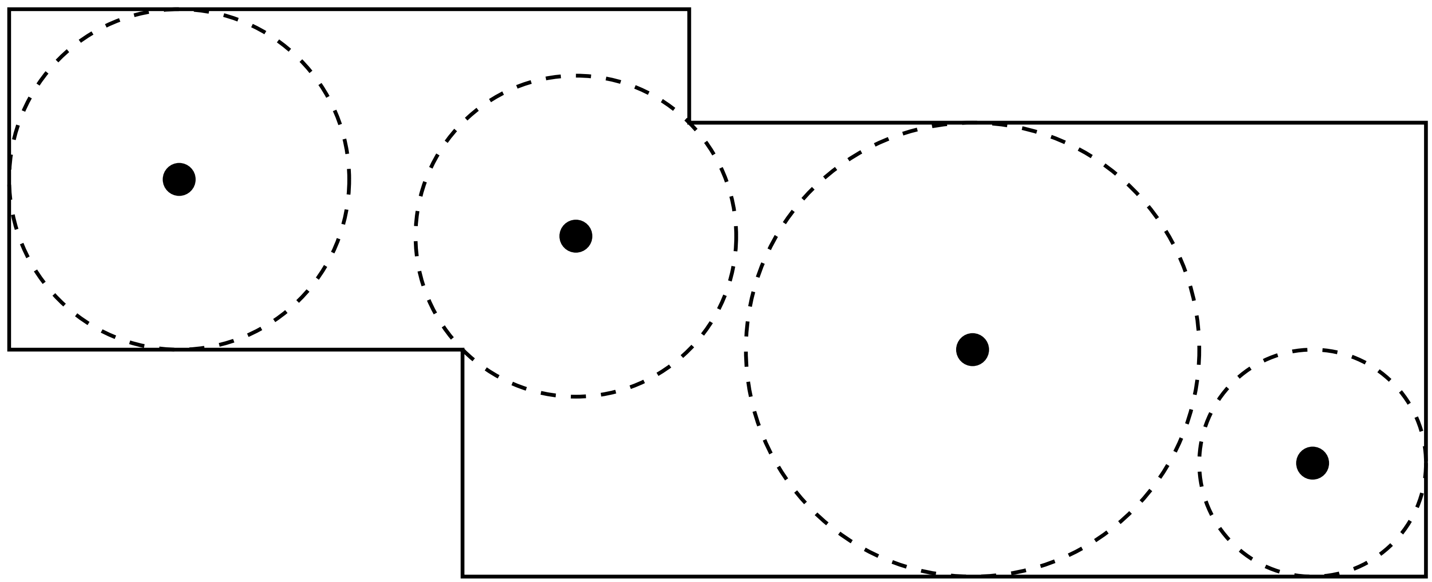
- **fast** comes from computational geometry.
- **close** comes from hyperbolic geometry.

Medial axis:

centers of disks that hit boundary in at least two points.



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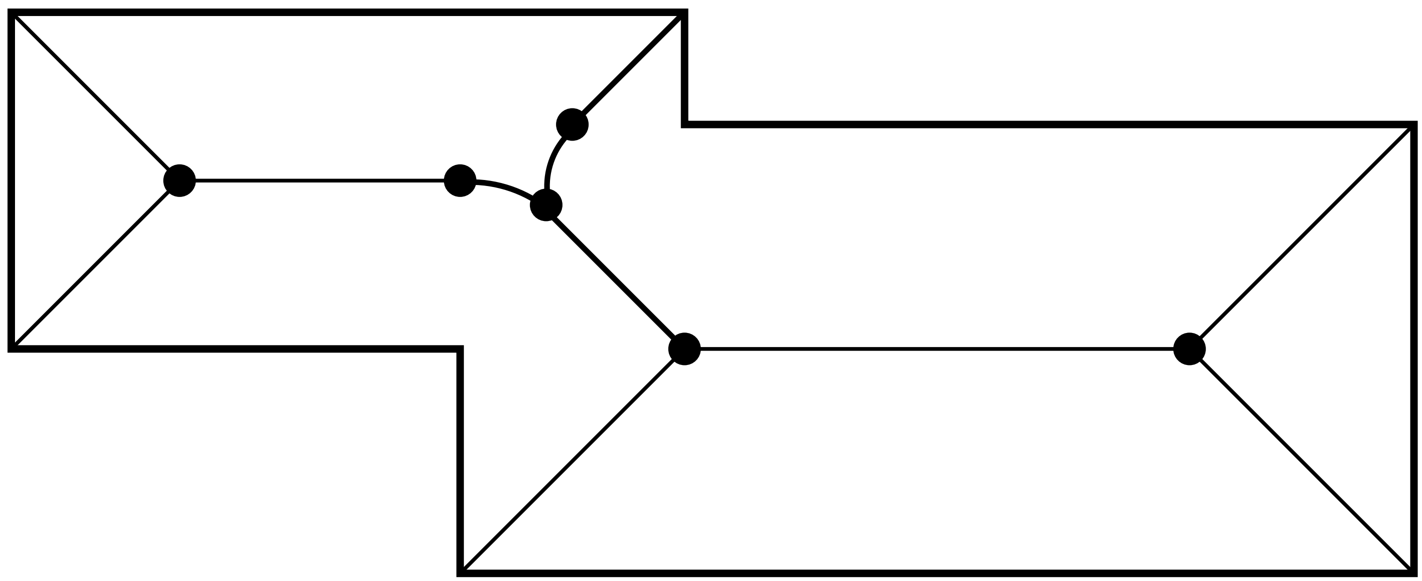
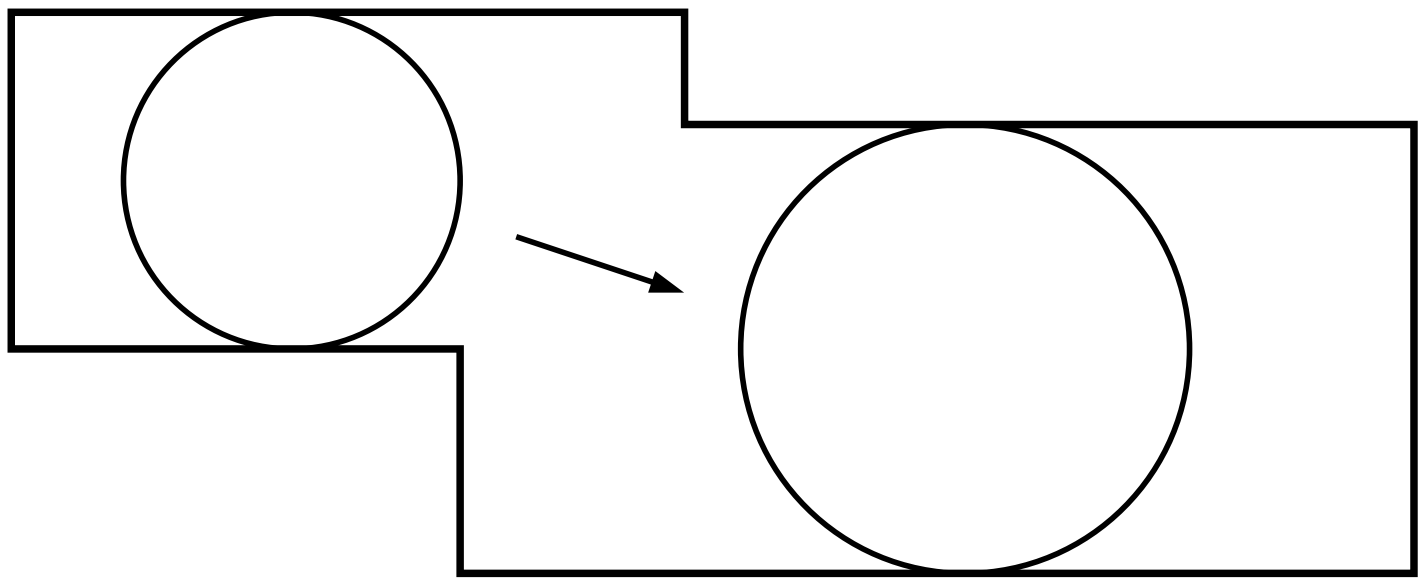


Medial axis of a polygon P is a finite tree.

Computable in O(n), Chin-Snoeyink-Wang (1999).

Voronoi diagram: divides P according to nearest concave arc.

Medial axis:
centers of disks that hit boundary in at least two points.



Claim: there is a "natural" map between any two medial axis disks.

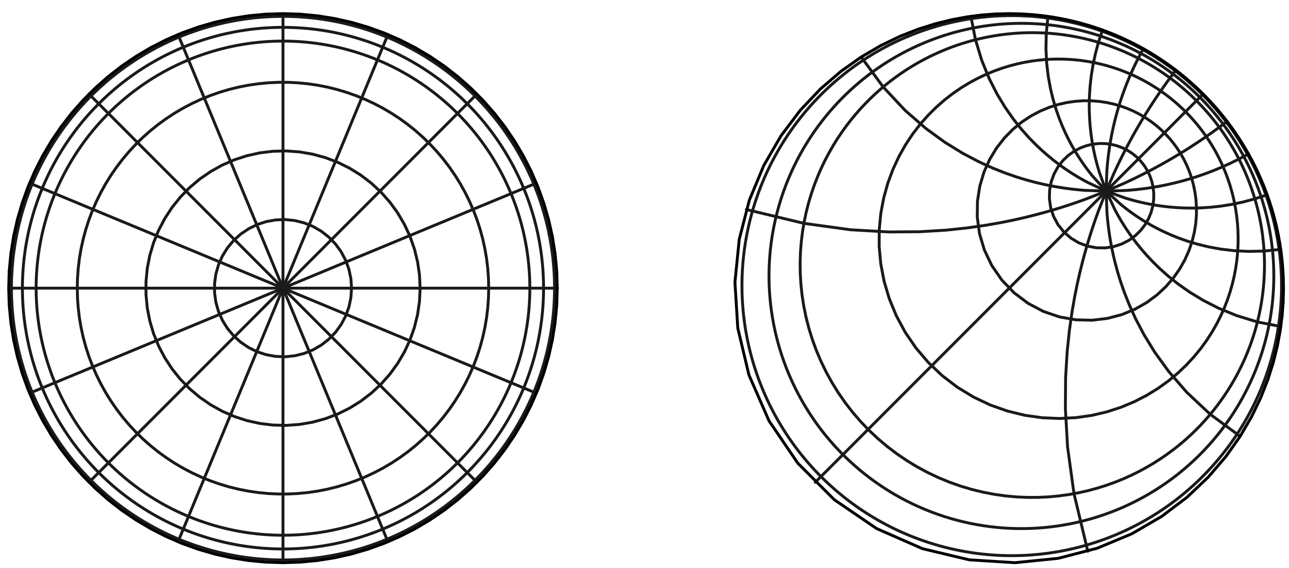
A **Möbius transformation** is a map of the form

$$z \to \frac{az+b}{cz+d}.$$

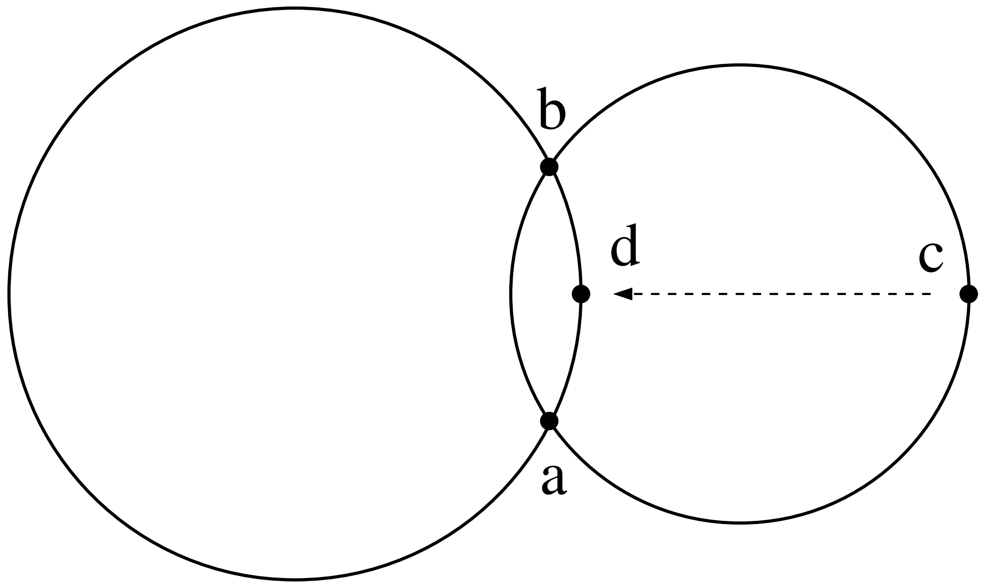
Conformally maps disks to disks (or half-planes).

Form a group under composition.

Uniquely determined by images of 3 distinct points.



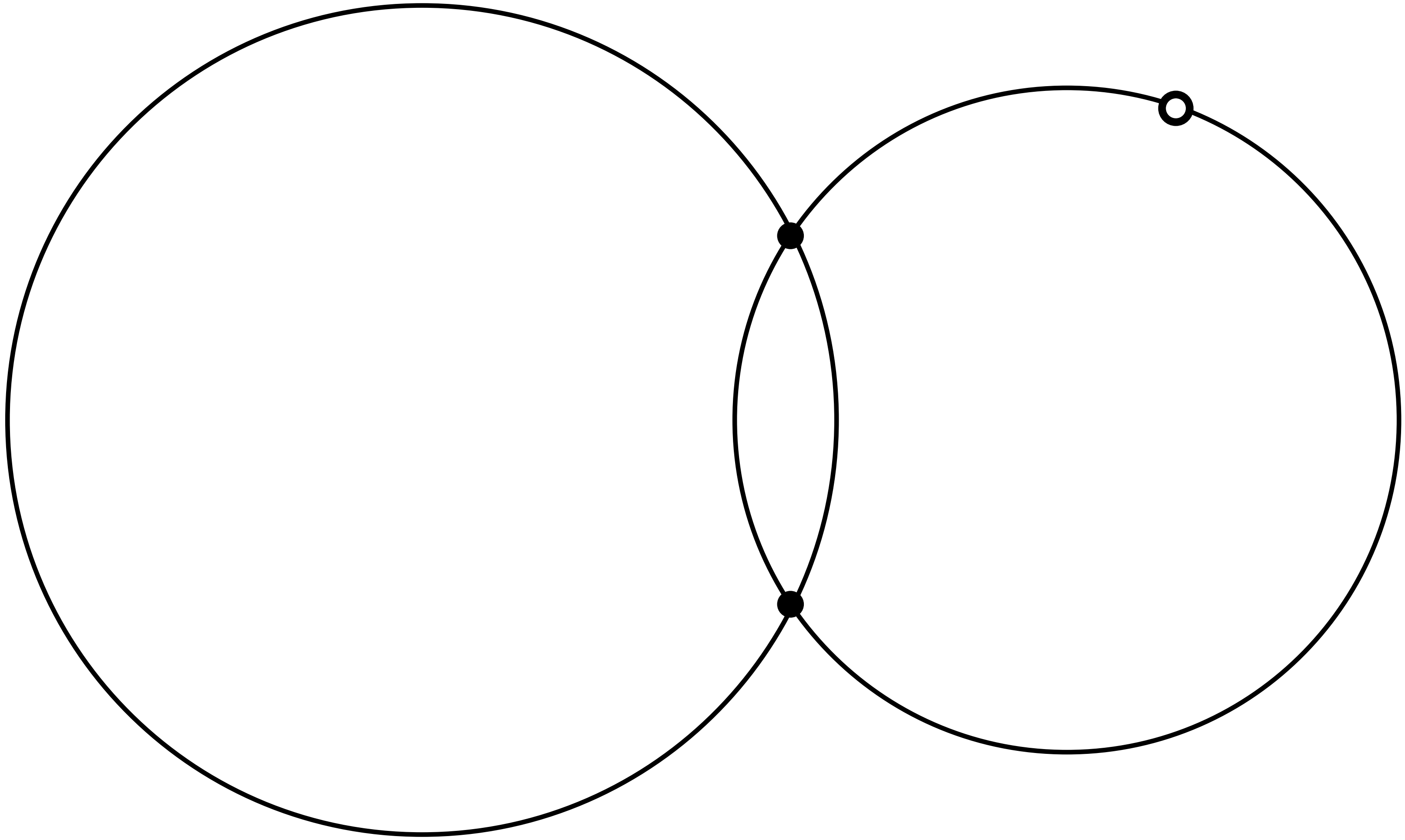
Intersecting circles:

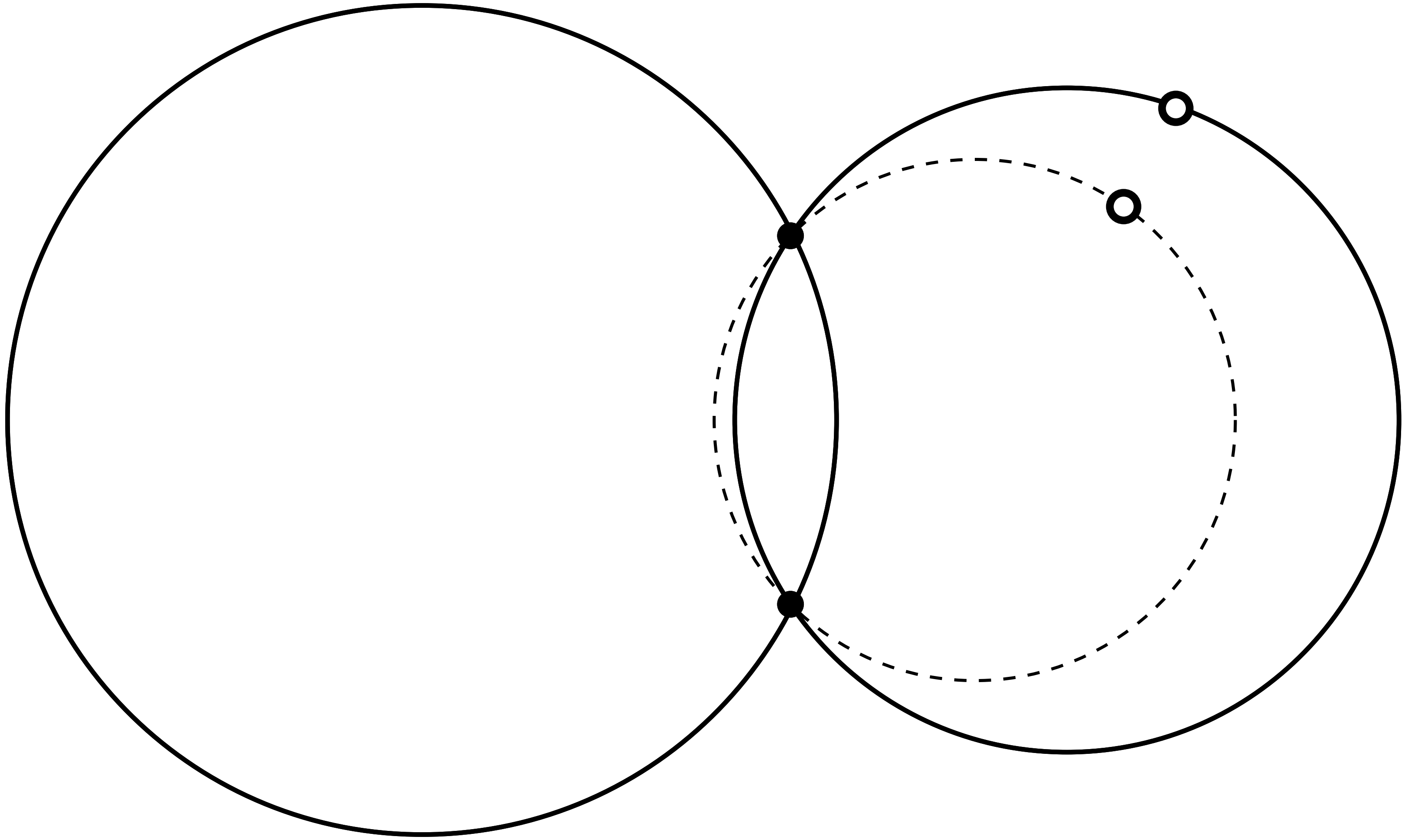


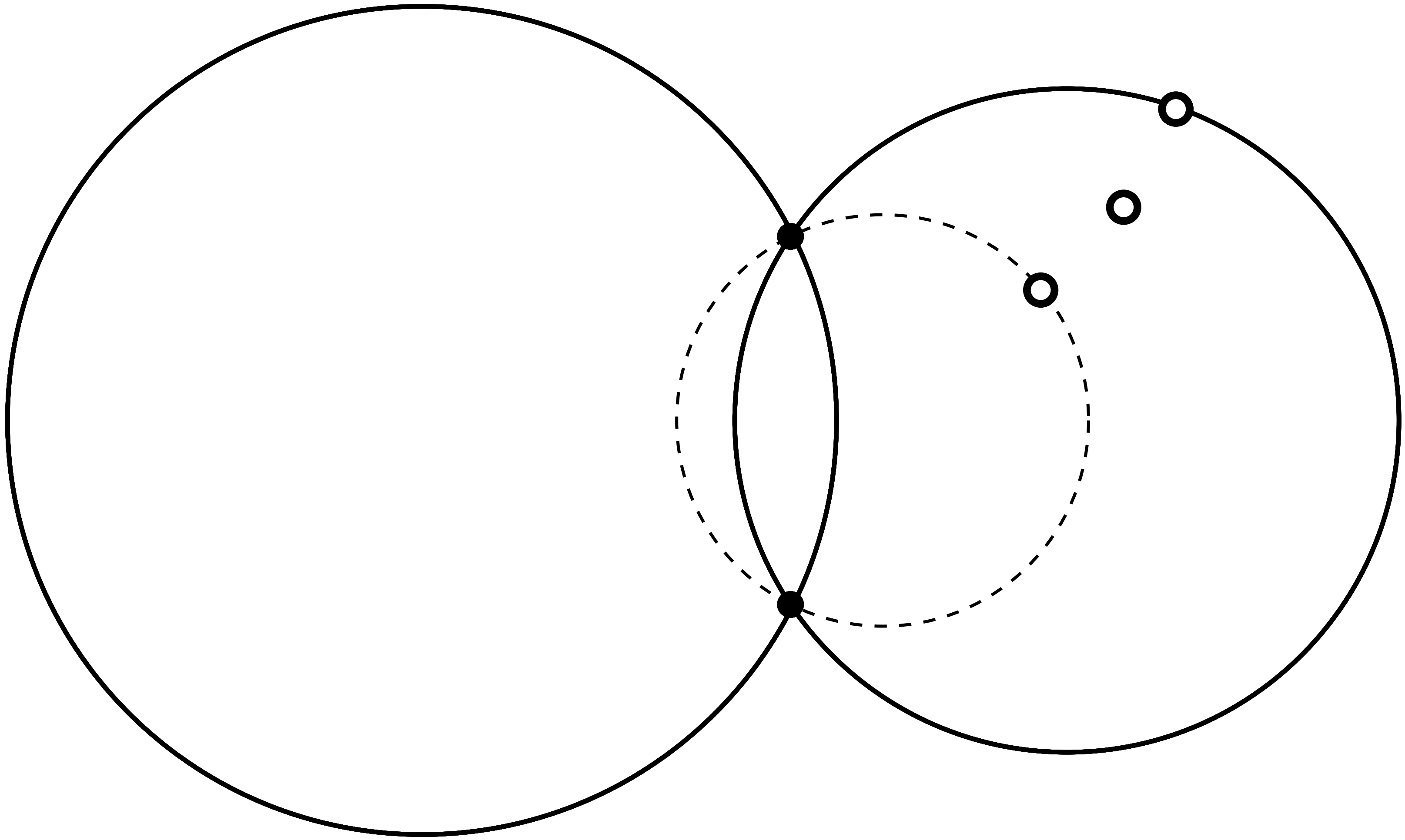
Fix intersection points a, b and map $c \to d$ as shown.

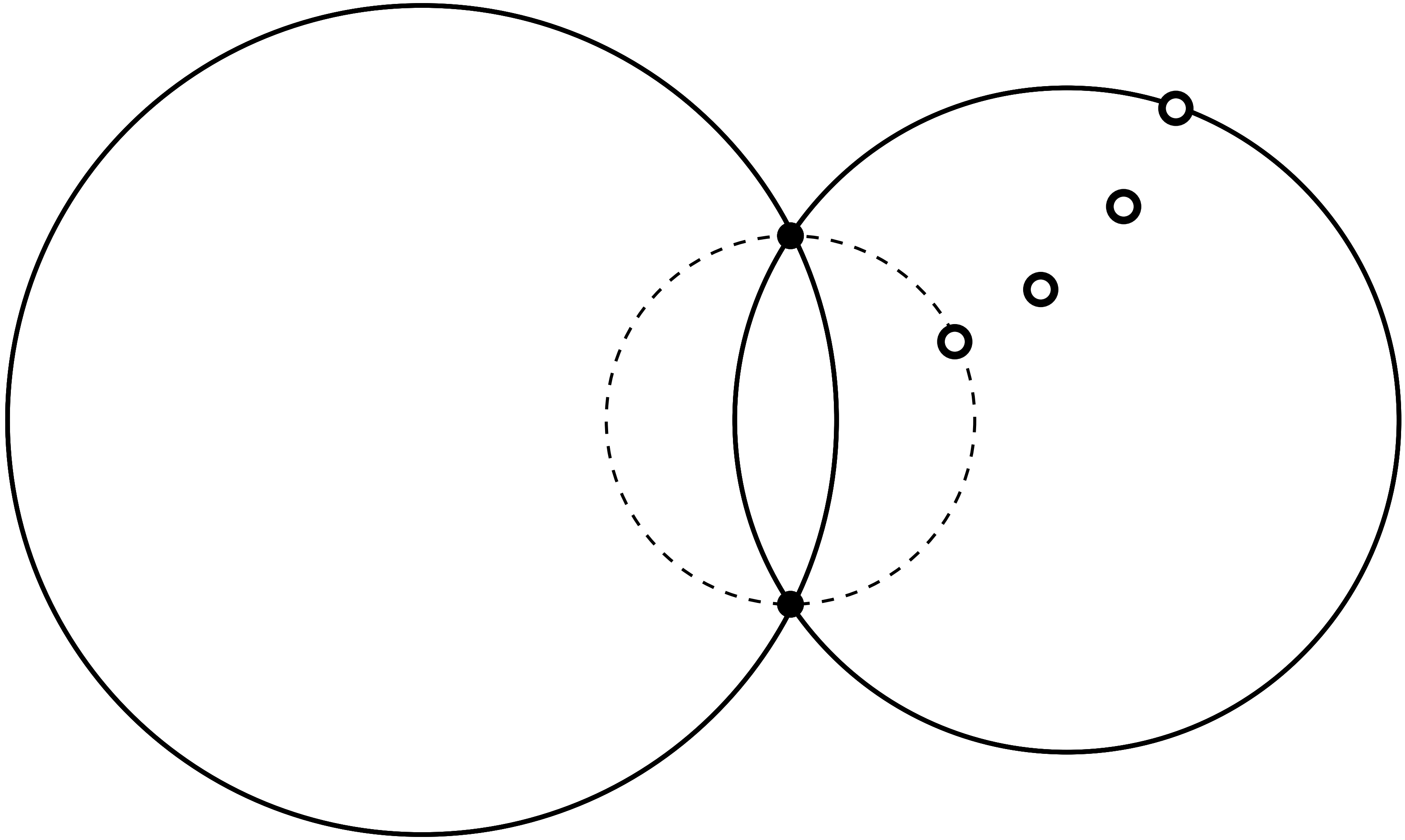
Determines unique Möbius map between disks.

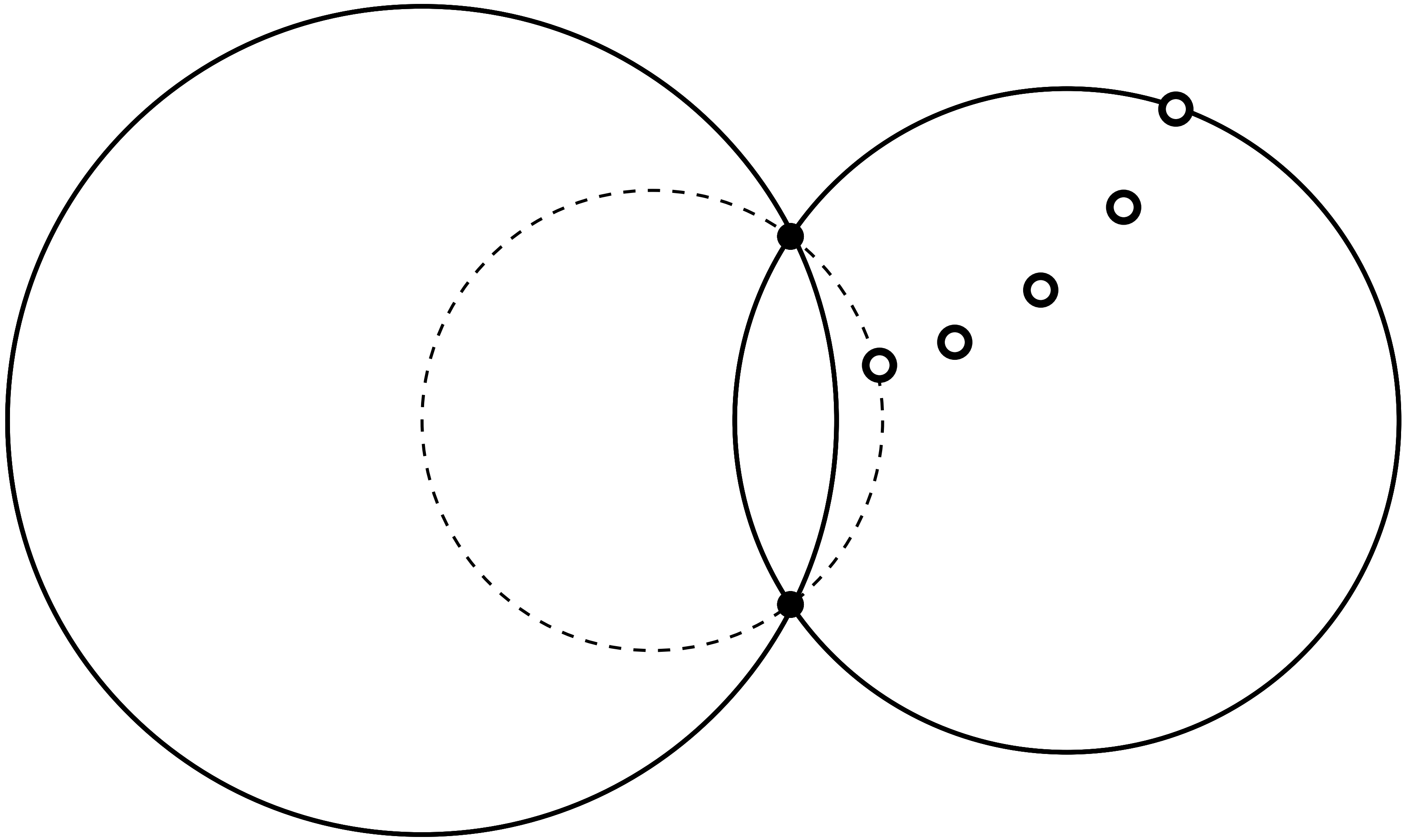
Part of 1-parameter symmetric family fixing a, b.

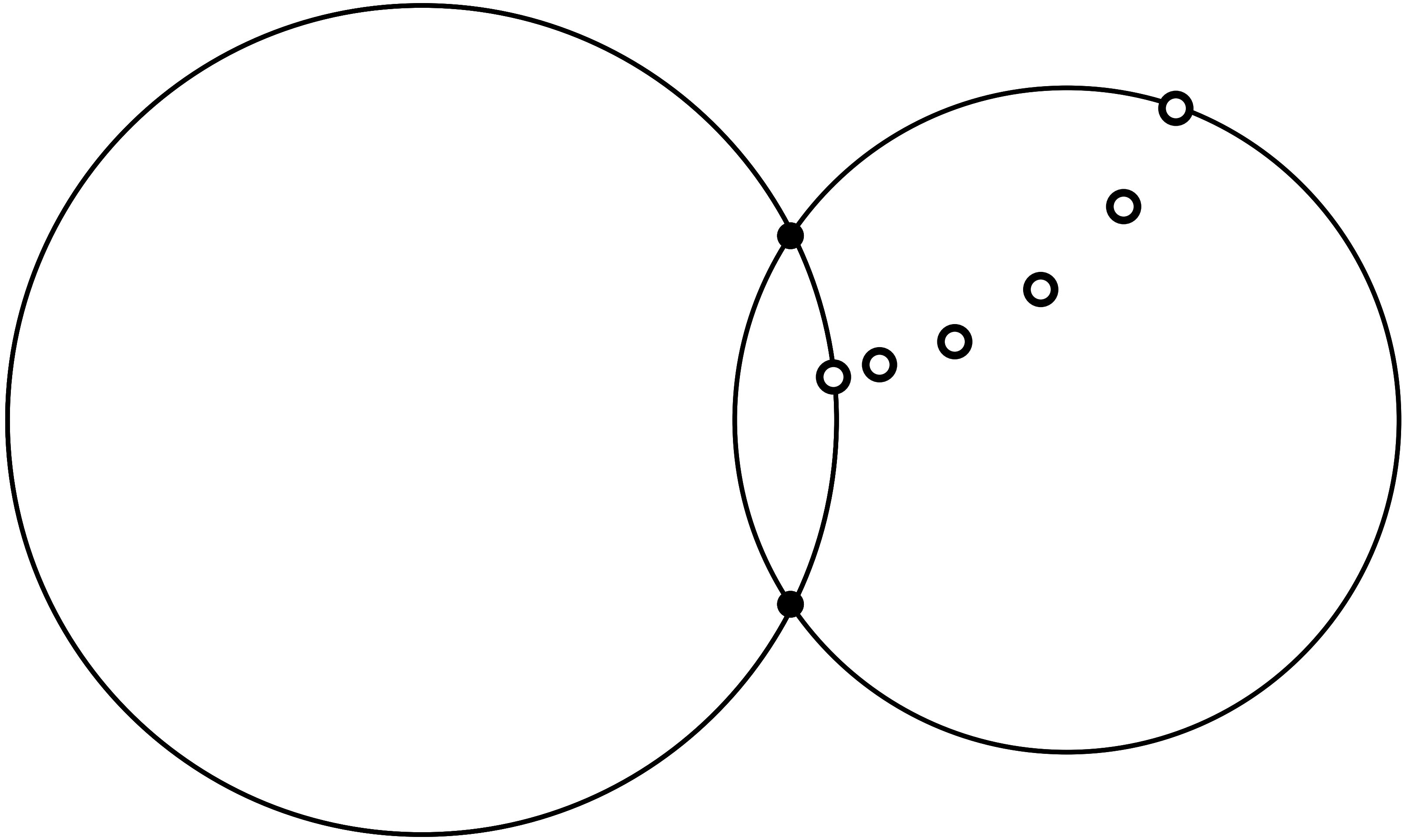


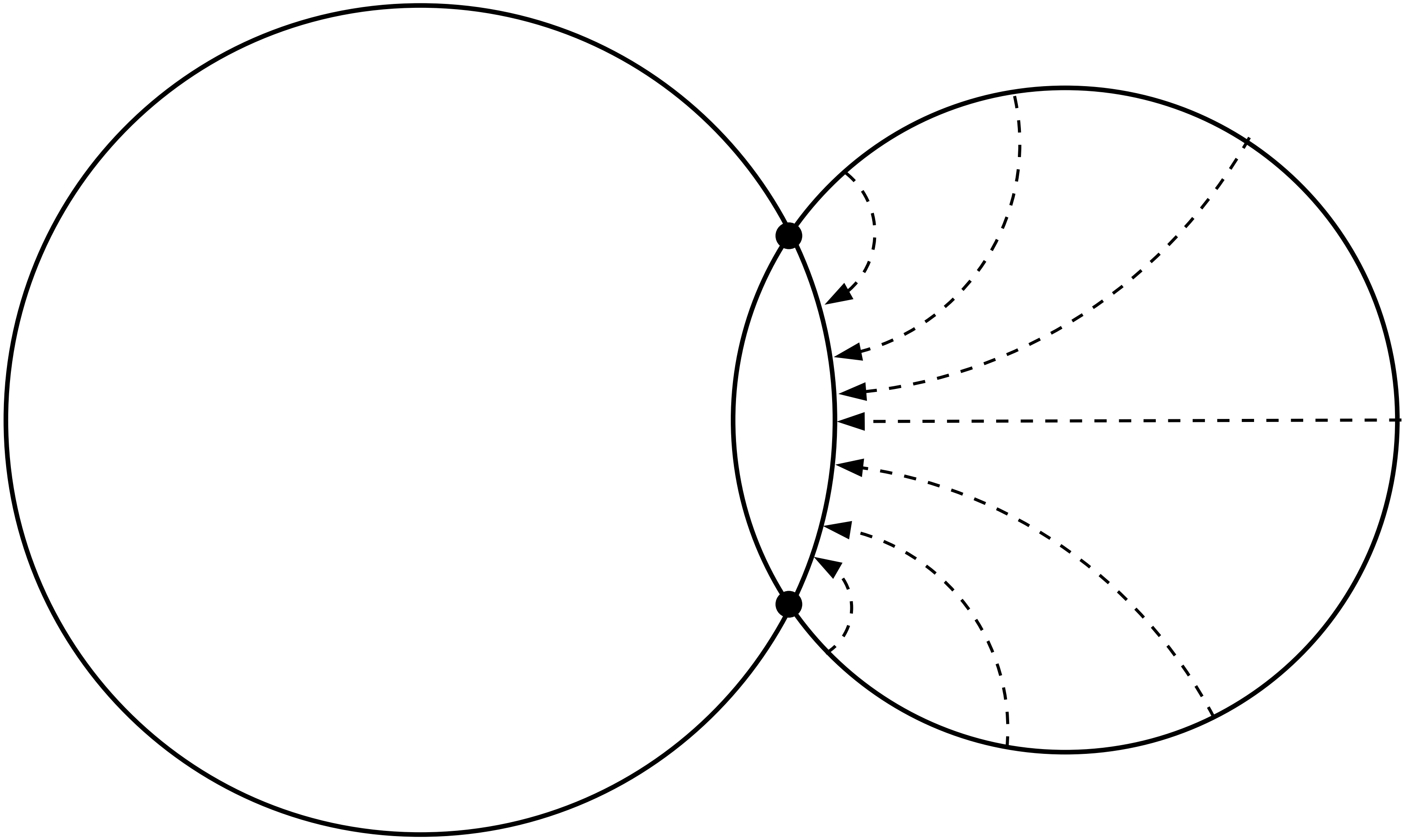






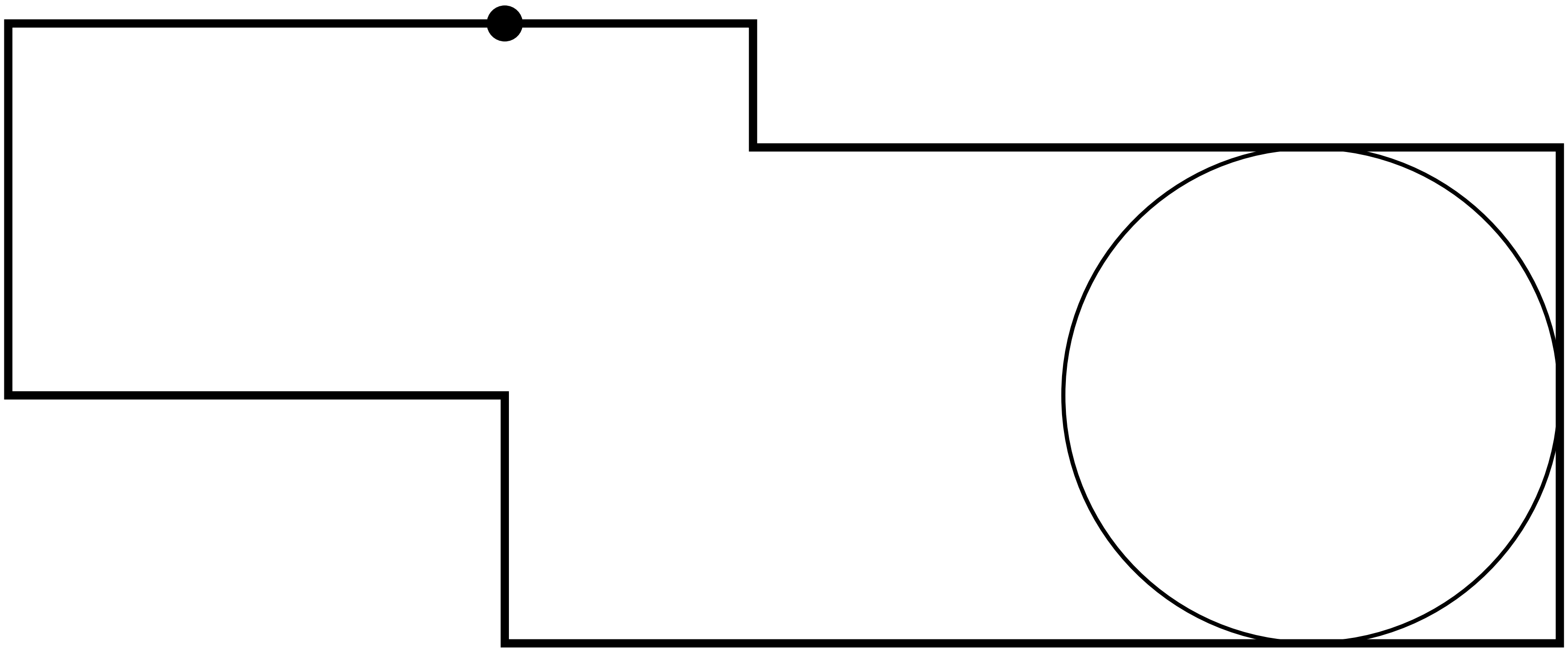




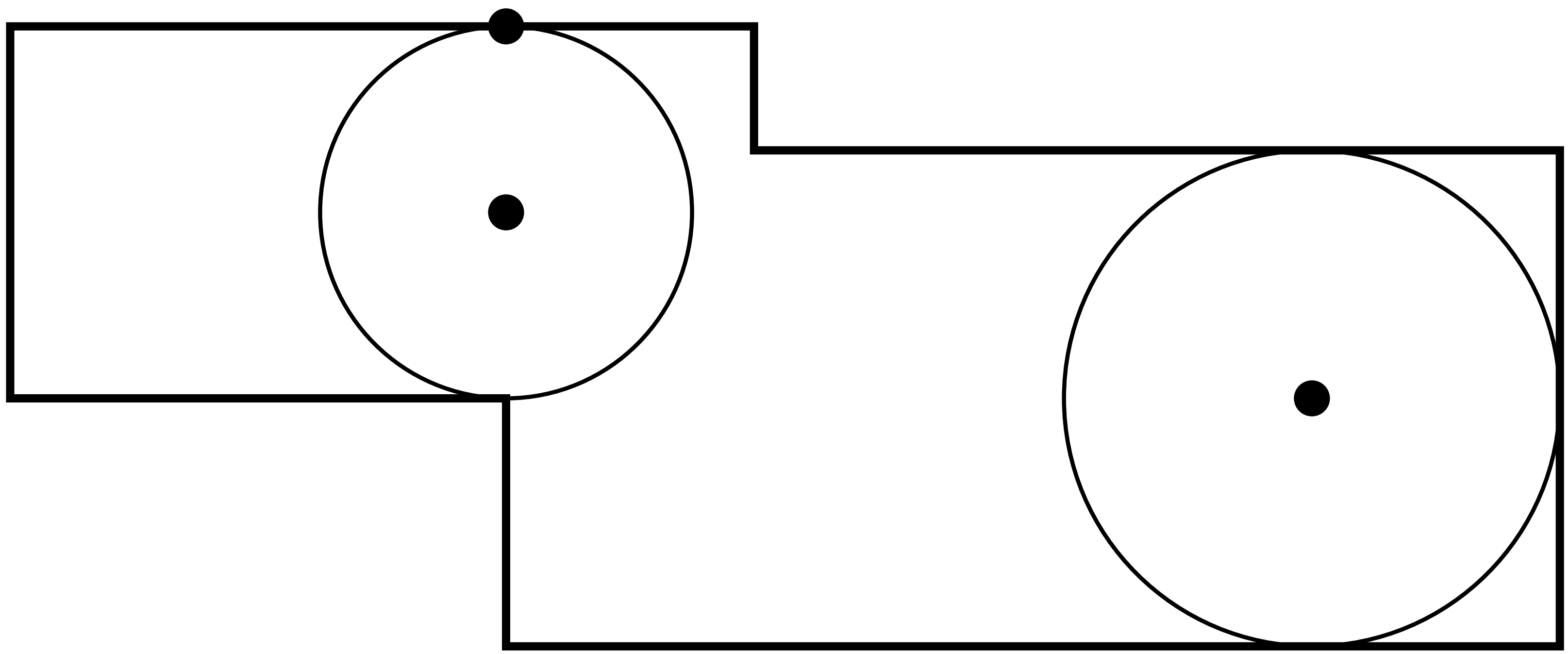


Points follow circular paths, perpendicular to boundary.

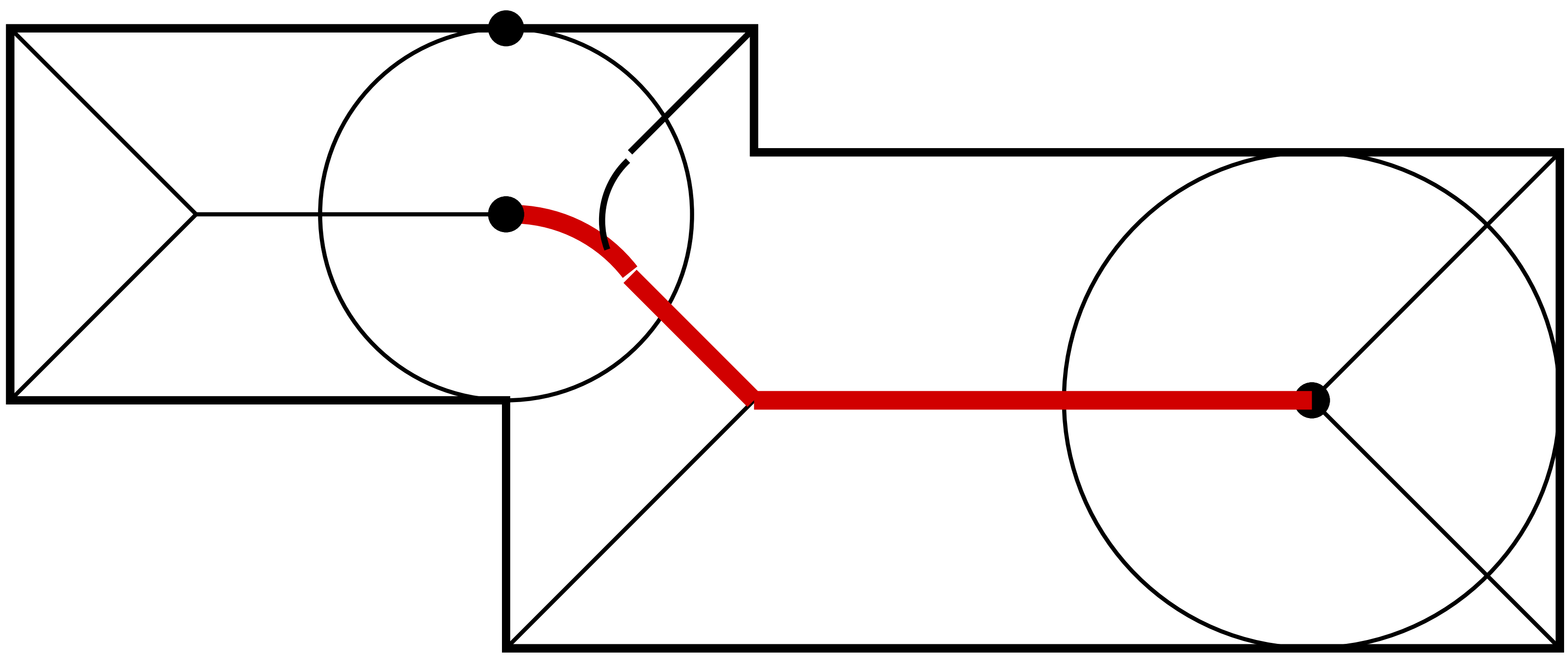
How does this give a map from polygon P to a circle?



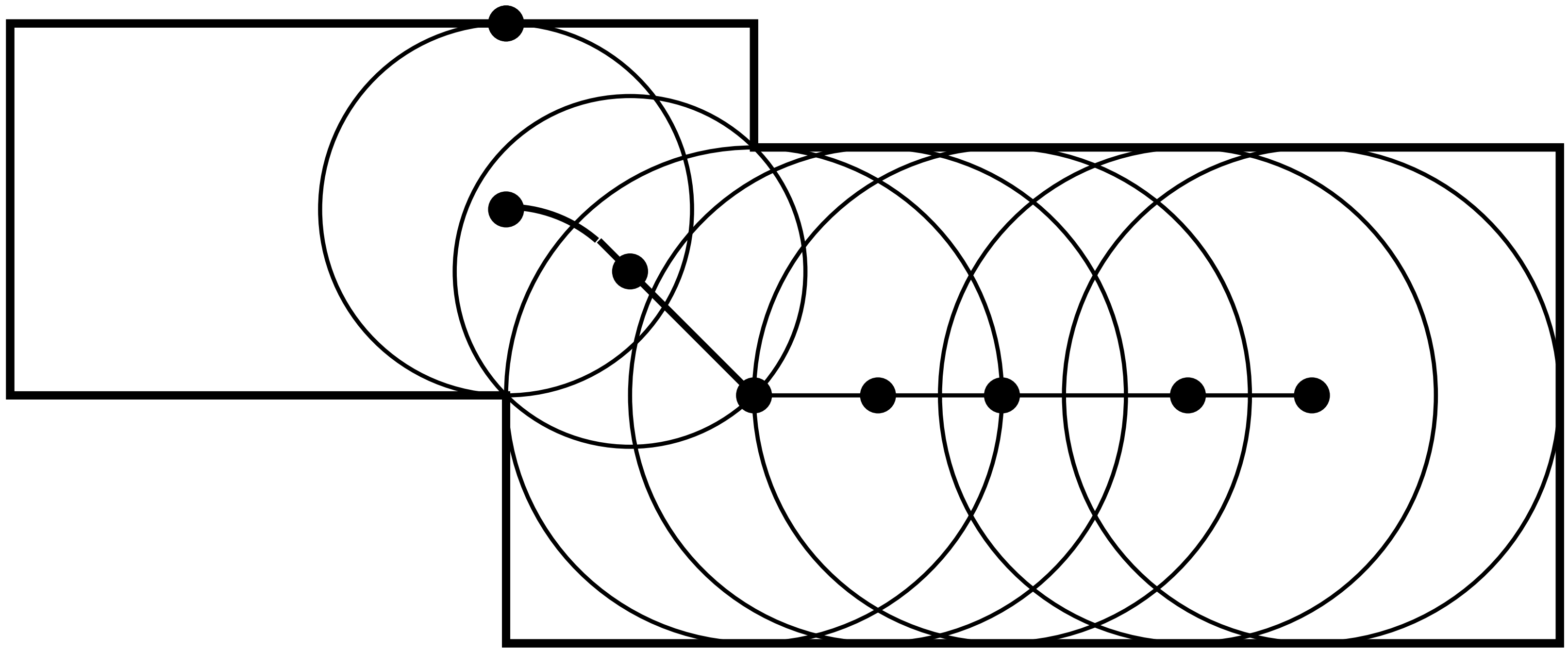
- Fix a “root” MA disk D.

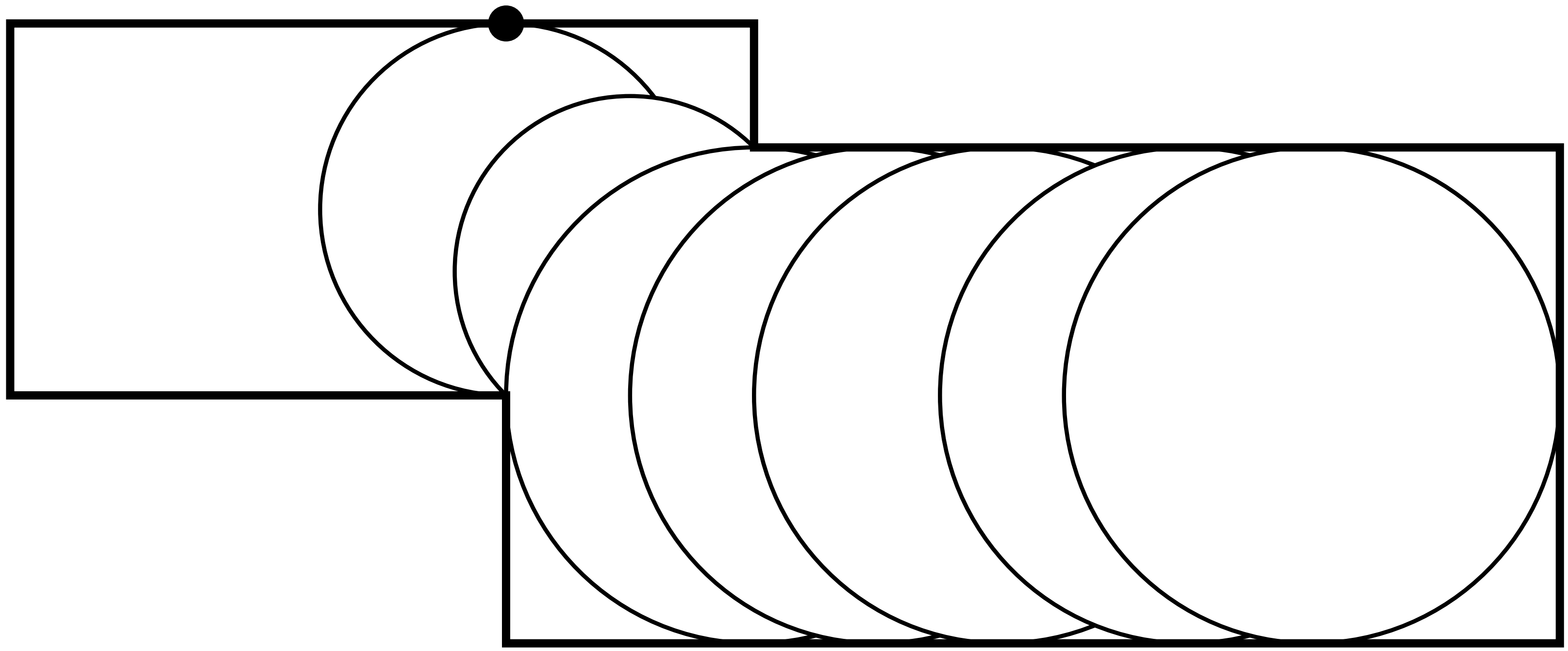


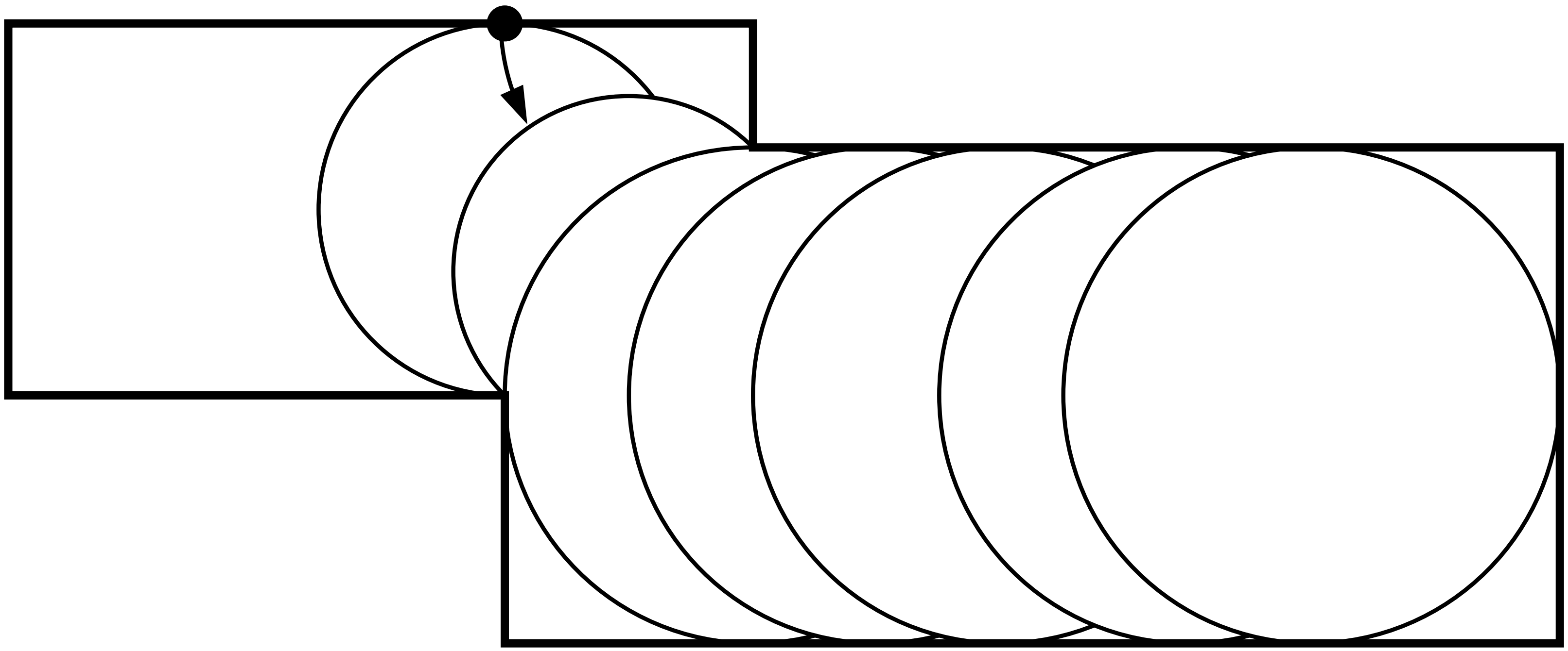
- For any $z \in P$, take MA disk D_z touching z.

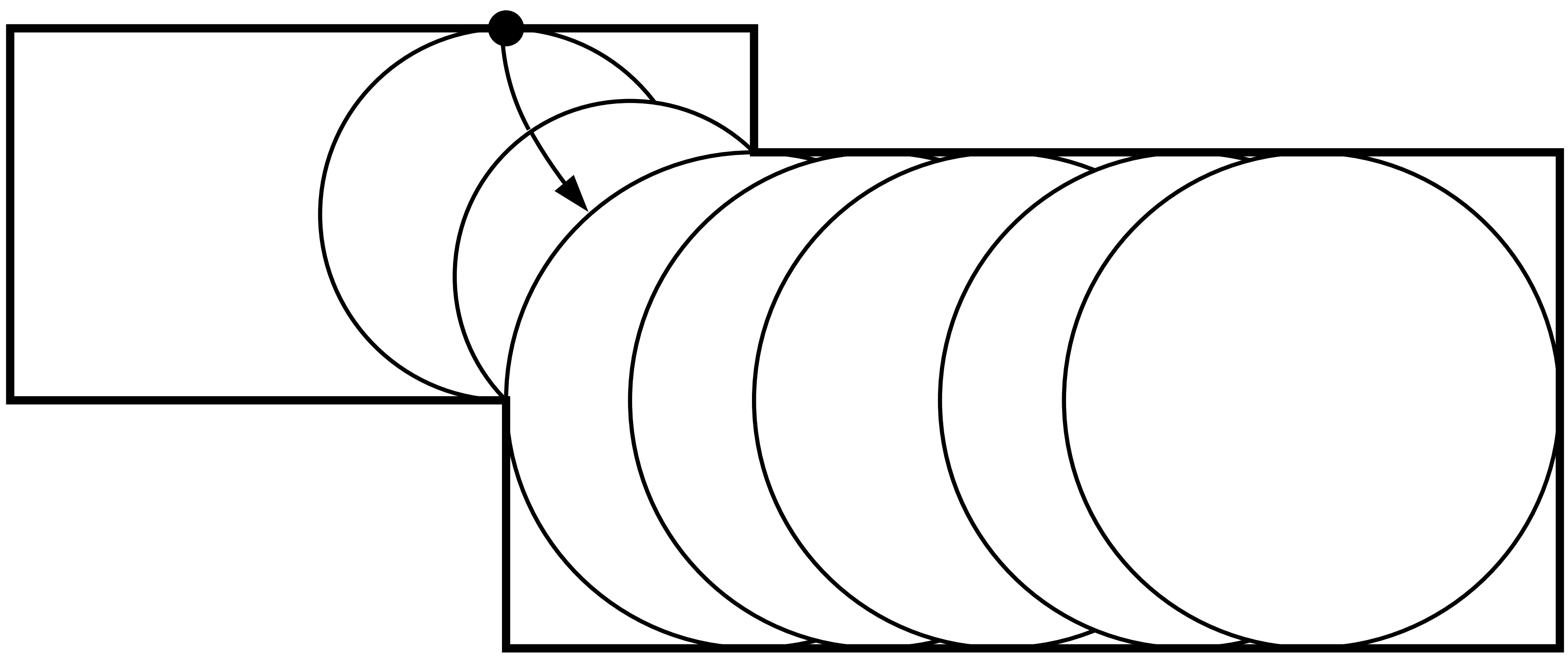


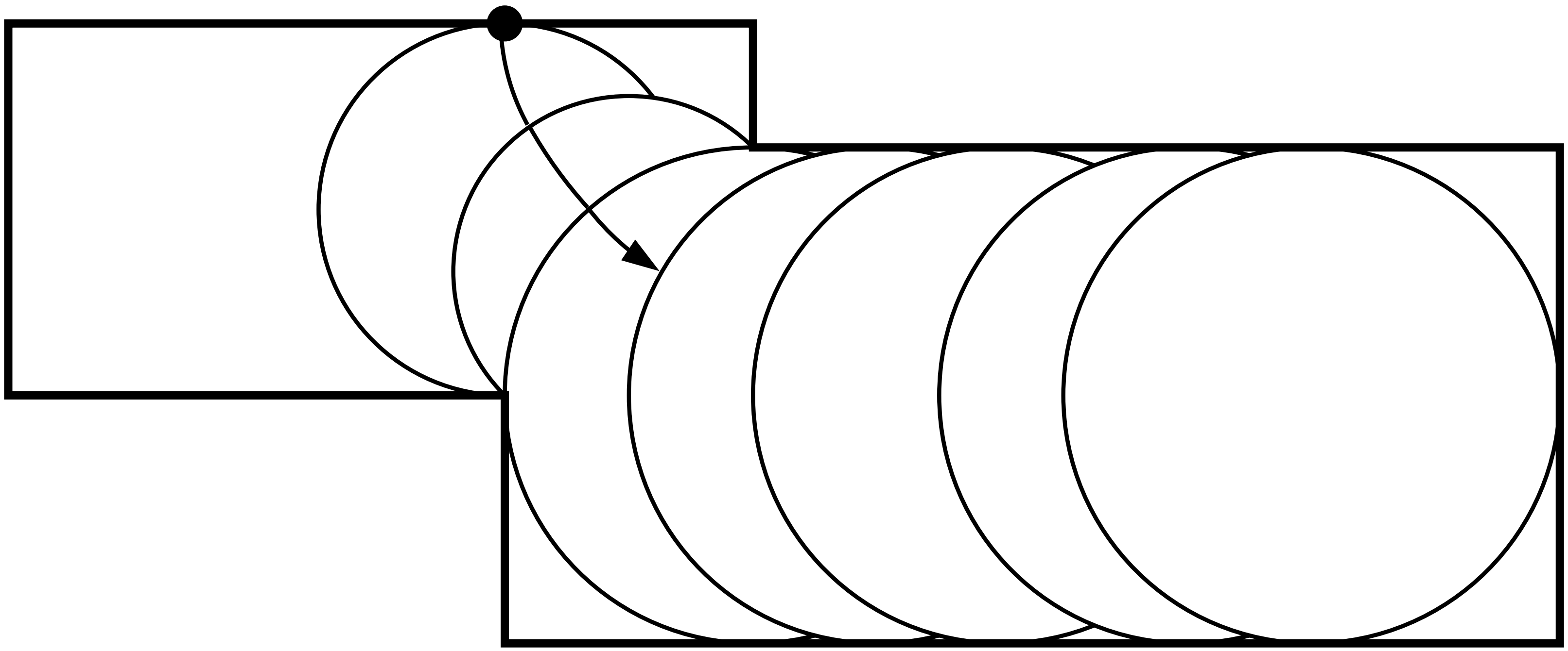
- Connect D_z to D on MA.

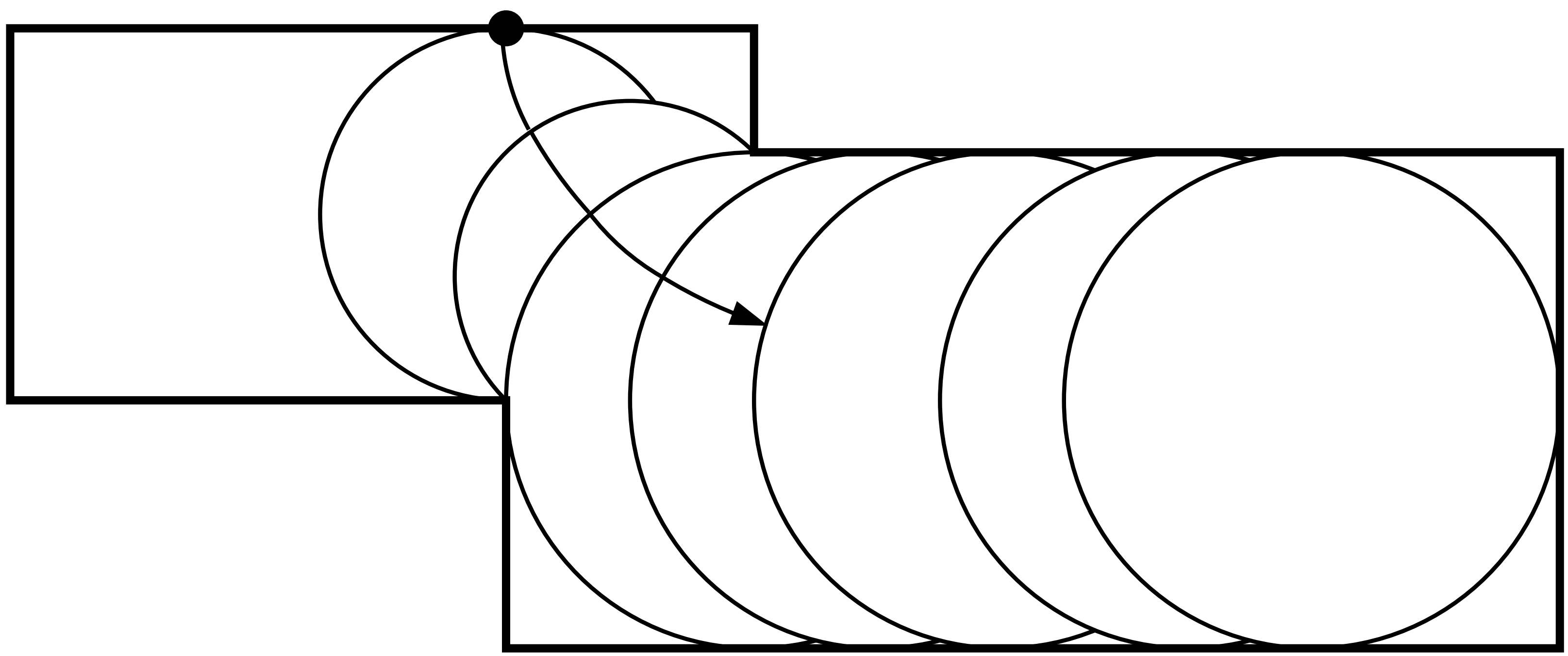


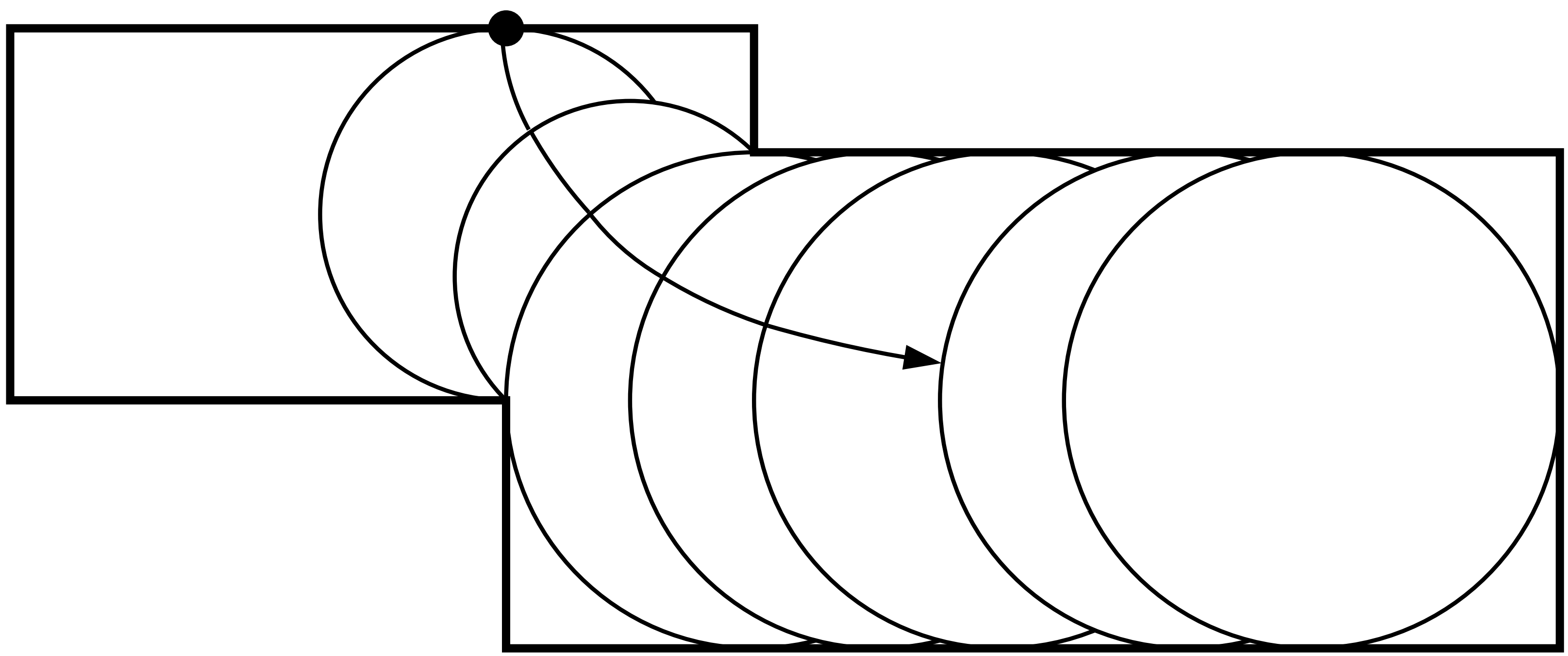


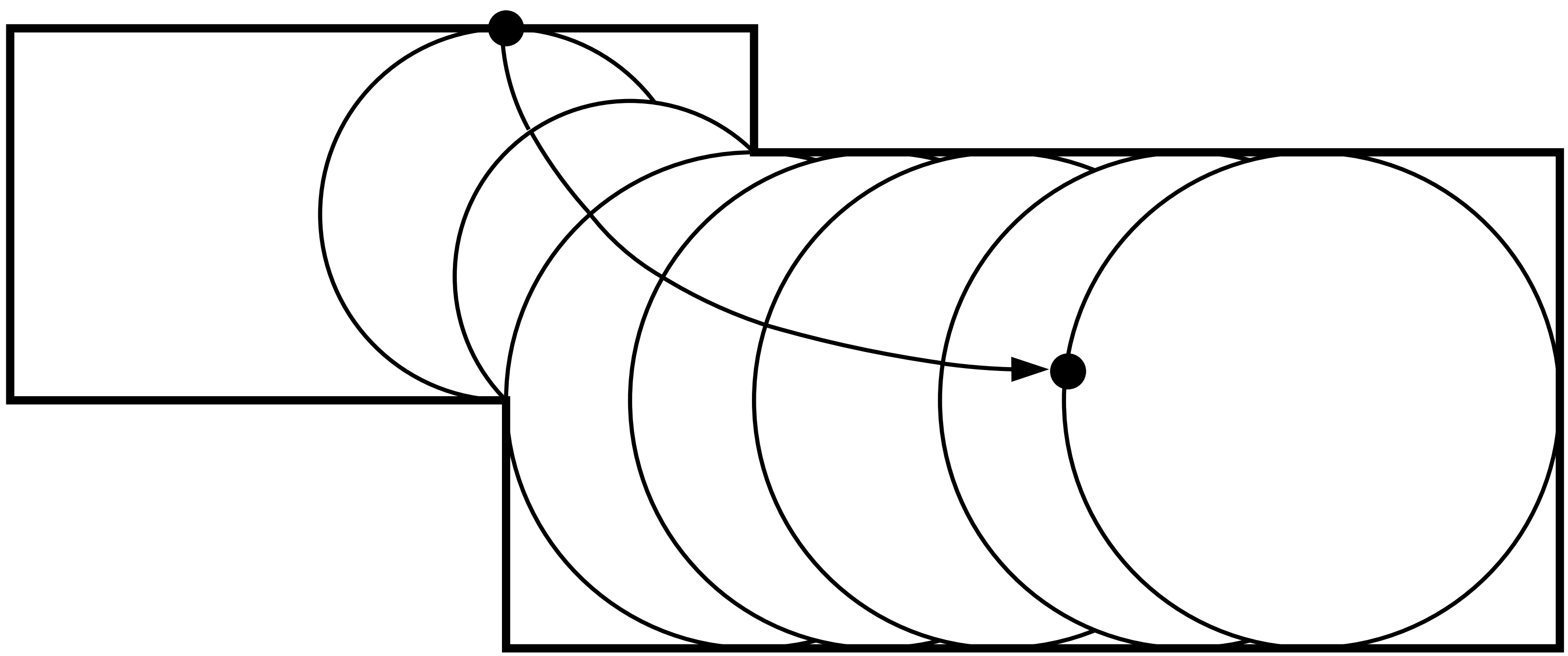








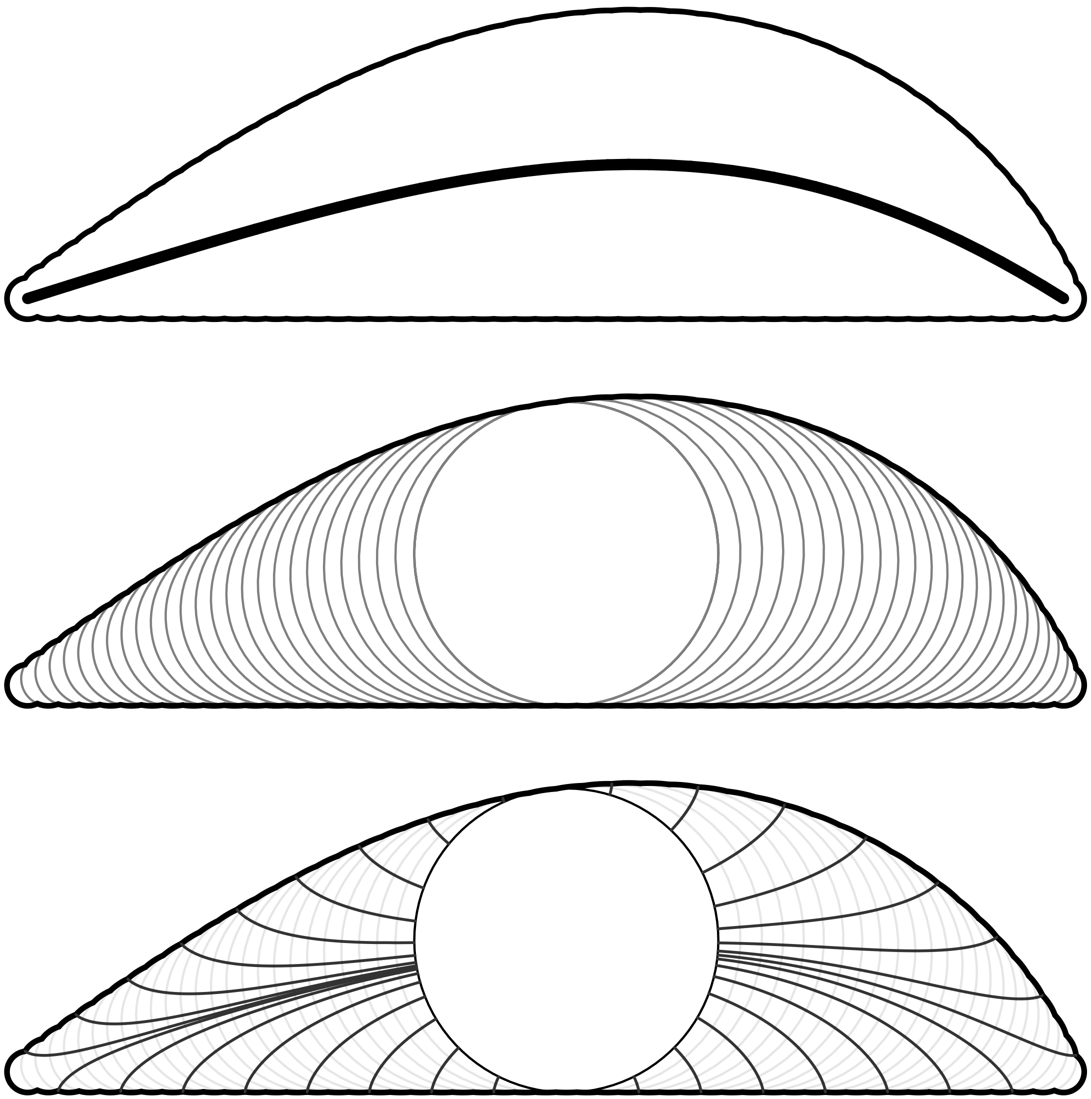


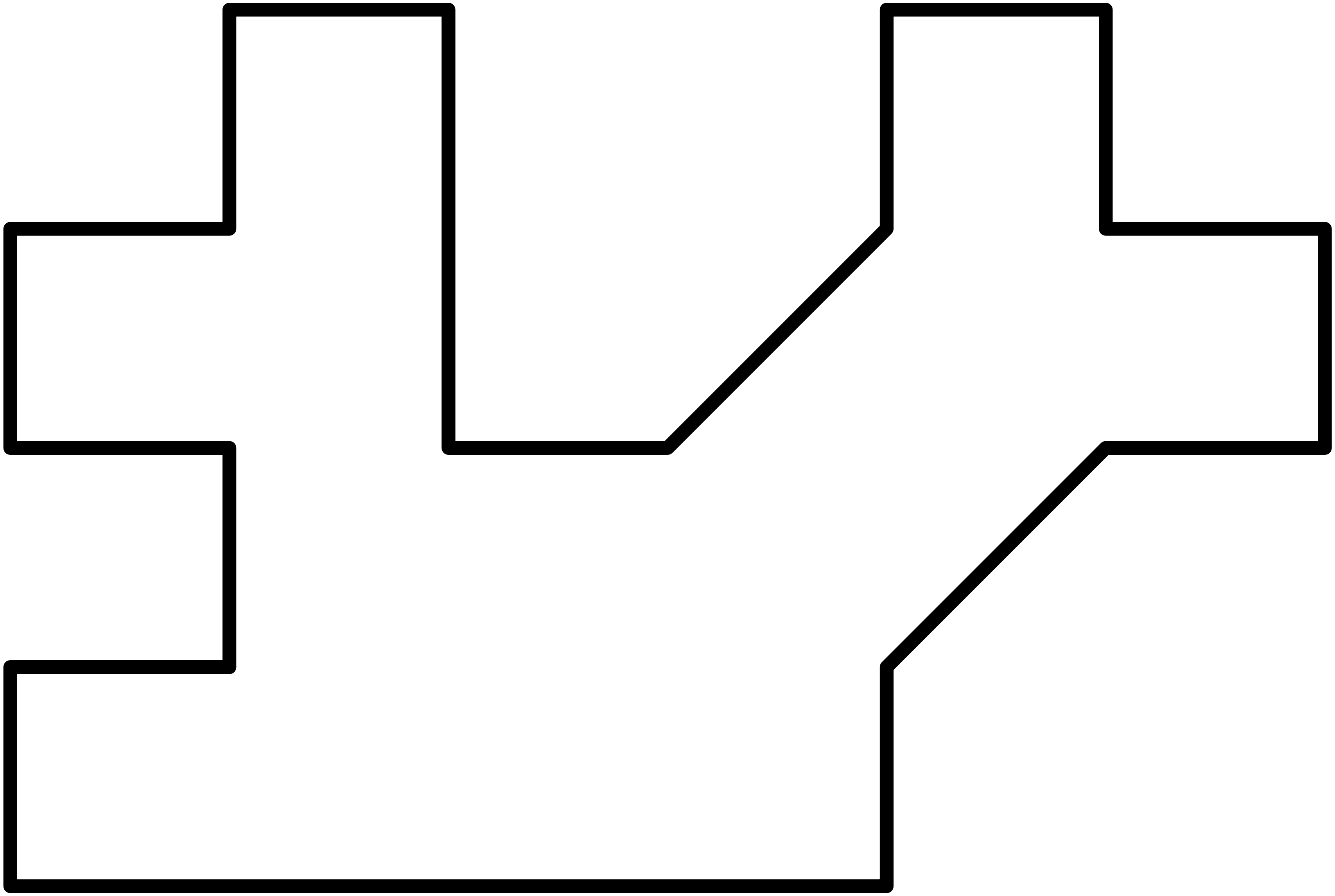


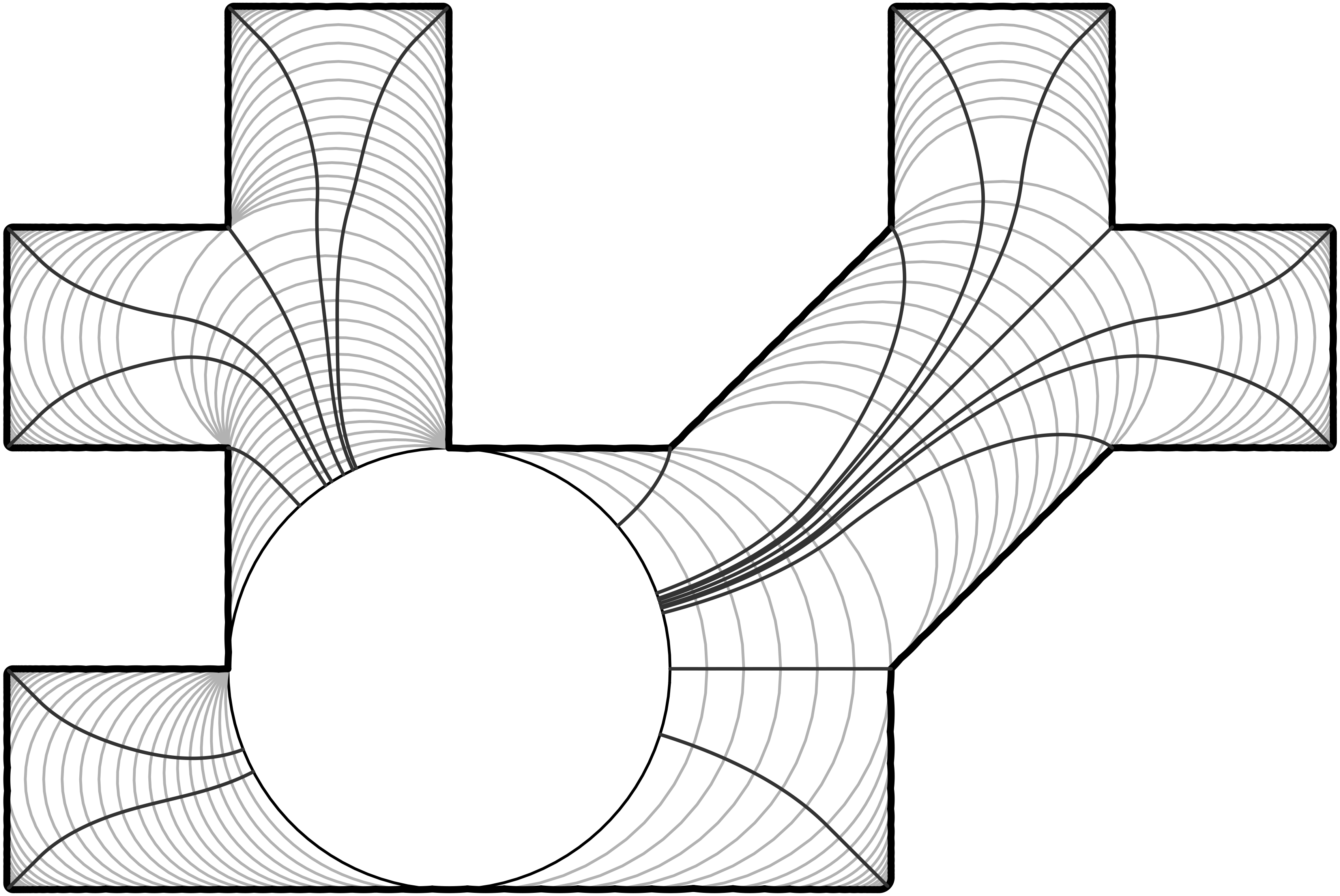
We discretize only to draw picture.

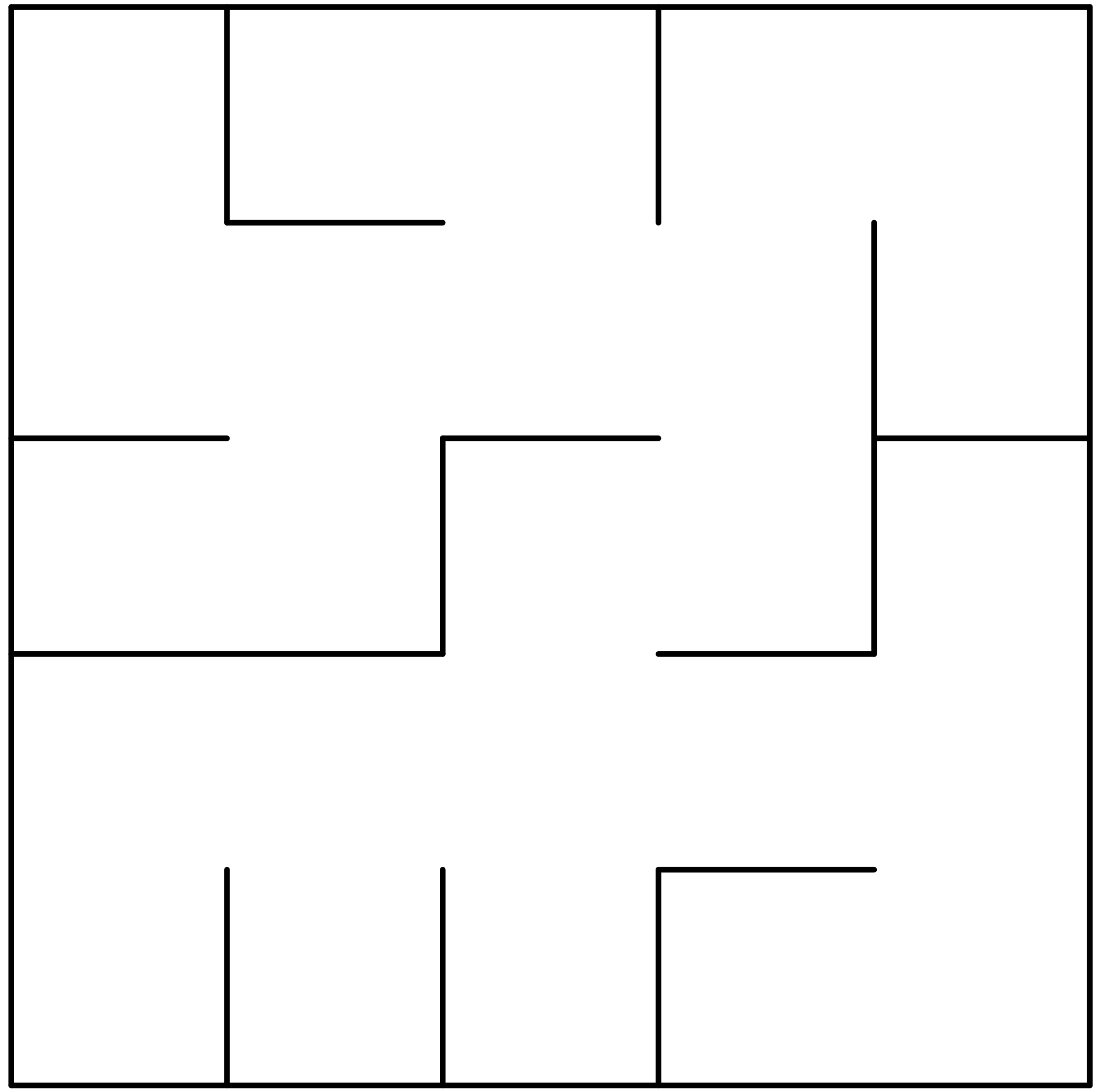
Limiting map has **formula** in terms of medial axis.

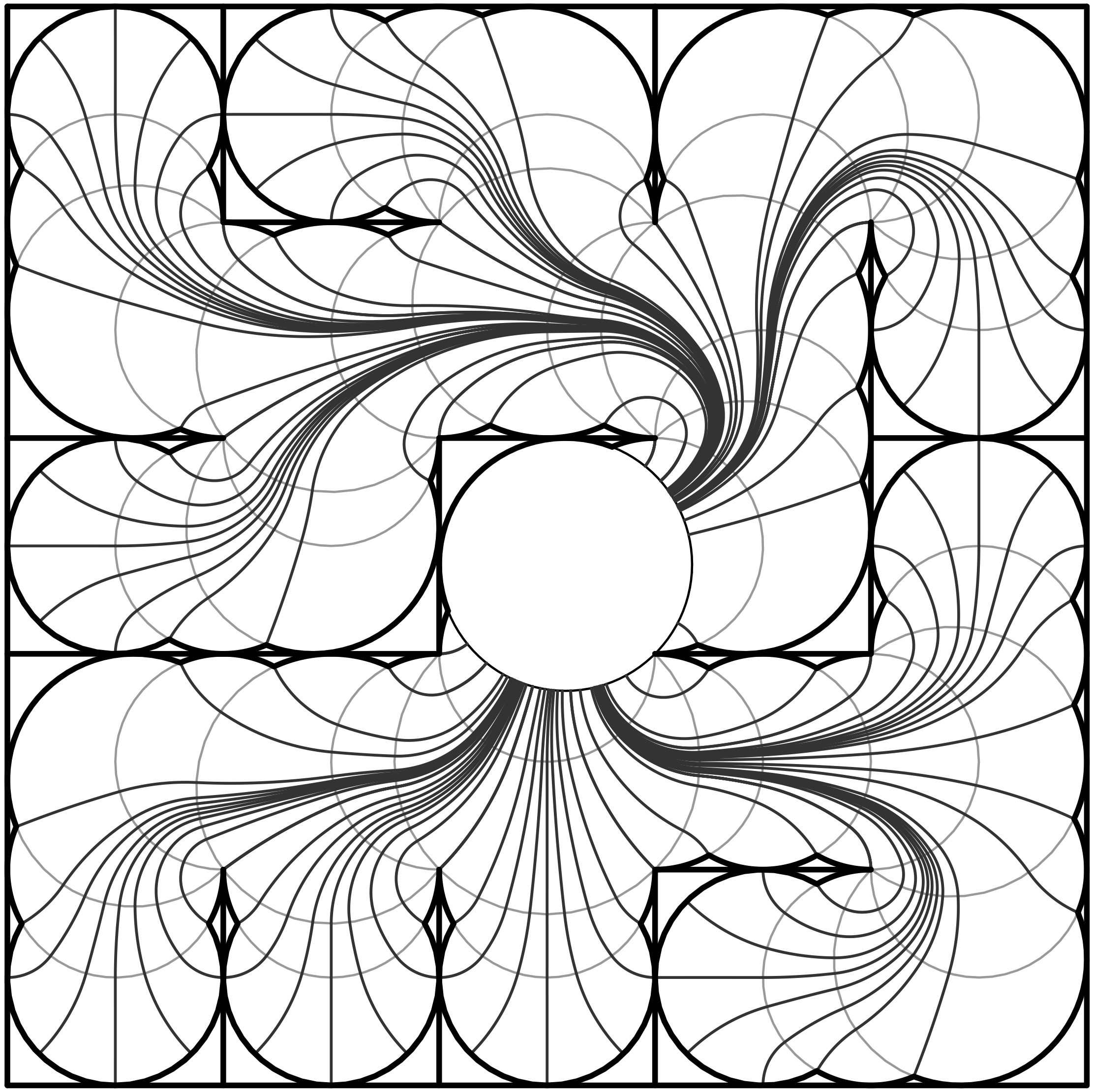
Similar flow for any simply connected domain.





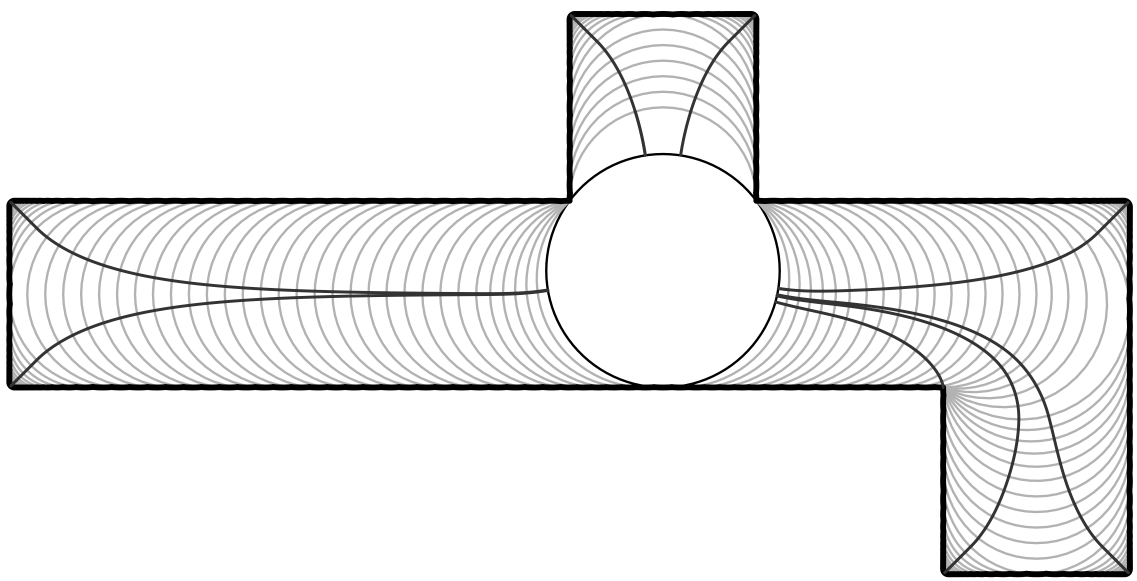






Theorem: Mapping all n vertices takes $O(n)$ time.

Uses linear time computation of MA (Chin-Snoeyink-Wang) and book-keeping with cross ratios.



How close is medial axis map to conformal map?

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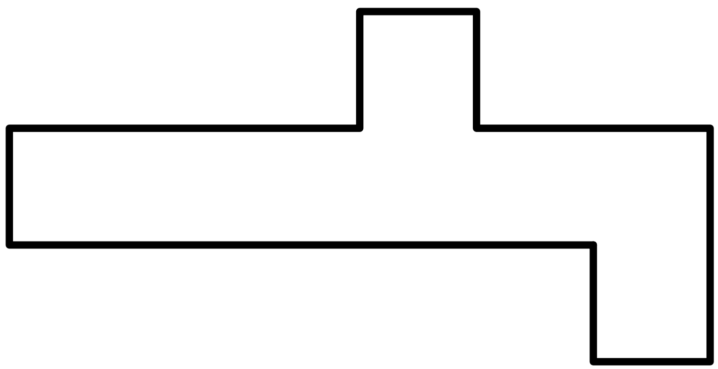
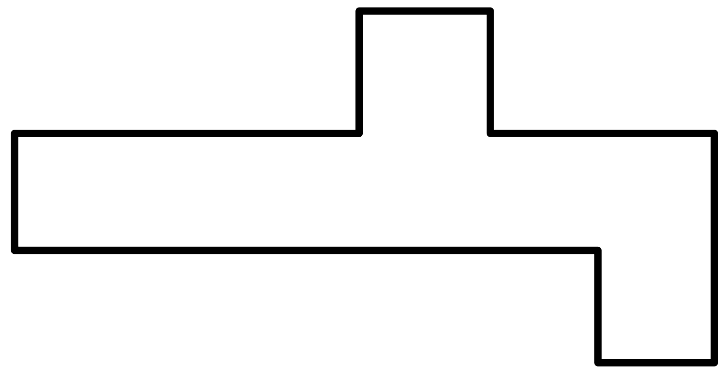
Use "MA-parameters" in Schwarz-Christoffel formula.



Target Polygon

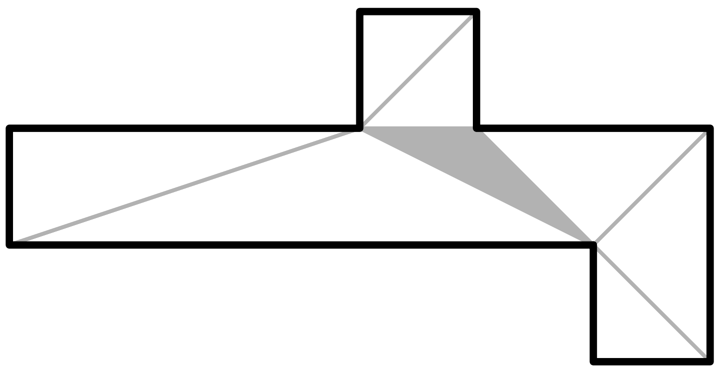


MA Parameters

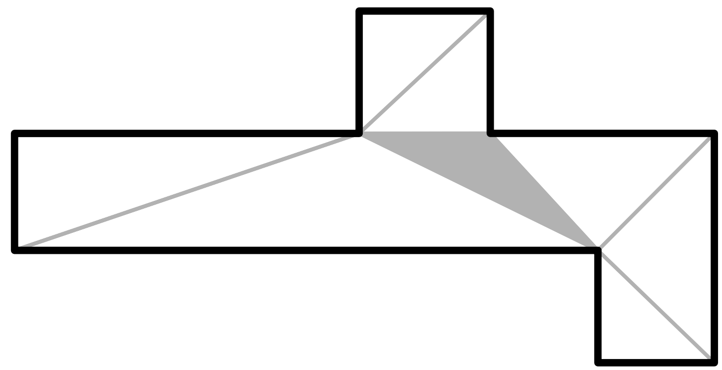
Looks pretty close.

How close is medial axis map to conformal map?

Use “MA-parameters” in Schwarz-Christoffel formula.



Target Polygon



MA Parameters

Looks pretty close.

There is a (1.24)-QC map between polygons preserving corners.

This implies MA-parameters are close to SC-parameters.

Theorem: Medial axis map always gives QC-error $K < 8$.

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Why is this theorem true?

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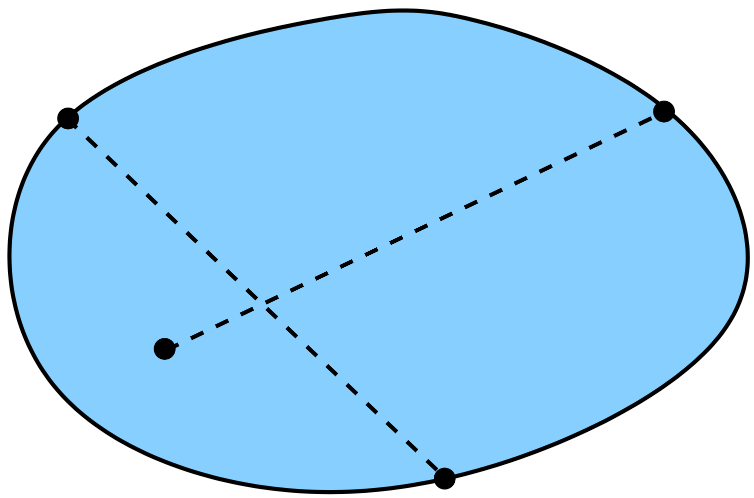
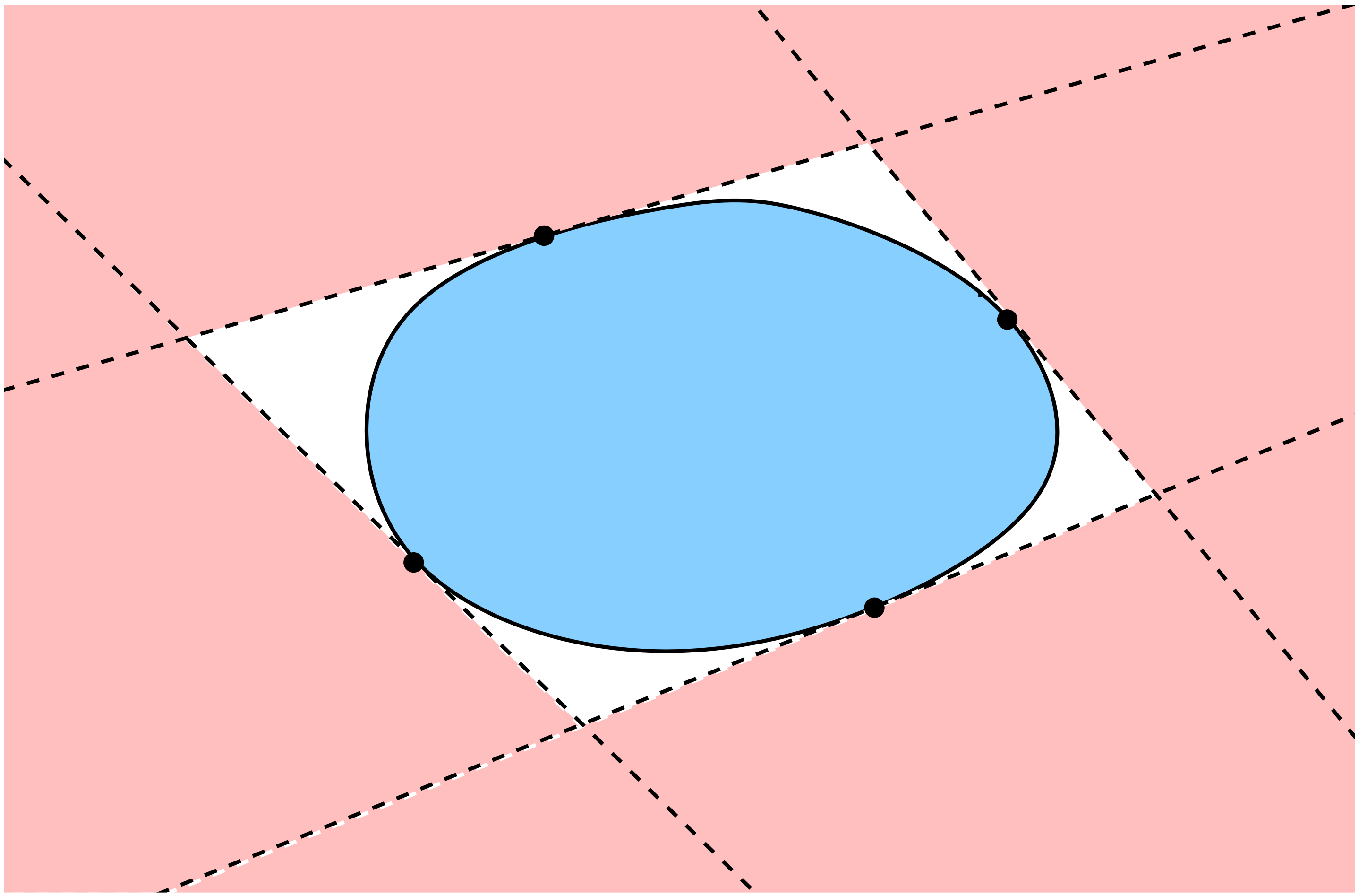
Why is this theorem true?

Short answer: convex sets in hyperbolic 3-space

Usual definition of convex: contains geodesic between any two points.



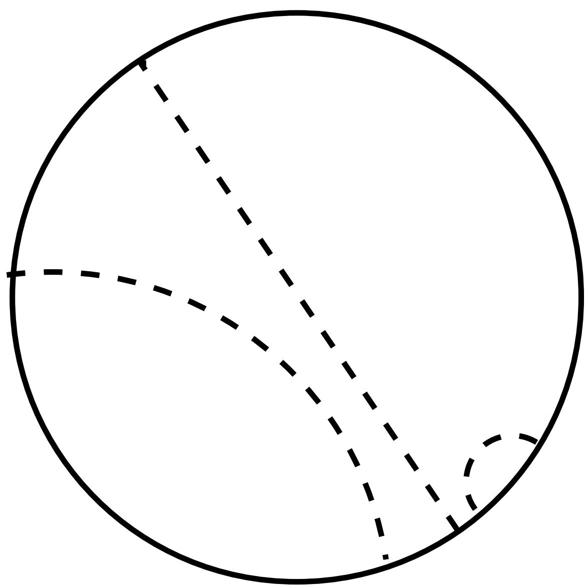
More useful for us: complement is a union of half-spaces.

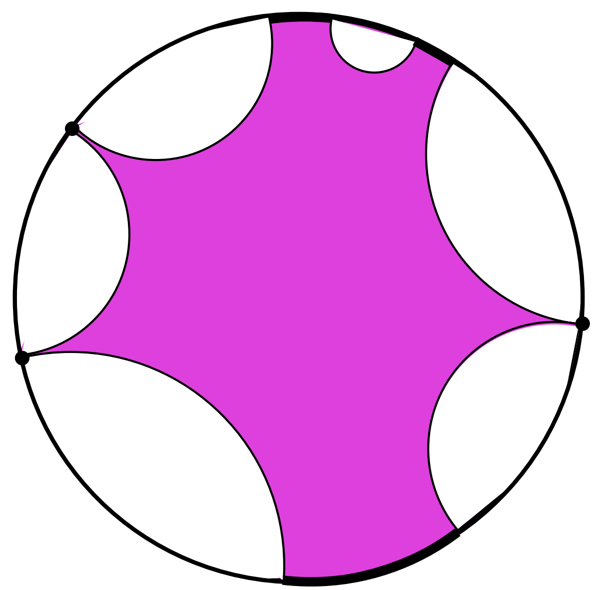


Hyperbolic metric on disk given by

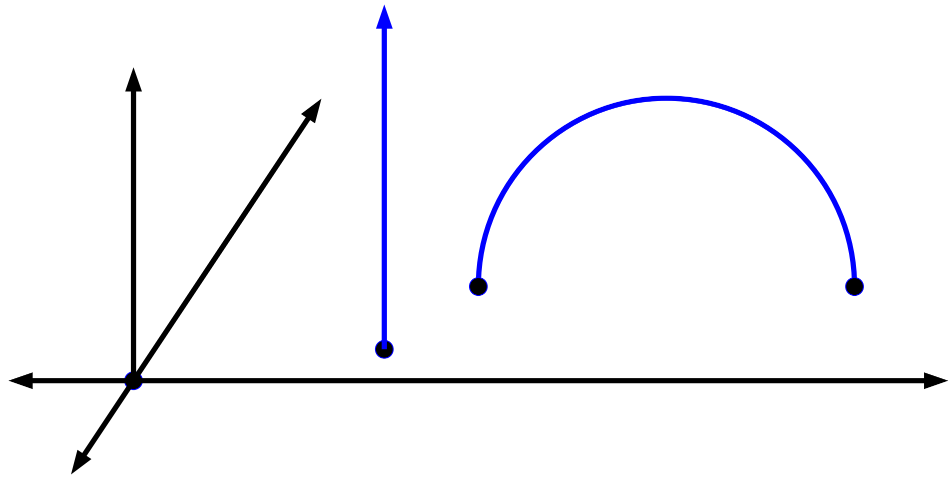
$$d\rho = \frac{ds}{1-|z|^2} \simeq \frac{ds}{\mathrm{dist}(z, \partial D)}.$$

- Geodesics are circles perpendicular to boundary.
- Shaded region is hyperbolically convex.
- Metric transfers via conformal maps to other domains.
- For simply connected regions $d\rho \simeq ds/\mathrm{dist}(z, \partial\Omega)$.

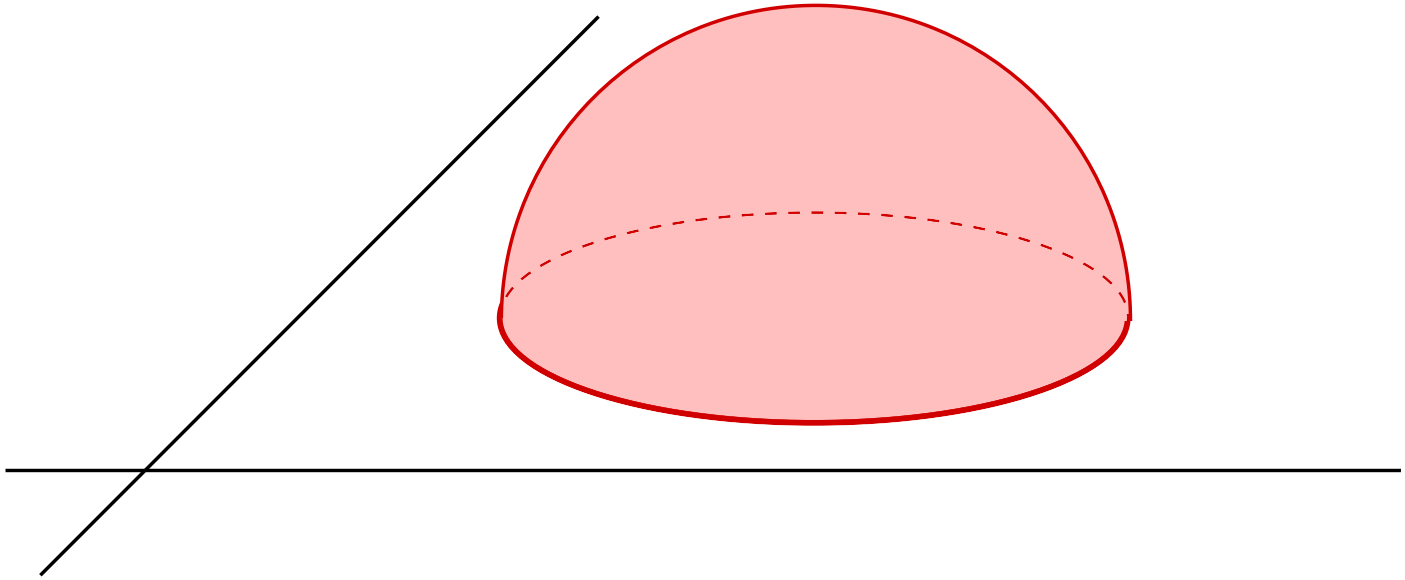




In the upper half-space $\mathbb{R}^3_+ = \{(x, y, t) : t > 0\}$, metric is $d\rho = ds/2t$.



Geodesics in $\mathbb{R}^3_+$ are vertical rays or semi-circles perpendicular to $\mathbb{R}^2$.

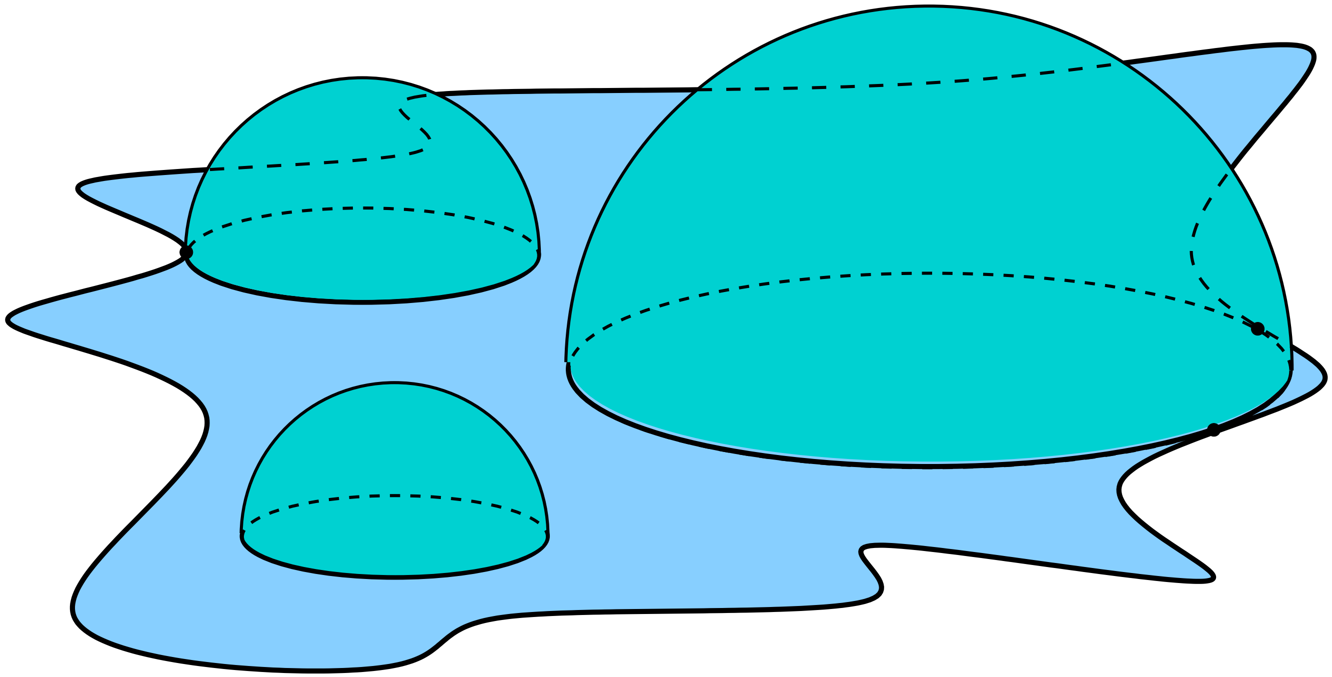


In $\mathbb{R}^3_+$, a hyperbolic half-space = hemisphere.

Given $\Omega \subset \mathbb{R}^2$, compute hyperbolic convex hull its complement.

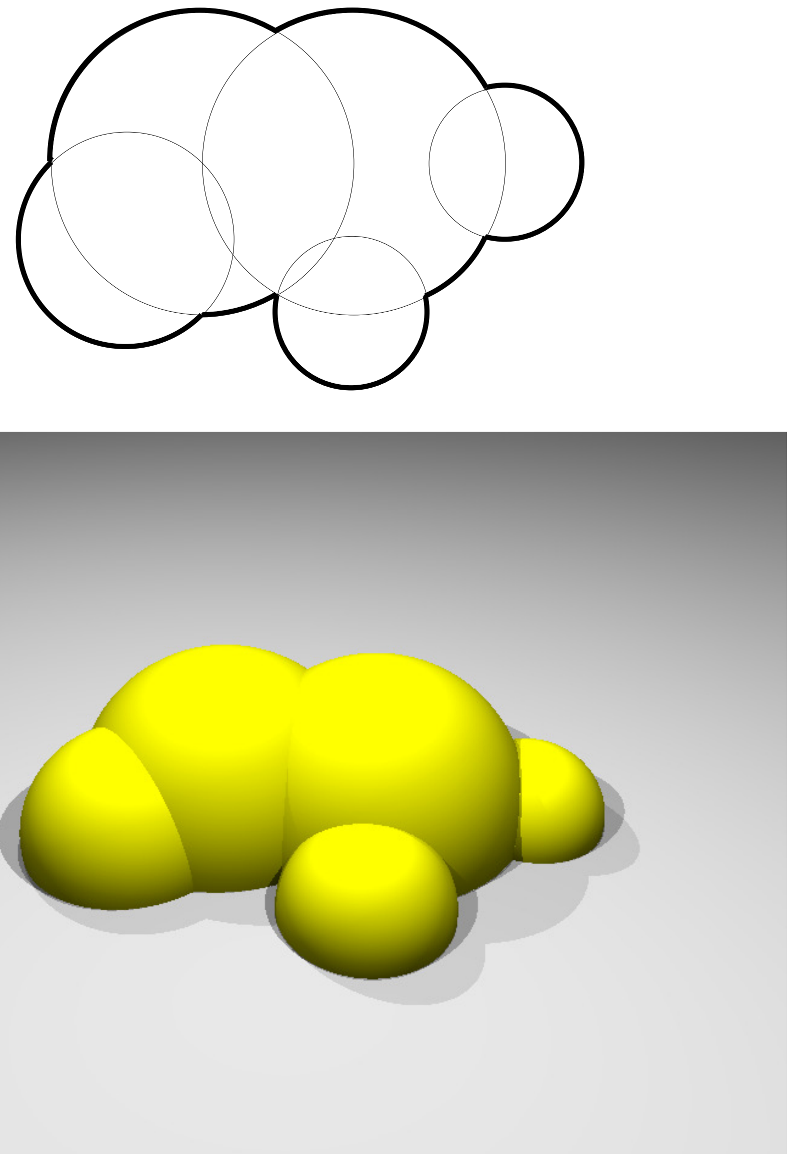
Easier to visualize the complement of convex hull = union of hemispheres.

Dome(Ω) is union of hemi-spheres with base disks in Ω.

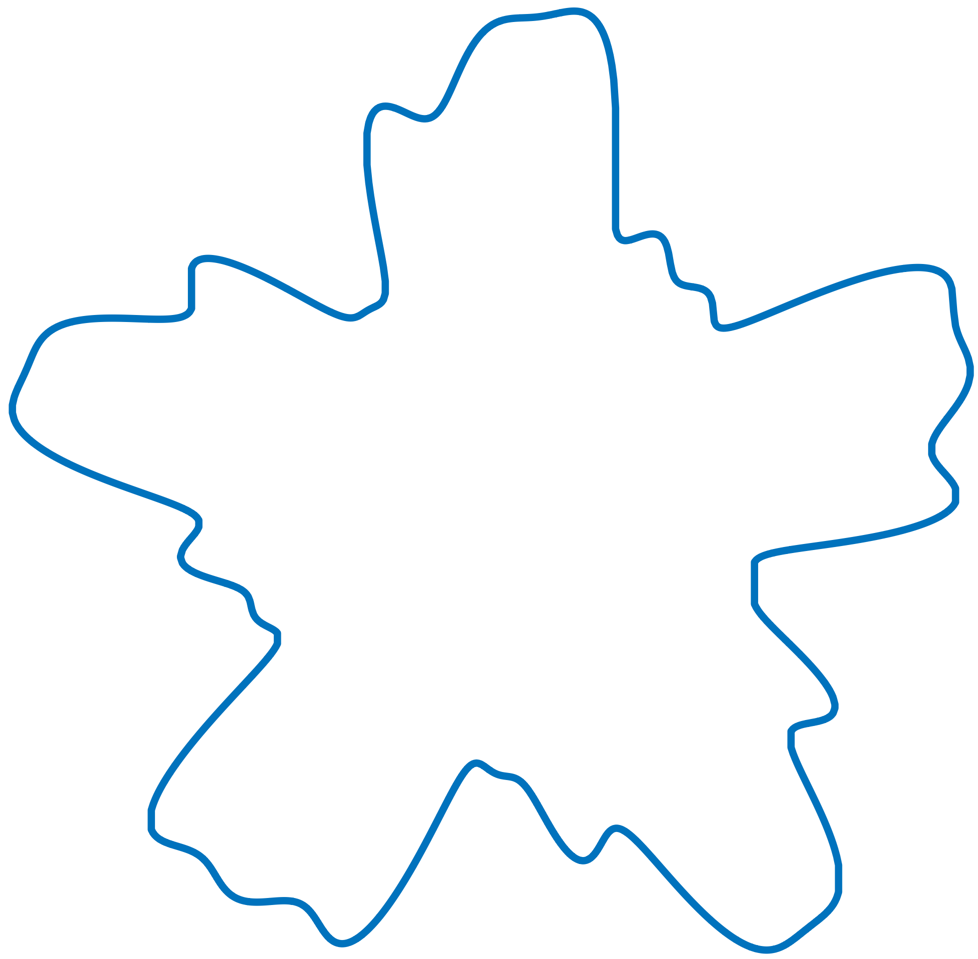


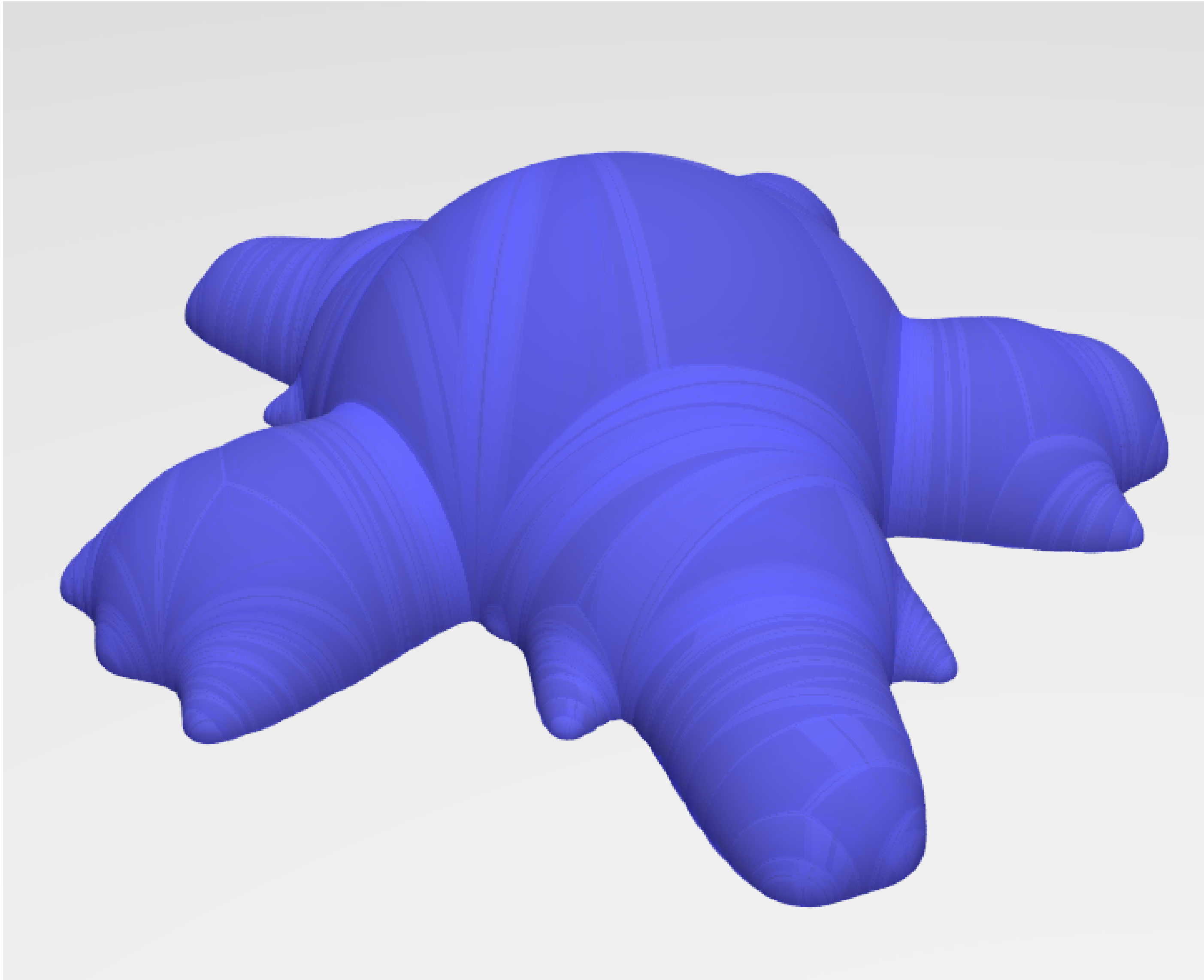
Region above dome is intersection of half-spaces, hence convex.

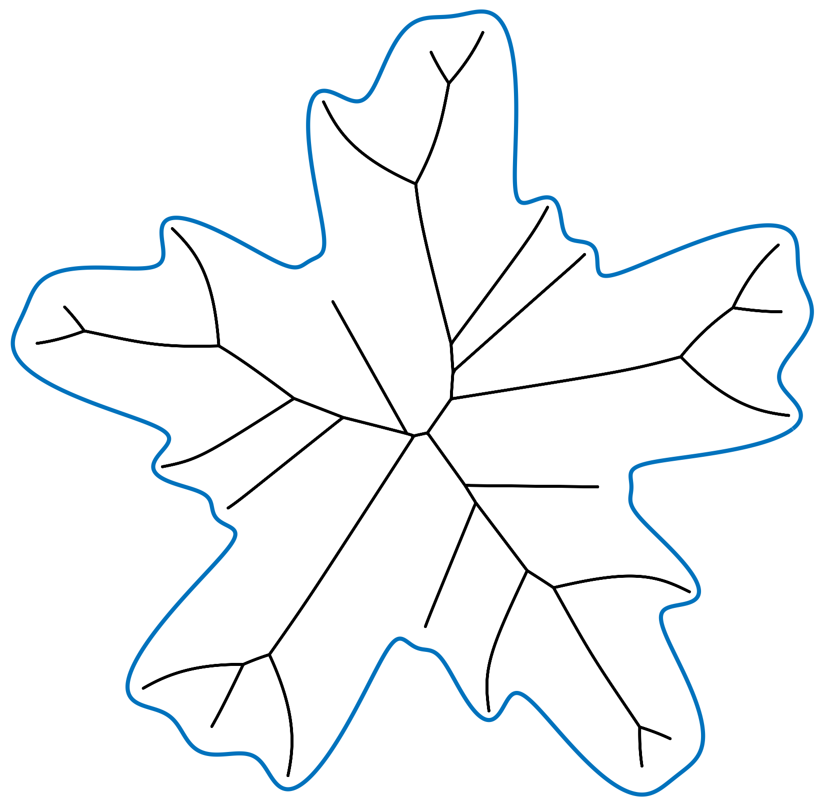
Upper boundary S of dome is a surface in $\mathbb{R}^3_+$ with $\partial S = \partial\Omega$.



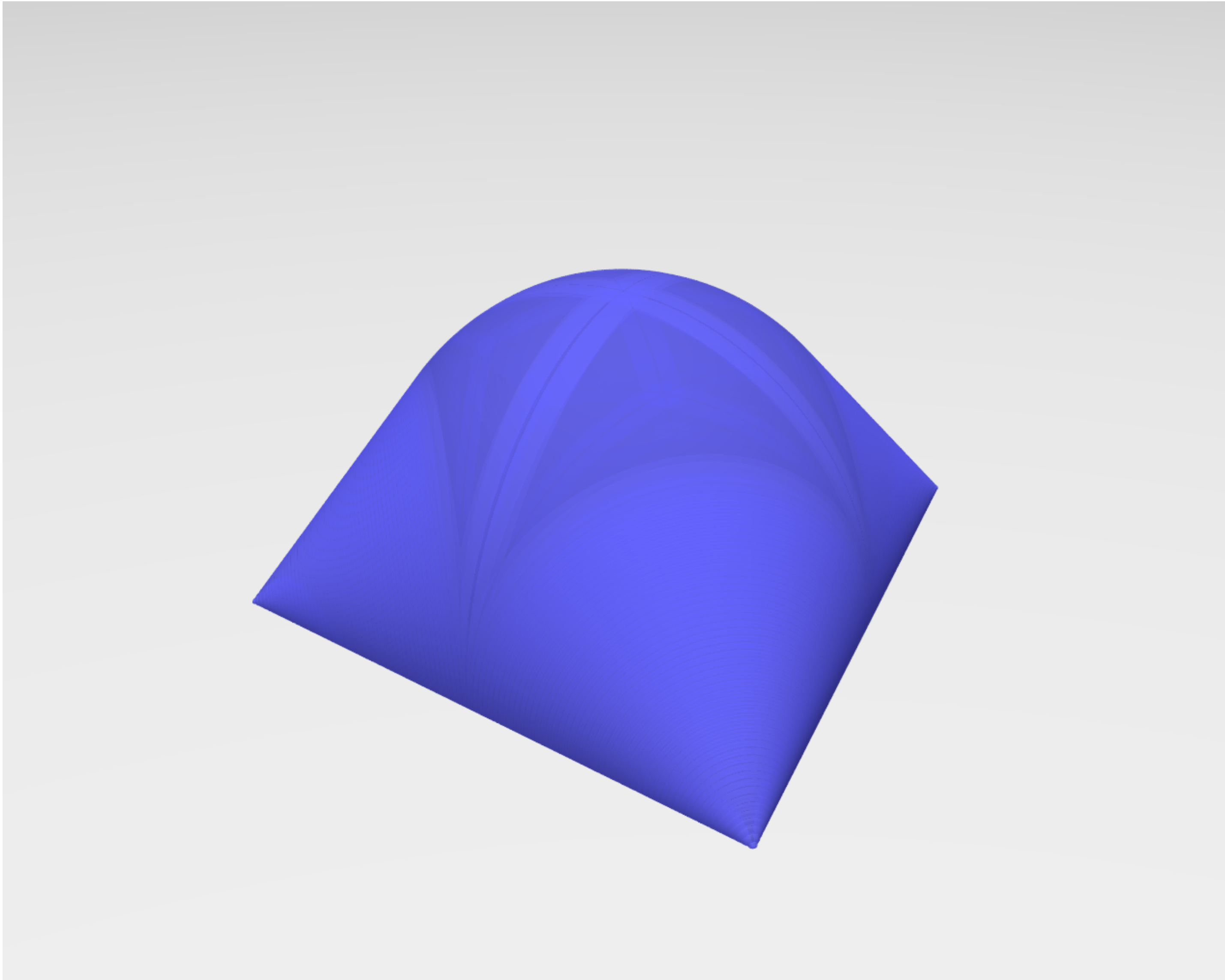
Finite dome

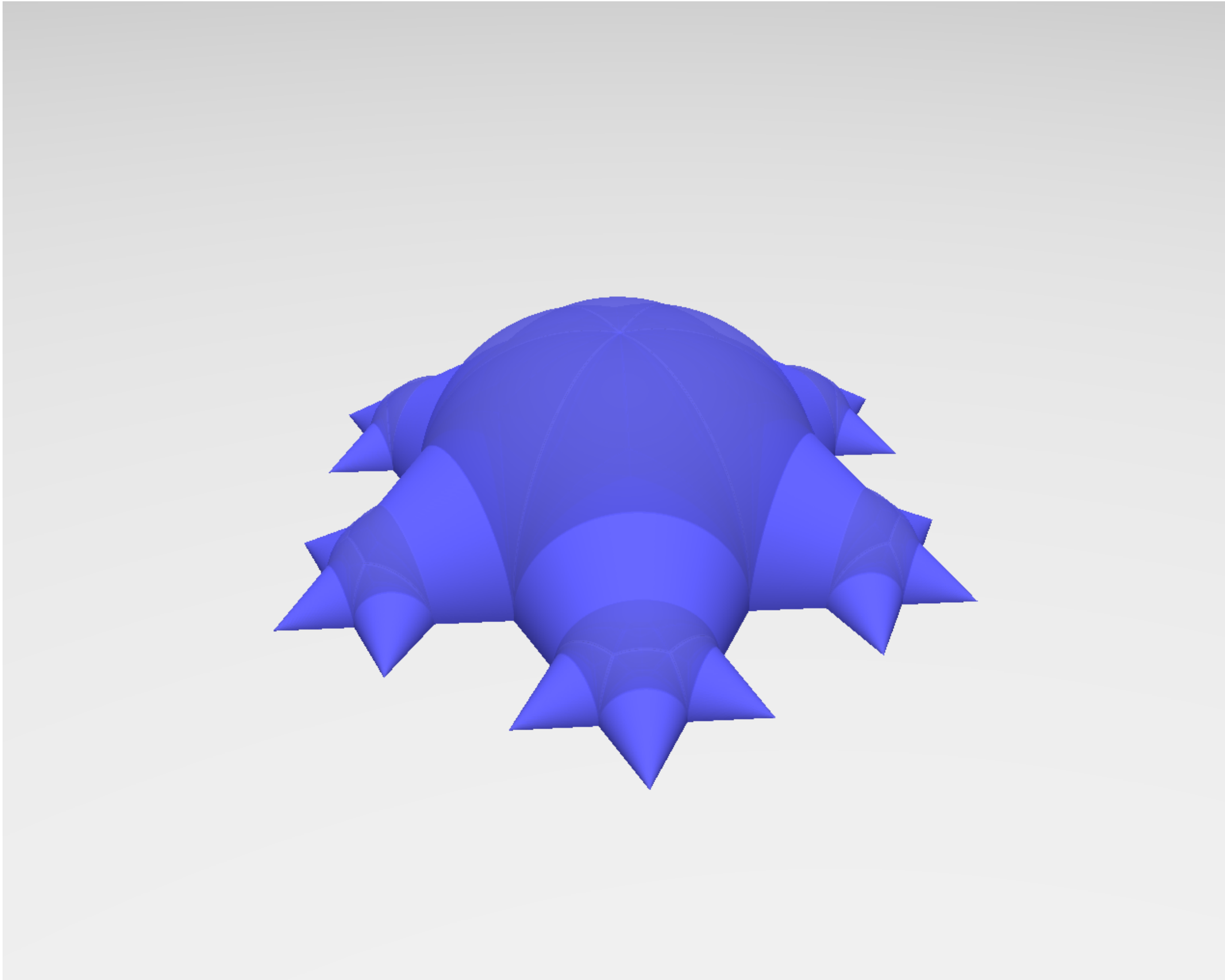






The medial axis. Equidistant from at least two boundary points.
Corresponding hemispheres give the dome.





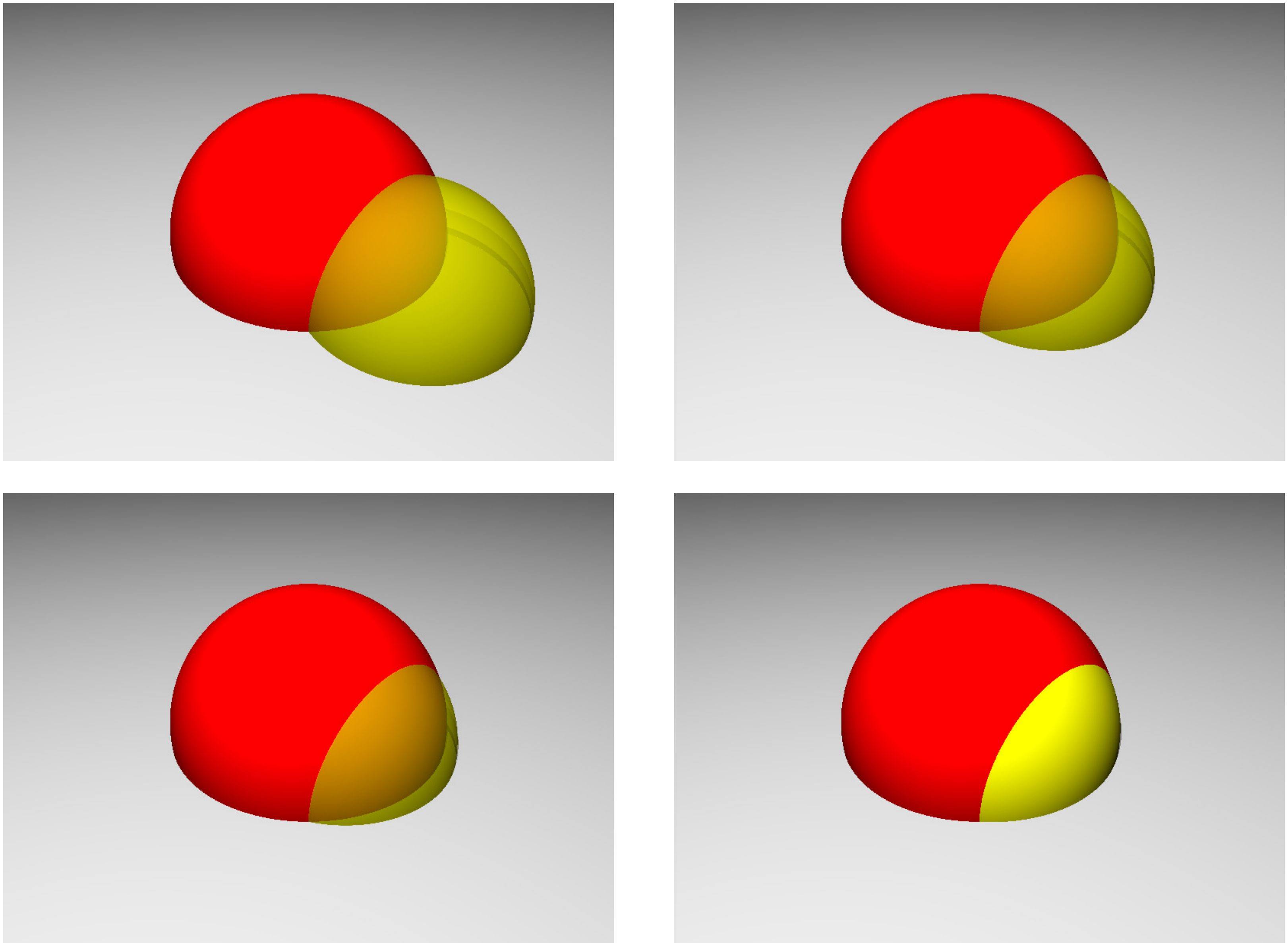
Thm: Simply connected domes are isometric to hyperbolic disk.

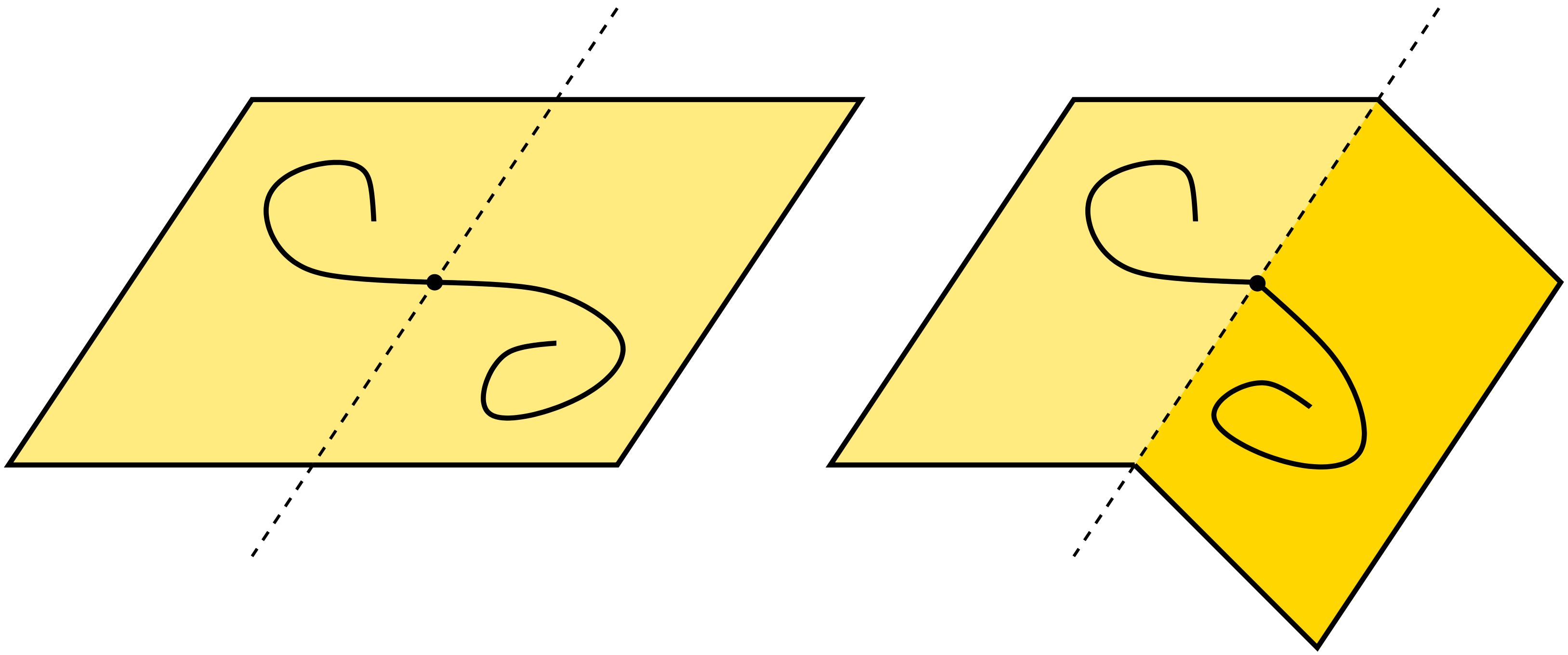
We are taking hyperbolic path metric on dome.

- Prove for finite unions of disks.
- Every dome is a limit of finite domes.
- Limit of isometries is an isometry.

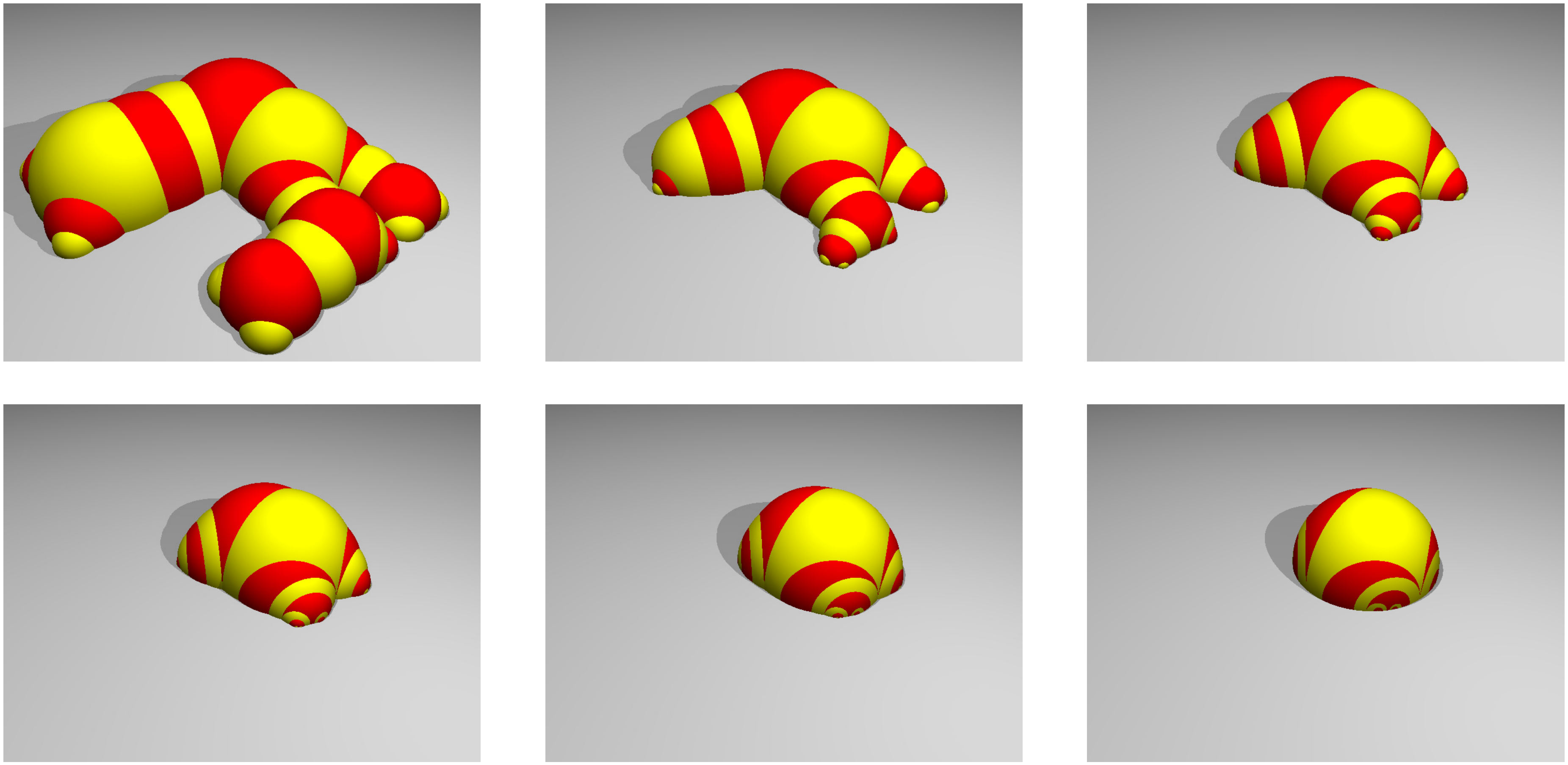
Isometry on boundary Γ defines a map Γ to circle.

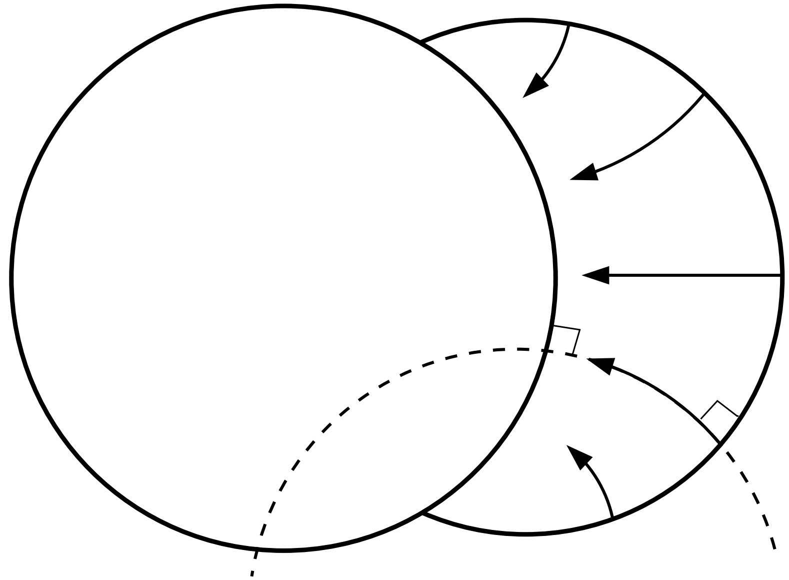
Every dome has conformal map to disk by “flattening”.





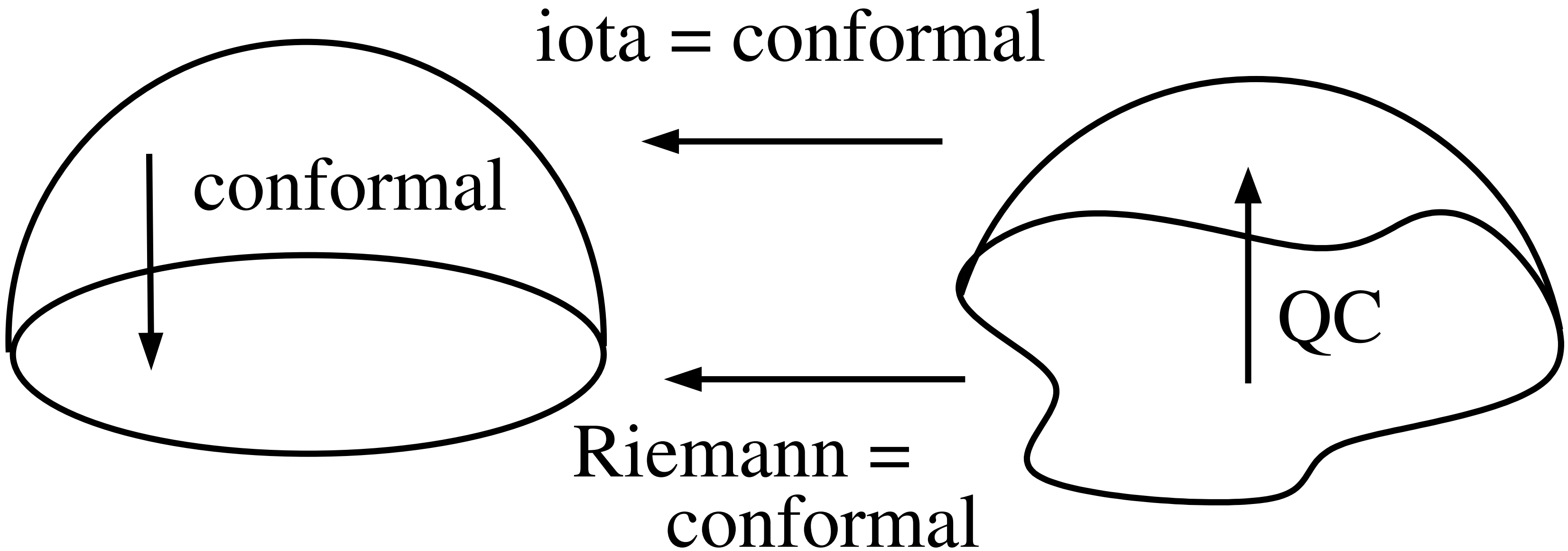
Folding plane along geodesic does not change length.
Pleated surface (folded along disjoint geodesics) = Flat plane





Medial axis map = boundary of flattening map (iota)

= boundary of conformal map of dome to hemisphere



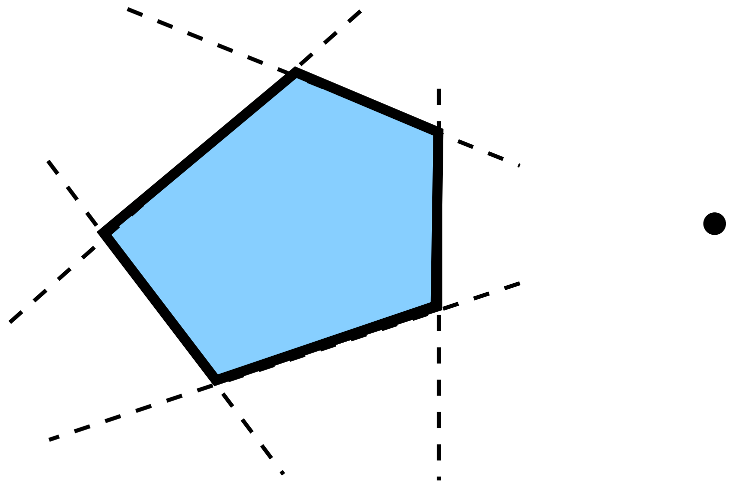
Iota = conformal from dome to disk.

Medial axis flow = boundary values of iota

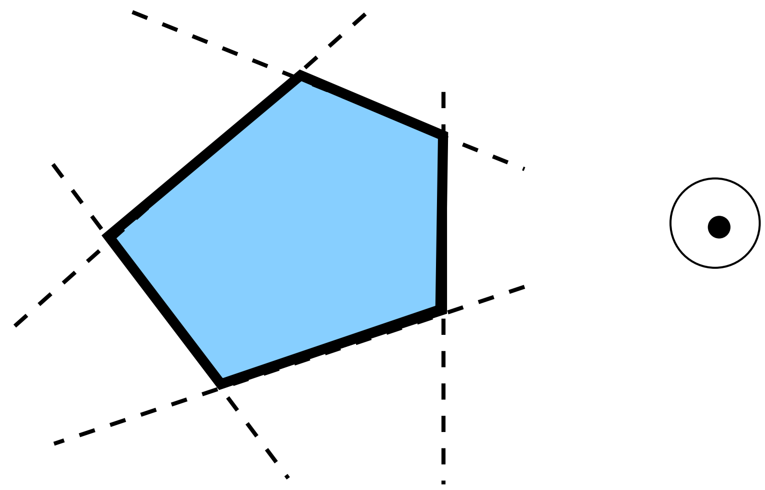
Claim: There is QC map base $\to$ dome fixing boundary pointwise.

Implies that medial axis map has QC extension $\Omega \to \mathbb{D}$.

Nearest point map in $\mathbb{R}^n$ is Lipschitz.



Nearest point map in $\mathbb{R}^n$ is Lipschitz.



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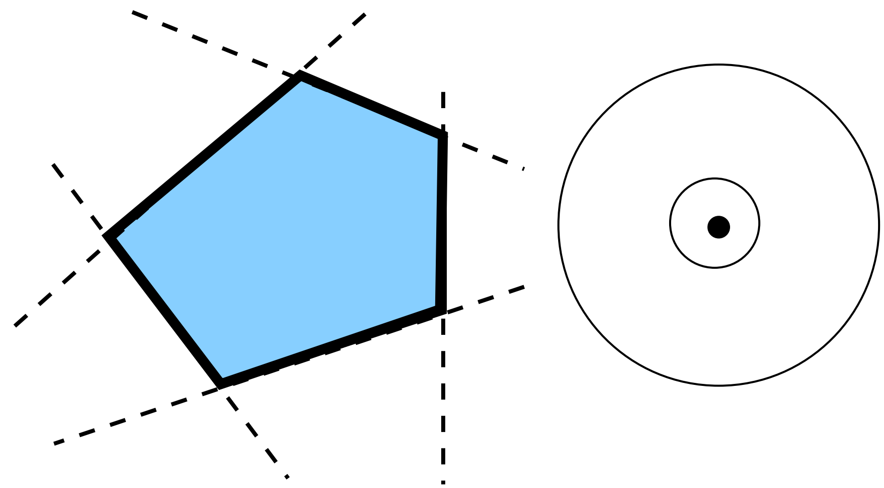

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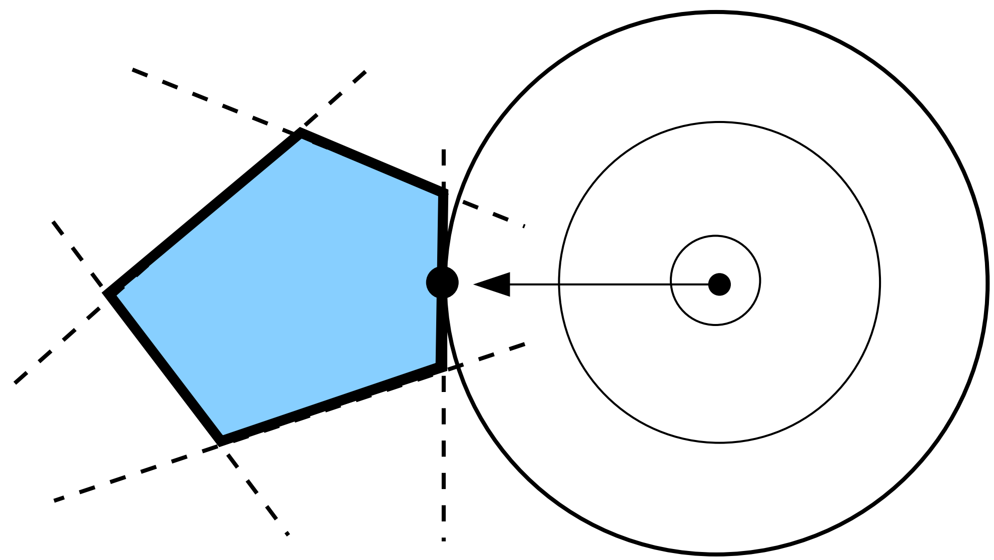

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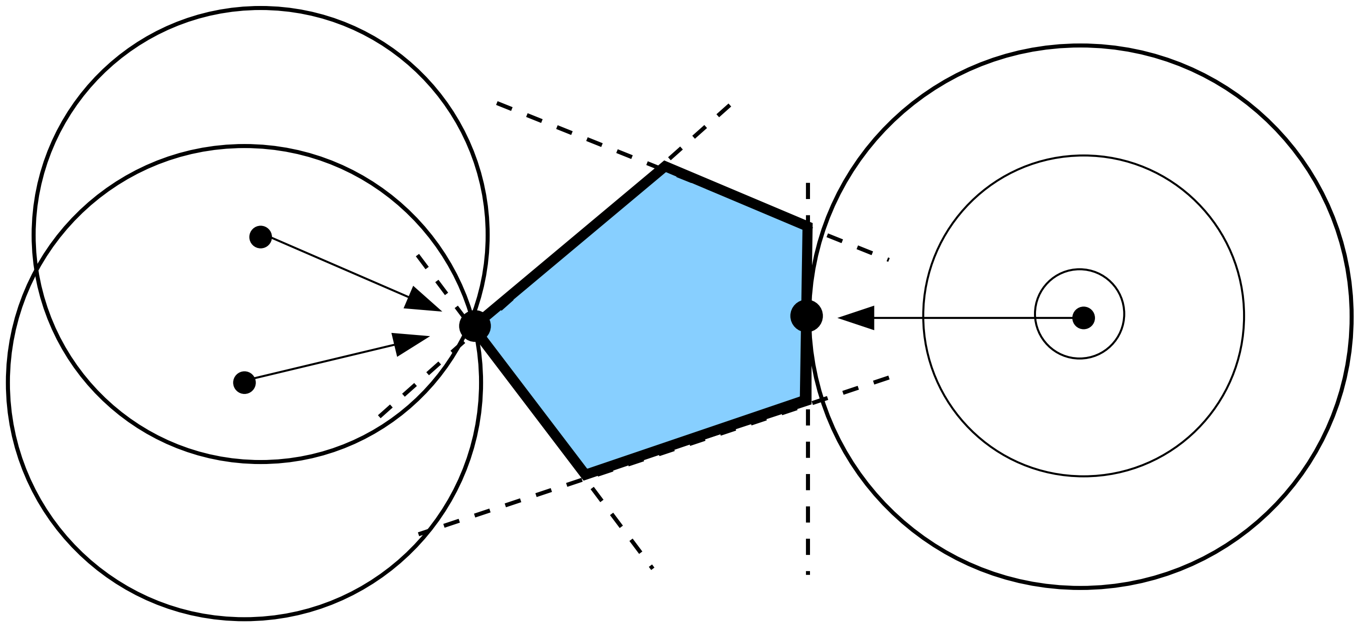
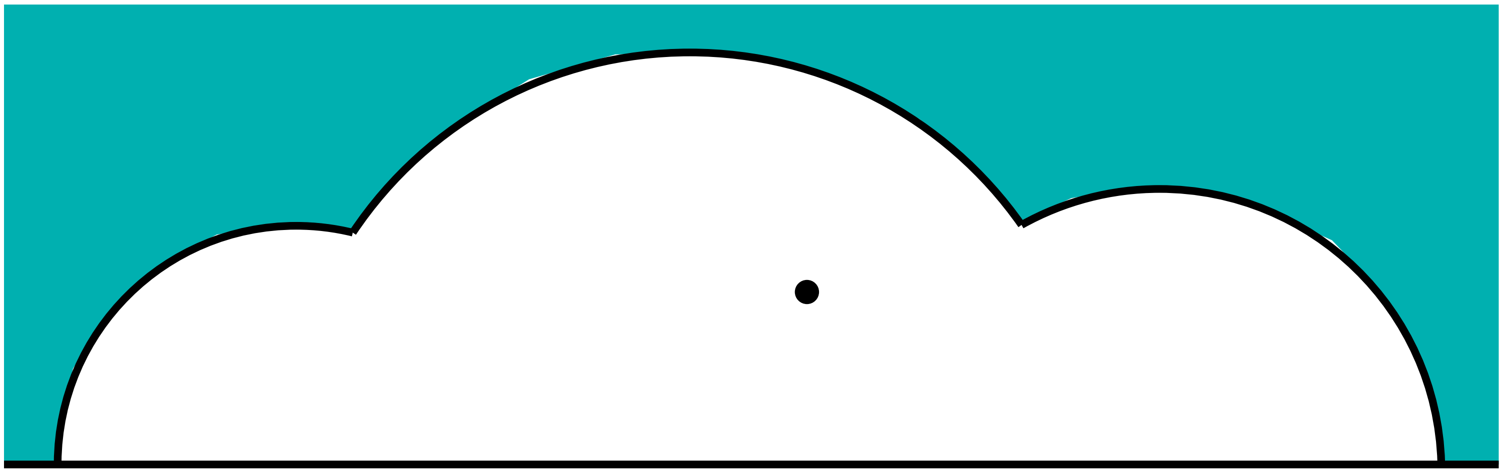


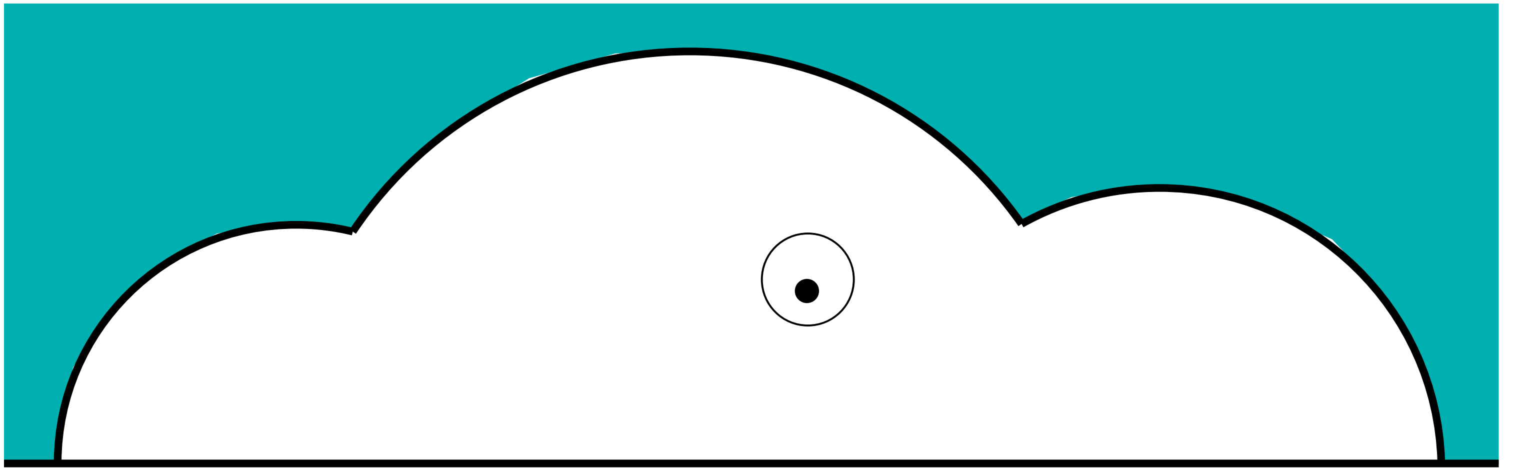


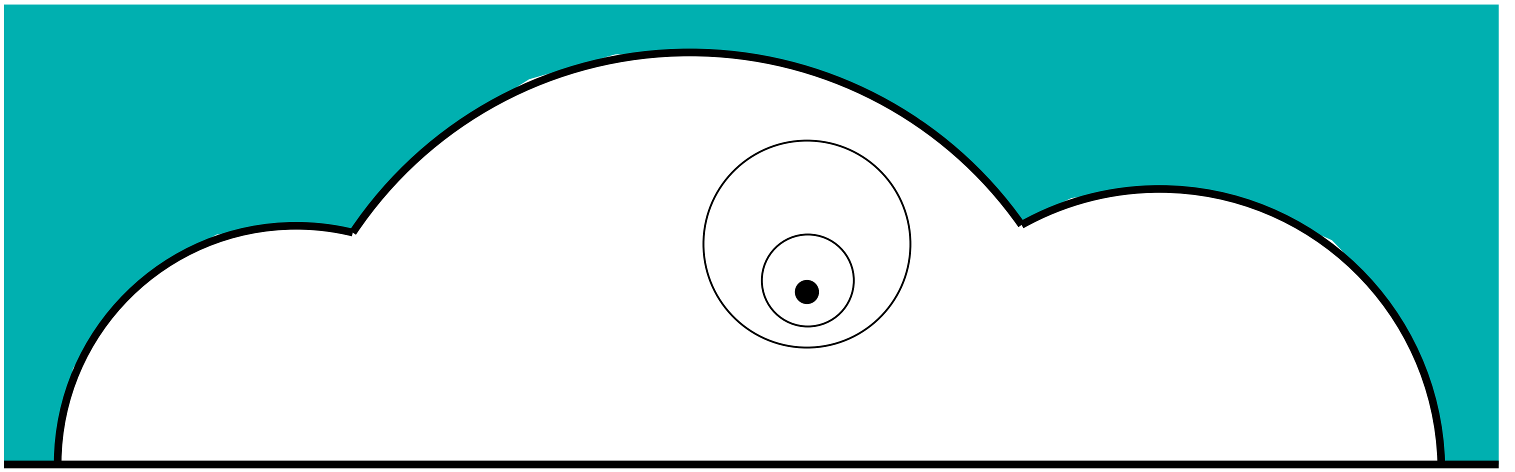
Region below dome is union of hemispheres

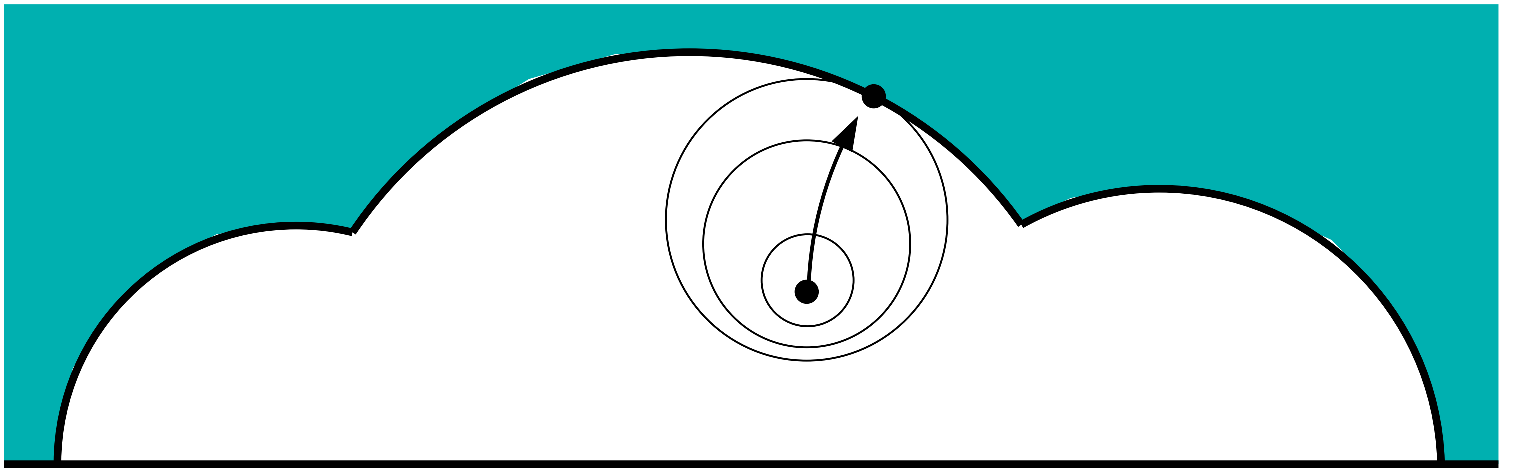
Hemispheres = hyperbolic half-spaces.

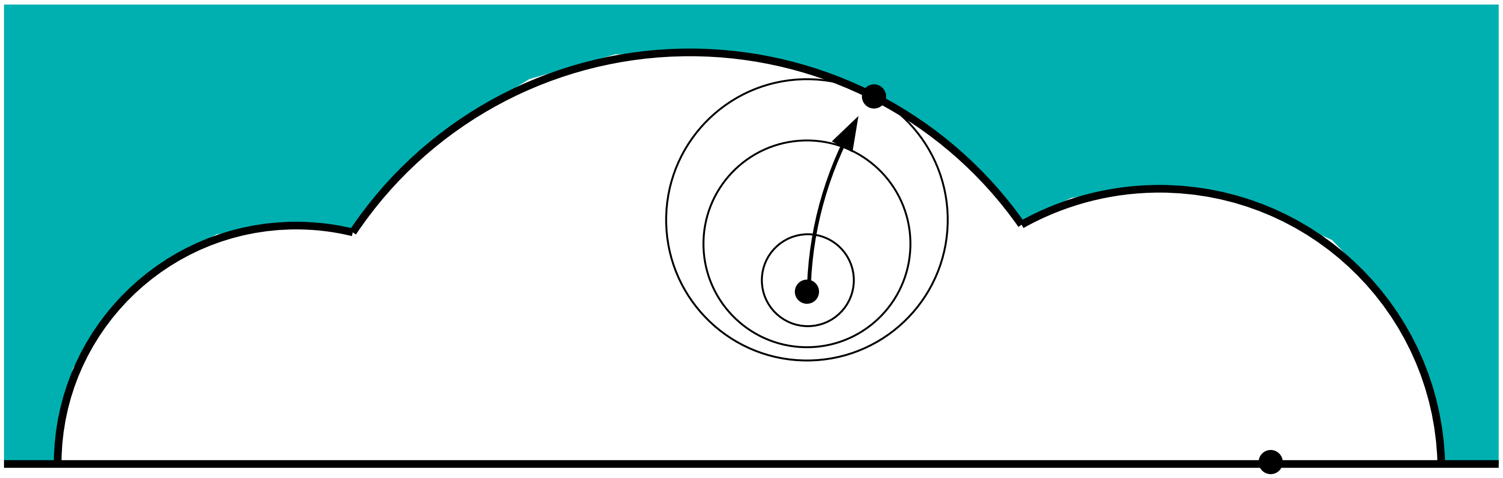
Region above dome is hyperbolically convex.

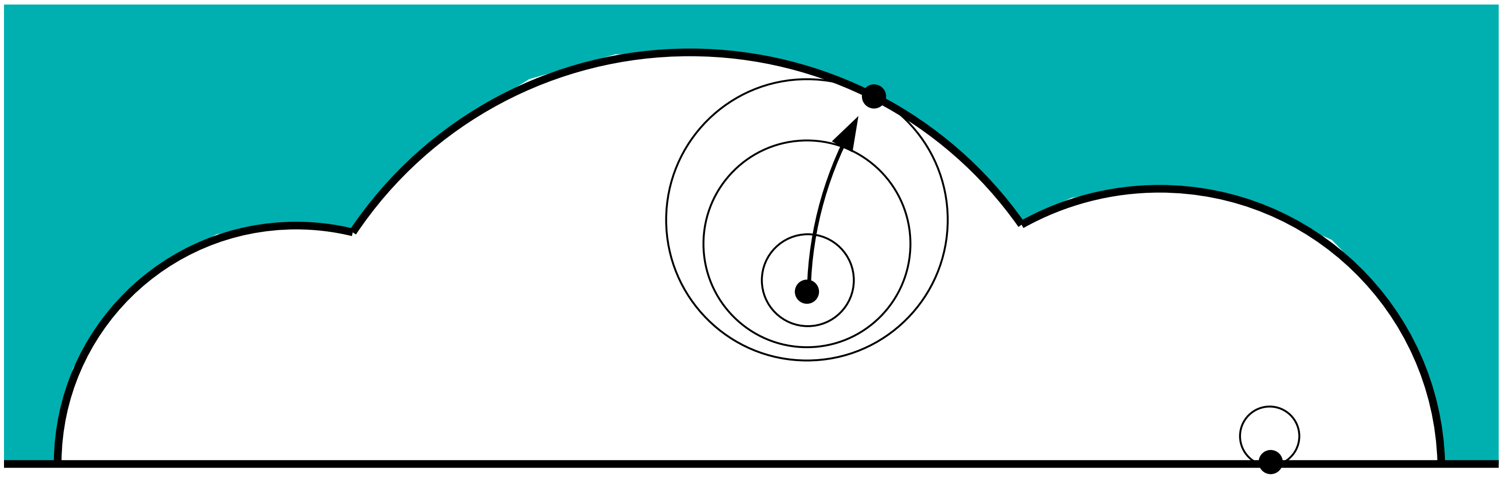
Consider nearest point retraction onto this convex set.

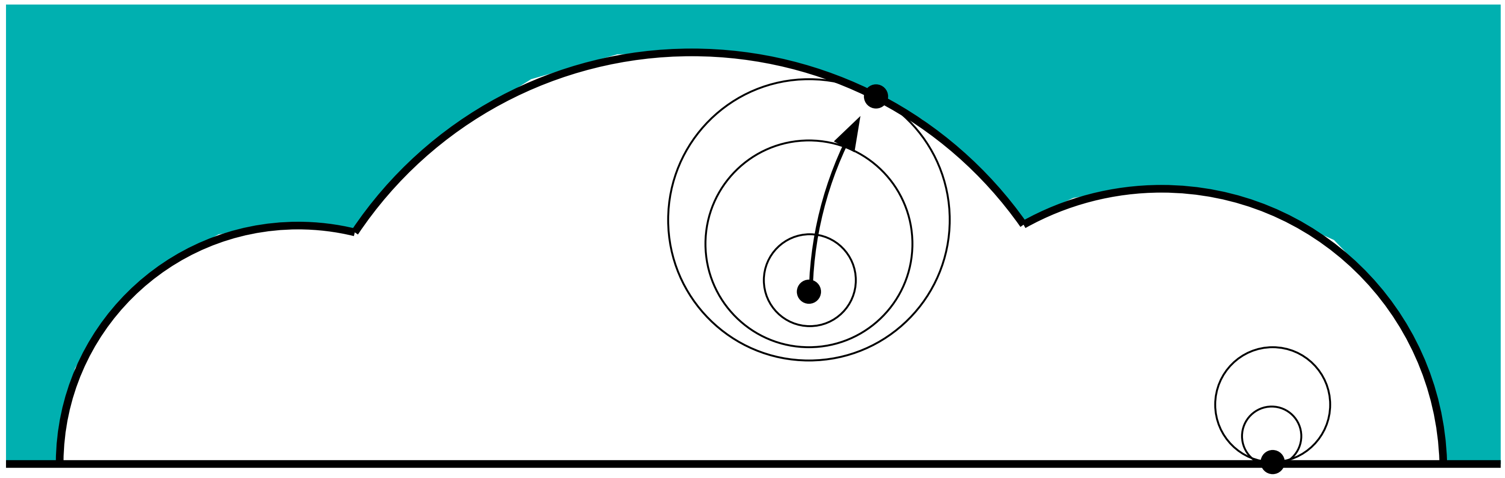


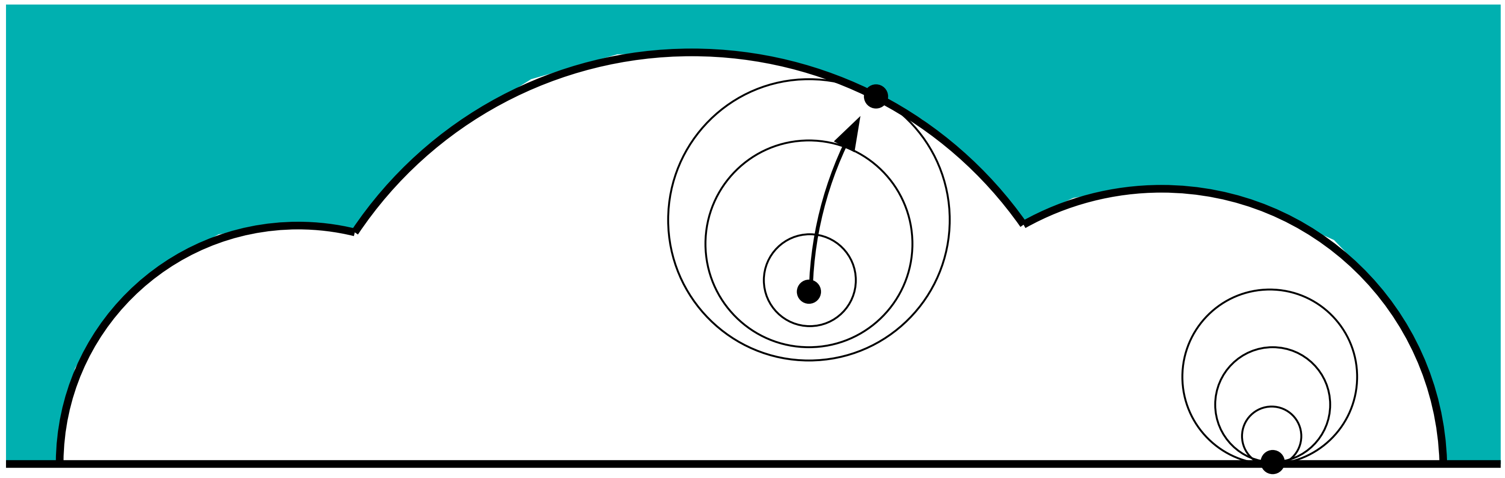


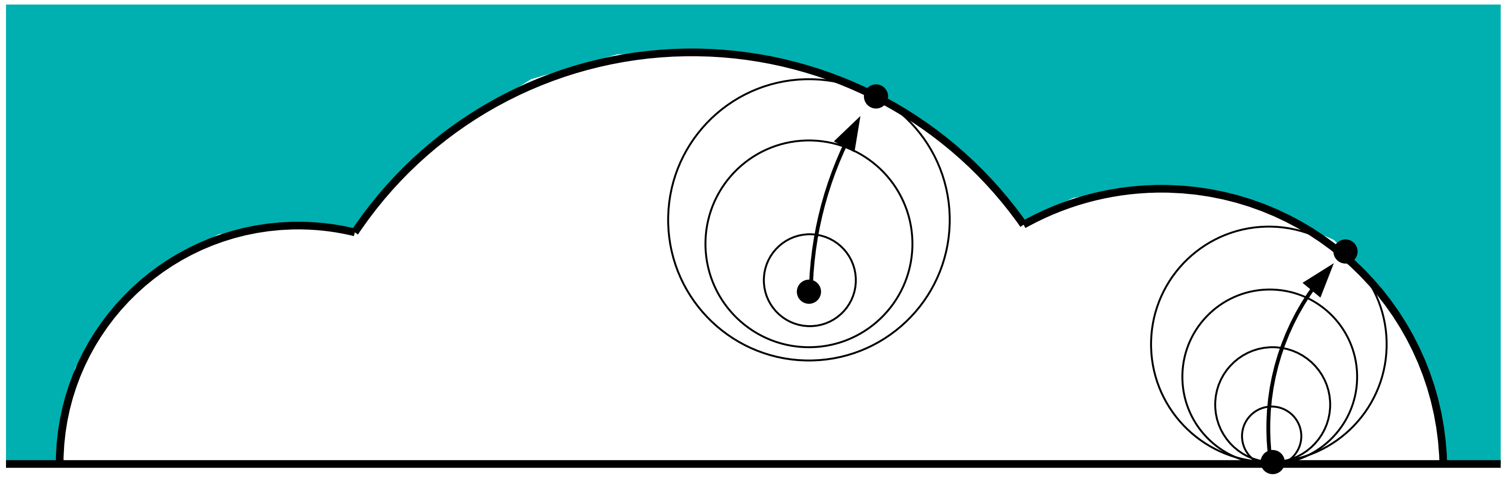


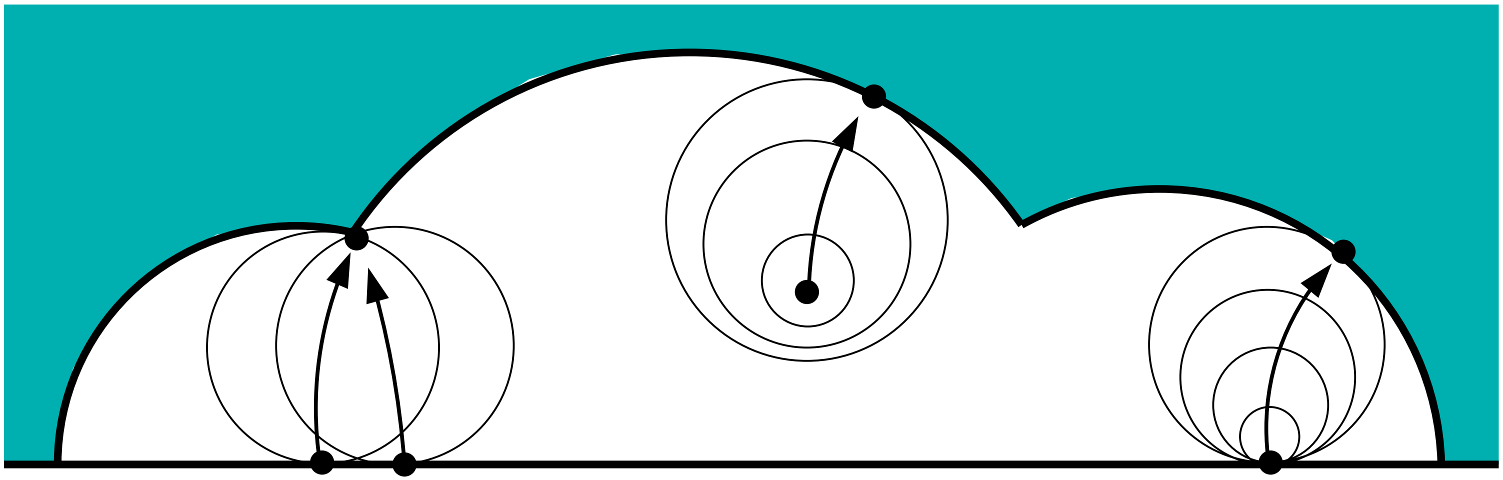




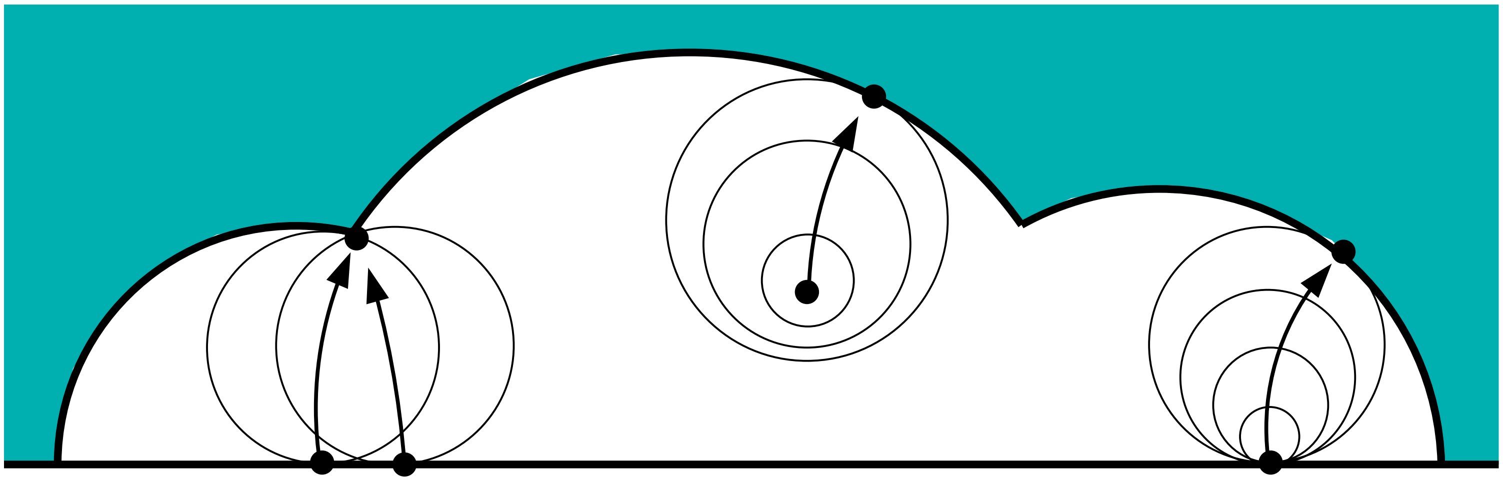








Need not be a homeomorphism, but . . .

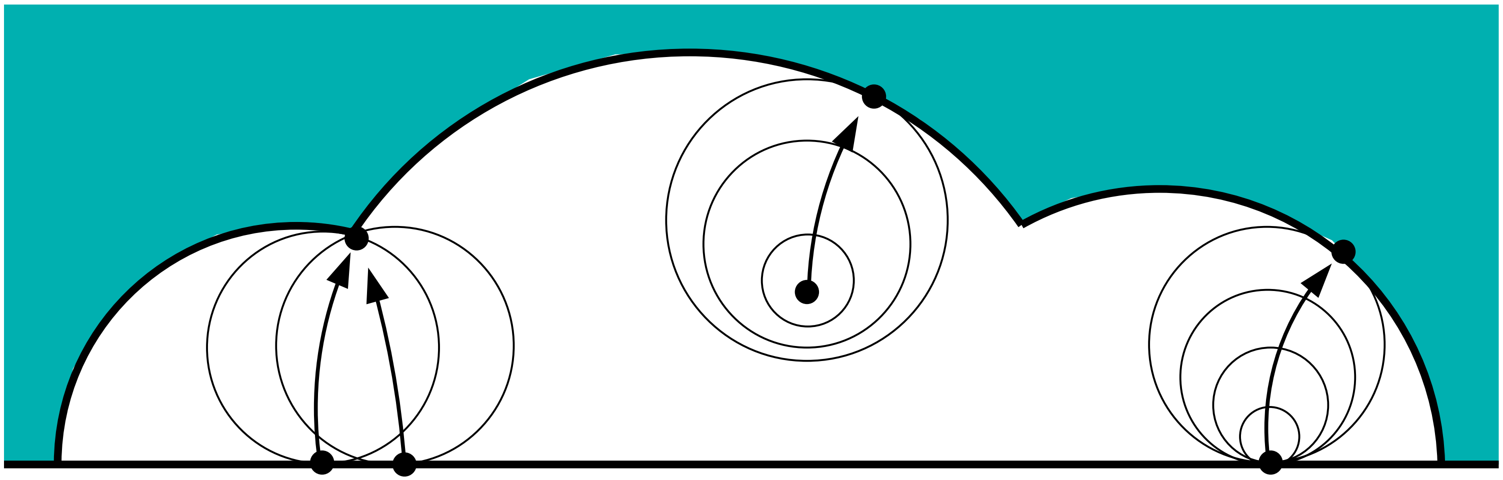


Need not be a homeomorphism, but it is a **quasi-isometry**

$$\frac{1}{A} \le \frac{\rho(R(x), R(y))}{\rho(x,y)} \le A, \qquad \text{if } \rho(x,y) \ge B.$$

i.e., R is bi-Lipschitz on large scales.

Metrics are hyperbolic metrics on Ω and S.



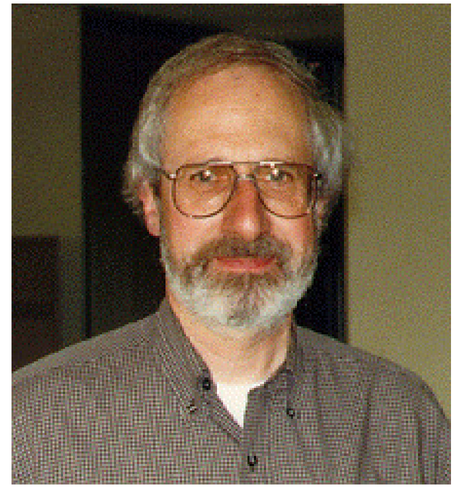
“Smoothing” gives K-QC map fixing boundary points.

Sullivan’s convex hull theorem: K is independent of domain.

Dennis Sullivan, David Epstein and Al Marden, C.B.

Dennis Sullivan

David Epstein



Al Marden

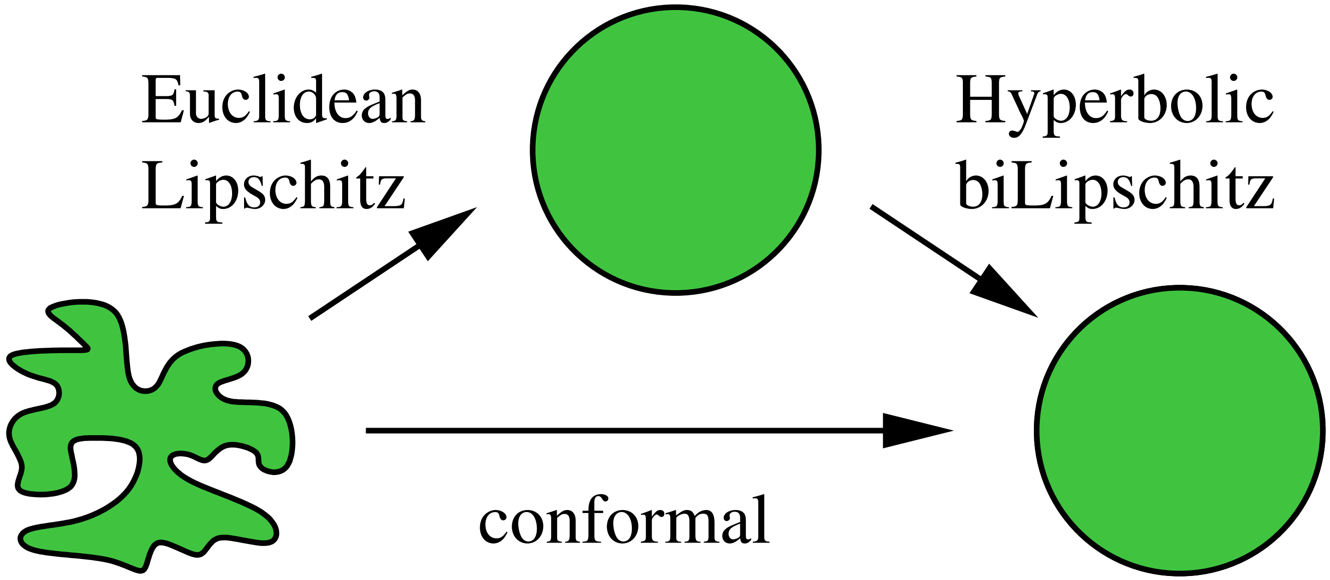
Dennis Sullivan proved this assuming invariance under a group of Möbius transformations. This was used by William Thurston to prove certain 3-manifolds have a hyperbolic metric.

Epstein and Marden extended to general simply connected Ω. $K \approx 85$.

Best value unknown, but $2.1 < K < 7.82$.

Application: factorization Riemann map $f = h \circ g$ where

- $g : \Omega \to \mathbb{D}$ is Lipschitz in Euclidean path metrics,
- $h : \mathbb{D} \to \mathbb{D}$ is biLipschitz in hyperbolic metric



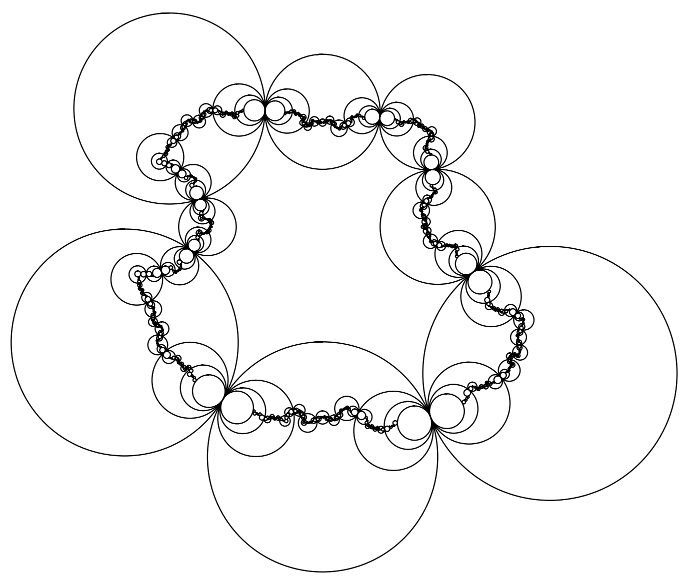
Cor: Any simply connected domain can be mapped 1-1, onto a disk D by a contraction for the internal path metric.

Application: Bowen's dichotomy

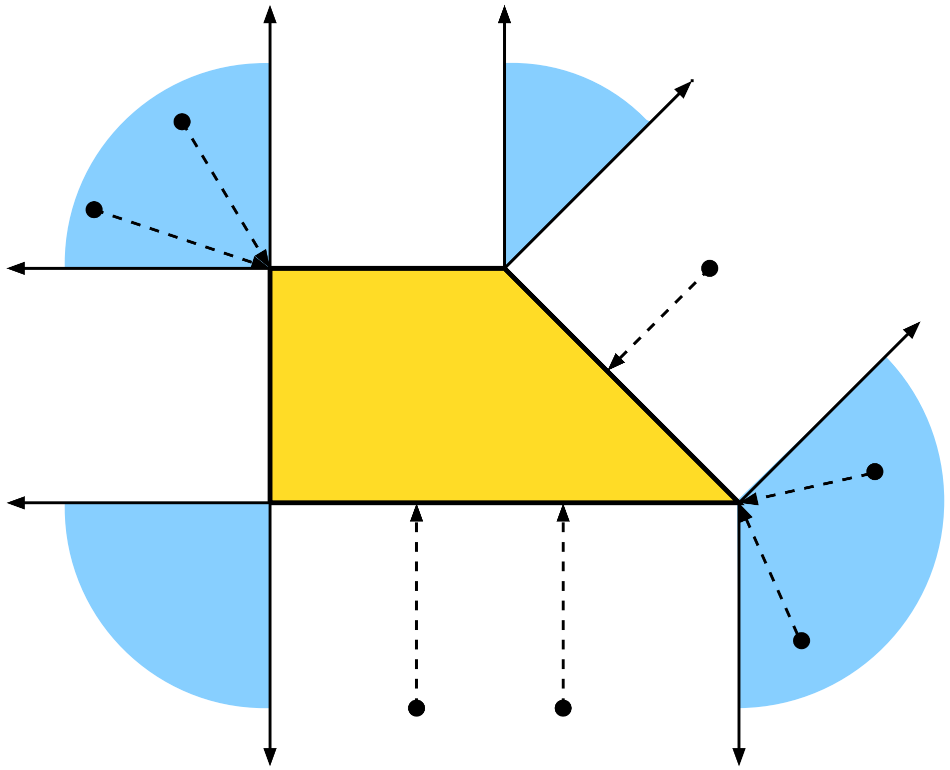
Thm: any quasi-Fuchsian limit set is either a circle or has dimension > 1.

G = Möbius group, Γ = limit set, $R = (\mathbb{C} \setminus \Gamma)/G$ = Riemann surface

- Rufus Bowen (79): true if R is compact
- Dennis Sullivan (84): true if R has finite area
- Astala-Zinsmeister (90): false if Brownian motion transcient on R
- CB (01): true iff Brownian motion recurrent on R



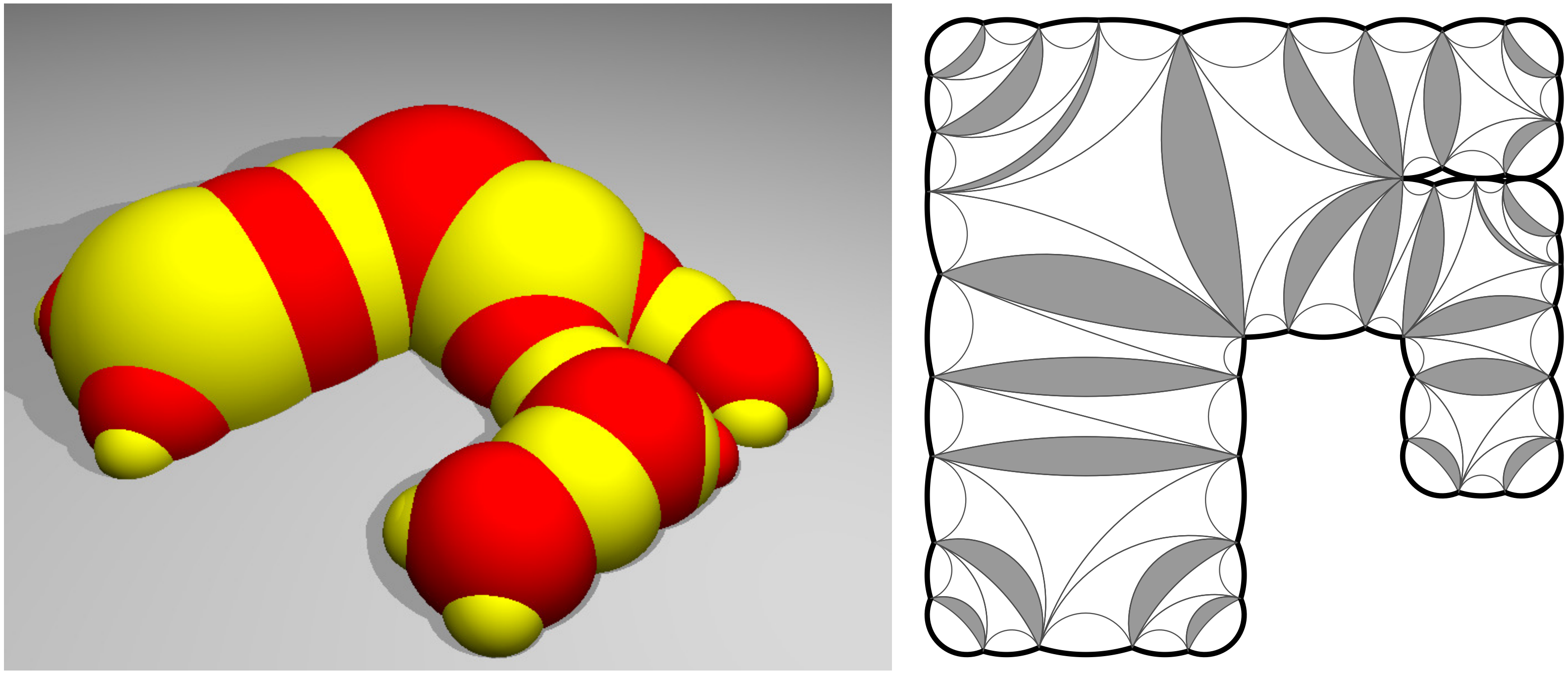
Application: Angle scaling



For nearest point retraction in Euclidean space,
lines map to face points, sectors map to corners.

Application: Angle scaling

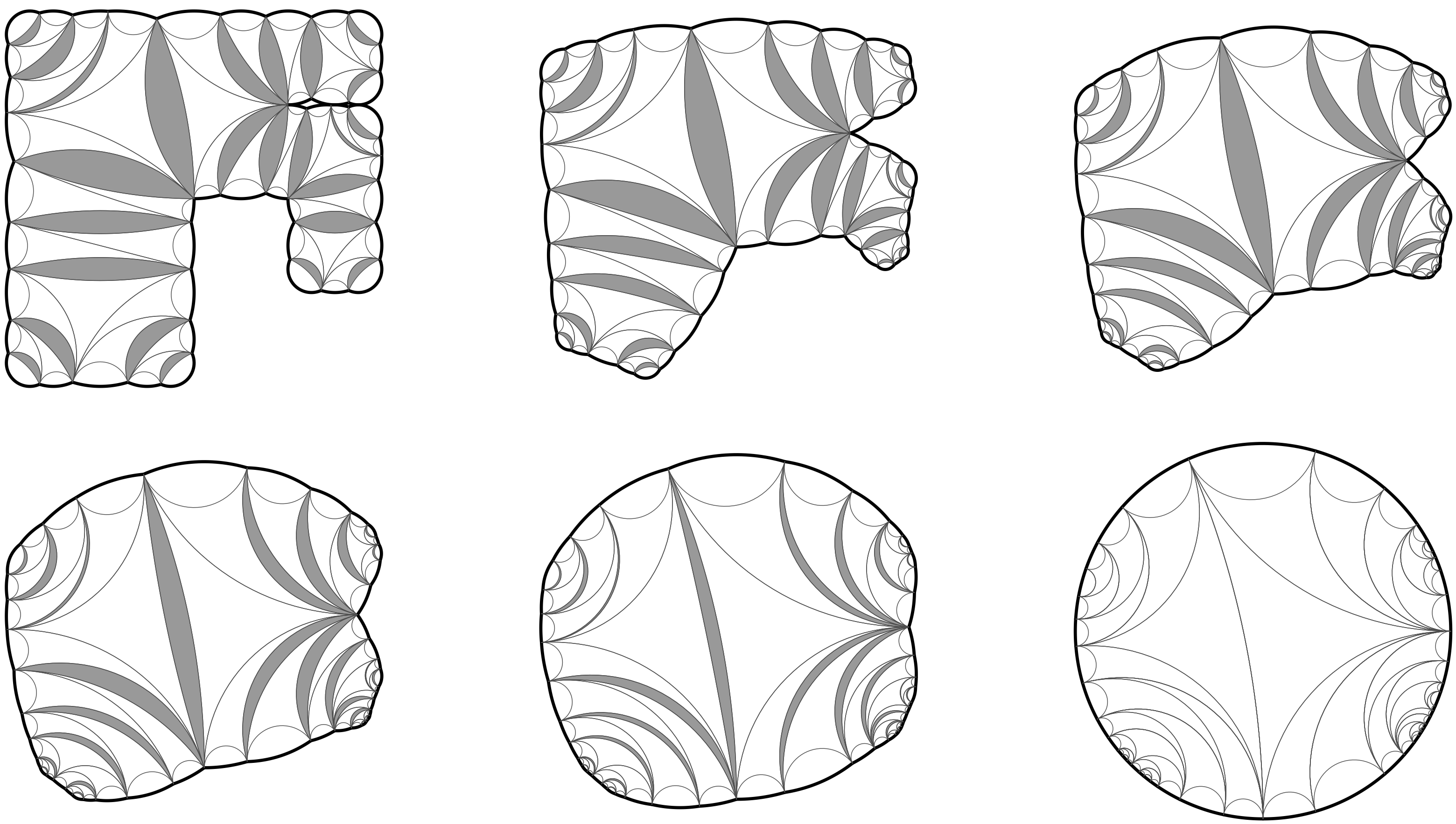
Crescents in base can map to folding geodesics on surface.



Gray collapses to bending lines, “width = angle”.

White maps isometrically to dome.

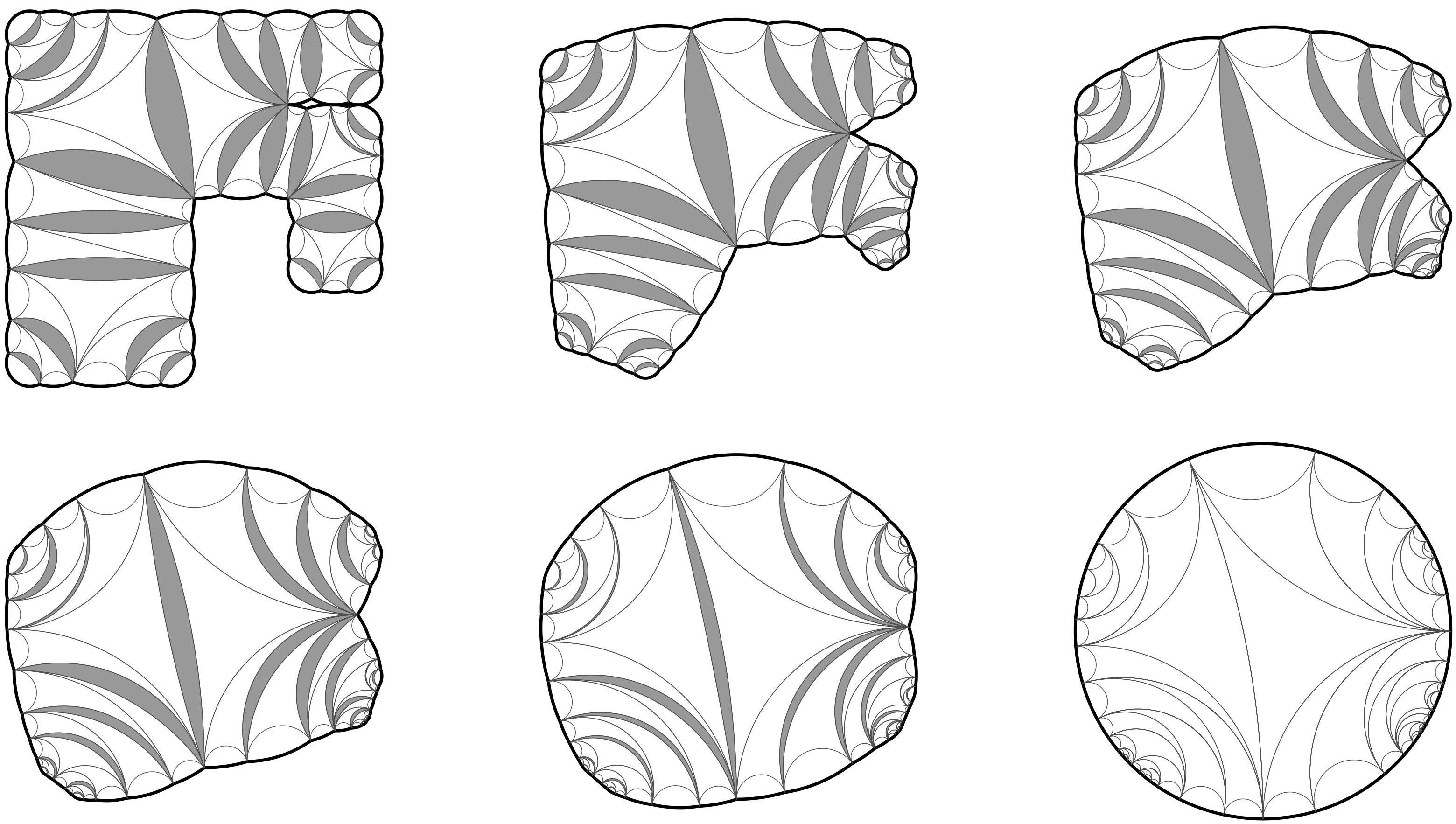
Discrete Riemann map: collapses crescents (gray), Möbius elsewhere (white).



Angle scaling family - crescent angles decrease

"Morphs" region to disk.

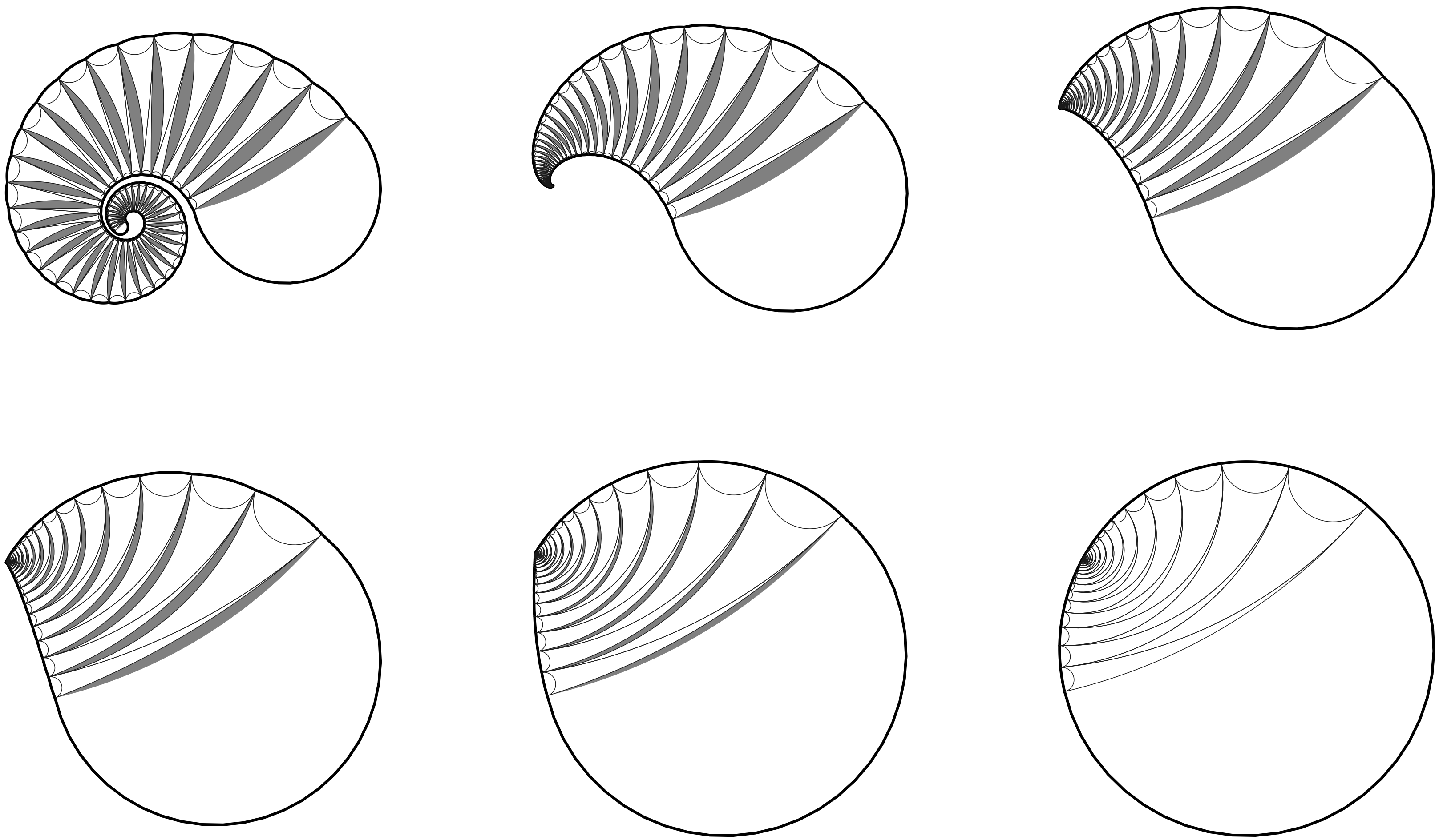
Reduces solving Beltrami equation to case of small dilatations.



Riemann map approximated by cutting into simple pieces and rearranging.

Gray pieces collapse orthogonally.

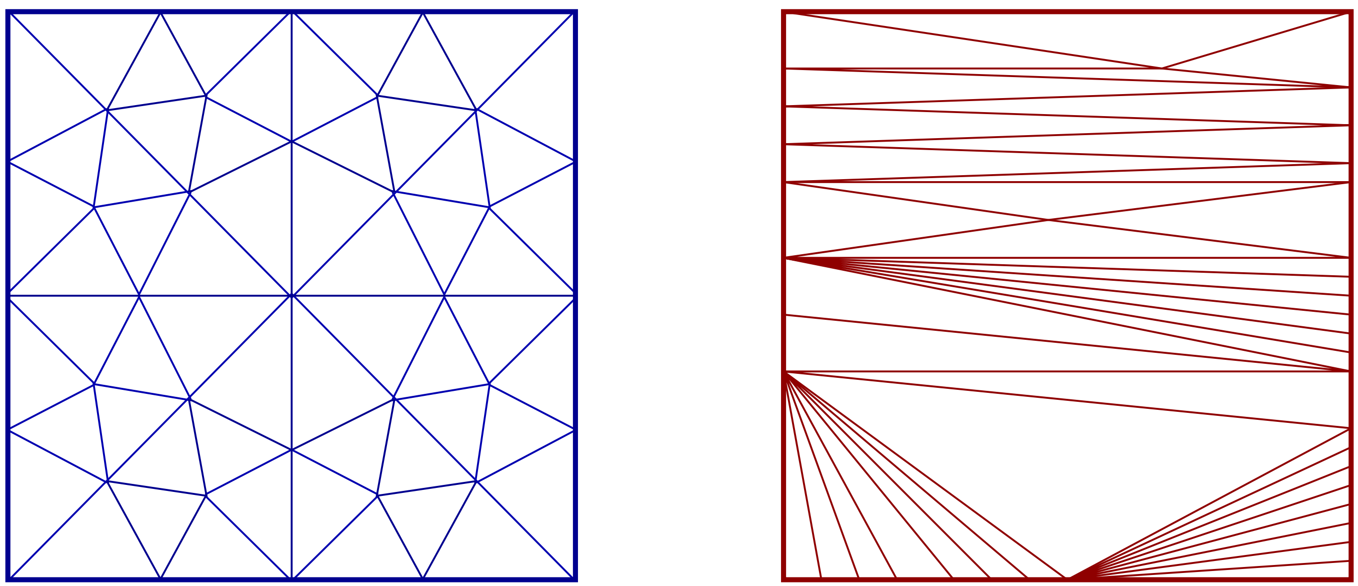
White pieces map by Möbius transformations.



Optimal triangulation of polygons

Cut a polygon into triangles so that each is as "round" as possible.

More precisely, minimize the maximum angle used.



Useful for finite element methods and other applications.

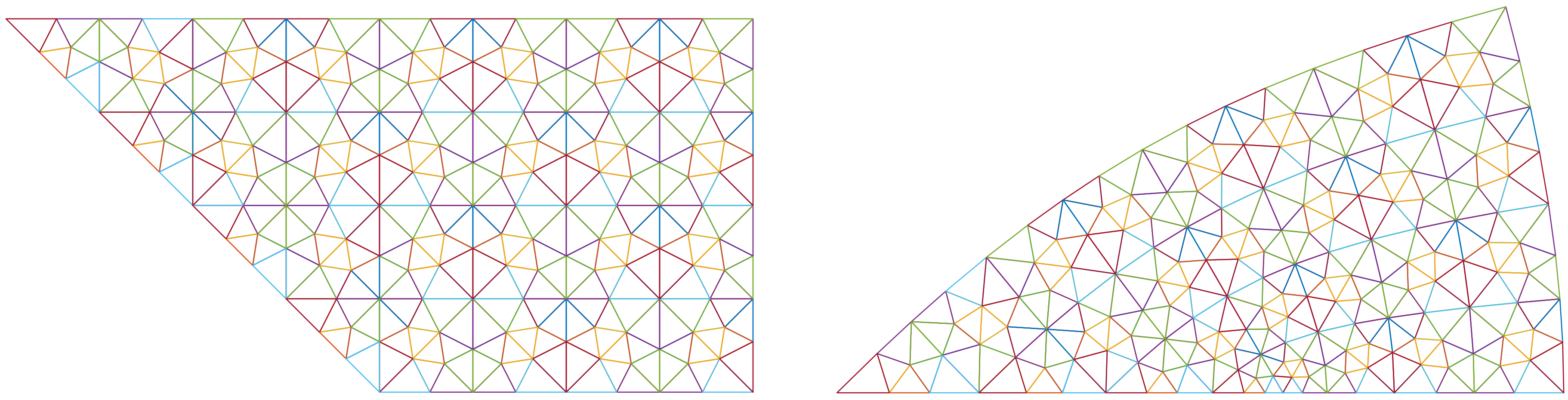
Some questions:

- What angle bound works for every polygon?
- Given P, can we compute the optimal angle bound for P?
- Is this bound attained or only approximated?
- Is this related to conformal and QC maps?

Some questions:

- What angle bound works for every polygon? **90 degrees**
- Given P, can we compute the optimal angle bound for P? **Yes**
- Is this bound attained or only approximated? **Usually attained**
- Is this related to conformal and QC maps? **Of course**

More about all this in the next lecture.



THANKS

QUESTIONS?

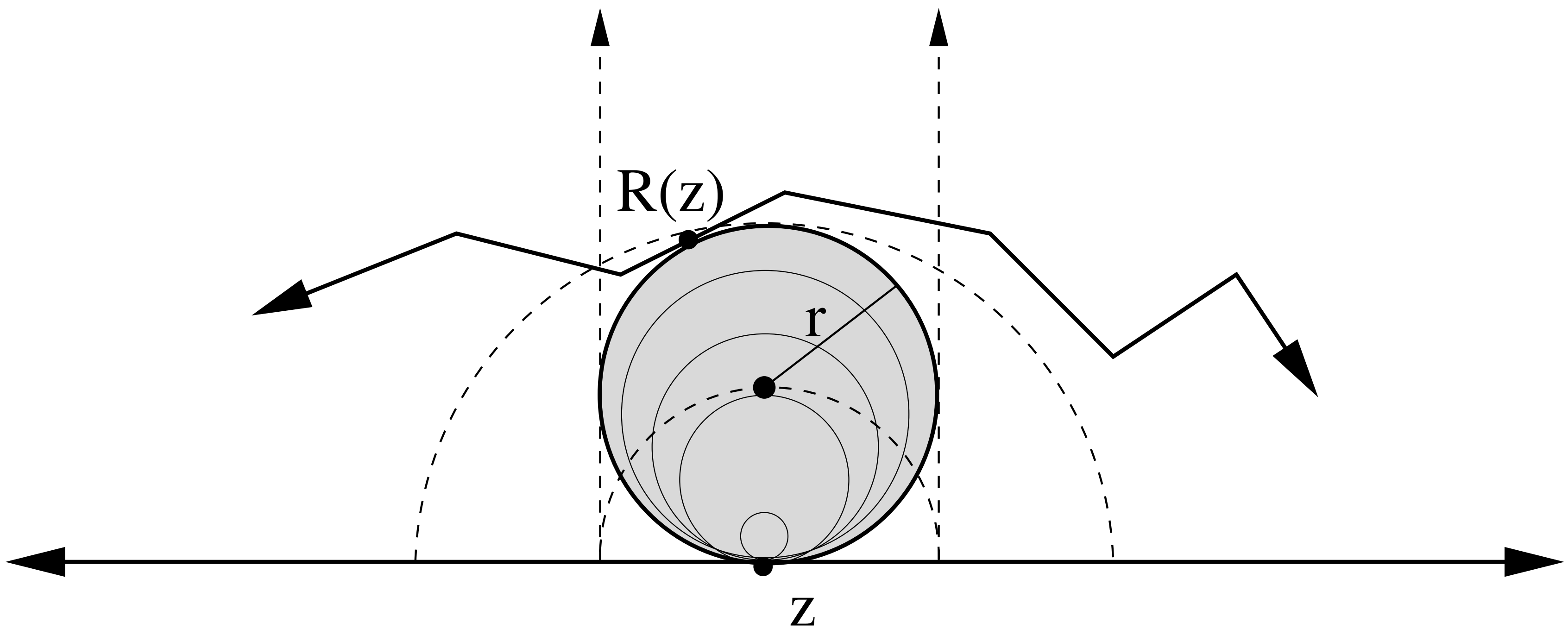
Sketch of proof that R is quasi-isometry

One direction: R is Lipschitz.

Other direction: R^{-1} is Lipschitz at distances ≥ 1.

Fact 1: If $z \in \Omega$, $\infty \notin \Omega$,

$$r \simeq \mathrm{dist}(z, \partial\Omega) \simeq \mathrm{dist}(R(z), \mathbb{R}^2) \simeq |z - R(z)|.$$



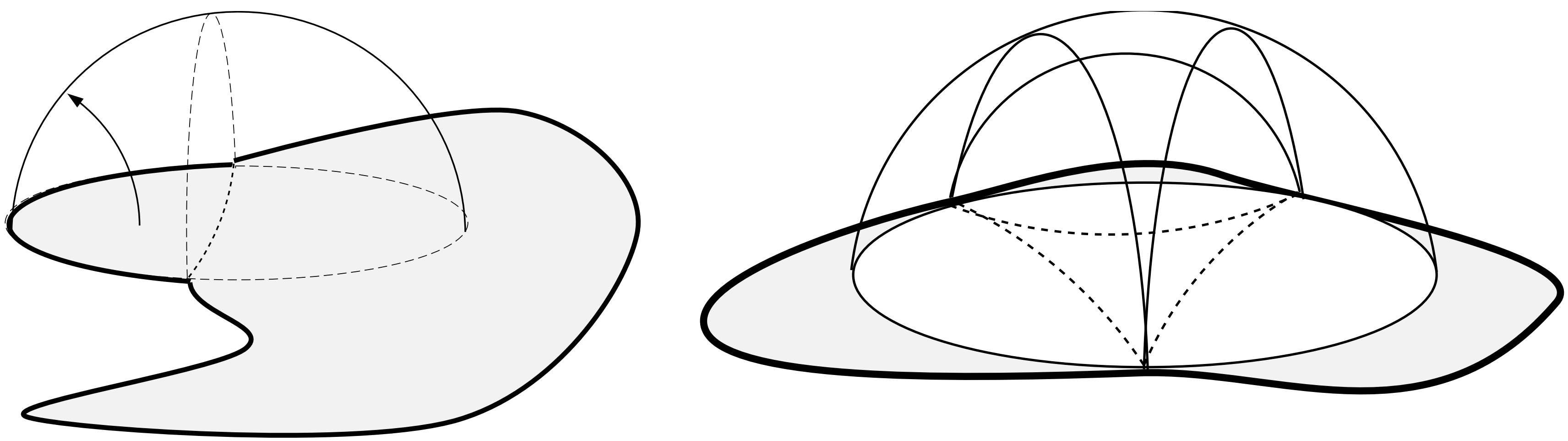
Fact 2: R is Lipschitz.

- Ω simply connected $\Rightarrow d\rho \simeq |dz|/\text{dist}(z, \partial\Omega)$.

- $z \in D \subset \Omega$ and $R(z) \in \text{Dome(D)} \Rightarrow z$ in hyperbolic convex hull of $\partial\Omega \cap \partial D$ in D.

$$\Rightarrow \text{dist}(z, \partial\Omega)/\sqrt{2} \le \text{dist}(z, \partial D) \le \text{dist}(z, \partial\Omega)$$

$$\Rightarrow \quad \rho_\Omega(z) \simeq \rho_D(z) = \rho_{\text{Dome}}(R(z)).$$

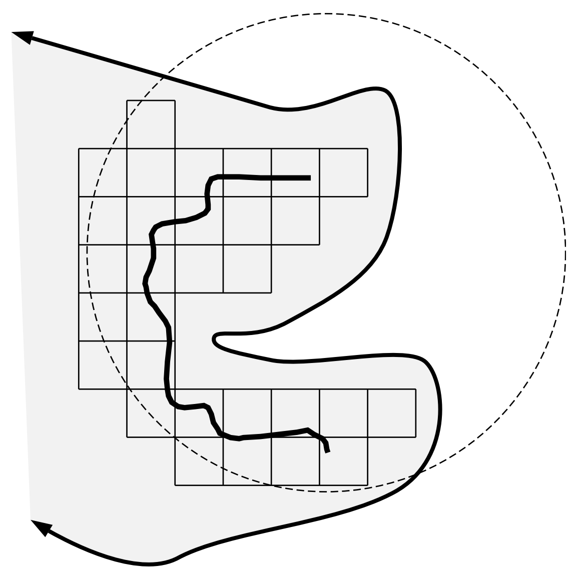


Fact 3: $\rho_S(R(z), R(w)) \le 1 \Rightarrow \rho_\Omega(z,w) \le C$.

Suppose $\mathrm{dist}(R(z), \mathbb{R}^2) = r$.

Suppose γ is geodesic on dome from $R(z)$ to $R(w)$.

$$\Rightarrow \qquad \mathrm{dist}(\gamma, \mathbb{R}^2) \simeq r$$
$$\Rightarrow \qquad \mathrm{dist}(R^{-1}(\gamma), \partial\Omega) \simeq r, \qquad R^{-1}(\gamma) \subset D(z, Cr)$$
$$\Rightarrow \qquad \rho_\Omega(z,w) \le C$$



Moreover, $g = \iota \circ \sigma : \Omega \to \mathbb{D}$ is locally Euclidean Lipschitz.

$$|g'(z)| \simeq \frac{\mathrm{dist}(g(z), \partial\mathbb{D})}{\mathrm{dist}(z, \partial\Omega)}.$$

Use Fact 1

$$\begin{aligned}
\mathrm{dist}(z, \partial\Omega) &\simeq \mathrm{dist}(R(z), \mathbb{R}^2) \\
&\simeq \exp(-\rho_{\mathbb{R}^3_+}(R(z), z_0)) \\
&\gtrsim \exp(-\rho_S(R(z), z_0)) \\
&= \exp(-\rho_D(g(z), 0)) \\
&\simeq \mathrm{dist}(g(z), \partial D)
\end{aligned}$$